Some geometric comments on a "quantum" theorem of Kirby and Melvin

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Chapter 1

Introduction

This work is a modified version of a work I presented during my exams for the obtention of the DEA of Pure Mathematics in Toulouse in June 1997.
It contains a detailed presentation of the invariants of knots and 3-manifolds that intervene in the proof of theorem 6.3 of [**KM1**], and their relation to Spin structures. Since the proof of this theorem contains all the material that is necessary to read the other results of [**KM1**] concerning homology spheres (extended by Ohtsuki with the help of its polynomial invariants of rational homology spheres, cf. [**Oh**]), I decided to restrict my attention to this proof only in this version.

The reader is supposed to know the construction of the quantum $sl(2, \mathbf{C})$ Reshetikhin-Turaev invariants (RT-invariants), e.g. their treatment by Kirby and Melvin in [**KM1**] and Kirby's calculus for framed links in S^3 [**K**] for the reading of section 6.The reader will find in [**Saw**] an extensive bibliography on the subject ... up to 1995. Moreover, some definitions and classical results from fiber bundle and characteritic classes theories are used from the beginning; see [**MS**] for more details.
The rest is a matter of basic algebraic topology and low dimensional geometry, that the reader will find in [**H**] and [**GH**] for example.

The exhaustive treatment I give here to *Pin*-structures and characteristic surfaces in 4-manifolds is due to their constant appearance as a topological background for the splitting formulas of the RT-invariants $(\tau_r)_{r\in\mathbf{N}}$ into refined invariants that are in [**KM1**]. All involve *Pin*-structures, or the formalism of even and characteristic links in a 3-manifold.In particular, τ_3 appears as a (kind of) "trivial" factor of τ_r for r odd, whose complement has a representation-theoretic interpretation by quantum groups as shown in [**BHMV**].
This is also shown in the work of Justin Roberts [**Rob**] where the refined Spin invariants are presented in the framework of skein theory and furnish a global relation with refinements of the Turaev-Viro invariants and quantum invariants of compact oriented 4-manifolds of Broda (which are in fact classical invariants). In particular, the cup Product of 2-cohomology classes,the signature and Pontrjagin squares of such manifolds may be determined with the help of these refined invariants. Moreover, the Spin-$TQFT$ presented in [**BlM**] appears to bring *new* informations both on the topological viewpoint and on the categorical viewpoint.

I would like to thank Claude Hayat-Legrand for guiding me enthousiastically and with a constant disponibility into geometry, and for having pointing out to me the difference between intuition and proof.

Chapter 2

Embeddings of non orientable surfaces in closed 3-manifolds

2.1 Geometric background

We will denote by $(\tau_r)_{r\in \mathbf{N}}$ the family of $U_q(sl(2,\ \mathbf{C}))$-invariants of closed orientable 3-manifolds, defined by Reshetikhin and Turaev in [**RT**], where each element τ_r corresponds to the specialization $q = exp(2i\pi/r)$ of the deformation parameter of the quantum group $U_q(sl(2,\ \mathbf{C}))$.
Let M be a closed orientable 3-manifold. Recall (see [**KM1**, 8.9] and [**BHMV**, remark 1.17]) that for *odd* r, there is a splitting formula:

$$\tau_r(M) = \{ \begin{array}{ll} \tau_3(M)\ \tau'_r(M) & if\ r \equiv 3 \pmod 4 \\ \overline{\tau_3(M)}\ \tau'_r(M) & if\ r \equiv 1 \pmod 4 \end{array},$$

where τ'_r is the r^{th} $U_q(SO(3))$-invariant of closed oriented 3-manifolds obtained by the same procedure as in [**RT**]. This formula has been generalized by Kohno and Takata in [**KoTa**, 4.2.3] into a splitting formula for the $U_q(sl(n,\ \mathbf{C}))$-invariants, which factor into quantum $PSU(n)$-invariants and an invariant obtained in [**MOO**] that generalizes τ_3.
Besides these results, we have the following theorem of Kirby and Melvin [**KM1**, p. 522] (from now on denoted by KM1):

Theorem KM1: *We have $\tau_3(M) = 0$ if and only if one of the following conditions is verified :*

i) There exists two Spin structures for M whose μ-invariants are distincts $\pmod 4$;

ii) There exists an embedded non orientable closed surface F in M with odd Euler characteristic;

iii) There exists $\alpha \in H^1(M,\ \mathbf{Z}_2)$ such that $\alpha \cup \alpha \cup \alpha \neq 0 \pmod 2$;

Otherwise we have $\tau_3(M) = \sqrt{2}^{b(M)} c^{\beta(M)}$, where $b(M) = |H^1(M,\ \mathbf{Z}_2)|$, and $c = exp(-(i\pi)/4)$ and $\beta(M)$ is the Brown invariant of M (see Appendix).

The aim of this chapter is to discuss the existence of a closed non orientable surface in a given orientable 3-manifold, such as involved in condition $ii)$ above, and to give a somewhat elementary proof of the equivalence $ii) \Longleftrightarrow iii)$.

Let us denote by $\mathbf{RP}^3$ the real 3-dimensional projective space. We have:

$$iii) \Longleftrightarrow ii) \Longleftrightarrow iv) \text{ There exists a degree one map } f: \ M \to \mathbf{RP}^3.$$

The abstract point of view of [**BW**], p. 88, which uses Thom's L-equivalence, is sufficient to prove $iii) \Longleftrightarrow ii)$, but we add a geometric counterpart learned from [**HWZ1**].

Sketch of proof:
$iii) \Longleftrightarrow ii)$: Given an embedded surface F^2 in M representing a class $\alpha \in H^1(M^3, \ \mathbf{Z}_2)$, the reduction $\pmod 2$ of its Euler characteristic $\chi(F)$ is the non oriented cobordism class of F^2 (recall that the non oriented cobordism group in dimension 2 is generated by $\mathbf{RP^2}$) .
Consider the element $\phi \in H^2(M, \ \mathbf{Z}_2)$ which sends the homology class of F in M to its non oriented cobordism class. A characteristic class calculus in [**BW**, p. 88], proves that ϕ is a homomorphism since all the Stiefel Whitney numbers of M^3 vanish; moreover it shows that (denoting by D_P the Poincar duality isomorphism) we have $Ker(\phi) = \{[F] \in H_2(M, \mathbf{Z}_2) \ s.t. \ D_P([F])^3 = 0\}$.
Then $\chi(F)$ is even ($\phi([F]) = 0$) if and only if $\alpha \cup \alpha \cup \alpha = 0$.

$ii) \Longleftrightarrow iv)$: Suppose with the same datas as above that $\chi(F) \equiv 1 \pmod 2$; then we have $F = F' \# \mathbf{P^2} = \mathbf{F_0} \cup_{\partial \mathbf{F_0} = \partial \mathcal{M}} \mathcal{M}$ where F' is closed , the symbol $\mathcal{M}$ denotes a Möbius band and $F_0 = F' \setminus \overset{\circ}{B}{}^2$ (where $\overset{\circ}{B}$ is an open disk in F').
Take a regular neighborhood $\mathcal{N}(F)$ of F: it is a product over F_0 and it is twisted over $\mathcal{M}$. Considering the standard embedding of the projective plane $\mathbf{RP}^2 = B^2 \cup_{\partial B^2 = \partial \mathcal{M}} \mathcal{M}$ in $\mathbf{RP^3}$ and a regular neighborhood $\mathcal{N}(\mathbf{RP}^2)$ of it, we can construct a proper degree one map (d1m) :

$$f: \ \mathcal{N}(F) \to \mathcal{N}(\mathbf{R}P^2)$$

as follows. First send $\mathcal{N}(F)_{|\mathcal{M}}$ homeomorphically to the standard neighborhood $\mathcal{N}(\mathcal{M})$ of the Möbius band $\mathcal{M} \subset \mathbf{RP}^2 \subset \mathbf{RP}^3$. Since we can always find a d1m (that "pinchs some handles") from a closed surface F_1 to another one F_2 when the genus of F_1 is not inferior to that of F_2 [**Ed**], send $F_0 \times I$ onto $B^2 \times I \subset \mathcal{N}(\mathbf{RP}^2)$ by taking the cartesian product of a map from F_0 to B^2 induced by such a d1m (from $F \subset M$ to a sphere $S^2 \subset \mathbf{RP}^3$ with the identity on I.
This is a proper degree one map, in particular on the boundary. It may be extended to the exterior $M \setminus \overset{\circ}{\mathcal{N}}(F)$ of F by mapping a collar of $\partial \mathcal{N}(F)$ in M to the 3-Ball $B^3 = \mathbf{RP}^3 \setminus \overset{\circ}{\mathcal{N}}(\mathbf{RP}^2)$; this may be done by filling the holes of the collar. The rest is sent onto the center of B^3. The local behaviour of the degree of a map implies that this extended map f is still a degree one map, from M to $\mathbf{RP}^3$.

Conversally, suppose that an f as in $iv)$ exists: we can deform $f: \ M \to \mathbf{P}^3$ so as it is transverse to $\mathbf{P}^2 \subset \mathbf{P}^3$ and the surface $F = f^{-1}(\mathbf{P}^2)$ is connected. Note that $deg(f|\mathcal{N}(F)) = deg(f|\partial \mathcal{N}(F)) = 1$, and the boundary of $\mathcal{N}(\mathbf{P}^2) \subset \mathbf{P}^3$ is a 2-sphere.

Suppose that $\mathcal{N}(F) = F \times I$: then the degrees of the maps $f|F \times \{0\} : F \times \{0\} \to \partial\mathcal{N}(\mathbf{RP}^2)$ and $f|F \times \{1\} : F \times \{1\} \to \partial\mathcal{N}(\mathbf{RP}^2)$ have the same absolute values since they are sent onto S^2 as the two leaves of the non-connected oriented 2-covering of F. This implies that $deg(f|\partial\mathcal{N}(F)) \equiv 0 \pmod 2$. By contradiction with the degree of f, the surface F is non orientable.

Now, if $\chi(F) \equiv 0 \pmod 2$, F is a connected sum of a Klein Bottle Kl and an orientable closed surface F' and $\mathcal{N}(F)|F'$ is a product. Moreover $\mathcal{N}(F)|Kl$ is twisted over "the" reversing orientation curve of Kl: there is only one isotopy class β of such a curve in Kl, which runs twice around the S^1 longitude and bounds a Möbius band. Consider both generators α and β of $\pi_1(Kl)$, and their images (still denoted by α and β) in $H_1(Kl, \mathbf{Z}_2)$ when injected in $H_1(M, \mathbf{Z}_2)$. Move slightly F away from itself so that $F \cap F$ is α; notice that α can be remoted out of F since $\mathcal{N}(F)_{|\alpha}$ is a trivial I-bundle: then $\alpha \cdot F = 0$. But this is in contradiction with iii) (where $\alpha \in H^1(M, \mathbf{Z}_2)$ must be seen as the Poincar dual to the homology class of F in M) which is by the first step equivalent to ii). □

Remark: Suppose that there is an embedded surface $F \subset M^3$ with $D_P([F]) \neq 0 \in H^1(M,\ \mathbf{Z}_2)$: it does not separate in M and we can connect its different components without changing the homology class $[F] \in H_2(M,\ \mathbf{Z}_2)$. Furthermore we can glue a non orientable handle to F *inside* M, by thickening an arc, going through the complement of $F \subset M^3$, with a single intersection point with F. Again this operation does not alter the homology class $[F] \in H_2(M,\ \mathbf{Z}_2)$. It follows that any non zero class $\alpha \in H^1(M,\ \mathbf{Z}_2)$ may be dually represented by an embedded connected non orientable surface in M, and F is a connected sum of an odd (resp. even) number h of $\mathbf{RP}^2$ if and only if $D_P([F])^3 = \alpha^3 \neq 0$ (resp. $= 0$) by the proof of $iii) \Longleftrightarrow ii)$. Now recall that adding an orientable handle to F (which is always possible inside M) is the same as adding a non orientable one. Then we can embed the surface $F \sharp Kl$ in M, and h may be increased by two in the homology class of F.
Therefore the geometric representation problem of a fixed homology class in $H_2(M, \mathbf{Z}_2)$ reduces to the embedding problem of a connected non orientable surface in M, and it breaks into the two cases h even or h odd (with the above notations). Moreover, the determination of the minimum h for which $U_h \cong \mathbf{RP}^2 \sharp \dots \sharp \mathbf{RP}^2$ can be embedded in M within a fixed homology class arises naturally, as shown in the preceeding discussion. There is obviously a relative version of this problem, which relies heavily on the norm on the homology of a 3-manifold, as defined in [**S**] for instance.

Here is a simple result which illustrates our interest in this problem : recall [**H**] that an incompressible, connected, compact, properly embedded (or included in ∂M) surface in a 3-manifold is a surface that is neither :

- 1) a 2-sphere bounding an homotopy 3-cell;
- 2) nor a 2-cell F^2 with either $F^2 \subset \partial M$, or there is an embedded homotopy 3-cell X in M with $\partial X \subset F \cup \partial M$.
- 3) nor there exists a 2-cell $D^2 \subset M$ with $D \cap F = \partial D$ and ∂D not contractible in F.

Lemma: *let M be a 3 dimensional orientable manifold with an embedded closed surface whose Euler characteristic is odd. Take such an F with minimal genus. Then F is incompressible. If $F = \mathbf{P}^2$, then $\pi_1(M)$ is a free product $\pi_1(M) \cong \mathbf{Z}_2 \star G$ (where G could be trivial).*

Proof: Suppose on the contrary that F is compressible, then a surgery on a compression 2-cell B^2 would either give a surface F' with $\chi(F') = \chi(F) + 2$ or two distinct surfaces F' and F'' with $\chi(F') + \chi(F'') = \chi(F) + 2$, which depends on the fact that ∂B^2 separates F or not. But in both cases it contradicts the hypothesis. The second claim is obtained by noticing that $\mathcal{N}_M(\mathbf{RP}^2) \cong S^2$. $\square$

Incompressible surfaces embedded in a 3-manifold are representative of its homotopic and *geometric* properties. For basic results about the two sided case and Haken manifolds, we refer the reader to [**H**] or [**Jaco**]. Our case is the one sided, lesser well-known, references being [**Rub**] (and [**BW**]). As exemples of the results you could find there, let's cite the followings:

1) Given two closed orientable 3-manifolds M_1 and M_2 with connected non orientable embedded surfaces $F_i \subset M_i,\ i = 1,\ 2$, one obtains an embedding $F' := F_1 \sharp F_2 \subset M_1 \sharp M_2$ in *standard* position by taking the connected sum of M_1 and M_2 at points on these surfaces; this operation is well defined up to homeomorphisms on the pair $(M_1 \sharp M_2, F')$. Conversally, if M_1 and M_2 are closed orientable 3-manifolds and if a non orientable connected surface $F \subset M_1 \sharp M_2$ $(:= M)$ is given, then there exists a surface $F' \subset M$ in standard position such that $F' \cong F$ and $[F'] = [F] \in H_2(M, \mathbf{Z}_2)$ (see [**BW, 5.1**]). In particular it follows that, with the notations of the preceeding remark, the surface U_h embeds in $M_1 \sharp M_2$ if and only if there exists non negative integers $h_1,\ h_2$ with $h = h_1 + h_2$ and U_{h_i} embeds in M_i. If F and F' were orientable surfaces, this result turns out to be true but still with $\mathbf{Z}_2$ coefficients.
   Consequently, one can reduce the embedding problem of one-sided surfaces in a 3-manifold M to the case when M is irreducible, since the case of lens spaces and manifolds of the type $M^2 \times S^1$ (M^2 being a closed orientable surface) is completely solved in [**BW, 4.8**].

2) Let F be a non orientable surface in a closed orientable manifold M. The image of $\pi_1(F)$ into $\pi_1(M)$ under the inclusion homomorphism is its own normalizer (see [**BW, 4.3**]); in particular it contains the center of $\pi_1(M)$.Recall that in the two sided case $\pi_1(F)$ injects into $\pi_1(M)$.

3) Given any class $0 \neq \alpha \in H_2(M,\ \mathbf{Z}_2)$ there is a one sided Heegaard splitting (OSHS) $(M,\ K)$ with $[K] = \alpha$, which means that the closed non orientable embedded surface $K \subset M$ homologically represents α and $M - K$ is an open handlebody. /break Obviously two OSHS $(M,\ K)$ and $(M,\ K')$ are distinct as soon as $[K] \neq [K'] \in H_2(M,\ \mathbf{Z}_2)$.
   Conversally, let $(M,\ K)\sharp(S^3,\ L)$ be the connected sum of a $OSHS$ $(M,\ K)$ of M with the standard Heegard splitting (S^3, L) of genus 1 of S^3, at points on K and L (see 1)). Then we have:
   Suppose that $(M,\ K)$ and $(M,\ K')$ are $OSHS$ with $[K] = [K']$. Then they are

stably equivalent, i.e. $(M,\ K)\sharp\ n(S^3,\ L)$ is equivalent to $(M,\ K')\sharp\ m(S^3,\ L)$ [**Rub**, p.196].

4) Let's say that $(M,\ F)$ is an *incompressible* OSHS (denoted by IOSHS) if F is incompressible in M. Suppose that M is an irreducible closed orientable 3-manifold with an IOSHS: then the map $i_* : \pi_1(F) \to \pi_1(M)$ induced by the embedding of F in M is onto, and there is no incompressible orientable surface in M which is disjoint from F (see [**Rub**, p.189]).

5) Suppose that M is irreducible and not sufficiently large; then there is an IOSHS associated with any non zero class in $H_2(M, \mathbf{Z}_2)$.

6) The preceeding equivalence relation may be strongered by requiring that $(M,\ K)$ is equivalent to $(M,\ K')$ if and only if K is isotopic to K'. Then it is shown in [**Rub**,p.193] that in lens spaces written as $L(2k,\ q)$, two embedded incompressible surfaces are isotopic: so two incompressible $OSHS$ are isotopic;

7) Let $f:\ M' \to M$ be an odd degree map between closed orientable 3-manifolds. If $(M,\ K)$ is a $OSHS$ of M, there is a $OSHS$ $(M',\ K')$ of M' and a map f' homotopic to f such that ${f'}^{-1}(K) = K'$. In case f is a degree one map, one can chose f' and K' so that $f' : M' \setminus int\mathcal{N}(K') \to M \setminus int\mathcal{N}(K)$ is a standard mapping between handlebodies and $f^{-1}(\mathcal{N}(K)) = \mathcal{N}(K')$, where the symbol $\mathcal{N}$ denotes a tubular neighborhood. Recall that a standard mapping between two handlebodies H and K is a proper degree one map such that there exists an embedded disk in H that separates it into two handlebodies H' and H'', with H' being sent onto a disk in ∂K and with $f_{|\bar{H}''}$ being a homeomorphism onto K.

8) Suppose that M has an incompressible $OSHS$; Then M is irreducible. Indeed, take an embedded 2-sphere S in M transverse to a one sided Heegaard splitting surface K of M; since $M \setminus K$ is irreducible, an induction on the number of curves of $S \cap K$ shows that S bounds a 3-cell.

Let M be a closed orientable (irreducible) 3-manifold; properties $4),\ 5)$ suggest that OSHS are useful to study M when it is *small*, and properties $6),\ 7)$ suggest that considering degree one maps from M onto lens spaces might help to find obstructions to the existence of IOSHS in M. Now, KM1 can be considered as a first step in the effective recognition of such obstructions, using quantum invariants of closed orientable 3-manifolds.

2.2 Let's see a little bit further

Let us denote by $(C_q)_{q\in\mathbf{Z}}$ a cellular decomposition of the closed orientable 3-dimensional manifold M; recall the definition of the n-Bockstein:

$$B_n = \mu_n \circ \beta \circ j_n :\ H^1(M,\ \mathbf{Z}_n) \to H^2(M,\ \mathbf{Z}_n)$$

given by the following composition of group homomorphisms:

$$H^1(M,\ \mathbf{Z}_n) \stackrel{j_n}{\to}\ H^1(M,\ \mathbf{Q}/\mathbf{Z})\ \stackrel{\beta}{\to}\ H^2(M,\ \mathbf{Z})\ \stackrel{\mu_n}{\to}\ H^2(M,\ \mathbf{Z}_n)$$

$$x + n\mathbf{Z} \mapsto x/n + \mathbf{Z} \mapsto B(x/n) \mapsto\ \ B(x/n) \mod n\mathbf{Z}$$

where B is the Bockstein operator of the short exact sequence of abelian groups:

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$$

defined by:

$$H^q(M,\ \mathbf{Q}/\mathbf{Z}) \to H^{q+1}(M,\ \mathbf{Z}) :\ [\phi] \to [\partial\hat{\phi}]$$

where $\phi \in Hom(C_q,\ \mathbf{Q}/\mathbf{Z})$ and $\hat{\phi}$ is the lift of ϕ in $Hom(C_q,\ \mathbf{Q})$. One can easily see that $B_2(\hat{\alpha}) = \hat{\alpha} \cup \hat{\alpha}$.

Recall the definition of the (non degenerate) linking pairing:

$$TorH_q(M,\ \mathbf{Z}) \times TorH_{n-q-1}(M,\ \mathbf{Z}) \to \mathbf{Q}/Z$$

$$(\alpha,\ \beta) \mapsto \alpha \odot \beta = 1/a\ (c \cdot \beta)$$

where $c \in C_{q+1}$, $\partial c = a\ \alpha'$ (where α' is a cycle that represents α in $TorH_q(M,\ \mathbf{Z})$) and $\cdot$ is the intersection form:

$$H_q(M, \mathbf{Z}) \otimes H_{n-q}(M, \mathbf{Z}) \to \mathbf{Z}$$

We then have the following relation between the cup product and the linking pairing:

Theorem[**SZ**, §14]: *Given an n-dimensional orientable manifold M, and two classes $\alpha \in TorH_q(M,\ \mathbf{Z})$, $\beta \in TorH_{n-q-1}(M,\ \mathbf{Z})$, $k \neq 0,\ 1$ such that $k\alpha = k\beta = 0$, there exists $\alpha' \in H^{n-q-1}(M,\ \mathbf{Z}_k)$ and $\beta' \in H^q(M,\ \mathbf{Z}_k)$ such that the Poincar duals of $B \circ j_k(\alpha')$ and $B \circ j_k(\beta')$ are respectively equal to α and β. We then have the relation:*

$$\alpha \odot \beta = j_k \circ < \beta' \cup \beta_k(\alpha'),\ [M] >$$

We can then deduce the following formula for the condition iii) in KM1:

$$1/2\ \langle \hat{\alpha} \cup \hat{\alpha} \cup \hat{\alpha},\ [M] \rangle = \alpha \odot \alpha$$

Generalizing this remark we have (denoting by $[r/s]$ the class in $\mathbf{Q}/\mathbf{Z}$ of the rational number r/s):

Theorem [HWZ1] *Let M be a closed orientable 3-dimensional manifold.*

- 1) *If there exists $\alpha \in H^1(M,\ \mathbf{Z})$ of order n¿1 with $\alpha \odot \alpha = [r/n]$ where r is prime to n, then there exists a degree one map*

$$f :\ M \to L(n,\ s)$$

where $rs \equiv 1 \mod n$.

- 2) *If there exists a degree one map* $f: M \to L(n, m)$ *we can find* $\alpha \in TorH_q(M, \mathbf{Z})$ *such that:*
$$\alpha \odot \alpha = [m^{-1}/n]$$
where m^{-1} *denotes an* $m' \in \mathbf{Z}$ *verifying* $m'm \equiv 1 \pmod n$*,* n *is the order of* α *and* α *generates a direct summand of* $H^1(M, \mathbf{Z})$*.*

The main feature of this work was to understand the topological background of KM1 in view to answer the question: is it possible to generalize theorem KM1 by detecting some geometric properties (as discussed in 2.1) of 3-manifolds with degree one maps onto lens spaces, using quantum invariants and the above characterizations?
Beware that we are not interested only in homotopy invariants such as the linking pairing: we know for example that the invariant $Z_N(M, q)$ of a closed orientable 3-manifold M constructed in [**MOO**], which depends on the choice of a N^{th}-root of unity q, is a homotopy invariant. Moreover, it is equal to zero if and only if there exists an x in $H_1(M, \mathbf{Z})$ of order 2^m with $x \odot x = c/2^m$, where $N = 2^m b$, with odd integers b and c. We would like to go further than these results, keeping track of geometric properties that are not homotopy invariant: for instance, could we distinguish the lens spaces $L(7, 1)$ and $L(7, 2)$, which are already different from the point of view of the existence of degree one maps onto them ?

Following [**HWZ2**], consider now the partial order relation between 3-manifolds M and N induced by the existence of a degree one map:

$$f: M \to N$$

where M is non homeomorphic to N. Then we say that $M > N$; M is minimal if the existence of such an f towards N implies that $N \approx S^3$. One can effectively list the minimal Seifert fibered spaces, particularly minimal lens spaces [**HWZ2**], and $\mathbf{P}^3$ is one of them. Could we explain this minimality property in a geometric way, and detect it with quantum invariants ?
Notice that this relation between 3 manifolds dominates lots of known geometric and algebraic invariants: e. g the number of disjoints non parallel embedded incompressible surfaces (as shown in [**W2**]), the order and the rank of the homology and of the fundamental group, or the simplicial volume of Gromov.
In view to these remarks the condition $\tau_3(M) = 0$ gives an explicit lower bound for all of these invariants of M by calculating those of $\mathbf{RP}^3$.

We will now turn to Spin structures which will naturally introduce geometric methods to catch the linking pairing.

Chapter 3

Spin and Pin structures

In this section we will develop some caracterizations of Spin structures (and their Pin generalization) on low dimensional manifolds, and their algebraic formulation. We start with very general definitions from homotopy and obstruction theories, then we slide towards a more geometric viewpoint.

We will often refer to arbitrary CW-decompositions of manifolds without giving any precisions, since the structures we define here do not depend on their explicit choices. All 4-manifolds we consider are smooth.

3.1 Definitions

A principal $SO(n)$-bundle $\zeta \xrightarrow{\pi} X$ on a CW-complex X is a *Spin bundle* if its total space $E(\zeta)$ can be 2-covered (denoting the 2-sheeted covering space by $\widetilde{E(\zeta)}$) with a Spin(n)-action, commuting with the action of $SO(n)$ on $E(\zeta)$; in other words, the following diagram is required to commute:

$$\begin{tikzcd}
\widetilde{E(\zeta)} \times Spin(n) \arrow[r] \arrow[d, "p \quad \downarrow\lambda"] & \widetilde{E(\zeta)} \arrow[d, "p"] \arrow[dr, "\pi'"] & \\
E(\zeta) \times SO(n) \arrow[r] & E(\zeta) \arrow[r, "\pi"] & X
\end{tikzcd}$$

where $p: \widetilde{E(\zeta)} \to E(\zeta)$ verifies:

$$p(ag) = p(a)\lambda(g) \ \forall \ g \in Spin(n), \forall a \in \widetilde{E(\zeta)}$$

and λ is *the* non trivial $\mathbf{Z}_2$-covering of $SO(n)$ by the group $Spin(n)$ (see Appendix B).

This double covering $\tilde{\zeta}$ of ζ is consequently asked to be non trivial onto the fibers of X, and conversally, every double covering of ζ which is non trivial onto the fibers arises as a Spin bundle. Indeed, using the previous notations, given λ and p we can set $\pi' = \pi \circ p$ and one can extend the action of $SO(n)$ on $E(\zeta)$ to an action of $Spin(n)$ (which is 1-connected) on $\widetilde{E(\zeta)}$ using the relation ($\star$) above. Defining a *Spin structure* on ζ as the choice of a 2-covering such as $\tilde{\zeta}$ above, we have therefore a bijective (non canonical) correspondance:

$$\{Spin\ structures\ on\ \zeta\} \longleftrightarrow \{\sigma \in H^1(E(\zeta),\ \mathbf{Z}_2) \text{ s. t. } \langle \sigma_{|SO(n)} \rangle = H^1(SO(n),\ \mathbf{Z}_2)\}$$

Notice that this definition is parallel to that of the orientation of an $O(n)$-principal bundle; but here we don't reduce the structural group $O(n)$ to $SO(n)$, we rather $\mathbf{Z}_2$-cover $SO(n)$ by the simply connected Lie group $Spin(n)$. Since a 1-connected Lie group G is contractible as soon as $\pi_3(G) = 0$, and $\pi_2(G)$ is always 0 (see [**B-TD**]), a Spin structure on a fibre bundle looks like a stiffening of its structural group in a way that removes most of its homotopy.

Now, the fibration:

$$SO(n) \xrightarrow{i} E(\zeta) \xrightarrow{\pi} X$$

induces the following (Leray-Serre) long exact sequence of the fibration [**W**]:

$$0 \to H^1(X,\ \mathbf{Z}_2) \xrightarrow{\pi^\star} H^1(E(\zeta),\ \mathbf{Z}_2) \xrightarrow{i^\star} H^1(SO(n),\ \mathbf{Z}_2) \xrightarrow{\delta} H^2(X,\ \mathbf{Z}_2)$$

The class $\delta(1)$ may easily be identified with the second Stiefel Whitney class $\omega_2(\zeta)$ of the oriented bundle ζ. Indeed this definition of ω_2 is natural (by naturality of the sequence !), that is $\omega_2(f^\star\zeta) = f^\star\omega_2(\zeta)$, and it gives the non zero element in $H^2(BSO_n, \mathbf{Z}_2) \cong \mathbf{Z}_2$ when ζ is the universal oriented n-plane bundle γ_n over the classifying space BSO_n (which is the infinite Grassmanian of oriented n-planes in $\mathbf{R}^\infty$). The last claim follows from the exactness of the sequence and the fact that γ_n is contractible (see [**MS**]).

As for the existence of a Spin structure on ζ, we can therefore write:

$$\omega_2(\zeta) = 0 \Longleftrightarrow \textit{there exists } \sigma \in H^1(E(\zeta),\ \mathbf{Z}_2) \text{ such that } i^*(\sigma) = 1$$

and moreover, there is a (still non canonical) bijective correspondance between the set of Spin structures on ζ and $H^1(X,\ \mathbf{Z}_2)$; more precisely the group $H^1(X,\ \mathbf{Z}_2)$ acts in a simply transitive manner on the set of Spin structures.

A *Spin manifold* X is, by definition, a manifold for which the tangeant bundle TX is a Spin bundle.

Using the last equivalence, obstruction theory tells us that a Spin structure on ζ bijectively corresponds to a homotopy class of trivializations of ζ over the 2-skeleton of X (see 3.2 and [**St**]); note that when X is an orientable 2-manifold (which is always Spin, since $\omega_2(TX) \equiv \chi(TX) \pmod 2 \equiv 0 \pmod 2$), this condition must be turned into a homotopy class of trivializations of $TX \oplus \rho^k$ over the 2-skeleton of X, where ρ^k is a $\mathbf{R}^k$-trivial bundle that "stabilizes" TX. Otherwise, only the torus could be considered as a Spin surface from the point of view of obstruction theory.

Now we give the parallel results for *Pin structures* (see Appendix B). Let us denote by B the classifying space of a fiber bundle ζ over a CW-complex Y. Suppose that you want to deal with an $O(n)$-principal bundle ζ, arising for example as the restriction of a $SO(n)$ bundle.
Considering the two *non trivial* central extensions $p_\pm :\ Pin(n)^\pm \to O(n)$ of $O(n)$ by $\mathbf{Z}_2$ one may wish to get, as above, corresponding structures on ζ.

Then we say that $\zeta :\ E \to B$ has a $Pin(n)^\pm$ structure provided that there exists a $Pin(n)^\pm$-bundle $\zeta' :\ E' \to B$ that may fit into the above commutative diagram, where $\widetilde{E(\zeta)}$, the group $Spin(n)$ and the covering map λ are respectively replaced by $E(\zeta')$, the groups $Pin(n)^{\pm}$ and the covering maps $p_{\pm}$.

In view to give criteria for the existence and the classification of the set of $Pin(n)^{\pm}$-structures on a given fiber bundle, we will take an alternative approach rather than considering a Leray-Serre spectral sequence; then we need to recall some more definitions and basic results from bundle theory, generalizing what we have just constructed. Given Lie groups H and G, a continuous homomorphism $\Psi : H \to G$, a manifold X and an atlas $\mathcal{U}_i$ with its transition functions r_{ij} for a H-principal bundle ζ with projection $\pi : E(\zeta) \to X$, consider the G-bundle over X obtained by applying Ψ to r_{ij}. Let us denote it by $\pi_\Psi : E(\zeta) \times_H G \to X$. We say that a G-bundle $\pi : E(\zeta) \to X$ has an *H-structure* provided that there exists an H-bundle $\pi' : E(\zeta') \to X$ with an associated G-bundle $\pi'_\Psi : E(\zeta') \times_H G \to X$ equivalent with ζ. Moreover, two H-structures on a G-bundle ζ are *equivalent* if the equivalences of their associated G-bundles with ζ only differ by an equivalence of H-bundles. Then we have (see [**KT**, §2] and [**Hus**] for more details):

Fact: *Given a continuous homomorphism $\Psi : H \to G$ between Lie groups, there is an induced map $B_\Psi : B_H \to B_G$ between their classifying spaces. It can be deformed, without changing the homotopy type of B_H, into a fibration; then, given a G-bundle ζ with a classifying map $f : B \to B_G$, the set of H-structures on ζ are in $1-1$ correspondance with lifts of f to B_H.*

Consider the "cocycle presentation" $(\{\mathcal{U}_i\}, \{r_{ij}\})$ of a G-bundle on X, where $\mathcal{U} = \{\mathcal{U}_i\}$ is an open covering of X which has the continuous maps $r_{ij} : \mathcal{U}_i \cap \mathcal{U}_j \to G$ for transition functions. It is by its very definition a Cech 1-cocycle with coefficients in the sheaf of germs of continuous maps to G: we will consequently denote the set of equivalence classes of cocycle presentations of G-bundles over X *with atlas $\mathcal{U}$* by $H^1(\mathcal{U}, G)$. It is a standard result that we can consider the direct limit $H^1(X, G)$ (induced by refinements on the coverings) of the sets $H^1(\mathcal{U}, G)$. Therefore, the set of equivalence classes of G-bundles is naturally represented by $H^1(X, G)$.

When G is abelian $H^1(X, G)$ is the well-known first Cech cohomology group of X with coefficients in G, but in general $H^1(X, G)$ is not even a group, besides it has the trivial G-bundle as a distinguished element. However, given an exact sequence

$$1 \to K \xrightarrow{i} G \xrightarrow{j} G' \to 1$$

of topological groups, where K is abelian the arguments used in the construction of Cech cohomology theory allow us to write the exact sequence:

$$\to H^1(X, K) \xrightarrow{i^\star} H^1(X, G) \xrightarrow{j^\star} H^1(X, G') \to H^2(X, K)$$

since $H^2(X, K)$ is well defined. Note that we don't care about the smoothness of X and of the G-bundles, since one may show that there is a $1-1$ correspondance between the sets $H^1(X, G)$ in the two cases.

In a similar way, the short exact sequence:

$$0 \to \mathbf{Z}_2 \to Pin^{\pm}(n) \xrightarrow{p^{\pm}} O(n) \to 1$$

gives an exact sequence

$$H^0(X,O(n)) \xrightarrow{\delta^0} H^1(X,\mathbf{Z}_2) \rightarrow H^1(X,Pin^{\pm}(n)) \xrightarrow{(p^{\pm})^{\star}} H^1(X,O(n)) \xrightarrow{\delta} H^2(X,\mathbf{Z}_2).$$

We could define $\omega_2(\zeta) = \delta([\zeta])$ as above in the Spin(n) situation, but we rather make the link with the homotopy theoretic preceding Fact.
Recall both the definitions of principal bundles by classifying spaces and the (obstruction theoretic) interpretation of the group $H^2(X,\mathbf{Z}_2)$ as the set of homotopy classes of maps from X to the Eilenberg Mac-Lane space $K(\mathbf{Z}_2,\ 2)$. The last exact sequence induces the fibration

$$BPin^{\pm}(n) \xrightarrow{Bp^{\pm}} BO(n) \xrightarrow{\omega} K(\mathbf{Z}_2,\ 2)$$

where the symbol B denotes a classifying space. Now the classifying map $f_\zeta :\ X \rightarrow BO(n)$ of an $O(n)$-bundle ζ has a lifting to $BPin^{\pm}(n)$ if and only if $\omega \circ f_\zeta$ is homotopic to zero. Since $[X,\ K(\mathbf{Z}_2,\ 2)] \cong H^2(X,\mathbf{Z}_2)$, this is true when the pull-back classes by f of the generators of $H^2(BO(n),\mathbf{Z}_2)$ in $H^2(X,\mathbf{Z}_2)$ are zero. By the very definition of characteristic classes, these pull-back classes are $\omega_2(\zeta)$ and $\omega_1^2(\zeta)+\omega_2(\zeta)$.

Consequently, if there is a $Pin^{\pm}$-structure on our $O(n)$-bundle ζ over X, then $H^1(X,\ \mathbf{Z}_2)$ acts on the set of $Pin^{\pm}$ structures in a simply transitive manner (see the exact sequence above). Furthermore, the obstruction to the existence of such a structure is either $\omega_2(\zeta)$ or $\omega_2(\zeta)+{\omega_1}^2(\zeta)$.
We will be only interested in Pin^- structures, thus we now consider an example to determine to which of these two classes corresponds the group $Pin^{\pm}$. Denoting by λ a line bundle over X, the transition functions of $\oplus_{i=1}^3\lambda$ are given by taking transition functions for λ and composing with the homomorphism $O(1) \rightarrow O(3)$ which sends ± 1 to $\pm id \in O(3)$. Clearly this homomorphism lifts through $O(1) \rightarrow Pin^-(3)$ (see Appendix B for some similar calculus), and also equivalent $O(1)$ bundles give equivalent $Pin^-(3)$ bundles. Then $\oplus_{i=1}^3\lambda$ has a *canonical* Pin^- structure. Since we always have $\omega_2(\oplus_{i=1}^3\lambda) = {\omega_1}^2(\oplus_{i=1}^3\lambda)$ (see [**MS**]), we can find examples where $\omega_2(\oplus_{i=1}^3\lambda) \neq 0$, but $\omega_2(\oplus_{i=1}^3\lambda)+{\omega_1}^2(\oplus_{i=1}^3\lambda) = 0$. A simple one is the canonical (line) bundle over $\mathbf{P}_2$.

It follows that the obstruction to having a Pin^- structure on a fiber bundle ζ is $\omega_2(\zeta)+{\omega_1}^2(\zeta)$.

3.2 The Wu formula, geometric consequences

3.2.1 Preliminaries

Given a smooth closed manifold X, the Wu formula relates the homotopic and algebraic definitions of the Stiefel Whitney characteristic classes $\omega_k(TX)$ [**MS**, chapt. 11]; in particular, they depend only on the homotopy type of X (beware that different Spin structures are in general not equivalent under homotopy equivalences !).
The Wu formula may be written as follows:

$$\omega_k = \sum_{i=j=k} Sq^i(v_j)$$

where Sq^i is the i^{th}-*Steenrod Square automorphism* on the cohomology ring of X. The total Wu class $v = 1 + v_1 + v_2 + \ldots$ of X is defined by

$$\langle v \cup x,\ [X]\rangle = \langle Sq(x),\ [X]\rangle$$

with $x,\ v = 1+v_1+v_2+\ldots\ldots\ldots \in H^*(X,\ \mathbf{Z}_2)$, the symbol $Sq = id+Sq^1+Sq^2+\ldots$ is the total Steerod square automorphism and $[X]$ denotes the fundamental class of X.
Since $Sq^i(a) = 0$ if $i > deg(a)$, we get $v_k = 0\ if\ k > [dim(X)/2]$.

When X is a 3-manifold, we then have $v = 1+v_1 = 1+w_1$, so $\omega_2 = {\omega_1}^2$: it follows that when X is orientable, X has Spin structures (and then Pin^- structures).

When X is a closed smooth 4-manifold and $\omega_1 = 0$, the equation $v = 1 + v_1 = 1 + w_2$ implies:

$$\forall a \in H^2(X,\ \mathbf{Z}_2),\ \langle \omega_2 \cup a,\ [x]\rangle = \langle a \cup a,\ [X]\rangle$$

Using the intersection pairing on X we will deal on this result later. Recall [**MS**, § 12] that when $dim(X) = n$ and $k < n$ is even, then $\omega_k(TX) \in H^k(X,\ \pi_{k-1}(V(n,\ n-k+1))$ is the obstruction to find $n-k+1$ independant vector fields on the k-skeleton of X (where $V(n,\ k)$ is the $(n,\ k)$-*Stiefel manifold* consisting of all k frames in $\mathbf{R}^n$); if k is odd or $k = n$, then $\omega_k(TX)$ is only the (mod 2) reduction of this obstruction.

When X is a closed smooth 4-dimensional manifold, we thus have:

- $k = 1$: X is orientable $\Longleftrightarrow \omega_1 = 0$
- $k = 2$: ω_2 is the sole obstruction to find 3-framings on the 2-skeleton (i.e. fields of 3 independant vectors in the tangeant bundle); choosing a 4^{th} one on the 2-skeleton (take an orientation field for instance), it is then also the sole obstruction to get 4-framings on the 2-skeleton. If $\omega_2 \neq 0$ and F is the Poincare dual to ω_2, then TX can be trivialized on $X \setminus F$ (we will see another way to get this result in §4; this will notably allows us to fix the trivialization of F). Notice that the Spin structure corresponding to the trivialization of TX over the 2-skeleton $X_{(2)}$ is easy to determine: if $\sigma_0 = (0,\ 1) \in H^1(X_{(2)}, \mathbf{Z}_2) \oplus \mathbf{Z}_2 \cong H^1(X_{(2)} \times SO(n), \mathbf{Z}_2)$ and $\phi :\ E(TX_{|X_{(2)}}) \to X_{(2)} \times SO(n)$ is the trivialization, just take $\sigma = \phi^\star \sigma_0$.
- Finally, $\omega_3 = 0$ since $\pi_2(SO(4)) = 0$ (as a Lie group); hence every Spin 4-manifold admits a trivialization of the tangeant bundle over the 3-skeleton, and consequently also a so-called *almost framing* over the 4-skeleton minus an arbitrary point (whose extension obstruction is ω_4). Indeed, a celebrated theorem of Whitehead says that a 4-manifold may be presented as a CW-complex with a single 4-cell (glued without torsion).

Following [**Lic**], every closed oriented connected 3-manifold can be obtained by surgery on a framed link L embedded in S^3, which means that it is the boundary of a 4-manifold W^4 constructed by adding 2-handles to ∂B^4 along the components of

L, with the corresponding framings.
Since we now deal with compact 4-manifolds with boundary, we see in the same way as above that the sole obstruction to extend a given Spin structure on M^3 onto a simply connected 4-manifold W^4 is the *relative* Stiefel-Whitney class $\omega_2(W_L,\ M_L)$; see [**Ka**] for some explicit constructions to remove it. Notice that the *index* and the *Euler class* of W^4 are also obstructions to turn an almost framing of W^4 into a genuine one, see [**K1**, p. 43].

The geometry of Pin^- structures on compact (smooth) 4-manifolds bounded by a given orientable 3-manifold is much more hackward, except in the case where we consider only *orientable* 4-manifolds. Then it restricts to the preceding situation. From now on we will consider only this case.

3.2.2 The relation with the intersection form

Let V be a free $\mathbf{Z}$-module and Φ an integral unimodular symetric bilinear form (i. e. we have $det(\Phi(x_i,\ x_j)_{i,\ j}) = \pm 1$). The form Φ admits the following 3 remarkable invariants:

- its *rank* $r = dim_{\mathbf{Z}} V$,
- its *parity* defined by: Φ is even iff $\Phi(x,\ x) := x \cdot x = 0 \mod 2$ and odd otherwise,
- its *index* $\sigma(\Phi)$, which is defined as the number of positive entries minus the number of negative entries in a $\mathbf{Q}$-diagonalization of a matrix representing Φ.

The form Φ is said to be positive (resp. negative) definite if $\forall x \in V,$ we have $x \cdot x > 0$ (*resp.* $x \cdot x < 0$), and indefinite otherwise. It is well-known (see e. g. [**MH**] or [**HNK**]) that if Φ is indefinite, then it is *determined* by the preceding list of 3 invariants.

We say that $\omega \in V$ is a *characteristic* element for Φ if

$$\forall x \in V,\ \omega \cdot x \equiv x \cdot x \pmod 2.$$

If we take a closed orientable 4-manifold M, we set $V = H_2(M^4,\ \mathbf{Z})$ and the intersection form is identified with Φ, a characteristic element ω for Φ is an integral dual to $\omega_2(M^4)$, and to $\omega_2(M^4, \partial M^4)$ if M^4 has a boundary with a fixed Spin structure. Let us now turn to the choice of characteristic elements in a general free $\mathbf{Z}$-module V.

First note that the characteristic elements are well defined $\pmod 8$. Indeed, consider the $\mathbf{Z}_2$-vector space $\overline{V}$ which is spanned by the $\pmod 2$ reduction of the elements of V: the $\pmod 2$ reduction $\overline{\Phi}$ of Φ onto $\overline{V}$ is well-defined, and $\overline{\Phi}$ *is a definite bilinear form*. Then its adjoint homomorphism $ad(\overline{\Phi}) :\ \overline{V} \to \mathbf{Z}_2$ may be written as:

$$\forall\ \overline{x} \in \overline{V},\ \exists\ \overline{w} \in \overline{V} :\ ad(\overline{\Phi})(\overline{x}) = \overline{\Phi}(\overline{x},\ \overline{\omega}).$$

If $\omega \in V$ verifies $\omega \mod 2 = \overline{\omega} \in \overline{V}$, we now get (denoting by $\cdot$ the maps Φ and $\overline{\Phi}$):

$$x \cdot \omega \equiv x \cdot x \pmod 2.$$

Moreover, if ω' is a characteristic element distinct from ω, then:

$$\omega' \cdot \omega' = (\omega + 2x) \cdot (\omega + 2x) = \omega \cdot \omega + 4(\omega \cdot x) + 4(x \cdot x).$$

Thus ω is well-defined $\pmod 8$.

Now we prove that: $\sigma(\Phi) \equiv \omega \cdot \omega \pmod 8$.
Indeed an elementary linear algebra calculus shows that every odd indefinite form decomposes into a direct sum $\Phi = \overset{p}{\oplus} (+1) \overset{q}{\oplus} (-1)$, where (± 1) denotes the two (normalized) non trivial bilinear forms on one-dimensional vector spaces. Then $\sigma(\Phi) = p - q = \omega \cdot \omega$, where ω is the sum of the generators of each factor.
But an arbitrary Φ may be turned into an odd indefinite form $\Phi \oplus (+1) \oplus (-1)$, then finally:

$$\sigma(\Phi) = \sigma(\Phi \oplus (1) \oplus (-1)) \equiv (\omega_\Phi + \alpha + \beta) \cdot (\omega_\Phi + \alpha + \beta)\ (8) \equiv \omega_\Phi \cdot \omega_\Phi \pmod 8$$

where α and β respectively generate the summands (1) and (-1), and ω_Φ is characteristic for Φ. This formula proves our claim.

Hence when Φ is even, we can set $\omega = 0$ and $\sigma(\Phi) \equiv 0 \pmod 8$. A theorem of Rohlin will precise this result (see the following chapters).

When M^4 is a closed connected oriented smooth 4-manifold with $H_1(M^4,\ \mathbf{Z}) = 0$, we have an epimorphism $H^2(M,\ \mathbf{Z}) \to H^2(M,\ \mathbf{Z}_2)$; then $\omega_2(M)$ may be lifted to an integral class $\omega(\hat{M}) \in H^2(M,\ \mathbf{Z})$ such that

$$\forall\ \hat{x} \in H^2(M,\ \mathbf{Z}),\ \hat{\omega}(M) \cup \hat{x} = \hat{x} \cup \hat{x} \pmod 2$$

and dually:

$$\forall x \in H_2(M,\ \mathbf{Z}),\ \omega \cdot x = x \cdot x \pmod 2,$$

where ω is the Poincar dual to $\hat{\omega}$ and $\cdot$ is the intersection product over $H_2(M,\ \mathbf{Z})$. We deduce that

$$if\ H^1(M,\ \mathbf{Z}) = 0,\ then\ \omega_2 = 0 \Longleftrightarrow \cdot\ is\ an\ even\ intersection\ form.$$

Notice that the converse is wrong without the condition $H_1(M,\ \mathbf{Z}) = 0$, as shows the following example of N. Habegger, quoted in **[K]**, §2: take $M = S^2 \times S^2/\mathbf{Z}_2$. Since we have $\chi(M) = 2$, the rank of $H^2(M, \mathbf{Z})$ is 0. But the diagonal embedding of the 2-sphere in $S^2 \times S^2$ is turned into an $\mathbf{RP}^2$ under the $\mathbf{Z}_2$ action, with self intersection 1. Now its dual in $H^2(M, \mathbf{Z}_2)$ forces $\omega_2(M)$ to be different from zero.

In view to define the Rohlin invariant of a given closed oriented 3-manifold M, we will mainly consider oriented smooth 4-manifolds W with (oriented) boundary M, for which this formalism may be applied. The obstruction to the extension of a chosen Spin structure on M onto the whole of W will be represented by an orientable surface called "characteristic", whose properties will be studied in §4.

3.3 A "descent" theorem

Next consider an arbitrary smooth oriented 4-manifold W^4 with (oriented) boundary M^3. We will try to understand the relationship between the geometry of 2-manifolds embedded in W^4 and the obstruction for W to be Pin^-. All what follows may be easily transposed to $SO(n)$ (and then to Spin structures).

Let us denote by $TO(k)$ the Thom space of the universal bundle over $BO(k)$. If W is a n-manifold , the Pontrjagin-Thom construction shows that an element $a \in H^k(W,\ \mathbf{Z}_2)$ is dual to a (properly embedded) $(n-k)$-submanifold $F \subset W$ if and only if the map $W \to K(\mathbf{Z}_2,\ k)$ that represents a may be lifted to a map $W \xrightarrow{t} TO(k)$.
The geometric meaning of the lift is the identification of the universal bundle over $BO(k)$ with the normal bundle of F in W, in such a way that the Thom class $u \in H^k(BO(k), \mathbf{Z}_2)$ pulls back to $a = t^*(u)$. The map $(W,\ W \setminus F) \to (TO(k),\ *)$ associated to this identification not only induces a monomorphism on $H^k(\ \ ,\ \mathbf{Z}_2)$ by excision, but the Thom isomorphism theorem shows that we have $H^k(W,\ W \setminus F;\ \mathbf{Z}_2) \simeq H^0(F;\ \mathbf{Z}_2)$ (see [bf MS, § 10]).
Then a restricts to the product of the generators, by the very definition of the Thom class u. Furthermore we have $a_{|M\setminus F} = 0$ and $a_{|F}$ *is the Euler class of the normal bundle of* F *in* W.
Notice that besides any integral 2-cohomology class has a dual submanifold with oriented normal bundle (since $TSO(2) = \mathbf{C}P^2 = K(\mathbf{Z},\ 2)$), the preceding theory is of interest for the map $TO(2) \to K(\mathbf{Z}_2,\ 2)$, which is not a homotopy equivalence. We quote in a general context:

Theorem [**KT**, 2. 4]: *Let M be an oriented paracompact manifold, with or without boundary, and F a codimension 2 properly embedded submanifold with finitely many components. Then F is Poincar dual to $\omega_2(M)$ if and only if there is a Pin^- structure on $M\setminus F$ which does not extend across any component of F. Furthermore $H^1(M,\ \mathbf{Z}_2)$ acts simply transitively on the set of Pin^- structures of $M \setminus F$ which do not extend across any component of F.*

Proof: Denote the disk, sphere bundle tubular neighborhoods to each component F_i of F in M by $(D(\nu_i),\ S(\nu_i))$; moreover, set $i:\ M \hookrightarrow (M,\ M \setminus F)$ the inclusion. Suppose that $M \setminus F$ has a fixed Pin^- structure which does not extends across any component of F. In general, one may easily show that there is an equivariant correspondance

$$Pin^-(\zeta) \to Pin^-(\zeta \oplus_{i=1}^r \epsilon^1)$$

which commutes with the action of $H^1(B,\ \mathbf{Z}_2)$, where ζ is any vector bundle over a CW-complex B and ϵ^1 a trivial line bundle over B. Since $M \setminus\ (\amalg D(\nu_i)) \subset M \setminus F$ inherits the Pin^- structure of $M \setminus F$ by restriction, the last remark shows that we get a Pin^- structure on $\amalg S(\nu_i)$ considered as the boundary of $M \setminus \amalg D(\nu_i)$ (see also the following sections).

We now want to define a cohomology class for M, associated with the normal bundle to F, that extends $\omega_2(M)$. Let $b \in H^2(M,\ M\setminus F;\ \mathbf{Z}_2) \simeq \oplus H^2(D(\nu_i),\ S(\nu_i);\ \mathbf{Z}_2) \simeq \oplus\mathbf{Z}_2$ be equal to 1 on each summand F_i if the Pin^- structure on $S(\nu_i)$ extends across

$D(\nu_i)$, and -1 if it does not. Consider the embedding $j: \ G \to M$ of a surface G, such that $j(G)$ either misses F or intersects it transversally in a collection of points (p_l) with neighborhoods in $j(G)$ some disks that are fibres in $\amalg D(\nu_i)$.
The Pin^- structure on the normal bundle $\nu_{j(G)\subset \ M}$ restricted to $j(G) \setminus \amalg_i D_i$, which is induced by the Pin^- structure of $M \setminus F$, does not extend over the 2-disks D_l that lie over the points p_l. However, $j(G)$ will have a globally defined Pin^- structure if there are an even number of such disks. This follows from the equality:

$$\langle \omega_2(M),\ [j(G) \backslash \amalg_i D_i] \rangle = \langle \omega_2(M),\ [j(G) \backslash \amalg_{i\neq j,\ k} D_i] \rangle + \langle \omega_2(M),\ D_j \rangle + \langle \omega_2(M),\ D_k \rangle =$$

$$= \langle \omega_2(M),\ [j(G) \setminus \amalg_{i\neq j,\ k} D_i] \rangle$$

Now the definition of b implies that we have:

$$\langle i^*(b),\ j_*[G] \rangle = \begin{cases} 1 & \text{if } TM_{|j(G)} \text{ has a } Pin^- \text{ structure,} \\ -1 & \text{otherwise.} \end{cases}$$

But this is the defining property of ω_2, so we conclude that $i^*(b) = \omega_2(M) \in H^1(M, \mathbf{Z}_2)$.
Since the Pin^- structure of $M \setminus F$ does not extend across any component of F, the class b is equal to -1 over each of them and it is the image $t^*(u)$ of the Thom class u (see the preceding discussion). Therefore $i^*t^*(u) = i^*(b) = \omega_2(M)$; hence F is dual to $\omega_2(M)$.

Conversally, suppose that F is dual to ω_2; then $M \setminus F$ has a Pin^- structure. Fix one of them: we can alter the corresponding $i^*(b)$ (in the above construction) by the action of a class $c \in H^1(M \setminus F, \mathbf{Z}_2)$, so that $b + \delta^*(c)$ is the image of the Thom class (where δ is the coboundary operator of the long exact sequence of the pair $(M \setminus F,\ M)$). In other words, we can alter the Pin^- structure on $M \setminus F$ by the action of c, so that we get a new Pin^- structure which does not extend across any component of F (by the argument of the preceding paragraph). $\square$

3.4 Splittings of Pin^--bundles, other geometric characterizations, examples

Extension lemma: *If an $O(n)$-principal bundle ζ on a CW complex X has a Pin^- structure σ and Y is a subset of X with $H^1(X,\ \mathbf{Z}_2) \xrightarrow{\sim} H^1(Y,\ \mathbf{Z}_2)$, then a Pin^- structure on ζ is determined by the choice of a Pin^- structure τ on $\zeta|Y$.*

Indeed, we can define a 1-cochain c on Y with values in $\pi_1(SO(n)) = \mathbf{Z}_2$ by sending a closed curve $l \subset Y$ on 0 (resp. 1) if σ and τ (defined respectively on ζ and $\zeta|Y$) are equal (resp. different) on l. Then c determines a class $\gamma \in H^1(Y,\ \mathbf{Z}_2) \simeq H^1(X,\ \mathbf{Z}_2)$ and $\sigma + \tau$ determines a Pin^- structure on Y equal to τ. $\square$

Stability lemma [**ML**, p. 85]: *Given 3 vector bundles $\zeta',\ \zeta''$ with $\zeta \simeq \zeta' \oplus \zeta''$ on a manifold X, a choice of Spin structures on two of them determines a unique Spin structure on the third.*

Notice that this lemma implies directly the counterpart of the descent theorem in the case where the ambiant bundle splits. Let F be a smoothly embedded orientable surface F with trivial normal bundle ν in a Spin smooth oriented 4-manifold M: then the simply transitive action of the group $H^1(F, \mathbf{Z}_2)$ on the set of trivializations of ν implies that any Spin structure on F is obtained from the Spin structure on M and the adequate choice of trivialization of ν.
In the generality of the hypothesis of the lemma, the unicity is not easy to obtain. But for our purposes (aside of the preceding remark and the descent theorem) it suffices to see that given a codimension-1 submanifold N of a manifold M, such that M, N and the normal line bundle ν of $N \subset M$ are coherently oriented, we have (using the canonical Spin structure of ν):

$$\{Spin\ structures\ on\ TM_{|N}\} \longleftrightarrow \{Spin\ structures\ on\ TN\}.$$

For example we can complete a framing in N by the normal vector field that coincides with the orientation.

In the context of Pin^- structures, the restriction of structures with not necessarily trivial normal bundle ν can be formulated as follows:

Restriction lemma: *Let N denote a codimension one submanifold of an orientable manifold M with a* Pin^- *structure and with normal line bundle* ν*; if the determinant line bundle* $\nu = det(TN)$ *is not trivial, then there is a* Pin^- *structure on* N *inherited from the* Pin^- *structure of M.*

Indeed, $TN \oplus det(TN)$ is naturally oriented, so identifying ν and $det(TN)$ we have the natural isomorphism $TN \oplus \nu = TM_{|N}$. Then N gets a Pin^- structure from M. □

Next we want to relate the triviality of the normal bundle to any embedded or immersed surface in a given manifold and Pin^- structures:

First characterization: *A given vector bundle* ζ *of rank* ≥ 3 *on an oriented manifold M is* Pin^- *if and only if for every compact surface F and any continuous map* $f: \ F \to M$ *, the pullback bundle* $f^*(\zeta)$ *is trivial. If furthermore M is simply connected (resp. and* $dimM > 4$*), the bundle* ζ *has a* Pin^- *structure if and only if the restriction of* ζ *to any immersed (resp. embedded) 2-sphere is trivial.*

It suffices to see that $H^2(M, \ \mathbf{Z}_2)$ is generated by maps $f: \ F \to M$, where F denotes any compact surface. Then we have:

$$\begin{aligned} \omega_2(\zeta) = 0 &\Longleftrightarrow \forall\ f \text{ as above}: f^*\omega_2(\zeta) = \omega_2(f^*(\zeta)) = 0 \\ &\Longleftrightarrow \forall f \text{ as above } f^*(\zeta) \text{ is trivial.} \end{aligned}$$

Now an oriented bundle of rank ≥ 3 on a surface is trivial if and only if its second Stiefel-Whitney class is zero. Moreover, if $dimM > 4$, any immersion of a surface may be deformed into an embedding (by the Whitney trick), and if $\pi_1(M) = 0$ then $H^2(M, \ \mathbf{Z}_2)$ is generated by maps of the 2-sphere into M (by the Hurewicz isomorphism). □

Let us precise this result:

Second characterization : *If M is an oriented n-manifold such that* $H^2(M, \ \mathbf{Z}) \to H^2(M, \ \mathbf{Z}_2)$ *is an epimorphism, then:*

- *i) if $n \geq 5$, M is Spin if and only if every embedded compact orientable surface in M has a trivial normal bundle in M.*
- *ii) if $n = 4$, M is Spin if and only if the normal bundle of any embedded compact orientable surface in M has an even Euler class.*

The proof is similar to the preceding one: the hypothesis implies that $H_2(M,\ \mathbf{Z}_2) \simeq H_2(M,\ \mathbf{Z}) \otimes \mathbf{Z}_2$ so $H_2(M,\ \mathbf{Z}_2)$ is generated by inclusion maps of compact orientable surfaces. But any map from a surface into M is homotopic to an embedding if $n \geq 5$, and to a transverse immersion if $n \geq 4$. In the last case one may drop a little 2-cell from the surface in a neighborhood of each self-intersection point, and add an embedded handle. This induces an embedded surface in the same homology class (the last argument will be explained in more details later). Then $H^2(M,\ \mathbf{Z}_2)$ is generated by smooth embeddings of compact orientable surfaces. If $i:\ F \hookrightarrow M$ is such an embedding and ν still denotes the normal bundle to F in M, we have:

$$i^*\omega_2(M) := i^*\omega_2(TM) = \omega_2(i^*TM) = \omega_2(TF) + \omega_2(\nu) = \omega_2(\nu).$$

This equality induces:

$$\langle \omega_2(M),\ i^*[F] \rangle = \langle \omega_2(\nu),\ [F] \rangle.$$

If $\omega_2(M) = 0$, it follows that $\omega_2(\nu) = 0$; the converse is true because $H^2(M,\ \mathbf{Z}_2)$ is generated by such compact embedded orientable surfaces.
Moreover ν is orientable, which implies that if $dim(\nu) \geq 3$ (i. e. if $n \geq 5$) the nullity of $\omega_2(\nu)$ is equivalent to the fact that ν is trivial.
When $dim(\nu) = 2$, then we know that $\omega_2(\nu)$ is the mod 2 reduction of the Euler class.
Notice in particular that every 1-connected manifold M of dimension $n \geq 5$ is Spin if and only if any embedded 2-sphere in M has a trivial normal bundle in M. □

Some other examples:

i) any 2-connected manifold has a unique Spin structure : homology (or homotopy) spheres, Stiefel manifolds, simply-connected Lie groups.
ii) any manifold whose tangeant bundle is stably parallelizable is Spin: examples are the inverse image of regular values of a smooth map $f:\ \mathbf{R}^{n+p} \to \mathbf{R}^p$, Lie groups and orientable manifolds of $dim \leq 3$.

Remark The restriction lemma permits not only to frame (by a field of independant lines (resp. vectors)) the pullback of the tangeant bundle of a Pin^- (resp. Spin) 3-manifold on a non orientable (resp. orientable) embedded surface F, but also to get an induced Pin^- (resp. Spin) structure on F; the choice of a framing in the ambiant manifold is a necessary addition of structure to turn the linking pairing into a quadratic form, and to develop the associated theory of invariants of cobordism class of surfaces (see the next chapter).

3.5 Spin cobordism

We say that the manifold X^n is a Spin boundary of the manifold W^{n+1} if X is diffeomorphic to ∂W^{n+1} and the diffeomorphism makes the Spin structure of X and the Spin structure of ∂W^{n+1} correspond (this relation goes up in the Spin category). We note Ω_n^{Spin} the equivalence classes of Spin manifolds, where M_1^n and M_1^n are equivalent Spin manifolds if $M_1 \coprod -M_2$ is a Spin boundary for some Spin manifold W^{n+1}. Let us now take a look at the low dimensional Spin and Pin^- cobordism groups.

In the one dimensional case, note that a trivialization of a bundle is the same thing as an orientation for it. Hence there are 2 Spin structures on S^1. They correspond respectively to the trivial and the twisted $\mathbf{Z}_2$-bundles over S^1.
Explicitely, denote by $\sigma \in H^1(S^1,\ \mathbf{Z}_2)$ the class corresponding to the chosen Spin structure of TS^1. We have:
i) for $\sigma = 0$ we get the "Lie group Spin structure", that is the translation-invariant trivialization of $TS^1 \oplus \zeta^1$ (where ζ is a trivial line bundle stabilizing TS^1).
ii) for $\sigma = 1$ we take the trivialization of $\mathbf{R}^2$ restricted to S^1.
The last trivialization makes S^1 bounds the disk B^2, respecting the Spin structures of both manifolds.

Now the 2-disk has an orientation reversing involution that gives an equivalence between S^1 with Lie group Spin structure and S^1 with the reversed orientation and the Lie group Spin structure. Hence Ω_1^{Spin} and $\Omega_1^{pin^-}$ are each 0 or $\mathbf{Z}_2$.
Suppose that S^1 is the boundary of an orientable surface F: all Spin structures on F induce the same Spin structure on S^1, as follows from the stability lemma and the simply transitive action of $H^1(F,\ \mathbf{Z}_2)$ on the set of Spin structures of F (notice that the same argument works with Pin^- structures in case F is a non orientable surface). But any Spin (resp. Pin^-) structure on F extends uniquely to one on $\hat{F} = F \cup D^2$, and then the structure induced on S^1 is the one which extends over the 2-disk. So S^1 with the Lie group Spin structure does not bound, and we get:

$$\Omega_1^{Spin} \simeq \Omega_1^{pin^-} \simeq \mathbf{Z}^2$$

The same arguments show that we have 4 Spin structures on T^2, determined by the cyclic decomposition of the group $H^1(T^2,\ \mathbf{Z}_2)$ into $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Only extend those which are non zero for both standard longitude-meridian generators, and give $S^1 \times B^2$ (where B^2 is glued to a circle S^1 on which $\sigma \neq 0$).
Then any connected sum of k tori with the so-called non-everywhere Lie framings bounds a Spin compatible manifold. When we have the Lie framing on p of them,

say the p first tori $T_i,\ i = 1,\ \ldots,\ p$ (i.e. those where the Spin structure affinely corresponds to a class $\sigma = (0,\ 0) \in H^1(T^2, \mathbf{Z}_2)$), this connected sum clearly bounds $(\overset{p}{\sharp}_{i=1} (T_i^2 \setminus B_i^2)\ \overset{k}{\sharp}_{j=p+1} T_j^2) \times I$ and the Spin structure on this 3-manifold extends to the Lie framing when restricted to the boundary. Then $\Omega^{Spin}{}_2 \simeq \mathbf{Z}^2$.

This isomorphism can be viewed as follows (see Appendix A for more details): let $q:\ H_1(F^2,\ \mathbf{Z}_2) \to \mathbf{Z}_2$ be a quadratic enhancement of the intersection form, where F^2 is a compact oriented surface with a given Spin structure τ, such that:

$$q(x) := \begin{cases} 0 & \text{if } \tau(F^2)_{|x} \text{ is the bounding Spin structure} \\ 1 & \text{otherwise} \end{cases}$$

Then q is equal to 1 on the generators of T^2 with the Lie Spin structure, and we get an isomorphism:

$$\Omega_2^{Spin} \overset{Arf}{\to} \mathbf{Z}_2$$

We will deal with this isomorphism in more details in the next chapter, extending it to Pin^- structures on arbitrary 2-dimensional manifolds. The isomorphisms will still be realized by quadratic invariants.

It is wellknown that a connected Spin 3-manifold M bounds a Spin 4-manifold W with only handles of indices 0 and 2 (see [**K1**, p. 47]). Moreover one can control the rank of $H^2(W^4,\ \mathbf{Z})$ and the index of W^4, using for exemple Kirby's calculus and a clever elimination of characteristic surfaces, which are dual to $\omega_2(W,\ M)$ (see [**Ka**]).

Finally, we notice that we are able to compute the cyclic decomposition of Ω_4^{Spin} without even the determination of a generator for it, supposing only the knowledge of the (non elementary) following fact: suppose that a closed smooth oriented 4-manifold M has a Spin structure and that $p_1(M) = 0$, then there exists a Spin 5-manifold W with M as a Spin boundary (cf. [**K1**, §8]). We refer to §4 for details. Given a smooth closed connected oriented 4-manifold M, we have Hirzebruch 's formula (see [**K1**] or [**MS**]) $p_1(M) = 3\sigma(M)$ (where $p_1(M)$ is the first Pontrjagin class of M and σ is its index). .
Since $\sigma(\mathbf{C}P^2) = 1$, the index of intersection forms then induces a map $\Omega^{Spin}{}_4 \overset{\sigma}{\longrightarrow} \mathbf{Z}$.

Moreover, in case M^4 is Spin, its intersection form is even (as we have already seen), so $\sigma(M) \equiv 0 \pmod 8$. Consequently $\sigma/8$ is a monomorphism into $\mathbf{Z}$ or $\mathbf{Z}_2$. We can now state a theorem of Rohlin:

$$\Omega_4^{Spin} \overset{\sigma/16}{\longrightarrow} \mathbf{Z}$$

This will follow in the next chapter as an easy corollary of a more general result for smooth oriented 4-manifolds, in the form of an extension of the Guillou-Marin formula.

Chapter 4

Characteristic surfaces and quadratic forms

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Let M be a smooth oriented 4-manifold; recall that a properly embedded surface $F^2 \subset M^4$ is called a *characteristic surface* if it is an obstruction cycle to the trivialization of TM over the two skeleton: as we saw above, for a given trivialization t of TM over $M \setminus F$, the obstruction to the extension of the field of frames t over a disk D transverse to F is the non zero element of $\pi_1(SO(4)) = \mathbf{Z}_2$, where $SO(4)$ plays the role of the structural group of TM. This geometric point of view gives us the Wu formula,seen as an equality between indices of vector fields (cf. [**GM**]) : $\forall \Sigma \in H_2(M,\ \mathbf{Z}_2)$, we have $\Sigma \cdot \Sigma \equiv F \cdot \Sigma \pmod 2$, where $\cdot$ denotes the intersection product on $H^2(M, \mathbf{Z}_2)$.

The classical Rohlin theorem we cited at the end of chapter 3 aims at understanding the problem of the realization of quadratic forms as intersection pairings $H_2(M,\ \mathbf{Z}) \times H_2(M,\ \mathbf{Z}) \to \mathbf{Z}$ of a smooth orientable compact or closed manifold M^4. It is completed by Wall's version of the h-cobordism theorem in dimension 4 (see [**K**, §10] and the recent and much more hackward theories of M. Freedman [**F-Q**] and S. K. Donaldson [**D-K**], which in particular tell us that:
1) Given an even (resp. odd) quadratic form, there exists exactly one (resp. two) closed topological simply connected 4-manifold(s) that realizes it as its intersection pairing. In the odd case, the PL-invariants of Kirby and Siebenmann [**KS**] can distinguish the two manifolds.
2) When considering smooth simply connected 4-manifolds, the only realizable definite forms are the trivial ones: $\pm \oplus_{i=1}^{p} (+1)$, and by Friedman's theorem we know that it represents $\pm \sharp_{i=1}^{p} (\mathbf{C}P^2)$.

Let us denote by $\sigma(M)$ the index of M; we saw in Chapter 3 that algebra adds to the preceding results the formula $\sigma(M) - \omega \cdot \omega \equiv 0 \pmod 8$, where ω is a characteristic element. Now, Rohlin's theorem states that a closed smooth Spin and simply connected 4-manifold verifies $\sigma(M) \equiv 0 \pmod{16}$; hence half of the even form cannot be represented by such manifolds. This section deals with the proof of an extension of this formula, and with its geometric meaning.

4.1 Characteristic pairs

We say that a pair $(M,\ F)$ of manifolds is a *characteristic pair* if M is oriented, F is properly embedded in M and is dual to its second Stiefel Whitney class $\omega_2(M)$. The pair is said to be *characterized* provided we have fixed a Pin^- structure on $M \setminus F$ which does not extend across any component of F. We have already seen that such Pin^- structures are in one to one correspondance with $H^1(M,\ \mathbf{Z}_2)$. We denote by $Char(M,\ F)$ the set of characterizations of the pair $(M,\ F)$.
Following [**KT**], we say that two characterized pair $(M_1^m,\ F_1^{m-2})$ and $(M_2^m,\ F_2^{m-2})$ characteristically cobound if there exists a smooth characteristic pair $(W^{m+1},\ Y^{m-1})$ and a fixed Pin^- structure on $W^{m+1} \setminus Y^{m-1}$ which does not extend over Y^{m-1}, such that $\partial(W,\ Y) = (M_1,\ F_1) - (M_2,\ F_2)$ (as an oriented boundary) and the Pin^- structures are coherent. This is an equivalence relation between characteristic pairs (the choice of fixed structures is necessary to show transitivity), that induces the m^{th}*-cobordism group of Guillou and Marin*, denoted by Ω_r^{char}.

First we generalize the descent theorem of the preceding section showing geometrically how to descent the structure of an ambiant manifold.

Descent Theorem [**KT**, 6. 2]: *Let M^n be an oriented manifold with a codimension two submanifold which is dual to $\omega_2(M)$. There exists a function*

$$Char(M,\ F) \xrightarrow{\Psi} Pin^-(F)$$

The group $H^1(M,\ \mathbf{Z}_2)$ acts on $Char(M,\ F)$, the group $H^1(F,\ \mathbf{Z}_2)$ acts on $Pin^-(F)$ and the map Ψ is equivariant with respect to these actions and to the map $i^\star$ induced on $H^1(\ ,\ \mathbf{Z}_2)$ by the inclusion $i\ :\ F \subset M$; precisely we have, denoting by θ a characteristic pair in $Char(M,\ F)$:

$$\forall a \in H^1(M, \mathbf{Z}_2),\ \Psi(a.\theta) = i^\star(a)\Psi(\theta)$$

Sketch of proof: We may restrict our attention to the case where E is the total space of the normal bundle ν of $F \hookrightarrow M$ and F is connected, since there is an obvious restriction map of structures from the general case to this one. We generically denote by the symbol $\nu_\star$ a normal bundle, the index indicating its base.
Suppose that $n \neq 3$ and that we have defined Ψ on a set U of disjoints embedded tubular neighborhoods of embedded curves, such that $H_1(U,\ \mathbf{Z}_2) \xrightarrow{\sim} H_1(F,\ \mathbf{Z}_2)$. Then we may extend uniquely our Pin^- structure on U to all of F by the extension lemma. But is this structure independent of the choice of U ?
To see this, consider a tubular neighborhood V in F of an embedded circle c in F, with normal bundle $\nu' := \nu_{c \subset M}$ in M: we can either restrict the Pin^- structure of $\nu := \nu_{F \subset M}$ on ν', or define the Pin^- structure of ν' directly with the help of the geometry of $F \subset M$. Let us explain this construction: chose a section ϵ^1 of ν' (it always exists because V has the homotopy type of a circle), and write $\nu' = \lambda \oplus \epsilon^1$, where we orient the trivial line bundle ϵ^1 and use it to embed V in ∂E (the total space of the sphere bundle of $F \hookrightarrow M$). This orientation plus the orientation of E (hence of ∂E) induce a preferred isomorphism between the determinant line bundle $det(TV)$ of the tangeant bundle to V and the normal bundle to the embedding

$V \hookrightarrow \partial E$, that we have called λ; using the Restriction lemma, V inherits a Pin^- structure from the one of ∂E.
Here are now the problems:
1) to show that this construction for V is independent of the section we chose,
2) if yes, to show that this direct geometric construction of the Pin^- structure on V and the induced one (from ν) does correspond.

The second statement is easy to prove: suppose first that $n-2>2$; take an embedded surface W^2 in F which has for Pin^--boundary components the core circle c of V and some of the cores of U as the others; consider also a tubular neighborhood X of $W^2 \subset F$. The bundle ν restricted to X (which has the homotopy type of a wedge of circles) has a section and, using it as an embedding, we can induce a Pin^- structure on X from the Pin^- structure on ν. In particular, it does have to correspond to the Pin^- structures constructed directly on V (a construction that we suppose to be well defined by 1)).
Hence in this case, these two constructions coincide and every normal bundle $\nu_{c\subset F}$ of a circle c in F has a well defined Pin^- structure induced by the Pin^- structures over a set U as above. Moreover, the Pin^- structure of F is independent of the choice of U since Pin^- structures can be detected by restriction to circles, using the extension lemma.
If $n-2=2$, take a section over $F\setminus pt$ (pt denotes a point in F), embed $F\setminus pt$ in ∂E and give as above a Pin^- structure on $F\setminus pt$ which extends uniquely to a Pin^- structure on F (this is possible since we have the bounding Pin^- structure on the boundary S^1 of $F\setminus pt$!). In case F has a boundary, the section already defined for V (in the above direct construction) may be taken for the section on the boundary. Now the restriction of this last structure on F to a neighborhood of an embedded circle gives a structure that we have supposed to be independant of the section (by 1)), so the structure on the whole of F is independent of the section, and the argument of the preceding paragraph finish that case.

Now we prove 1). Suppose that the bundle ν' is trivial: it contains the case $n=3$ since we have $V=S^1$ and the bundle has oriented total space. Let us deal with this case first.
We have $\partial E = T^2$ and this induces the meridian m as a preferred generator of $H^1(T^2,\ \mathbf{Z}_2)$. Let x be another generator. The Spin structure of ν on V, i. e. its chosen trivialization, is determined by the number $q(V)$ of twists $\pmod 2$ that the homological image of V in $H^1(T^2,\ \mathbf{Z}_2)$, which is either x or $x+m$, makes in a complete traverse of V. As the Spin structure does not extend over disk transverse to S^1 (by the hypothesis of the theorem) we have $q(m)=1$, so obviously $q(x+m) \equiv q(x)+q(m)+x\cdot m \pmod 2 \equiv q(x) \pmod 2$.
In the higher dimensional case, we consider an embedded $S^1 \hookrightarrow V$ with a trivial normal bundle. There is an embedded T^2 in ∂E over this circle, and we can identify the normal bundle to T^2 in ∂E with the normal bundle to $S^1 \hookrightarrow V$ by the bundle projection p. The Spin structure on ∂E restricts to one on T^2, and we can use p to put a Spin structure on one of the normal bundles if we have already fixed a Spin

structure on the other. As the Pin^- structure we want to put on V is determined by using a section over S^1, we are brought back to the preceding case.
Finally we turn to the non trivial case: since V is the total space of a bundle over S^1 it is necessarily of dimension ≥ 2. We can reduce the case $dim(V) \geq 2$ to the case $dim(V) = 2$, because in the higher dimensional case, $\nu_{F \subset M}|S^1$ is isomorphic to $det(\nu) \oplus \epsilon$ (where ϵ is a trivial bundle over S^1), which is the form of bundles over a Möbius band (since also E is oriented). We then have a Klein bottle K^2 over our S^1 and the normal bundle to K^2 in ∂E is the pullback of ν. *As for the torus*, we can see that there is an induced Pin^- structure on K^2, so that the Pin^- structure we want to put on V is determined with the help of the section applied to S^1 as a longitude of K^2. We have proved 1).

Let us finally summerize what we have shown: first, we can construct *canonically* a Pin^- structure on a tubular neighborhood V of any embedded circle in E. Given a characterization of $(E,\ F)$, this allows us to get a canonical Pin^- structure on any embedded circle in F (part 1)) which does coincide with the Pin^- structure on V induced from the Pin^- structure on F (part 2)). Using the remark that the Pin^- structure on F is determined by any immersed collection of circles in F homologically equivalent to F, we get the result. $\square$

The function Ψ defined in the above theorem commutes with taking boundary, as follows from the naturality of the descent of structure in the stability lemma. Then we get a well defined homomorphism :

$$\beta : \ \Omega_r^{char} \to \Omega_{r-2}^{Pin^-}$$

Note that the Pin^- structure on a codimension 0 subset $X \subset F$ depends only on the Pin^- structure on the circle bundle lying over X.

All this discussion is clear in the particular case of Spin structures defined on an orientable pair $(M^4,\ F^2)$: the normal circle bundle ν to F in M inherits a Spin structure from $M \setminus F$. We can push a neighborhood of the generating circles of $H^1(F,\ \mathbf{Z}_2)$ with any section of ν to get a Spin structure on it, hence on F by H^1-isomorphism. The independence of the section follows from the fact that any normal circle to F has the Lie Spin structure $\sigma = 0$, and the difference between two sections is a multiple of a normal circle. The same argument could be applied to a bordism$(W^5,\ Y^3)$, and we would obtain a Spin structure on Y^3 bounding the previous ones. So the element of ${\Omega_2}^{Spin}$ represented by $F \subset M$ does only depend on the class of $(M^4,\ F^2)$ in ${\Omega_4}^{char}$.

Now that we have seen how to get well defined Pin^- structures from characterized pair, we will formalize the algebraic tools in the preceding proof to get direct informations about embedded surfaces.

4.2 Pin^- structures on surfaces and quadratic forms

What follows is the "axiomatization" of methods mostly used in [**GM**], [**T**] and [**Tu2**]. Recall that a quadratic enhancement of the intersection form of a surface is a function :

$$q: \ H_1(F^2, \ \mathbf{Z}_2) \rightarrow \mathbf{Z}_4$$

satisfying :

$$\forall x, \ y \in H_1(F^2, \ \mathbf{Z}_2): \ q(x+y) = q(x) + q(y) + 2x \cdot y,$$

where $\cdot$ is the intersection pairing on F^2. Fixing a Pin^- structure on F, we want to get a quadratic enhancement of its intersection pairing. The core of this problem is the following description of sufficient conditions of existence:

Lemma [**KT** 3. 4]: *Let $\hat{q}$ be a function which assigns an element in $\mathbf{Z}_4$ to each embedded disjoint union of circles in a surface F and is subject to the following conditions:*
(a) $\hat{q}$ is additive on disjoint union,
(b) In case of transversal intersection of two embedded collections of circles L_1 and L_2 in r *points, then we can get a third embedded collection L_3 of circles in F obtained from L_2 and L_3 by surgery on each crossing, and we then require that $\hat{q}(L_3) = \hat{q}(L_1) + \hat{q}(L_2) + 2 \cdot r$;*
(c) $\hat{q}$ is zero on any single embedded circle which bounds a disk in F*.*
Then $\hat{q}(L)$ depends only on the underlying homology class of L, and the induced function $q: \ H_1(F, \ \mathbf{Z}_2) \rightarrow \mathbf{Z}_4$ is a quadratic enhancement.

The second condition is the property we are looking for and the others give the necessary invariance of the quadratic enhancement.

Sketch of proof: The following drawings show how to replace a given L by a single embedded circle K with the same homology class in $H^1(F, \ \mathbf{Z}_2)$, by drawing an arc between some components, and with the same value under the map $\hat{q}$:

The corresponding equation is: $\hat{q}(K) \overset{(a)}{=} \hat{q}(K \amalg K_2 \amalg K_3) \overset{(b)}{=} \hat{q}(L) + \hat{q}(K_1) \overset{(c)}{=} \hat{q}(L)$;

The isotopy invariance follows once we notice that choosing a neighborhood W of K in F, a small perturbation of K by an isotopy h_t does not remove it out of W, and $K_t = h_t(K)$ represents the core in (mod 2) homology either of an embedded annulus or a Möbius band (resp. denoted by A and M). Then if $\hat{q}(K_t)$ is constant on each such small perturbation, then by connectedness we should have $\{t \in [0, \ 1] | \hat{q}(K_t) = \hat{q}(K)\} \simeq [0, \ 1]$, which would imply the invariance under isotopy. Suppose then that K_0 is the core of an embedded annulus $A \subset F$ with boundary $K_2 \amalg$

K_1; by applying the arguments of the above drawings (smoothing the intersection points of a connecting arc between components so that we obtain two new bounding disks) we have $\hat{q}(K_2) = \hat{q}(K_1) = \hat{q}(K_0)$; hence $\hat{q}$ does not depend on the representant of the isotopy class of the core.
Furthermore $\hat{q}(K)$ is even, so *any curve in F with trivial normal bundle has even $\hat{q}$*. Indeed, let C be a copy of K pushed off itself in the annular structure, then (a) and (b) give (where the third equality is obtained again using the splitting procedure) $2\hat{q}(K) = \hat{q}(K) + \hat{q}(C) = \hat{q}(K \amalg C) = 0$.
Finally, taking two different representants K_1 and K_2 of the homology class $[K] \in H_1(A, \mathbf{Z}_2)$ of K, intersecting each other transversally, we see that (b), when applied to K_1 and K, gives: r is even since $\hat{q}(K)$ and $\hat{q}(K_1)$ are even; hence $\hat{q}(K) = \hat{q}(K_1)$. Similarly $\hat{q}(K) = \hat{q}(K_2)$, so $\hat{q}(K_2) = \hat{q}(K_1)$
The proof for an embedded Möbius band M is similar: take representing curves K_0 of the core, and K_1 and K_2 in the same homology class in $H_1(M, \mathbf{Z}_2)$, intersecting each other transversally in distinct points. Applying (b) to the three pairs of circles gives:

$$\forall 0 \leq i,\ j \leq 2,\ i \neq j:\ \hat{q}(K_i) + \hat{q}(K_j) = 2.$$

Adding these equations and comparing with the initial ones shows that $\hat{q}(K_2) = \hat{q}(K_1) = \hat{q}(K_0)$ and each is odd, hence *whenever the normal bundle to K is non trivial $\hat{q}(K)$ is odd.*
Now the condition $b)$ applied as in the proof for an annulus gives $\hat{q}(K) = \hat{q}(K_1)$.

The homology invariance in F is a bit more complicated. Given transversally intersecting (by isotopy) links L_1 and L_2 representing the same class in $H_1(F, \mathbf{Z}_2)$, (b) implies the existence of a null homologuous link L_3 such that $\hat{q}(L_3) = \hat{q}(L_1) + \hat{q}(L_2) + 2 \cdot r$; since the parity of $\hat{q}$ is preserved for links in the same homology class, we have only to show that $\hat{q}(L_3) = 0$. But L_3 (that we suppose to be connected) is null-homologuous, hence L_3 has trivial normal bundle in F so that $\hat{q}(L_3)$ is even, and there exists a 2-manifold $W \subset F$ with $\partial W = L_3$. If W is not a disk, write $W = W_1 \overset{\partial}{\cup} V$, where V is either a twice punctured torus or a punctured Möbius band and W_1 has larger Euler characteristic than W. Denoting the boundary of V by $\partial_l V = \partial_0 V \amalg \partial_1 V$, an induction on the Euler characteristic implies that we are done if we show $\hat{q}(\partial_0 V) = \hat{q}(\partial_1 V)$. But it follows respectively (using (b) and (c)) from $\hat{q}(\partial_0 V) = \hat{q}(\partial_1 V) = \hat{q}(m_1) + \hat{q}(m_2)$ (where m_1 and m_2 are meridians on either sides of the holes) in the toral case, and from $\hat{q}(\partial_0 V) + \hat{q}(\partial_1 V) = 0$ in the Möbius band case. Hence the induced $q:\ H_1(F,\ \mathbf{Z}_2) \to \mathbf{Z}_4$ inherits the quadratic properties of $\hat{q}$. $\square$

Description of $\hat{q}$: we use some facts stated in the preceding chapter. Let K be an embedded circle in a surface F, and embed F as the zero section of a line bundle λ with $\omega_1(\lambda) = \omega_1(F)$ over F. The total space $E(\lambda)$ is a Spin 3-manifold. Fix a homotopy class of trivialization of $\tau = TE(\lambda)_{|K} = TS^1 \oplus \nu_{K \subset F} \oplus \nu_{F \subset E(\lambda)}$, where ν denotes normal bundles. Since TS^1 is trivial, an orientation of τ picks out a trivialization, hence we orient each line bundles in a point $p \in K$ so that the orientation on τ agrees with that coming from the Spin structure. Note that

the framing of $\nu_{K\subset F} \oplus \nu_{F\subset E}$ (a 2-plane bundle) corresponding with this chosen orientation is acted on by $\pi_1(S0(2)) \simeq \mathbf{Z}$ although the framings of a 3-plane bundle corresponds to $\pi_1(SO(3)) \simeq \mathbf{Z}_2$: consequently we will say that τ picks out an *even* set of framings of $\tau' = \nu_{K\subset F} \oplus \nu_{F\subset E}$.
Now the obstruction to extend a given vector field on K to τ' as a subline bundle can be measured by the number of *right half twist* that $\nu_{K\subset F}$ makes in a complete traverse of K; it is only well defined $\pmod 4$ as we have seen in the proof. Choose an odd framing on τ' and use it to compute this number: this is $\hat{q}(K)$. Another choice of odd framing will change the count by a multiple of 4, so the specific choice of odd framing is irrelevant. It is elementary to verify that $\hat{q}$ does not depend on the point p or on the local orientations made at p. Finally, we have to verify that $\hat{q}$ satisfy the conditions (b) and (c) in the lemma. As for (c), the framing of the three trivial line bundles (given by the local orientations at p) induces the stable Lie group framing over K; so it is an odd framing. To show (b), it suffices to see that we can remove a crossing without changing the count in the framing coming from a small disk neighborhood of it. Hence the choice of an odd framing (which introduces a full twist) gives a contribution of 2, and this end the construction of $\hat{q}$.

Consequences: Let us denote by $Quadr(\cdot)$ the set of quadratic enhancements of the intersection form on $H_1(F,\ \mathbf{Z}_2)$; we have defined a function:

$$\phi :\ \{Pin^- \ structures\ on\ F\} \to Quadr(\cdot)$$

We can act simply transitively by $H^1(F,\ \mathbf{Z}_2)$ on the set of quadratic enhancements by defining $\gamma \cdot q(y) = q_\gamma(y) = q(y) + 2\cdot\gamma(y) \in \mathbf{Z}_4,\ \forall\gamma \in H^1(F, \mathbf{Z}_2)$.
Note that acting on Pin^- structures by $\gamma \in H^1(F,\ \mathbf{Z}_2)$, we reverse even and odd framings on embedded circles $K \subset F$ *for which* $\gamma(K) = -1$. Since the effect is to add 2 to $q(x)$ *if* $\gamma(x) = -1$, the induced action on $Quadr(\cdot)$ gives exactly the definition of q_γ, hence ϕ is natural for the action of $H^1(F, \mathbf{Z}_2)$.
Notice that ϕ is canonical; *this is different* from the $1-1$ correspondance between $H^1(F,\ \mathbf{Z}_2)$ and the set of Pin^- structures, which requires the choice of a base point in the first set. Then we have proved:

[**Theorem KT, 3. 2**]: *There is a canonical 1 - 1 correspondance between Pin^- structures on a surface F and quadratic enhancements of the intersection form.*

Now we can use Brown's work on the generalization of the Kervaire invariant to get the structure of $\Omega_2^{Pin^-}$: Brown [**Br**] establishes a method for constructing functions on the homology of a manifold M^{2n} endowed with some structure, satisfying:

$$\phi :\ H^n(M,\ \mathbf{Z}_2) \to \mathbf{Z}_2,\quad \phi(u+v) = \phi(u) + \phi(v) + (u \cup v)(M)$$

where $\cup$ denotes the cup product on the cohomology ring of M. In case $M \subset W$ is a boundary, Brown shows that every such map verifies the following property: if $i:\ M \hookrightarrow W$ is the inclusion, then $\phi i^* = 0$. It allows him (with a lot of algebraic machinery) to define an homomorphism $K:\ \Omega_{2n}(G) \to \mathbf{Z}_8$, where $\Omega_{2n}(G)$ denotes the cobordism group based on orientable G-manifolds (manifolds endowed with an action of the group G). For $G \simeq Pin^-$ and surfaces, we then have an homomorphism:

$\beta: \ \Omega_2^{Pin^-} \to \mathbf{Z}_8$. It is the "usual" Brown invariant that we shall use, and which is defined in Appendix A. Then

Theorem: *We have an isomorphism:*

$$\beta: \ \Omega_2^{Pin^-} \xrightarrow{\sim} \mathbf{Z}_8$$

The $\pmod 2$ *reduction of* β *is the* $\pmod 2$ *reduction of the Euler characteristic and hence determines the unoriented bordism class of the surface.*

Proof: the two trivial enhancements $\pm\gamma$ of the reduction $\pmod 2$ of the intersection form of $\mathbf{RP^2}$ generates the Witt group $\mathrm{W}(\mathbf{Z}_2, \ \mathbf{Z}_4)$, which is isomorphic to $\mathbf{Z}_8$; see Appendix A. Moreover $\beta(\pm\gamma) = \pm 1 \in \mathbf{Z}_8$, hence β is an epimorphism. But the reduction $\pmod 2$ of the induced homomorphism from the Witt group is the dimension $\pmod 2$ of the underlying vector space (see Appendix A); whence the second result. $\square$

Here is a simple geometric construction to see the injectivity of the map β. Suppose that, given a surface F, we have $\beta(F) = 0$: the 2-manifold F is an unoriented boundary of a 3-manifold W^3. We are going to look after a more simpler representant of the Pin^- bordism class of F in W^3: consider the Poincare-dual circle $K \subset W \setminus F$ of the obstruction class in $H^2(W, \ \partial W; \ \mathbf{Z}_2)$ to the extension of the Pin^-structure of F across W, and take the boundary of a neighborhood in W of K. Then F is Pin^- bordant to this surface S, a torus or a Klein bottle, with $\beta(S)$ equal to zero. The obstruction to extend the Pin^- structure of $W \setminus K$ across a neighborhood of K translates into a non zero value of q on $H_1(S, \mathbf{Z}_2)$.
In the Klein bottle case, the obstruction must lie over a curve l whose self intersection is even, that is a preserving orientation curve ($\mathbf{RP}^1$ is a Pin^- boundary): but this contradicts the fact that an enhancement q on a Klein bottle with such an obstruction has non zero Brown invariant (q does not split, with the terminology of the Appendix). Thus the boundary of K must be a torus. But for a torus β is zero if and only if q vanishes on the remaining classes of $H_1(T, \mathbf{Z}_2) \setminus l$, and a Pin^- boundary is obtained by filling the disk bundle of the torus with the help of the two generating sections.

Next we investigate how is the behaviour of β under change of Pin^- structures. With the usual notations, let q_a be the image of the quadratic enhancement q under the action of $a \in H^1(F, \ \mathbf{Z}_2)$. We have:

Lemma : $\beta(q_a) = \beta(q) + 2 \cdot q(a)$

Proof: We can check this formula for the two generators of the Witt group, and the result follows from the additivity of the Brown invariant. $\square$

In view to obtain the most general extension of Rohlin 's theorem, we need to investigate the structure of the characteristic cobordism group in dimension 4.

4.3 Calculus of Ω_4^{char}

Following the lines of [**GM**], we aim at showing that there is a short exact sequence:

$$0 \to \Omega_4^{char} \xrightarrow{\chi} \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_8 \xrightarrow{\pi} \mathbf{Z}_{16} \to 0$$

$$0 \mapsto (M,\ F,\ \mathcal{F}) \mapsto (\sigma(M),\ F \cdot F,\ \beta(M,\ F,\ \mathcal{F})) \mapsto \sigma(M) - (F \cdot F + 2\beta(M,\ F,\ \mathcal{F})$$

Here we denote by $\mathcal{F}$ the choice of a trivialization of $TM_{|M \setminus F}$ which does not extend over F. This sequence imply in particular that the map

$$(M,\ F,\ \mathcal{F}) \mapsto (\sigma(M),\ F \cdot F)$$

is an isomorphism onto a subgroup of index 2 of $\mathbf{Z} \oplus \mathbf{Z}$ defined by $\{(x, y) \in \mathbf{Z} \oplus \mathbf{Z} / x - y \equiv 0 \pmod 2)$. The proof will give the generators.

Proof:
1) Consider the pair $(\mathbf{C}P^2,\ \mathbf{C}P^1)$: we have $\sigma(\mathbf{C}P^2) = \mathbf{C}P^1 \cdot \mathbf{C}P^1 = 1$ and obviously (with the single trivialization $\mathcal{CP}^1$ that is possible) $\beta(\mathbf{C}P^2,\ \mathbf{C}P^1) = 0 \pmod 8$. Hence one can bring the study of

$$\{(x,\ y,\ z) \in \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_8 \text{ such that } x = y + 2z \pmod{16}\}$$

back to

$$\{(0,\ y,\ z) \in \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_8 \textit{ such that } y = -2z \pmod{16}\},$$

by the disjoint sum of the given pair $(M,\ F)$ with some $(\mathbf{C}P^2,\ \mathbf{C}P^1)$.

2) Consider the pair $(S^4,\ \mathbf{RP}^2_\pm)$, where we note $\mathbf{R}P^2_\pm$ the two distinct embeddings of the projective plane in $\mathbf{R}^4$, which correspond to the two distinct writhes of the Möbius trip $M \subset R^4$. We have $\sigma(S^4) = 0$, and $\beta(S^4,\ \mathbf{R}P^2) = \pm 1 \pmod 8$ (see below the proof of the Guillou-Marin formula) and $\mathbf{RP}^2_\pm \cdot \mathbf{RP}^2_\pm = \mp 2$.
Indeed, consider $\mathbf{RP}^2_+$, and M_+ (the positive Möbius trip) where $\mathbf{RP}^2_+$ is obtained by gluing a disk Δ_+ in $\mathbf{R}^4_- = \{(x,\ y,\ z,\ t)/t \geq 0\}$ perpendicularly along the central circle of M_+. Under a widering of $M_+ \subset \mathbf{R}^4$ (which gives M') and a slight vertical move, we obtain a vertical annulus $\partial M' \times [0,\ r] \subset \mathbf{R}^3 \times [0,\ r]$ whose sides are parallel closed curves to ∂M_+.
Attach a disk $\Delta' \subset \mathbf{R}^4$ to $\partial M' \times \{0\}$, isotopic to Δ_+ and intersecting it in general position. Finally, let $\mathbf{R}P' = M' \cup (\partial M' \times [0,\ r]) \cup \Delta'$. Now we have $\mathbf{R}P^2 \cdot \mathbf{R}P^2 = \mathbf{R}P' \cdot \mathbf{R}P^2 = \Delta' \cdot \Delta_+ = -lk(\partial \Delta',\ \partial \Delta_+) = -lk(\partial M' \times \{0\},\ \partial M_+) = -2$; the minus sign in the third equality results from the fact that the orientation of $\mathbf{R}^4_-$ (where lie both Δ and Δ') is induced from $\mathbf{R}^4 = \mathbf{R}^3 \times \{\textit{last coordinate}\}$ and is opposite to the orientation that makes $\mathbf{R}^3$ a boundary.

Since any non degenerate bilinear symmetric form on $\mathbf{Z}_2$ decomposes in a direct sum of factors [**HNK**] of the type:

$$(1) \textit{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(whose characteristic elements are respectively 1 and 0), we have

$$F \cdot F \equiv dim(H_2(M,\ \mathbf{Z}_2)) \equiv \sigma(M) \pmod 2.$$

Consequently, if $\sigma(M) = 0$, then $F \cdot F$ is even. Summing disjointly z times the result of the preceding operation 1) with the pair $(S^4,\ \mathbf{RP}^2_\pm)$ shows that π is an epimorphism.

Next we show that χ is a monomorphism. First note that we may obtain a 1-connected manifold $(M_1,\ F,\ \mathcal{F})$ from $(M,\ F,\ \mathcal{F})$ by adding handles of index 2 with even surgery coefficients on circles disjoints from F (then we do not modify characteristic elements), so that the resulting bordism between the two pairs is characteristic; then we can turn F into an *oriented* characteristic surface in the same homology class.

Let us give another way to define the quadratic enhancement q defined above (this is the initial treatment of [**GM**]):
We want to set another way to get a well-defined "even" framing of the normal bundle of the characterized surface $F \subset M_1^4$, *when* $i_\star(H_1(F, \mathbf{Z}_2)) = 0$, the map $i : F \subset M_1$ being the inclusion. Take an embedded curve k in F and cap it off with a surface B (not necessarily orientable nor embedded) in M_1.
The normal bundle $\nu_{B\subset M_1}$ to B in M_1 splits off a trivial line bundle since B is a punctured surface, with bundle complement the determinant line bundle for the tangeant bundle of B. The sections are acted on by $H^1(B,\ \mathbf{Z}_2^{\omega_1})$ where $\mathbf{Z}^{\omega_1}$ denotes $\mathbf{Z}$ cofficients twisted by $\omega_1(\nu_{B\subset M_1})$. Then the restriction ν_k of $\nu_{B\subset M_1}$ to the boundary circle k inherits an "even" framing. Recall that we defined it as follows in §4.2: it is the Pin^- structure of the total space of the normal bundle that gives the Pin^- structure of M_1^4 restricted to k, when added to the Pin^- structure on $S^1 = k$ which makes it into a Spin boundary. Now we have to verify that the quadratic enhancement of the intersection pairing on F may be calculated with the help of B: in other words, we have first to verify that these two ways to construct even framings are the same?
Suppose that $B \subset M_1$ does not intersect F except along ∂B. The total space of the determinant line bundle of the tangeant bundle to B is a Spin 3-manifold W embedded in M_1, in which k bounds B. Consider then the following general problem: given a Spin 3-manifold W with a knot k which class is zero in $H_1(W, \mathbf{Z}_2)$, any Spin structure on W induces the same Spin structure in a neighborhood of k. Hence the notion of even framing does not depend on the ambiant Spin structure for these classes. Is it true that any surface in W with boundary k selects a longitude for it that represents an even framing ? Let T^2 be the boundary of a tubular neighborhood of $k \subset W$: the image of $H_1(T^2,\ \mathbf{Z}_2) \to H_1(W \setminus k,\ \mathbf{Z}_2)$ is generated by the meridian, and the kernel contains a unique element, *the* $\pmod 2$ *longitude*. More generally, we say that $l \in T^2$ is an even longitude if its homological $\pmod 2$ reduction is the $\pmod 2$ longitude. Now, any embedded surface $B \subset W$ such that $\partial B = k$ can be chosen so that it intersects T^2 transversally in a given even longitude. Moreover, the restriction to k of the normal bundle to B in W is trivial, so that the surface frames the normal bundle to k in W. The stability lemma assigns a Pin^- structure to B from that of W, and the Spin structure on W restricted to k is the sum of the Spin

structure on k coming from the restriction of the Pin^- structure on B *plus* the Spin structure on the normal bundle $\nu_{k\in W}$ coming from the framing induced by B. Since the boundary circle receives the non-Lie structure, we get the previous definition of the even framing in § 4.2.
Hence we finally see that in our initial situation, the framing on ∂B is an even one in the sense of §4.2.

Remark also that if B also intersects F away from k (assumed transversally), the punctured surface $\hat{B} = B \amalg D^2$ lies in $M_1 \setminus F$ so each circle of this transverse intersection has the non bounding Spin structure. Then, the only way to have an even framing on ∂B is that the $\pmod 2$ intersection number of F and B is zero. Moreover, the number of right half twists mod 4 that $\nu_{k\subset F}$ makes in a complete traverse of k is just the obtruction to extending the section given by $\nu_{k\subset F}$ over all of B. Then the enhancement of the intersection pairing of F defined in §4.2 and above are the same. But notice that in §4.2 we did not need the condition that $i_\star(H_1(F, \mathbf{Z}_2)) = 0$, since we did not use any membrane to select the Pin^- structure on k.

This setup allows us to define the quadratic enhancement q in § 4.2 as:

$$\forall x \in H_1(F,\ \mathbf{Z}_2)\ q(x) = o + 2B \cdot F \pmod 4$$

where o is the obstruction to extend the normal line bundle to a representant of x in F to a rank one subbundle of the normal bundle in M_1 of any B such as above. It gives a convenient way to visualize and to calculate q. Note that each connected components of ∂B has an annulus as tubular neighborhood if and only if the obstruction o is even. In particular, if the characteristic surface F is orientable we have :

$$q(x) = 2(o(v) + B \cdot F) \pmod 4$$

where $o(v)$ is the obstruction to extend a normal vector field to ∂B in F into a normal vector field to ∂B in B without zero (the factor two comes from the fact that the natural row $S^1 \to \mathbf{RP}^1$ has degree 2).

Let us suppose that we have already done a surgery on the pair $(M,\ F)$ to the characteristically bordant one $(M_1,\ F)$. Denote the new pair as the former one.

Lemma [**GM**], p. 111: *Let c be a simple closed curve of F; the result $(M',\ F')$ of a surgery of pair on $(M,\ F)$ along c is a characterized pair if and only if q(c) is equal to* 0

Proof: Notice that $c \cdot c \equiv 0 \pmod 2$ since the surgery is possible. Take a membrane P for c and the 2-disk D^2 that is the core of the surgery (i. e. the core of the handle attached to B^4 along S^3 along c) to form the closed 2-cycle Σ. Then we have $q(c) \equiv 2(o(v) + P \cdot F) \equiv 2(\Sigma \cdot \Sigma + \Sigma \cdot F') \pmod 4$ with the above notations. the Wu formula concludes.
Conversely, let Σ be a 2-cycle of M' supposed to intersect transversally the co-core

of the surgery in n points. then we can deform Σ so as it is the union of n disjoints translated cores of the surgery and a membrane P for nc in M. Then we have:

$$2(\Sigma \cdot \Sigma + \Sigma \cdot F') \equiv 2(o(v) + P \cdot F) \equiv q(nc) \equiv nq(c) \equiv 0 \pmod 4$$

Hence F' is characteristic in M'. □

Proposition: *With the same setup, if c is a simple closed curve in F, we have $q(c) = 0$ if and only if the bordism of a surgery of pair on $(M,\ F,\ \mathcal{F})$ along c is characteristic.*

Proof: If $q(c) = 0$, the preceding lemma shows that the result $(M',\ F')$ of the surgery on $(M_1,\ F)$ along c admits F' as a characteristic surface in M'. We need to verify that it preserves the trivialization of $TM_{1|M\setminus F}$ which does not extend across F through the bordism. The complement $(W,\ N)$ of a neighborhood of c in $(M_1,\ F)$ is the complement of the dual sphere of the attaching sphere of the handle of surgery. The difference of two trivializations of $T(W \setminus N)_{|2-skeleton(W\setminus N)}$ coming from trivializations of $T(M_1 \setminus F)$ and $T(M' \setminus F')$ over their two skeletons is an $x \in H^1(W \setminus N,\ \mathbf{Z}_2)$. But W is 1-connected so $H_1(W \setminus N,\ \mathbf{Z}_2)$ is generated by the meridians of $F \cap N$, hence the difference is zero on the meridians of $F \cap N = F' \cap N$. Then the two trivializations over the two skeletons are homotopic, and the bordism between $(M_1,\ F)$ and $(M',\ F')$ is characteristic. Transitivity concludes with the characteristic bordism between M and M_1. □

The converse is the following lemma:

Lemma: *Let $(M^4,\ F^2)$ be a characteristically bounding pair of $(V^5,\ G^3)$, let $\Delta^2 \subset G^3$ be a surface and P be a membrane of F in M such that $c := \Delta \cap F = \partial\Delta = \partial P$; then $q(c) = 0$.*

Sketch of proof: The point is to find an appropriate characteristic surface H^2 for $N^4 = \partial(V \setminus \overset{\circ}{W})$, where $\overset{\circ}{W}$ is a tubular neighborhood of Δ^2 in V^5. Suppose that $\partial\Delta^2$ is connected.
Denoting by $o(v)$ the obstruction to extend a normal vector field v to ∂P in F in a normal vector field to P without zero, and using the Wu relation, we then try to show that $o(v) + P \cdot F = 0 \pmod 2$ since the self-intersection of $\partial\Delta$ in F (which is the boundary of the self intersection of Δ in G^3) is $0 \pmod 2$. Take two transverse (with each other and with Δ^2) sections s and s' of $\nu_{G^3 \subset V^5}|\Delta^2$ and consider a section t of $\mu_{\Delta^2 \subset G^3}$ which is identical to v on $\partial\Delta$ and without zero on a neighborhood of $\Delta^2 \cap (s(\Delta^2) \cup s'(\Delta^2))$. Use it to push P^2 out of itself with the help of Δ^2.
Set $\Sigma^2 = (P \setminus \overset{\circ}{W}) \cup (s \oplus \rho t)(\Delta^2)$, where ρ will be an adequatly chosen smooth function (see the end of the proof); take also $H^2 = N^4 \cap G^3 = (F \setminus \overset{\circ}{W}) \cup \partial U$, where U is a tubular neighborhood of Δ^2 in G^3. Finally, take a section of the normal bundle $\nu_{P \subset M}$ which coincide with v on Δ; Move slightly any of the preceeding objects to get transversality.
Then

$$\Sigma \cdot \Sigma = ((P \setminus \overset{\circ}{W}) \cup (s \oplus \rho t)(\Delta^2)) \cdot (u(P \setminus \overset{\circ}{W}) \cup (s' \oplus (\rho')t)(\Delta^2),$$

where ρ' is a function as above. Since $P \cdot u(P) \equiv o(v) \pmod 2$, we get

$$P \cdot u(P) + s(\Delta^2) \cdot s'(\Delta^2) = o(v) + s(\Delta^2) \cdot s'(\Delta^2) \pmod 2$$

Finally, the relation $\Sigma \cdot H = ((P \setminus \overset{\circ}{W}) \cup (s \oplus \rho t)(\Delta^2)) \cdot ((F \setminus \overset{\circ}{W}) \cup \partial U) = P \cdot F + s(\Delta^2) \cdot \Delta^2 \pmod 2$ with the Wu formula gives the result.
If $\partial \Delta^2$ is not connected, make tunnels along disjoints arcs from G^3 to connect the components of ∂V, iterate this consecutively to modify G^3 (in a neighborhood of connecting embedded arcs in ∂V) into a properly embedded G^3 with connected boundary and to the pair $(G^3,\ \Delta^2)$ to connect Δ^2. $\square$

Let us return to the proof of the injectivity of χ. First, $(M,\ F,\ \mathcal{F})$ characteristically cobounds $(N,\ S^2,\ \mathcal{G})$ if and only if $\beta(M,\ F,\ \mathcal{F}) = 0$. Indeed, this last condition imply that the associated quadratic form on $H_1(F,\ \mathbf{Z}_2)$ splits by an isotropic half dimensional subspace which may be seen as generated by disjoints simple closed curves in F. But a surgery along these curves is characteristic, and the obtained surface verifies $H_1(F',\ \mathbf{Z}_2) = 0$; so $F = S^2$.
The converse follows also from the proposition. Adding the condition that $F \cdot F = 0$, we see that we have a characteristic bordism with $(N,\ \emptyset,\ \mathcal{G})$ since $S^2 \cdot S^2 = 0$ implies that we can glue an handle to S^2 without changing the characteristic element. The condition $\sigma(M) = 0$ plus $\Omega_4 \equiv \mathbf{Z}$ and the inclusion $\Omega_4^{Spin} \hookrightarrow \Omega_4$, with this discussion give that $(M,\ F,\ \mathcal{F}) \in \Omega_4^{char}$ is zero. The following paragraph sketch how to do this in an even more general situation.

An argument of desingularization in Knesser's manner shows that if M^4 is a closed oriented 4-dimensional manifold and $\sigma(M) = 0 = F \cdot F$, then $(M,\ F) = \partial(V,\ G)$ where V^5 is a 5-dimensional compact oriented manifold and G^3 is a characteristic submanifold. The point is to modify a characteristic relative 3-cycle $\pmod 2$ denoted G^3 (given by the obstruction theory) such that $\partial G = F$ into an $\pmod 2$ homologuous submanifold G^3 relative to the boundary ∂G^3.
There we use the fact that the property $F \cdot F = 0$ is transfered (when V is orientable) to bordant singular surfaces L ($= link(singular\ tree)$ in G^3) in S^4 ($= link(singular\ tree)$ in V^5) which will consequently bound; indeed there is a non zero section s of the normal disk bundle E of L in S^4 that we can twist so that $s(L)$ bounds in $S^4 \setminus \overset{\circ}{E}$;hence also $L = \partial W$. But this gives the possibility to smooth G^3 in its $\pmod 2$ homology class, by the desingularization of the singular tree of G^3 with the help of W. For more details, we refer to [**GM**], p. 109.

In conclusion we have sketched that $\Omega_4^{char} \simeq \mathbf{Z} \oplus \mathbf{Z}$, with generators $(S^4,\ \mathbf{RP}^2)$ and $(\mathbf{CP}^2,\ \mathbf{CP}^1)$.$\square$

4.4 Guillou-Marin formula

.

All the needed material has already been done to give the more general extension of the Rohlin characteristic theorems, called the Guillou-Marin formula. Note that the version given here is different from [**GM**], p. 98 in that we do not need the nullity of $Im(i_\star)$, where $i_\star :\ H_1(F^2,\ \mathbf{Z}_2) \to H_1(M^4,\ \mathbf{Z}_2)$, i. e. their condition of existence of a well-defined quadratic form on F induced by the characterization of the pair $(M^4,\ F^2)$. It has been observed in [**KT**], using Pin^- structures and the descent theorem in § 3.

Theorem [**KT**, 6. 3]: *Let M^4 be an oriented 4-manifold, and suppose we have a characteristic structure on the pair $(M,\ F)$. The following formula holds:*

$$2 \cdot \beta(F) = F \cdot F - \sigma(M) \pmod{16}$$

where the Pin^- structure on F is the one induced by the characteristic structure on $(M,\ F)$ as in the "Descent Theorem".

Proof: The formula is trivially verified for $(\mathbf{CP}^2,\ \mathbf{CP}^1)$ and $(S^4,\ \mathbf{RP}^2)$, using the values of the self intersection and the Brown invariants for both pairs. Note that we can either do as follows: the right-handed $\mathbf{RP}^2$ can be constructed by capping off the "positive" Möbius trip Mb in the equatorial S^3 of S^4 with a ball in the northern hemisphere, our vector field is the north-pointing normal and so the even framing on the bundle $\nu_{Mb \subset S^3}|k$ is the one given by the 0-zero framing of S^3 (where k is the core of the Möbius band). The number of half twists may consequently be counted in S^3, and it is 1.$\square$

We list some corrollaries:
1) The first Rohlin theorem (1952): A smooth oriented Spin 4-manifold verifies: $\sigma(M^4) \equiv 0 \pmod{16}$.

2) Let M^4 be a Spin 4 manifold and F^2 a characteristic surface then: $F \cdot F \equiv -2\beta(M,\ F) \pmod{16}$.

This is a generalization of a theorem of Whitney: if F^2 is a surface in S^4 then we have: $F \cdot F \equiv -2\chi(F) \pmod{4}$.
3)In case F^2 is an *orientable* surface in M^4, we use $\beta(M,\ F) \equiv 4\ Arf(M,\ F) \pmod{4}$ (see Appendix A and the following chapter) to get: $Arf(M,\ F) \equiv (\sigma(M) - F \cdot F)/8 \pmod{2}$, where $Arf(M,\ F)$ is the Arf invariant of the quadratic enhancement q on F induced by $(M,\ F)$.

In view to link this theorem to the so called Rohlin invariant, we will now give a brief account of invariants of knots deduced from the Brown invariant, and then deal with relations between linking pairing, quadratic enhancements and the Brown invariant.

Chapter 5

Some classical invariants of knots and surfaces in 3-manifolds

Otherwise stated, all manifolds are supposed to be smooth.

5.1 Generalization of Robertello's invariant

Consider a 3-manifold M^3 with a given Spin structure, and $L: \amalg S^1 \hookrightarrow M^3$ a dual link to $\omega_2(M) = 0$. Note that $0 = [L] \in H_1(M,\ \mathbf{Z}_2)$. Let us fix a characterization $\mathcal{L}$ of $(M,\ L)$, i. e. a Spin structure over $M \setminus L$ which does not extend to any component of L.
In view to associate the Brown invariant of a surface to L, we need to define without ambiguity a Pin^- structure on a "spanning surface" of L, determined by the characterization $\mathcal{L}$.

1) We say that L is characterized if each component of L inherits from $\mathcal{L}$ the Pin^- bounding structure on his normal bundle.
2) $(M,\ L)$ is said to be characterized if and only if there exists an element $\lambda \in H^1(M \setminus L,\ \mathbf{Z}_2)$ that, when acting on the fixed Spin structure $\mathcal{L}$ of $M \setminus L$, gives the restriction of the one on M ($\delta^*\gamma \in H^2(M,\ M \setminus L,\ \mathbf{Z}_2)$ hits every generator by the Thom isomorphism (for a proof, see the preceding chapter)).
3) There is an embedded surface F in M such that γ is dual to $F \overset{embedding}{\hookrightarrow} M \setminus E$, where E is the total space of an open normal disk bundle to L. Then $\partial F \cap S$, where $S = \partial E$, is a longitude in the peripheral torus of each component of L. It will be called an even longitude (in coherence with the definition of γ, which gives an even framing on $\nu_{L \subset M}$ (see chapter 3).

How can we distinguish the characterizations of $(M,\ L)$? A characterization will be called even if the Pin^- structure induced on each component of L by the Descent Theorem is the bounding one. It is easy to verify that L is an even link (i.e. is endowed with an even characterization): since L is characterized, any $K \subset L$ has an even framing on ν_K which selects a (mod 2) longitude on the peripheral torus.

Hence L is even if and only if the sum of these even longitudes is $0 \in H_1(M \setminus L,\ \mathbf{Z}_2)$. For exemple the Hopf link in S^3 is not even; and you can have $[L] = 0$ althought L is not even.

The difference between having even characterizations and having L characterized is that the first condition *selects* a set of even longitudes in the peripheral torus lying over each component $K \subset L$. The second one only precise the $\pmod 2$ chosen longitudes, which have a whole set of even longitudes lying over them. For exemple, two surfaces homologically dual to γ induce the same $\pmod 2$ longitude, but acting on a component K_0 of L by even integers we can find infinitely many dual surfaces to γ, each selecting another even longitude over K_0, but with the same set of $\pmod 2$ longitudes. Any set of even longitudes is induced as a boundary by an embedded surface in $M \setminus L$.

Let us take a spanning surface F for L. We have $\nu_{F\subset M} \simeq det(TF)$ (the determinant line bundle of the tangeant bundle to F), so that the orientations of the total spaces agree; note that the total space of the determinant line bundle is naturally oriented, and the total space of $\nu_{F\subset M}$ is oriented by the orientation on M. The stability Lemma implies that F inherits a Pin^- structure from the Spin structure on M. If we had considered the Spin structure on $M \setminus L$ besides the Spin structure on M to define the one on F, the Pin^- structure on F would have differred by the action of $\gamma_{|F} = \omega_1(F)$.
In any case the induced Pin^- structure on any component of L is the bounding one, since this is obviously the case when we consider the Pin^- structure on F induced by the one of $M \setminus L$, and this property is equivariant under the action of $\omega_1(M) \in H^1(M, \mathbf{Z}_2)$. Hence we have on any spanning surface F of L a well defined Pin^- structure which extends uniquely to its embedded closure $\overline{F} \subset M$.

Given a characterized pair $(M,\ L)$ with a set of even longitudes l, pick a spanning surface for L which induces l. We define:

$$\beta(L,\ l,\ M) = \beta(\overline{F})$$

where $\overline{F}$ is F with a disk added to each component of F, the Pin^- structure is extended over each disk, and β is the usual Brown invariant.

Remarks: We do not require our link to be oriented; to see what happens in this last case, consider an integral homology 3-sphere Σ with an oriented embedded link L. Denote by l_i the linking number of the ith component with the rest of the link: it corresponds to a longitude since the 0-linked pushed off of L_i is a preferred longitude, determined by the kernel of the map: $H_1(\Sigma \setminus L, \mathbf{Z}) \to H_1(\Sigma, \mathbf{Z})$.
Now we would like to know when L is even. We saw that a knot is even if and only if it is homologically $\pmod 2$ trivial. if $\forall k \subset L,\ [k] = 0 \in H_1(M,\ \mathbf{Z}_2)$ (in particular when M is a $\mathbf{Z}_2$-homology sphere), the $\pmod 2$ linking number of a component $K \subset L$ with the rest of the link is well defined. Taking an embedded spanning surface F for L, we see that the longitude picked out for a component of L is even if and only if its $\pmod 2$ linking number with the rest of the link is 0.
In particular the link $L \subset \Sigma$ is even iff each l_i is even: this is Robertello's condition

to define its Arf invariant $\beta(L,\ -l,\ \Sigma)$ of a link L in an integral homology sphere. Note that the Spin structure on Σ is unique as in any $\mathbf{Z}_2$-homology sphere, and there is a unique way to characterize an even link. Here is a simple way to calculate $Arf(L)$: take a Seifert surface for L (i. e. an orientable spanning surface) and Let $q:\ H_1(F^2,\ \mathbf{Z}_2)$ be the quadratic enhancement of the (mod 2) intersection pairing of F ($q(\gamma) =$ the (mod 2) number of full twists of a push off of an embedded circle c representing γ in a neighborhood $\nu_{c\subset F}$ of c in F). Then $Arf(L) = Arf(q)$ is the sum over the generators γ_i of $H_1(F^2,\ \mathbf{Z}_2)$ of $q(\gamma_i)q(\gamma_{i+1})$.

We end this section by a statement showing how the invariant depends upon the characterizations: it is the "knot" counterpart to the result in chapter 3 concerning the effect on the Brown invariant of the action of $H_1(F, \mathbf{Z}_2$ on $Quadr(\cdot)$ We refer for the proof to the original source, and to the next sections for the definition of Rohlin's μ invariant:

Theorem [KT], 8. 2: *Let L_i be a characterized link, with l_i a collection of even longitudes, in a 3 manifold M_i supposed to have a given Spin structure. Let $(W^4,\ F^2)$ be a Spin bordism between $(M_1,\ L_1)$ and $(M_2,\ L_2)$, with W^4 oriented. Pick one section of the normal bundle $\nu_{F\subset W}$ on every non closed component of F (which splits a trivial line bundle) so that the longitudes l_i selected for each component L_i are even. With the Pin^- structure on F inherited from $(M,\ F)$, each component of ∂F bounds, and hence F has a Brown invariant, as we saw above.*
Orient W so that M_1 receives the reverse Spin structure; then we have:

$$\beta(L_2,\ l_2,\ M_2) - \beta(L_1,\ l_1,\ M_1) = -\beta(F) - \sigma(W) - \mu(M_2) + \mu(M_1)$$

Let us precise first what do we mean by "reversing the Spin structure" (similarly, reversing the orientation, with $SO(n)$ in place of $Spin(n)$ and $O(n)$ in place of $Pin^{\pm}(n)$) on a given vector bundle ζ. Suppose that ζ is given by transition functions $g_{i,\ j}$ defined into $Spin(n)$ and based on a numerable cover $\{\mathcal{U}_i\}$ of the base space. recall that as a set $Pin^{\pm}(n) = Spin(n) \amalg Spin(n)$, and chose maps $h_i:\ \mathcal{U}_i \to Pin^{\pm}(n) \setminus Spin(n)$. Consider the bundle with transition functions $h_i \circ g_{i,\ j} \circ h_i^{-1}$: it is, by definition, the bundle obtained by reversing the Spin structure over ζ. The choice of the maps h_i is obviously not unique but any two choices yield equivalent $Spin(n)$ bundle, and these maps yield a $Pin^{\pm}$ equivalence with the original bundle.

Theorem [KT], 8. 3: *Let $L \subset M^3$ be a characterized link with two sets of even longitudes l and l'. Let $2r$ be the sum of the integers which act on the set of longitudes l to give l' (there is one even integer for each component). Then we have:*

$$\beta(L,\ l',\ M) = \beta(L,\ l,\ M) + r \pmod 8$$

Proof: We can construct a spanning surface for l' given a spanning surface F_1 for the longitudes l. Take a neighborhood $W = T^2 \times [0,\ 1]$ of the peripheral torus , and embed a surface V in W, which intersects $T^2 \times 0$ in l and $T^2 \times 1$ in l' and with no boundary components in the interior of W. We want also that it induces the zero map $H_2(V,\ \partial V,\ \mathbf{Z}_2) \to H_2(W,\ \partial W,\ \mathbf{Z}_2)$. The restriction on W of the Spin structure on M is the stabilization of one on T^2, i. e. it has enhancement 0 on the longitude

and the meridian (since l is even). Since the Pin^- structure can be locally evaluated (due to the Extension Lemma in chapter 2), we see that the invariant β evaluated on $F_2 = F_1 \cup V$ is equal to $\beta(F_1) + \beta(V)$; but $\beta(V)$ depends only on the geometry of the surface V and the Spin structure in W, and these are independant of the link L. So we may calculate the difference between the $\beta' s$ using the unknot.
Then we have only to see what happens when you go from the zero longitude to the 2 longitude (2 twists added to the zero longitude), since the effect on β of successive addition of kinks to a given longitude is additive. The 2 longitude is given by the Möbius band, which inherits a Pin^- structure that extends uniquely to one on $\mathbf{R}P^2$, and $\beta(\mathbf{RP^2}) = 1$. □

As for Robertello's link invariant, β is a link concordance invariant; it will allow us to drop the (non canonical) choice of even longitudes:
Recall that a link concordance between characterized links $L_0 \subset M$ and $L_1 \subset M$ is an embedding of $(\amalg S^1) \times [0,1] \subset M \times [0,1]$, with $(\amalg S^1) \times i$ being L_i for $i = 1,\ 2$. The concordance picks out a set of longitudes l_1 on L_1 from a given even one l_0 on L_0, since there is a unique way to extend the initial even framing of $\nu_{L_0 \subset M}$ to a framing of $\nu_{(\amalg S^1)\times[0,1]} \subset M \times [0,1]$. By the same argument, we state the uniqueness of the extension of a characterization of $L_0 \subset M$ to a Spin structure on $M \times [0,\ 1] \setminus (\amalg S^1) \times [0,1]$, and hence to $M \setminus L_1$. Note that $(\amalg S^1) \times [0,1]$, when capped off with disks, is a union of spheres, so $\beta((\amalg S^1) \times [0,1]) = 0$.
We conclude that:

Corollary: *Let L_0 and L_1 be concordant links in M. Suppose L_0 is characterized and that l_0 is a set of even framings (i.e. even longitudes). Then the transport of even framings and Spin structures, described above, along the concordance gives a characterization of L_1,and a set l_1 of even framings. Furthermore $\beta(L_0,\ l_0,\ M) = \beta(L_1,\ l_1,\ M)$.*

We will use later these results in view to remove the set of even longitudes, and postpone any comments on their use by Kirby and Melvin as a geometric background for the splitting formulas of the quantum invariants τ_r.

5.2 Rohlin invariant

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Let us see how it works with an embedded link L in an integral homology 3-sphere Σ.
First consider $\Sigma = S^3$; the invariant β reduces $\pmod 2$ to the Spin bordism class of the orientable spanning surface selecting the unique set of even longitudes.
This can be seen as follows: consider $S^3 \times [0,\ 1]$ and add an 2-handle along L with odd index. The union $\hat{F} = F \cup B^2$, where B^2 is the core of the 2-handle, is a characteristic surface. Then fill $S^3 \times -1$ by a 4-Ball B^4 and kill the boundary with an orientable Spin 4-manifold: the new manifold is denoted by M^4. Then $\beta(L,\ S^3) \pmod 2 \equiv Arf(L,\ S^3) \equiv \Psi(M^4,\ \hat{F})$ (with the notations of the preceding chapters).

We see in this particular case that β is independant of the chosen Seifert surface F, since 2 Seifert surfaces F and F' for L are bordant in $S^3 \times [0,\ 1]$, and are to be identified when considered as characteristic pairs $(M^4, \hat{F})$ and $(M^4, \hat{F'})$ in Ω_4^{char} (using the above construction). Moreover, it shows that the bordism class $Arf(L,\ S^3) \in \Omega_2^{Spin}$ is obtained when doing surgery on L with an odd integral framing.
If we had taken L in an arbitrary $\mathbf{Z}$-homology sphere Σ, the above equality must be corrected by adding the characteristic bordism class $Arf(L')$, where L' is a framed link in S^3 such that Dehn surgery on L' gives Σ: $Arf(L, \Sigma) = Arf(L, S^3) + Arf(L', S^3) \in \mathbf{Z}_2$.

The Rohlin theorems takes place in this setup: in particular we have

$$Arf(L,\ \Sigma) = Arf(L,\ S^3) + Arf(L',\ S^3) = \sigma(M^4)/8 + \sigma(M'^4)/8 \pmod 2.$$

Note that if we consider a pair $(M^4,\ F^2)$ with M^4 a 1-connected manifold and the dual homology class to ω_2 may be represented by a smooth embedded 2-sphere, then $(F \cdot F - \sigma(M))/8 \equiv 0 \pmod 2$. This is a way to prove a version of the Guillou-Marin formula where F is oriented.

Next we define the Rohlin invariant for the 3-manifold N. This invariant is derived from the characterizations of the 1-connected characteristic pairs that have N as a boundary. It aims at determinating "how" behaves the Spin structure of N when extended to a Spin 4-manifold bounded by N.
First note that a $\mathbf{Z}$-homology sphere Σ only bounds 4-manifolds M^4 with even and unimodular intersection forms, since we can represent any $x \in H_2(M,\ N,\ \mathbf{Z})$ by an absolute class in M so that the $\pmod 2$ intersection form has zero diagonal; hence $\sigma(M^4) \equiv 0 \pmod 8$. Moreover, we have a unique Spin structure on Σ, so we can set:

$$\rho(\Sigma) = \sigma(M^4)/8 \pmod 2$$

where M^4 is Spin with boundary Σ. It is not possible to extend this definition to $\mathbf{Z}_2$-homology sphere Σ', for which we lose the congruence between the index of 4-manifolds bounded by Σ' and 8.
So we set, for an arbitrary closed oriented connected 3-manifold N with a given Spin structure θ:

$$\rho(N_\theta) \equiv \sigma(M^4) \pmod{16}$$

where M^4 is a smooth, oriented, compact and Spin manifold with boundary N^3.

Theorem: *Let M^4 be a closed oriented 4-manifold with a smooth embedded characteristic surface F^2 except in n points where the embedding is locally homeomorphic to the cone $(S^3,\ K_i)$ of a knot $K_i \subset S^3$, for $i = 1,\ 2, \dots,\ n$.*
Then surgery in the 4-dimensional characteristic bordism class induced from $(S^3,\ K_i)$ as above gives:

$$(K \cdot K - \sigma(M))/8 + \sum_{i=1}^{n} Arf(K_i) \equiv \Psi(M,\ F) \pmod 2$$

Proof: this theorem is an immediate consequence of the preceeding discussion; note that $K \cdot K = \hat{F} \cdot \hat{F}$. This generalizes the oriented version of the Guillou-Marin formula. $\square$

Suppose that F^2 is only smoothly immersed in M^4 with n double points, then F^2 looks locally like the cone of the Hopf link near these double points, that is two transverse 2-disks $D_1 \amalg D_2$ that we can smooth into an annulus by a surgery. This surgery may be done on a *linking torus* of the double point, which may be visualized as follows. Consider a parametrization of a small neighborhood of a double point in the form $(0,\ 0) \in \mathbf{R}^2 \times \mathbf{R}^2$: then the linking torus T^2 is $S^1 \times S^1$ and the surgery $(0,\ e^{2i\pi\theta}) \mapsto (e^{2i\pi\theta},\ e^{2i\pi\theta}),\ \theta \in [0,\ 1]$, turns $D_1 \amalg D_2$ into the annulus. The generator γ of $H_1(T^2, \mathbf{Z})$ corresponding to the core of the annulus has a twist, so we have $q(\gamma) = 1$.
However, a complementary generator of $H_1(T^2, \mathbf{Z})$ corresponds to an arc leaving the double point on a leaf and returning to the other leaf: its twist must be calculated with the help of the description of F^2 (cf. [**AK**] and [**KM2**]).

We can translate immediatelly the last theorem for the Rohlin invariant in the form of the Guillou-Marin formula:

Theorem: *Let N^3 be an oriented 3-manifold which bounds a compact oriented M^4, and let F^2 be an oriented surface in M^4 dual to the obstruction to the extension of the Spin structure of N to M; then*:

$$\mu(N^3) \equiv \sigma(M^4) - F \cdot F \pmod{16}$$

If F is smoothly embedded;
otherwise we correct for n singular points by $8\sum_{i=1}^{n} Arf(K_i)$.

Note that this statement has a particular interest, out of the Guillou-Marin formula, since it gives a 3-manifold invariant, for the determination of which we don't need to care about the chosen characteristic pair-so we can restrict to an adequate one to apply the theorem (e. g. when M^4 is simply connected). Using Kirby's calculus for links in S^3, we will see later how to work with this formula.

For example: given a $\mathbf{Z}_2$-homology 3-sphere Σ obtained by surgery on a link L, extract a sub-link K with the following property: it links any other component K' with the same parity as the framing of K'. Take the band connected sum of the components of K to get a knot still denoted by K. We have:

$$\mu(\Sigma) = \sigma(lk(L)) - framing(K) + 8Arf(K) \pmod{16}$$

where $lk(L)$ is the linking matrix of the link L (obtained by choosing an arbitrary orientation of L).
To see this, it suffices to apply the last theorem with M^4 constructed by gluing handles to B^4 along the components of L with the corresponding framing, F^2 being the closure of a Seifert surface of the characteristic link K by the cores of the corresponding handles. then we do not add intersection in F that we did not have before.
Consequently, $\mu(L(p,\ p-1)) = 1 - p \pmod{16}$ since $L(p,\ p-1)$ is obtained by surgery on the trivial knot with framing p and bounds for odd p the even (since 1-connected) following 4-manifolds:

and for an homological lens space L' obtained by p-surgery on an arbitrary knot K, we have $\mu(L') = 1 - p + 8Arf(S^3,\ K)$.

5.3 Characteristic links

This section deals with formalism equivalent to that of the characteristic pairs in Ω_4^{char}, especially *for the study of oriented* 3*-manifold*. Recall that a framed link $L \subset S^3$ determine a compact oriented simply connected 4-manifold obtained by adding 2-handles to B^4 along L with the corresponding attaching framing. If L is oriented, each component $L_i \subset L$ lifts to $[F_i] \in H_2(W_L,\ \mathbf{Z})$, where F_i can be chosen as an oriented Seifert surface in S^3 for L_i capped off with the core of the associated 2-handle. The set of homology classes of these surfaces F_i generate $H_2(W_L,\ \mathbf{Z})$; note that we have a non degenerate intersection pairing over $H_2(W_L,\ \mathbf{Z})$ (since W_L is 1-connected), which coincide with the linking pairing of L: i. e. we have $F_i \cdot F_j = lk(L_i,\ L_j)\ \forall i \neq j$.
Next we turn to the calculation of the $\pmod 4$ reduction of the self intersection of the surfaces F_i, as it appears in the demonstration of the Theorem KM1. Denote by $F_E \in H_2(W_L,\ \mathbf{Z})$ the class corresponding to a sublink E of L: $F_E \cdot F_E \pmod 4$ is independant of any fixed orientation on L (since we have $(A+B)\cdot(A+B) \equiv (A-B)\cdot(A-B) \pmod 4$). Moreover, denoting by $E_i \subset E$ any component of E, we have $E_i \cdot E_i \equiv 0 \pmod 2$ since this is equal to the intersection of E_i with a characteristic submanifold of S^3 (by definition), that is a dual link to $\omega_2(S^3) = 0$.
Denote by $\theta \in H^1(M,\ \mathbf{Z}_2)$ a one dimensional cohomology class verifying $\theta(\mu) \neq 0$, where μ is the homology class of a meridian of E_i (a $\mathbf{Z}_2$-reduced Hom-dual to E_i in M^3). The following formula holds:

$$F_E \cdot F_E \equiv E \cdot E \equiv 2\theta^3 \pmod 4$$

Indeed, we can modify (cf. [**Ka**]) the sublink $E \subset L$ by sliding and blowing up and down using Kirby moves so that E turns into an unknotted knot with framing $E \cdot E$. Then, the Poincare dual class to θ can be represented by a Seifert surface for E, disjoint from $L \setminus E$, and capped off with a connected sum of $e = E \cdot E/2$ projective plans (remember that $\mathbf{R}P^2{}_\pm \cdot \mathbf{R}P^2{}_\pm = \mp 2$) in the surgery solid torus. But the triple intersection of this surface is e.

We may in fact induce from this particular construction a general method for the computation of the triple self intersection of an immersed surface Poincare dual to a class $\theta \in H^1(M,\ \mathbf{Z}_n)$ (see [**MOO**] or [**Tu**]). This triple self intersection is then equal to half of the self intersection of a Hom-dual to θ.

Define a sublink $C \subset L$ as a characteristic sublink if and only if for any component $L_i \subset L$, we have the Wu formula: $C \cdot L_i \equiv L_i \cdot L_i \pmod 2$; then $(L,\ C)$ is called a characteristic pair.

Theorem: *There is a one to one correspondance between the Spin structures on M_L and the characteristic sublinks of L.*

Proof: Fix a Spin structure on M_L and let us associate to any $\theta \in H^1(M_L,\ \mathbf{Z}_2)$ the sublink $C \subset L$, where the Spin structure that corresponds affinely to θ does not extend to the 2-handles glued over C. To show that C is characteristic, it suffices to verify the Wu formula. Consider the surface $\hat{F} = F \cup B^2 \subset W_L$, obtained by capping off an immersed Seifert surface F for C by the cores B^2 of the 2-handles attached to B^4 along C, which gives W_L. Then $\hat{F}$ is the Poincare dual to $\omega_2(W_L, M_L)$, hence it verifies the Wu formula (see chapter 2). Since any closed surface in W_L is obtained by such a capping off procedure, we get $\forall[\Sigma],\ [\Sigma'] \in H_2(W_L, \mathbf{Z}_2),\ \exists K,\ K' \subset L :$ $[\Sigma]\cdot[\Sigma'] \equiv K \cdot L$; the result follows. A direct topological proof can be found using [**GM**, "Characteristic surfaces"].

This correspondance is a monomorphism since two Spin structures on M^3_L that induce the same characteristic sublink $K \subset L$ agree on the *core* $L' = f(\amalg(S^1 \times \{\star\}))$ of the surgery along L in S^3, where $f : \amalg(S^1 \times B^2 \to S^3 \subset B^4$ is the embedding that defines the attachment of the handles on B^4 along L. Now $H := H_1(M_L,\ \mathbf{Z}_2)$ has L' for support, so theory of obstruction concludes, by defining from $\theta,\ \theta' \in H^1(M_L, \mathbf{Z}_2)$ the same vector fields (up to homotopy) over the 2-skeleton of M_L, i.e. the same Spin structures.
It is an epimorphism: since the $\pmod 2$ reduction of the linking matrice is a presentation matrix for H, we have $|Ker(A)| = |H|$. But C is characteristic if and only if, written as a vector in the basis of definition of A, it preserve the framing: $AC = D$, where D is the diagonal matrix extracted from A. Then $\sharp\{characteristic\ sublinks\ C \subset L\} = |Ker(A)| = |H|$. $\square$

Let us denote $(M_L)_\theta = M_{L,\ C}$. There is a calculus for characteristic pairs of links as follows.
$M_{L,\ C} = M_{L',\ C'}$ iff $(L',\ C')$ is obtained from $(L,\ C)$ by isotopy or by a finite sequence of moves of the following forms:

- 1) Add (or extract) an unknotted disjoint component with framing ± 1 and replace C by $C \pm K$ (such as we keep the characteristic property);
- 2) $\forall i \neq j$, let L_i slides over L_j to give $L'_i = L_i + L_j$, and C turns into:

$$\left\{ \begin{array}{ll} C & \text{if } L_i \text{ is not a component of } C \\ C - (L_i + L_j) + L'_i & \text{if } L_i,\ L_j \subset C \\ C - L_i + (L_j + L'_i) & \text{if } L_i \subset C \text{ and } L_j \text{ is not a component of } C \end{array} \right.$$

For example, after applying the second move to L when $L_i \subset C,\ L_j \not\subset C$, the linking matrix A of L verifies: $C' \cdot L'_i \equiv L'_i \cdot L'_i \equiv A_{i,\ i} + A_{j,\ j} \pm 2A_{i,\ j}$ (the sign depends upon the orientation of the band connected sum of L_i and L_j).
We can change the orientation of L_j after a sliding: the second move becomes

$L'_i = L_i + L_j$ and $L'_j = -L_j$. Then, in case S is a link of S^3 containing L_i but not L_j, we have $[S'] = [S] - [L_i] + [(L'_i + L'_j)] = [S]$.
We can turn these moves into a single local move, as in [**FR**]:just add to the Fenn-Rourk move the following: if $C \cdot K$ is even, take $C' = C + K$.

Set $\mu_{L,\ C} = \sigma(M_{L,\ C}) - C \cdot C + 8Arf(C) \mod 16$, where Arf($C$) exists since C is a proper link (i.e. C is characteristic for itself) and the (mod 2) reduced intersection form of a Seifert surface for C has an Arf invariant.

Theorem [KM1, p. 543]: *$\mu_{L,\ C}$ is an invariant of $M_{L,\ C}$ and it is equal to $\mu_{M_{L,\ C}}$.*

Proof: The prove the invariance needs only elementary calculations with the preceeding generalized Kirby moves, and properties of the Arf invariant. Note that we can turn a given pair $(L,\ C)$ into $(L',\ \emptyset)$ with theses moves, hence the definitions gives the second claim. □

Appendix C of [**KM1**] shows how to get an elementary proof of the first "characteristic" Rohlin theorem with the help of this formalism.

5.4 More on the linking form

Let M^3 be a 3-manifold with a given Spin structure. We shall define a map that lift all quadratic enhancements of the intersection form on embedded surface in M^3 and is a quadratic enhancement of the linking pairing.

Given any surfaces $F_y,\ F_x \subset M^3$ in general position, with respective homology classes $y,\ x \in H_2(M,\ \mathbf{Z}_2)$, take the Poincare dual $\alpha \in H^1(M,\ \mathbf{Z}_2)$ of F_y and restrict it to F_x. This gives an element $\alpha_x \in H^1(F_x,\ \mathbf{Z}_2)$. Let $\hat{y}$ be an embedded collection of circles Poincare dual to α_x, and consider the quadratic enhancement q_x associated to the Pin^- structure on F_x induced by the Spin structure of M. Then we set:

$$f(F_x,\ F_y) = q_x(\hat{y})$$

In other words, f is defined by counting the number (mod 4) of half twists of (a section of) the normal bundle $\nu_{\hat{y}_i}$ of each $\hat{y}_i \subset F_x \cap F_y = \hat{y}$ in F_x. We may use the Spin structure of M to put even framings on $\hat{y}$; this is an embedded collection of circles in M, so no correction has to be added to the above definition of f- see the construction of q in chapter 3. We could define $\nu_{\hat{x}_i}$ in the same manner, and this definition is obviously symmetric.

This definition of f shows that it only depends on the homology class of F_y, as for q_x, and by symmetry on the homology class of F_x. Then we have defined a symmetric map:

$$f:\ H_2(M,\ \mathbf{Z}_2) \times H_2(M,\ \mathbf{Z}_2) \to \mathbf{Z}_4$$

This is not a bilinear map, but we can transfer the properties of $q_x,\ x \in H_2(M,\ \mathbf{Z}_2)$ to f, so that we get:

$$q_x(y \hat{+} z) = q_x(\hat{y}) + q_x(\hat{z}) + 2 \cdot \hat{y} \cdot_x \hat{z}$$

where $\cdot_x$ denotes the (mod 2) intersection pairing on F_x. Then $f(x,\ )$ is a quadratic enhancement of the map:

$$\tau_x : \ H_2(M,\ \mathbf{Z}_2) \times H_2(M,\ \mathbf{Z}_2) \to \mathbf{Z}$$

$$(y, z) \to \hat{y} \cdot_x \hat{z}$$

and the equality $\hat{y} \cdot_x \hat{z} = x \cap y \cap z[M]$ shows that the trilinear symmetric map:

$$\tau : \ H_2(M,\ \mathbf{Z}_2) \times H_2(M,\ \mathbf{Z}_2) \times H_2(M,\ \mathbf{Z}_2) \to \mathbf{Z}_2$$

$$(x, y, z) \mapsto x \cap y \cap z[M]$$

allows us to write the preceeding equation as :

$$f(x, y + z) = f(x, y) + f(x, z) + 2 \cdot \tau(x, y, z)$$

Let us postpone the discussion about τ, in view to link the properties of f to the Brown invariant.

If we act on the Spin structure of M (which intervene explicitely in the definition of q) by the element $\alpha \in H^1(M,\ \mathbf{Z}_2)$, we get a new map f_α:

$$f_\alpha(x,\ y) = f(x,\ y) + 2 \cdot \tau(x,\ y,\ a)$$

where $a \in H_2(M,\ \mathbf{Z}_2)$ is Poincare dual to α. This formula follows directly from $q_{x,\ \alpha}(\hat{y}) = q_x(\hat{y}) + 2 \cdot \tau(x,\ y,\ \alpha)$, a fact we noted in chapter 3. This can also be written as 2-difference equation:

$$f_\alpha(x,\ y) = f(x,\ y + a) - f(x,\ a)$$

We now may recover one of the main formula that appears in the theorem of Kirby and Melvin.
Define a map $\beta : \ H_2(M,\ \mathbf{Z}_2) \to \mathbf{Z}_8$ as follows: given any $x \in H_2(M,\ \mathbf{Z}_2)$, take the Pin^- bordism class (induced by the Spin structure of M) of an embedded surface that represents x, and using the Brown map identify it with an element of $\mathbf{Z}_8$. This is clearly well defined since the Brown invariant is a Pin^- bordism invariant, and a bordism $W^3 \subset M \times [0,\ 1]$ between two boundary components $F,\ F' \subset M$ with $[F] = [F'] \in H_2(M, \mathbf{Z}_2)$ inherits a Spin structure from M compatible with the Pin^- structure on the extremities.

Theorem [KT, 4. 11]: *Let M be a Spin 3-manifold with the induced Rohlin μ invariant and the function β. Let $\alpha \in H^1(M,\ \mathbf{Z}_2)$ act on the Spin structure, and let μ_α be the new invariant. Then*:

$$\mu - \mu_\alpha = 2 \cdot \beta(a) \pmod{16}$$

where $a \in H_2(M,\ \mathbf{Z}_2)$ is the Poincare dual to α.

Proof: We put on $M \times [0,\ 1]$ the original Spin structure on $M \times 0$ and the altered one to $M \times 1$. We can capp off the boundary components of $M \times [0,\ 1]$ by adding

Spin 4-manifolds to $M \times 0$ and $M \times 1$, and the resulting closed 4-manifold W has the index $\mu_\alpha - \mu \pmod{16}$. Consider $F^2 \subset W^4$ with $[F] = a \in H_2(M,\ \mathbf{Z}_2)$. Then $F \times 1/2$ is a Poincare dual to $\omega_2(W)$ and $F \cdot F = 0$ since F lives in a product. The enhancement of the intersection pairing on F used in the Guillou Marin formula is the same as the one we put on F to calculate β, so the Guillou Marin formula applies directly. □

If we act on the Spin structure of M by $\alpha \in H^1(M,\ \mathbf{Z}_2)$, the Pin^- structure of F is equivariantly acted on by α_F. The formula (see chapter 3) $\beta(q_a) = \beta(q) + 2 \cdot q(a)$, a being a Poincare dual to α, implies:

$$\beta_\alpha(x) = \beta(x) + 2 \cdot f(x,\ a)$$

Combining two equations $\mu_\alpha - \mu = 2 \cdot \beta(a)$ and $\mu - \mu_{\alpha_1} = 2 \cdot \beta(a_1)$ with $\mu_\alpha - \mu_{\alpha_1} = 2 \cdot \beta_\alpha(a_1 - a)$, we get $\beta_\alpha(a_1 - a) = \beta(a_1) - \beta(a)$. Setting $a_1 = x + a$ and we find:

$$\forall x,\ y \in H_2(M,\ \mathbf{Z}_2)\ \beta(x+y) = \beta(x) + \beta(y) + 2 \cdot f(x,\ y)$$

In particular we have $f(x,\ x) = -\beta(x) \pmod 4$ and $\beta(x+y) = \beta(x) + \beta(y) + 2 \cdot \tau(x,x,y) \pmod 4$.

As promised, we now investigate the map τ. First we define a bilinear symmetric map $\lambda :\ H_2(M,\ \mathbf{Z}_2) \times H_2(M,\ \mathbf{Z}_2) \to \mathbf{Z}_2$ as a kind of dilatation of the linking pairing: take as above two embedded surfaces F_x and F_y, in general position, that represent respectively $x,\ y \in H_2(M,\ \mathbf{Z}_2)$; then $\lambda(x,\ y)$ is defined to be the number of embedded circles $c_i \subset F_x \cap F_y$ with non trivial normal bundle $\nu_{c_i \subset F_x}$)to the surface F_x. But $[\amalg_i c_i] = x^* \cup y^* \cap [M]$, so it suffices to evaluate $\omega_1(F_x)$ of the normal bundle $\nu_{F_x \subset M}$ on these circles to compute $\lambda(x,\ y)$.
In general, given $a \in H^1(M, \mathbf{Z}_2)$, where M is a n-dimensional manifold, and $i : F^{n-1} \hookrightarrow M$ a $\pmod 2$ Poincare dual submanifold to a, we have $i^*(a) = \omega(\nu_{F \subset M})$. Then $\omega_1 = x^*$, and consequently $\lambda(x,y) = x^* \cup x^* \cup y^*[M] = \tau(x,\ x,\ y)$. This proves the bilinearity of λ, and the symmetry follows from the symmetry of f (for example). Moreover f is clearly an enhancement of λ.

The map λ is a dilatation of the linking pairing; we have already used this fact in chapter 1. In fact we have, denoting the Bockstein operator by B:
If $x \in H^1(M,\ \mathbf{Z}_n)$, then $x^2 = (n/2)\ B(x)$ and Poincare duality followed by the universal coefficient formula sends $B(x) \in H^2(M,\ \mathbf{Z}_n)$ to $\overline{x} \in H^1(M,\ \mathbf{Z}_n)$. Then $\forall x,\ y \in H^1(M,\ \mathbf{Z}_n)$ we have:

$$\Psi(\tau_n(x,\ x,\ y)) = \Psi((x^2 \cup y)([M])) = n/2\ \Psi((B_2(x) \cup y)([M]))$$

Hence:

$$\Psi(\tau_n(x,\ x,\ y)) = n/2\ \Psi(y(\overline{x})) = n/2\ \overline{x} \odot \overline{y}$$

where Ψ denotes the inclusion $k \pmod n \mapsto k/n$, and the last equality follows from the definition of the linking pairing.

5.5 Quadratic enhancements of the linking form

The aim of this section is to show that we can deduce the whole set of quadratic enhancements of the linking form from the set of Spin structures of M; we shall also give their relationship with the Rohlin invariant. First of all, here is a general method of construction of such quadratic forms.

Let M be a Spin 3-manifold. The finitely generated abelian group $Tors(H_1(M,\ \mathbf{Z}))$ is the torsion part of the cokernel of the map $H_2(V,\ \mathbf{Z}) \to H_2(V,\ \partial V;\ \mathbf{Z})$, where V^4 is an arbitrary compact orientable Spin 4-dimensional manifold such that $H_1(V,\ \mathbf{Z}) = H_1(V,\ \partial V,\ \mathbf{Z}) = 0$, and $\partial V^4 = M^3$ as a Spin boundary. Then $H_2(V,\ \mathbf{Z}) \simeq H^2(V,\ \partial V;\ \mathbf{Z})$ is a free abelian group. The adjoint of the intersection form A in $H_2(V,\ \mathbf{Z})$ is defined as:

$$ad(A):\ H_2(V,\ \mathbf{Z}) \to Hom(H_2(V,\ \mathbf{Z}),\ \mathbf{Z})$$

The composition of $ad(A)$ with the Hom duality, followed by the Poincare duality isomorphism gives the inclusion $H_2(V,\ \mathbf{Z}) \hookrightarrow H_2(V,\ \partial V;\ \mathbf{Z})$. Thus the groups $Tors(Coker(ad(A)))$ and $Tors(H_1(M,\ \mathbf{Z}))$ are canonically isomorphic.

Now take a look at the linear extension of $ad(A)$:

$$\overline{a} := \overline{ad(A)}:\ H_2(V,\ \mathbf{Z}) \otimes \mathbf{Q} \to Hom(H_2(V,\ \mathbf{Z}),\ \mathbf{Q})$$

Set $K = Hom(H_2(V,\ \mathbf{Z}),\ \mathbf{Z}) \cap Im(\overline{ad(A)})$. It is clear that

$$K/im(ad(A)) = Tors(Coker(ad(A))).$$

Then the linking pairing can be defined by the formula (with values in $\mathbf{Q}/\mathbf{Z}$):

$$\forall x,\ y \in K,\ L_A(x + Im(ad(A)),\ y + Im(ad(A))) = -x(\overline{a}^{-1}(y)) \pmod{\mathbf{Z}}$$

If A is an even intersection form (which is the case here since we have *chosen* V^4 to be a Spin 1-connected manifold), this last formula gives the following quadratic enhancement on $\mathbf{Q}/Z$ of the linking pairing $\odot$:

$$\eta_A(x + Im(ad(A))) = -1/2x(\overline{a}^{-1}(x)) \pmod{\mathbf{Z}}$$

It can be shown that η only depends on the Spin structure on M, see the paragraph before the remarks.
There is a simple geometric interpretation of the rational numbers whose class in $\mathbf{Q}/\mathbf{Z}$ is a self linking pairing: it describes the framings of the normal bundle to an embedded circle k in M with torsion homology class x, and gives a direct geometric interpretation to η.

Fact: *The framings on the normal bundle to k are in one-to-one correspondence with rational numbers q such that the class of q in $\mathbf{Q}/\mathbf{Z}$ is $x \odot x$.*

Proof: Let r be the order of x in $H_1(M,\ \mathbf{Z})$. Choose r copies of the longitude selected in the peripheral torus of k by the choice of a framing of $\nu_{k\subset M}$, and count the intersection number of k with an oriented surface F bounded by the r copies of k. If one gets $p \in \mathbf{Z}$, we assign the rational number p/r to this framing. To see that this is independant of the choice of the embedding of the r copies of k in M and of the choice of F, take an oriented surface F' embedded in $M \times [0,\ 1]$, such that $\partial F' = (\amalg_{i=1}^{r} S^1) \amalg S^1$ and these *two* sets of boundary components lie respectively in $M \times 0$ and $M \times 1$. The normal bundle to F' is trivial, so once we fix the sections of the normal bundle of the embedding $\amalg_{i=1}^{n} S^1 \to M \times 0$, there is only one choice of sections on the normal bundle $\nu_{k \subset M \times 1}$ which come from sections of $\nu_{F' \subset M \times [0,\ 1]}$. This shows the independance of p/r in the choice of the embedding of the r copies of k, and in the same manner we prove the independance from the choice of F. Then p/r is well defined once the framing is fixed, and $p/r \pmod{\mathbf{Z}} \equiv l(x,\ x)$ follows from the definition of the linking pairing. Finally, a full right-twist of the initial framing gives r new intersections between F and k, hence the correspondance is *one-to-one* from $\mathbf{Q}/\mathbf{Z}$ to the framings. □

This induce naturally an order on the framings of $\nu_{k \subset M}$ (k a knot), where $[k] \in H_1(M, \mathbf{Z})$ is torsion, that is useful to simplify the generalized Robertello's invariant of Section 1: indeed, we can remove the longitudes in its very definition.
Take the minimal rational number q_i for the ith component $L_i \subset L$ so that it gives an even framing, and call it a *minimal longitude*; then $0 \leq q_i \leq 2$. We may orient arbitrarily L_i, since the orientation does not intervene in the calculus below. Then we set:

Let L be a link in M so that each component $L_i \subset L$ represents a torsion class in $H_1(M,\ \mathbf{Z})$.
Suppose L is characterized; we define $\hat{\beta}(L,\ M) = \beta(L,\ l,\ M)$, where l is the set of even longitudes of l such that each one is minimal.

It is still a concordance invariant. Recall that an even framing on $\nu_{k \subset M}$ is framing which, when added to the bounding Spin structure of k gives the restricted Spin structure of M over k. In case L is a torsion link as above, the given Spin structure on M picks out half of the rational number for which the longitude gives a framing compatible with the Spin structure on the normal bundle. Hence:
$q/2 \in \mathbf{Q}/\mathbf{Z}$ determinates the class of compatible framings on $\nu_{k \subset M}$, using the fact above.
But this is the definition we took above for the quadratic enhancements of the linking form denoted γ. Note that we have allready defined (see Section 4.1) the notion of even framing for knots k such that $[k] = 0 \in H_1(M,\ \mathbf{Z}_2)$, but there it is independant of the Spin structure of M, since they all induce the same Spin structure in the neighborhood of k.

Remarks: 1) In S^3 with its unique Spin structure, the framing on a knot k designed by an even number in a surgery presentation *a la Kirby*, defines also an even framing in the above meaning. If the class in $H_1(M, \mathbf{Z})$ represented by k has odd (non zero) order, then $l(x,\ x) = p/r$ with r odd; the framings of $\nu_{k \subset M}$ that the Spin structure of S^3 will produce as even framings are the ones with even numerator.
Finally, here is a simple way to see the relationship between the framings $q \in \mathbf{Q}$ for $\nu_{k \subset M}$, where $[k]$ is a torsion class, and the signature of a 4 manifold W^4 constructed from M^3 and k. Consider M^3 as the boundary of a 4-manifold constructed

by attaching handles along the link $L \subset S^3 \subset B^4$, such that the resulting boundary $M_L \simeq M$, and attach another 2-handle along $k \in M^3$. We get a 4-manifold W^4 with $H_2(W,\ M,\ \mathbf{Z}) \simeq \mathbf{Z}$, and there is a unique class $x \in H_2(W,\ \mathbf{Q})$ which hits the generating relative class (e.g. represented by the embedded surface $\hat{F} = F \cup (\amalg_{i=1}^r B_i^2)$, where $F \subset M^3$ and $\partial F = r\partial k$ and B_i^2 is a copy of the core of the attached handle). If the framing of the attached handle is $q \in \mathbf{Q}$, $x \odot x = q$ and $sign(W^4) = \pm 1$ depending upon the sign of q.
2) The first definition of a quadratic enhancement of the linking form did not explicitely referred to the Spin structures on M: it is hidden in the choice of the 1-connected Spin 4-manifold V with even intersection form that is Spin bounded by M.

The only role of the Spin structure in the definition of γ is to determine a non zero section (i. e. a framing) of $\nu_{k \subset M}$, where k is a knot with torsion homology. Hence if we have two Spin structures on M^3 (denoted θ_1 and θ_2) which determine the same Spin structure in a neighborhood U of k, $\gamma_1([k]) = \gamma_2([k])$, where γ_i has been constructed as above from θ_i. This equality between the $\gamma_{i_{|U}}$'s corresponds to $(\theta_1 - \theta_2)([k]) = 0$.
Conversely, if $(\theta_1 - \theta_2)([k]) \neq 0$, we have different Spin structures on U, so there is a switch of Spin structures in $\nu_{k \subset M}$ (used in the definition of γ_i to get even framings); this imply that $\gamma_1(k) = \gamma_2(k) + 1/2$. Define $\psi = \gamma_1 - \gamma_2 :\ Tors(H_1(M,\ \mathbf{Z})) \to \mathbf{Q}/\mathbf{Z}$: this is a linear map, and $\psi([k]) = \psi(-[k])$. Hence ψ lands in $\{\pm 1\}$, and conversely, given such a ψ on a Spin 3-manifold M^3_θ, we may recover a quadratic linking form associated to θ' by taking $\gamma_\theta + \psi$.
Hence we have a bijective correspondance between quadratic linking forms and $Hom(Tors(H_1(M,\ \mathbf{Z})),\ \mathbf{Z}_2)$. An element $h \in Hom(Tors(H_1(M,\ \mathbf{Z}),\ \mathbf{Z}_2)$ acts on q by the formula:

$$\forall a \in Tors(H_1(M,\ \mathbf{Z})),\ (q+h)(a) = q(a) + \Psi(h(a))$$

where $\Psi : \mathbf{Z}_2 \hookrightarrow \mathbf{Q}/Z$.

We have proved that:

Theorem: *Let M be a closed compact oriented 3-manifold; consider the epimorphism $\Psi :\ H^1(M,\ \mathbf{Z}_2) \to Hom(Tors(H_1(M,\ \mathbf{Z})),\ \mathbf{Z}_2)$, which is natural for the action of $H^1(M,\ \mathbf{Z}_2)$ on itself (when $Hom(Tors(H_1(M,\ \mathbf{Z})),\ \mathbf{Z}_2)$ acts on the quadratic enhancements of the linking form as above). Given two Spin structures θ_1 and θ_2 on M with associated quadratic enhancements γ_1 and γ_2 of the linking form on $Tors(H_1(M,\ \mathbf{Z}))$, we have:*

$$\Psi(\theta_1 - \theta_2) = \gamma_1 - \gamma_2$$

Hence any quadratic enhancement of the linking form on $T = Tors(H_1(M,\ \mathbf{Z}))$ comes from a Spin structure by the above construction, and in case M is a $\mathbf{Z}_2$-homology 3-sphere, the set of quadratic enhancements, denoted $Quadr(\odot)$, determine the Spin structure *canonically.*

In view to link the Rohlin invariant with the quadratic enhancements of the linking form, we have to look at the Gauss-Brown sum formula. We will see later that it is the more general algebraic connection between the Rohlin invariant and the enhancement of a quadratic linking form of a Spin 3-manifold that is possible.

Recall that the Gauss-Brown map (cf. Appendix) $\beta : Quadr(\odot) \to \mathbf{Z}_8$ associates to a form q the residue $\beta(q) \in \mathbf{Z}_8$ such that:

$$exp((\beta(q)\pi i)/4) = (card\ T)^{-1/2} \sum_{a \in T} exp(2\pi i q(a))$$

where $T = Tors(H_1(M,\ \mathbf{Z}))$, as above. Now, we follow the lines of the construction of η in the preceeding pages, but we start with a $\mathbf{Q}$-bilinear symmetric pairing, i. e. the linear extension $\widehat{ad(A)}$ (with the preceeding notations). Except this, all is identical.

Suppose we have a bilinear symmetric non-degenerate pairing $A:\ S \times S \to \mathbf{Q}$, where S is a finite dimensional $\mathbf{Q}$-vector space, and that A is even on a restricted $\mathbf{Z}$-lattice L. The dual lattice $L^\sharp \simeq \{s \in S | A(s,\ l) \in \mathbf{Z},\ \forall l \in L\}$ contain L, and $L^\sharp/L$ is a finite abelian group. Then the function:

$$\hat{A}:\ L^\sharp/L \to \mathbf{Q}/Z$$

$$x \mapsto 1/2\ \eta(y,\ y) \in \mathbf{Q}/Z$$

where $y \in L^\sharp$, $x \in L^\sharp/L$, and η is defined at the beginning of this Section. This is a quadratic linking form on $L^\sharp/L$. Moreover, the Gauss-Brown map gives:

$$exp((\sigma(A)\pi i)/4) = (\beta(\hat{A})\pi i)/4) = (cardT)^{-1/2} \sum_{a \in T} exp(2\pi i q(a))$$

where $\sigma(A)$ is the index of the even intersection form A. The first equality follows from [**vdB**]. Let us apply this construction:

Theorem [**Ta**, 4. 5], [**TU2**, th5]: *Let M be a compact oriented 3-manifold without boundary, and let θ be a Spin structure on M. Denote the resulting quadratic linking form on T by γ. Then $\mu(M;\ \theta) = -\sigma(\gamma)$ (mod 8)*

Proof: We first suppose that M is a rational homology sphere; if not, chose a basis for the torsion free part of $H_1(M, \mathbf{Z})$ and do surgery to kill the circles, so that you get a resulting Spin coherent bordism W^4 from M to N with $H_1(N,\ \mathbf{Q}) = 0$. Then $(Tors(H_1(M,\ \mathbf{Z})),\ \gamma)$ is isomorphic to $(H_1(N),\ \gamma)$. Moreover $\sigma(W^4) \equiv 0$ (mod 8), and if we change both Spin structures on the two boundary components of W^4 by an element of $H^1(W^4,\ \mathbf{Z}_2)$ they are obviously still equal. Now any Spin structure on M may be obtained by acting on a *fixed* one with an element of the form $x+y$, where x is induced from $H^1(W,\ \mathbf{Z}_2)$ and $y \in H^1(M,\ \mathbf{Z})$; but this second kind of elements does not change the (mod 8) μ-invariant or the quadratic enhancement of the linking form. So we can suppose that $H_1(M,\ \mathbf{Q}) = 0$ without loss of generality. Take a 1-connected Spin 4-manifold V with $\partial V = M$, so that the unique Spin structure on V restricts to θ on M. Let A denote the intersection pairing:

$$H_2(V, \partial V;\ \mathbf{Q}) \otimes H_2(V,\ \partial V;\ \mathbf{Q}) \to \mathbf{Q}$$

The 1-connectedness of V shows that it is non degenerate; consider $L = H_2(V,\ \mathbf{Z})$ inside $H_2(V, \partial V;\ \mathbf{Q})$ and denote $L^\sharp = i_\star((H_2(V,\ \partial V;\ \mathbf{Z}))$, where we note the inclusion: $i_\star : (H_2(V,\ \partial V;\ \mathbf{Z}) \to (H_2(V,\ \partial V;\ \mathbf{Q})$, its Poincare dual lattice. We have easily: $L^\sharp/L \simeq H_1(M)$ (see the beginning of the Section).
The discussion preceeding this proof imply the result once we show that $-\gamma = \hat{A}$. In fact, the adjunction of structure in the geometry, due to the choice of a framing on M^3, allow us *to enhance the whole linking form* on M (that gives γ) to the quadratic linking form $\hat{A}$ on $(V,\ \partial V)$ (this method seems to have been inaugured in [**Mil**] and [**MO-Su**]). We want to lift $x \in H_1(M, \mathbf{Z})$ to $\overline{x}\ H_2(V,\ \partial V; \mathbf{Z})$ and then calculate $\overline{x} \odot \overline{x} = 1/r\ \overline{x} \cdot y$ with the help of an absolute class $y \in H_2(V, \mathbf{Z})$ which lift $r\overline{x}$. We proceed as follows: take an embedded oriented surface $F^2 \subset V$ with boundary a representative for x; using the Spin structure on M^3 and another arbitrary Spin structure on F^2, fix a section of $\nu_{F \subset V}$ that pushes F into a disjoint copy $\hat{F} \subset V$, so that we get an even framing on $\nu_{F^2 \subset V^4}$. Now glue an embedded surface $K^2 \subset M^3$,bounded by r parallel copies of $\partial\hat{F}^2$, to $r\partial\hat{F}$: the resulting closed surface represents a class $y = r\overline{x} \in H_2(V, \mathbf{Z})$. It is now clear that $\overline{x} \cdot \overline{x} = -x \odot x$ (the sign comes from the opposite orientations induced on the circles by K^2 and $\hat{F}^2$). □

As easy corollaries, we have that the (mod 8) reduction of the Rohlin invariant is a homotopy invariant, and the linking form of a $\mathbf{Z}_2$-homology sphere determines the Rohlin invariant (since $H_1(M^3,\ \mathbf{Z})$ is an odd torsion group). In [**Ta**], a generalization of this result to *relative* Rohlin invariants can be found.

5.6 Determination of the set of algebraic relations between the linking pairing, μ and the cohomology ring

Before starting with the proof of **KM1**, we want to exhaust all relations between the linking pairing, the Rohlin invariant, and the cohomology ring of a 3-manifold M. It is useful to point out the informations that the family of invariants τ_r may produce. We first want to prove that the (mod 2) cohomology ring of a 3-manifold is determined by its Rohlin invariant. Denote by $Spin(M)$ the set of Spin structures of M:

Theorem [TU, th. 3]:*Let M^3 be a closed oriented 3-manifold, $\alpha \in Spin(M)$ and $x_1,\ x_2,\ x_3 \in H^1(M,\ \mathbf{Z}_2)$; then we have* :

$$(\star)\ 8\ \tau_2(x_1,\ x_2,\ x_3) \pmod{16} \equiv \mu(M_\alpha) - \sum_{1\le i\le 3} \mu(M_{\alpha+x_i}) + \sum_{1\le i<j\le 3} \mu(M_{\alpha+x_i+x_j}) - \mu(M_{\alpha+x_1+x_2+x_3})$$

For exemple, the Rohlin invariant of the 3-torus, which is invariant by translation, is equal to 8.

Remarks: If we apply the relation of the theorem to α and $\alpha + x_4$, where $x_4 \in H^1(M,\ \mathbf{Z}_2)$, we see that the fourth difference of the Rohlin invariant is zero: in other words, it is of degree ≤ 3:

$$\mu(M_\alpha) - \sum_{1\leq i\leq 4} \mu(M_{\alpha+x_i}) + \sum_{1\leq i<j\leq 4} \mu(M_{\alpha+x_i+x_j}) - \sum_{1\leq ij<k\leq 4} \mu(M_{\alpha+x_i+x_j+x_k}) +$$

$$-\mu(M_{\alpha+x_1+x_2+x_3+x_4}) = 0$$

To simplify the notations, denote $\mu(M_\alpha) = \mu(\alpha)$. In general, maps $\mu(M):\ S \to \mathbf{Z}_{16}$ of degree ≤ 3, defined on a $\mathbf{Z}_2$-affine space S, are determined by their values on the elements $\alpha,\ \alpha + x_i,\ \alpha + x_i + x_j,\ \alpha + x_i + x_j + x_k$, where $i < j$ and $i < j < k$ respectively, $\alpha \in S$ is (arbitrarily) fixed and $\{x_l\}_l$ is a basis of the associated linear space. Substituting $x_1 = x_2,\ x_1 = x_2 = x_3,\ x_1 = x_2 = x_3 = x_4$ in the above equation, we get the following relations:

$$\mu(\alpha) - \mu(\alpha + x_i) \equiv 0 \pmod 2$$

$$\mu(\alpha) - \mu(\alpha + x_i) - \mu(\alpha + x_j) - \mu(\alpha + x_i + x_j) \equiv 0 \pmod 4$$

$$\mu(\alpha) - \mu(\alpha+x_i) - \mu(\alpha+x_j) - \mu(\alpha+x_k) + \mu(\alpha+x_i+x_j) + \mu(\alpha+x_i+x_k) + \alpha(\alpha+x_j+x_k) -$$

$$-\mu(\alpha + x_i + x_j + x_k) \equiv 0 \pmod 8$$

We see in theorem **KM1** that the obstruction to have: $\forall\alpha \in Spin(M),\ \forall x \in H^1(M, \mathbf{Z}_2):\ \mu(\alpha) - \mu(\alpha + x_i) \equiv 0 \pmod 2$ is *geometric*.

Proof: Recall that given a surface $F^2 \subset M^3$, we have:

$$(\star)\ \ \forall a \in H_1(F^2,\ \mathbf{Z}_2),\ \forall x \in H^1(M,\ \mathbf{Z}_2),\ q_{\alpha+x}(a) - q_\alpha(a) = 2x(a)$$

where q_α is the quadratic enhancement of the intersection pairing of F induced by $\alpha_{|F}$, and $2 : \mathbf{Z}_2 \to \mathbf{Z}_4$ denotes the inclusion.
Note that, when we consider directly the link l embedded in M^3 instead of embedded in a surface F^2, we may substitute the data of a one dimensional subbundle ν of $\nu_{l\subset M}$ to the data of the embedding $F^2 \subset M^3$. The theory of membranes and of characteristic pairs is convenient to give a geometric view of the formula $(\star)$, a view that will be useful later in the proof:
Consider $V,\ V'$ two 1-connected compact orientable 4-manifolds with boundary M^3, over which the two Spin structures α and $\alpha + x$ over M extends respectively. Take two membranes $L \subset V$ and $L' \subset V'$ with boundary an embedded link l of M, and glue $(V,\ L)$ and $(V',\ L')$ along their common boundary $(M,\ l)$. The resulting pair is denoted $(W^4,\ N^2)$. The left-hand-side (LHS) of $(\star)$ is the $\pmod 4$ reduction of the obstruction to the existence of a subline bundle of rank 1 of $\nu_{W^4}(N^2)$. Hence it is $2\omega_2(W^4,\ M^3)[N^2] = 2[N^2] \cap [C^2] = 2[l] \cap [C^2] = 2x[l]$, where $C^2 \subset M^3$ is a closed embedded surface dual to $x \in H^1(M^3, \mathbf{Z}_2)$, i. e. C^2 represents a class dual to $\omega_2(W^4,\ M^3)$.
To continue the proof, we need:

Let W^4 be endowed with an orientation compatible with the orientation of V^4; using the Guillou-Marin formula and the fact that $[F]\cdot[F] = O$, we get $\mu(\alpha) - \mu(\alpha +$

$x) = 2\beta(q_F) \pmod{16}$ (the LHS is $\sigma(W^4) \pmod{16}$, by additivity of the signature). Note that we have allready obtained this formula. We now have to prove:

Lemma [Tu, p. 72]: *Suppose that F^2 and $(F')^2$ are oriented closed surfaces of M^3 which intersect transversally in a link l; denote by $x,\ x' \in H^1(M,\ \mathbf{Z}_2)$ the dual classes to F and F' in M^3. Then:*

$$\forall\ \alpha \in Spin(M),\ \mu(\alpha) - \mu(\alpha + x) - \mu(\alpha + x_2) + \mu(\alpha + x + x_2) = 4q_\alpha([l])$$

Proof of the lemma: We can suppose that l is connected, by surgery of index 1 on F and F' (this does not modify $q_F([l])$). Setting $\alpha' = \alpha + x'$, the LHS turns into $2\beta(q_\alpha) - 2\beta(q_{\alpha'})$: so the problem is now to show that $\beta(q_\alpha) - \beta(q_{\alpha'}) = 2q_\alpha([l])$, where q_α denotes the quadratic enhancement of the $\mathbf{Z}_2$-intersection form on F induced by $\alpha \in Spin(M)$.
The preceeding discussion in the proof shows it is trivially verified when $[l] = 0 \in H^1(F^2,\ \mathbf{Z}_2)$. When l is a non-orientable curve in F^2, setting C for a tubular neighborhood of l in F^2 (a Möbius band) and D for $\overline{F^2 \setminus C}$, we have $q_\alpha = q_\alpha|C \oplus q_\alpha|D$ and $q_{\alpha'} = q_{\alpha'}|C \oplus q_{\alpha'}|D$. The preceeding discussion shows that $q_\alpha|D = q_{\alpha'}|D$. By additivity of the Brown invariant we have $\beta(q_\alpha) - \beta(q_{\alpha'}) = \beta(q_\alpha|C) - \beta(q_{\alpha'}|C) = 2\beta(q_\alpha|C) = 2q_\alpha|C([l])$, since $q_{\alpha'}|C([l]) = q_\alpha|C([l]) + 2$ and $H_1(C, \mathbf{Z}_2) = \mathbf{Z}$; this concludes. If l is a non-zero orientable curve in F^2, we use the same argument with a punctured torus containing l in place of C. □

End of the proof of the theorem: Now take two embedded surfaces F_1 and F_2 which intersect trasversally in a link l and which represent dual classes to $x_1,\ x_2 \in H^1(M,\ \mathbf{Z}_2)$. Substracting $\beta(q_{\alpha+x_3}) - \beta(q_{\alpha'+x_3}) = 2q_{\alpha+x_3}([l])$ to $\beta(q_\alpha) - \beta(q_{\alpha'}) = 2q_\alpha([l])$ (where $\alpha' = \alpha + x_2$), we get the RHS of the theorem equal to $4q_\alpha([l]) - 4q_{\alpha+x_3}([l]) = 8x_3([l]) = 8(x_1 \cup x_2 \cup x_3)([M])$. □

Remark: if the kernel of an homomorphism $x : H_1(M,\ \mathbf{Z}) \to \mathbf{Z}_2$ contains $Tors(H_1(M,\ \mathbf{Z}))$, the relation $\beta(\gamma_M) \equiv -\mu_M \pmod 8$ implies that:

$$\forall \alpha \in Spin(M),\ \mu_M(\alpha) \equiv \mu_M(\alpha + x) \pmod 8$$

With the theorem we get: if μ_M takes the value $k \pmod{16}$ and if there exists $x_1,\ x_2,\ x_3 \in H^1(M,\ \mathbf{Z})$ such that$(x_1 \cup x_2 \cup x_3)([M]) \equiv 1 \pmod 2$, then μ_M also takes the value $k + 8$. We already get an exemple with the 3-torus, for which this value corresponds to the 3 Lie Spin structures on its generic fibers. This fact was first noted by Kaplan in **[Ka]** and will later be used when discussing the first point of theorem KM1. Moreover, The third difference of the Rohlin function is a homotopy invariant, just as its $\pmod 8$ reduction.

Next we turn to the relations between the cohomology ring of a closed oriented 3-manifold and its linking pairing. We list some results proved in **[Tu]**, p. 73-77.
Denote $u_n(M) :\ \{(x,\ y,\ z) \mapsto (x \cup y \cup z)([M]) :\ H^1(M,\ \mathbf{Z}_n)^3 \to \mathbf{Z}_n\}_n$, where $[M]$ is the fundamental class of the 3-manifold M.
Given an abelian group H, we denotes its $\mathbf{Z}_n$-adjoint $Hom(H, \mathbf{Z}_n)$ by $H^\star$. Then we say that a sequence of forms $\{u_n : (H_n^\star)^3 \to \mathbf{Z}_n\}$ is compatible if we have the

following condition: $\forall m,\ n,\ n_1,\ \ldots,\ n_4 \in \mathbf{N},\ \forall \eta_i \in Hom(H, \mathbf{Z}_{n_i})(i = 1,\ 2,\ 3,$ and for any commutative diagram:

$$\mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \mathbf{Z}_{n_3}$$

$$\alpha_1 \times \alpha_2 \times \alpha_3 \qquad\qquad\qquad \beta_1 \times \beta_2 \times \beta_3$$

$$(\mathbf{Z}_m)^3 \qquad\qquad\qquad (\mathbf{Z}_n)^3$$

$$\mathbf{Z}_m \overset{\alpha_4}{\rightarrow} \mathbf{Z}_{n_4} \overset{\beta_4}{\leftarrow} \mathbf{Z}_n$$

where the vertical arrows are induced by the multiplication on the quotient rings and $\alpha_1, \ldots,\ \alpha_4$ and $\beta_1, \ldots,\ \beta_4$ are linear (not necessarily ring) homomorphisms, the following identity holds:

$$\alpha_4(u_m(\alpha_1 \circ \eta_1,\ \alpha_2 \circ \eta_2,\ \alpha_3 \circ \eta_3)) = \beta_4(u_n(\beta_1 \circ \eta_1,\ \beta_2 \circ \eta_2,\ \beta_3 \circ \eta_3))$$

This compatibility reflects the fact not only the forms $\{u_n(M)\}_{n\geq 0}$ (with M a compact orientable closed 3-manifold) are related to one another by homomorphisms of the coefficients, but also to the maps:

$$H^1(M, \mathbf{Z}_{n_1}) \times H^1(M, \mathbf{Z}_{n_2}) \times H^1(M, \mathbf{Z}_{n_3}) \to \mathbf{Z}_{n_4}$$

that corresponds to the various trilinear forms:

$$\mathbf{Z}_{n_1} \times \mathbf{Z}_{n_2} \times \mathbf{Z}_{n_3} \to \mathbf{Z}_{n_4}$$

Given a finitely generated abelian group H with a skew-symmetric trilinear form $u:\ (H_n^*)^3 \to \mathbf{Z}_n$, $n \geq 0$; if $n = 0$ or if n is odd, then there is a compact orientable 3-manifold M such that $(H,\ u)$ and $(H_1(M,\ \mathbf{Z}),\ u_n(M))$ are isomorphic in a natural way. If n is even and $n \geq 2$, there exists such an M^3 if and only there exists a non degenerate symmetric bilinear pairing $L:\ (Tors(H))^2 \to \mathbf{Q}/Z$ such that:

$$(1)\ (\star\star)\ \forall x,\ y \in H_n^*,\ \Psi(u(x,\ x,\ y)) = n/2\ L(\overline{x},\ \overline{y})$$

where $\Psi : \mathbf{Z}_n \to \mathbf{Q}/Z$ is the inclusion and $\overline{z} \in Tors(H)$ is the element for which $L(\overline{z},\ a) = \Psi(z(a))\forall a \in Tors(H)$.

In fact, [**KaKo**] proved that if T is a finite abelian group, then an arbitrary non degenerate symmetric bilinear form $T \times T \to \mathbf{Q}/Z$ can be realized as the linking form $Tors(H_1(M,\ \mathbf{Z}_2))^2 \to \mathbf{Q}/Z$ of a compact closed oriented 3-manifold M. Then there is no obstruction to find an adequate L, except that it does not necessarily verify condition (1). Precisely we have:

The condition (1) *for all $n \geq 2$ and n even (with u_n in place of u) is a necessary and sufficient condition to realize $(H,\ L,\ u_n)$, where $u_n : (H_n^\star)^3 \to \mathbf{Z}_n$ is a* compatible *sequence of skew symmetric trilinear forms, as $(H_1(M,\ \mathbf{Z}),\ \odot,\ u_n(M))$.*

Note that (1) imply that $u_n(x,\ x,\ y) = u_n(y,\ y,\ x)$, a fact we already used when $n = 2$ in Section 2, in view to define τ. We also allready saw that (1) is necessary at this stage.
Consequently, the above relation is the more restrictive that we can find between $\odot$ and $u_n(M)$, given any M as above.
Note that u_n and $H_1(M,\ \mathbf{Z})$, determine the cohomology ring of M.

Adding to the hypothesis an affine H_2^*-space S and a map $R:\ S \to \mathbf{Z}_{16}$:

We can realize (H, L, S, R) *as above as* $(H_1(M,\ \mathbf{Z}),\ \odot,\ Spin(M),\ \mu_M)$ *for a compact closed oriented* M^3 *if and only if* $degR \leq 3$ *and the Brown-Taylor relation* $\beta \circ \phi = -R \pmod 8$ *is verified, with a map* $\phi:\ S \to\ Quadr(L)$*, equivariant under the action of* H_2^* *on* $Quadr(L)$ *by:*

$$\forall q \in Quadr(L),\ \forall h \in H_2^\star, \forall a \in Tors(H),\ (q+h)(a) = q(a) + \Psi(h(a))$$

where $\Psi : \mathbf{Z}_2 \to \mathbf{Q}/Z$ *is the inclusion.*

Looking in details the abstract construction we did of the quadratic enhancements of the linking pairing, you can see that the canonical isomorphism (with the natations we adopted there) $Coker(ad(A)) \to H_1(M, \mathbf{Z})$ carries η to a quadratic $\odot$ form, which depends only on the Spin structure α we fixed on M (this dependance being equivariant under $H_1(M, \mathbf{Z}_2)$ as above). Then ϕ in the preceeding fact does indeed corresponds canonically to η.
Finally, we have:

Theorem [Tu, p.70]:*Let* $H,\ S,\ L$ *and* R *be as above and let* $\{u_n : (H_n^\star)^3\mathbf{Z}_n\}$ *be a compatible sequence of trilinear skew symmetric forms. Then there exists a closed oriented* 3*-manifold* M *such that* $(H,\ L,\ S,\ R,\ \{u_n\}_{n\geq 0})$ *is isomorphic to* $(H_1(M, \mathbf{Z}),\ \odot,\ Spin(M),\ \mu_M,\ \{u_n(M)\}_{n\geq 0})$ *if and only if condition* (1) *holds for all even* $n \geq 2$ *(with* u_n *in place of* u*), relation* $(\star)$ *in the first theorem of the Section holds* $\forall \alpha \in S,\ \forall x_1,\ x_2,\ x_3 \in H^\star$*, and there exists an* $H_2^\star$*-map* $\phi : S \to Quadr(L)$ *such that* $\beta \circ \phi = -R \pmod 8$

The proof of the first and the second assertion follows from this theorem. The proof of this theorem uses subtle manipulations with the Kirby operations *and* some transformations on a surgery presentation l of a 3-manifold M (which modify M) that preserves the "Milnor residue" over all 3-uplets of components of l. This allows to put $u_n(M)(x,\ y,\ z)$ in a kind of canonical form, in which u_n can always be written. The identification of R is a bit more complicated. For more details, see **[Tu], p. 76**. Let us finally note a direction in which we could try (and use) to extend the theorem, in view to exclude other invariants. Let M be a closed orientable 3-manifold, and $\tilde{M} \to M$ be an m-sheeted cyclic cover of M. If $f : H_1(M, \mathbf{Z})$ is the corresponding homomorphism, the total Atiyah-Singer invariant $\sum_{r=1}^{m-1} \sigma_r(M,\ f)$ of the cover (in the notations of **[CG]**) is an integer whose $\pmod 1$6 reduction is equal to $m\mu_{M_\theta} - \mu_{\tilde{M}_{\tilde{\theta}}}$, where θ is an arbitrary Spin structure on M and $\tilde{\theta}$ its lift to $\tilde{M}$.

Chapter 6

Proof of theorem KM1

6.1 Stable equivalence and quadratic forms

The stable equivalence between integral matrices specify (from our geometric point of view) the class of the 4-manifolds for which the intersection forms are represented by the matrices we consider, modulo connected sums with copies of $\pm CP^2$. Given a 3-manifold M_L^3 obtained by surgery on a framed link L in S^3 with linking matrix A, consider the 4-manifold $W_L^4 s.\ t.\ \partial W_L^4 = M_L^3$ obtained by attaching handles along $L \subset S^3 \subset B^4$ with the corresponding framing.
Any element in the class (defined just above) of W_L has the same intersection pairing as W_L, *when restricted* out of some 4-balls of W_L, since the equivalence relation may be geometrically interpreted as overtaking supports of vector fields in the neighborhood of isolated non-zero index points over W_L^4 (or also as the places of blowing up or blowing down of isolated singular points). We formalize it as follows:

$$B \sim B' \Longleftrightarrow \exists S\ unimodular,\ B' = S^t BS\ or\ B' = B \oplus (\pm 1) or B = B' \oplus (\pm 1)$$

This is a little bit stronger than the linking pairing, since we have the following well known result [**KaKo**]:

Theorem:*Stable equivalence classes of linking matrices of framed links are determined by:*
1) the first Betti number of the manifold M obtained by surgery on the link,
2) the isomorphism class of the pair $(Tors(H_1(M,\ \mathbf{Z}),\ \odot)$.

Note that the stable equivalence classification of 3-manifolds induced by this theorem dominates the existence of degree one maps onto lens spaces, and that homotopy equivalent manifolds have the same stable equivalence class.
This relation between linking pairings is an algebraic counterpart of the Kirby moves. Indeed:

- 1) A blowing up or a blowing down, i. e. adding or removing an isolated unknotted component k with framing ± 1 from a link L, may be viewed on the linking matrix of L as the above second condition;

- 2) a handle-slide of L_i over L_j, i. e. transforming L_i into $L'_i = L_i \sharp_b f_j$ (where f_j is a framing curve for L_j and $\sharp_b$ means band connected sum), corresponds to the transformation of $A = (\lambda_{ij})$ into $A' = (\lambda'_{ij})$, with the relations:

$$\lambda'_{ii} = \lambda_{ii} + \lambda_{jj} \pm 2\lambda_{ij}$$

$$\lambda'_{ki} = \lambda_{ki} \pm \lambda_{kj}(i \neq k)$$
$$\lambda'_{il} = \lambda_{il} \pm \lambda_{jl}(l \neq i)$$
$$\lambda'_{kl} = \lambda_{kl}(k,\ l \neq i)$$

  Then $A' = T^t A T$ where $T_{kk} = 1,\ T_{ki} = \pm 1,\ T_{kl} = 0$ elsewhere.

6.2 A simplification of the expression of τ_3

Formula 1. 7 of [**KM1**] is written as:

$$\tau_r(M) = \alpha_L \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{r}-\mathbf{1}} [\mathbf{k}] J_{L,\ \mathbf{k}}$$

where M^3 is a closed oriented manifold obtained by surgery on the framed link L with n_L components, σ_L is the signature of the linking pairing A of L, and:

$$\alpha_L := \underbrace{\sqrt{2/r}\sin\pi/r^{n_L}}_{b} \times \underbrace{\exp 2i\pi(2-r)^3/8r^{\sigma_L}}_{c}$$

Taking $r = 3$ we have $b = 1/\sqrt{2}$ and $c = (1-i)/\sqrt{2}$; furthermore, in case $\mathbf{k} \leq \mathbf{r} - 1$, $J_{L,\ \mathbf{k}} = J_{S,\ \mathbf{2}} = i^{S \cdot S}$.
Here we use corollary [**KM1**], th. 4. 14 to extract from the colored link $(L,\ \mathbf{k})$ a colored sublink S by elimination of the one-colored components (this keeps $J_{L,\ \mathbf{k}}$ unchanged) and [**KM1**], p. 4. 11 expressing $J_{L,\ \mathbf{2}}$ in function of the conjugate of the Jones polynomial $\tilde{V}$ by:

$$J_{L,\ \mathbf{2}} = [\mathbf{2}] \exp(i\pi/2r)^{3L \cdot L} \tilde{V_L}$$

Remember that the quantification is over $q = \exp(2i\pi/r)$, $[\mathbf{k}] = [1][2]\dots[k]$ and $[k] = \sin(\pi k/r)/\sin(\pi/r)$; moreover $\tilde{V_L}(\exp(2i\pi/3)) = V_L(\exp(-2i\pi/3) = 1$.
Denoting $S < L$ when S is a sublink of L, we then have:

$$\tau_3(M_L) = (1/\sqrt{2})^{n_L}((1-i)/\sqrt{2})^{\sigma(L)} \sum_{S<F} i^{S \cdot S}$$

where $\emptyset \cdot \emptyset \equiv 0 \pmod 4$.

Murakami, Ohtsuki and Okada showed in [**MOO**] how to get a family of quantum 3-manifold invariants out of τ_3 using the preceeding remarks about stable equivalence classification of 3-manifolds; their definition is:

$$Z_N(M,\ q) = (G_N(q)/|G_N(q)|)^{-\sigma(L)} |G_N(q)|^{-n} \sum_{l \in (\mathbf{Z}_n)^n} q^{l^t A l}$$

where $q = \exp(d\pi i/N)$ with $d+N$ odd, $N \geq 2,\ d \geq 1,\ (d,\ N) = 1$ and $G_N(q) = \sum_{h \in \mathbf{Z}_n} q^{h^2}$.
Note that the quadratic form $l \mapsto l^t A l$ must be viewed as a $\mathbf{Z}_{2N}$-quadratic form to be well defined when q is an $2N^{th}$ root of unity. These invariants generalize τ_3 for which $q = \exp(2i\pi/r),\ N = r,\ d = 1$, and are invariant under stable equivalence for A, as it is easily seen by applying the above matricial form of the Kirby moves.
It is shown in [**Ko-Ta**] that the splitting formula for τ_r, with r odd, in τ_3 and $\tau^{SO(3)}$ generalizes to the case of $PSU(N)$ in place of $SO(3)$ and $Z_N(M,\ q)$ in place of τ_3.

6.3 Proof of KM1

Consider the reduction (mod 4) of the linking matrix A of L along the diagonal and (mod 2) *out* of the diagonal. Besides it does not necessarily represent a $\mathbf{Z}_4$-quadratic form on a $\mathbf{Z}_2$-vector space, it is Witt equivalent to a diagonal matrix (still denoted A) with n_j entries congruent to $j \pmod 4$. This follows easily from the extension of the results shown in the Appendix.
The above formula for τ_3 splits in 3 parts due to the multiplicativity of $i^{S \cdot S}$ under the block sum decomposition of A, induced by (n_j). Denoting $\omega = \tau_3((1-i/\sqrt{2})^{\sigma(L)}$, we have:

$$\omega = \amalg_{j=0}^{3} \omega_j^{n_j}$$

where $\omega_0 = 1/\sqrt{2}\ (i^0 + i^0) = \sqrt{2},\ \omega_1 = 1/\sqrt{2}\ (i^0 + i^1) = \overline{((1-i)/\sqrt{2})},\ \omega_2 = 1/\sqrt{2}\ (1-1) = 0$ and $\omega_3 = (1-i)/\sqrt{2}$.Then:

$$\tau_3(M_L) = c^{\sigma(A)+n_3-n_1} \sqrt{2}^{n_0} \text{if } \mathrm{n}_2 = 0$$

$$\tau_3(M_L) = 0 \text{ otherwise}$$

Now if $n_2 = 0$, A is the matrix representation of a $\mathbf{Z}_4$-quadratic form, since the (mod 2)-linking of L is well defined. It is associated to a spanning surface F for L, capped off with the cores of the surgery that gives M.
Set as in the appendix $\beta(A) \equiv n_1 - n_3 \pmod 8 \equiv \sigma(A \pmod 4) \pmod 8$ for the Brown invariant of A; $\beta(M_L) = \sigma(A) - \beta(A) \pmod 8$ is an invariant of M_L, since $\sigma(A)$ and $\beta(A)$ modify identically under blowing up and down. Note that $|H^1(M, \mathbf{Z}_2)| \equiv n_0 + n_2 \pmod 4$ since A is a presentation matrix for $H_1(M, \mathbf{Z})$.
Then if $n_2 = 0$, we get the expected formula.

In case $n_2 \neq 0$, there exists a surface $F' \subset F$ over 2-framing components, hence with a $\mathbf{RP}^2$ inside. Any characteristic link $C \subset L$ contains all components with odd framing since by choice of a representant of the Witt class of A we have $\forall\ i \neq j\ L_i \cdot L_j \equiv 0 \pmod 2$.
Furthermore, $C \cdot C \equiv \beta(A) \pmod 4$ if C contains an odd number of 2-framing components, but $C \cdot C \equiv \beta \pm 2 \mod 4$ otherwise.
Using $\mu_{L,\ C} = \sigma(A) - C \cdot C + 8\ Arf(C) \pmod{16}$ and the $1-1$ correspondance

between characteristic links and Spin structures of M_L, we see that the μ invariants of M_L have all the same $\pmod 4$ reduction if and only if $n_2 = 0$:

$$n_2 = 0 \Longleftrightarrow \mu_{L,\ C} \equiv \sigma(A) - \beta(A) \pmod 4$$

Finally, taking two Spin structures θ_1 and θ_2 with distincts $\pmod 4$-μ invariants, the formula $\mu_{M_{\theta_1}} - \mu M_{\theta_2} = 2\beta(a) \pmod{16}$, where $a \in H_2(M, \mathbf{Z}_2)$ is a Poincare dual class to $\theta_1 - \theta_2 \in H^1(M, \mathbf{Z}_2$ (see chapter 4), shows that there exists an embedded surface in M_L, e.g. the above surface F', whose Pin^- class generates $\Omega_2^{Pin^-}$.It has obviously odd Euler characteristic since $\beta \pmod 2 \equiv \chi$.

Remarks

- With the notations of the preceeding chapters we have: $\beta(F') = \beta(C)$, where C is by definition the even link on which F' is capped off, and the even longitudes are selected by the choice of F'.The link between all these objects is: the Poincare dual to F' is the difference Spin structure considered in i) of $KM1$, and the Hom dual to the last one is C.

- In case $\tau_3(M_L) \neq 0$, the square of its modulus and a surgery presentation (in fact it suffices to have the index of W_L) of $M = M_L$ gives the $\pmod 4$ reduction of μ_θ for any Spin structure θ of M_L.
  Using the last remark of Chapter 4 about a theorem of **[Ka]**, we may complete condition i) as follows: if all μ invariants are distincts $\pmod 8$, there does not exists an embedded surface in M_L with odd Euler characteristic since then:

  $$\nexists\ x_1, x_2, x_3 \in H^1(M_L, \mathbf{Z})\ s.t.\ (x_1 \cup x_2 \cup x_3)([M]) \equiv 1 \pmod 2$$

  It is possible to interprete the non zero reduction $\pmod 4$ of $\mu_{\theta_1} - \mu_{\theta_2}$ as a normal surgery data of $f:\ M_L \to \mathbf{RP}^3$ towards $L(6,1)$ **[Ta]**.

- The splitting theorem **[KM1]**,§8, between $\tau_r^{SU(2)}$ and $\tau_r^{SO(3)}$ shows that $\tau_3 = O$ represents a somewhat geometric trivial obstruction.

Appendix A

Some invariants of quadratic forms

For more details,we refer to [**MH**] or [**GM**].

A.1 Arf invariant

This $\mathbf{Z}_2$-quadratic invariant aims at showing for any $\mathbf{Z}_2$-quadratic form q if more of an half of the elements of V are sich that $q = 1$; in that case $Arf(q)$ is equal to 1. Let V be a finite dimensional $\mathbf{Z}_2$-vector space with a given inner product $x \cdot y \in \mathbf{Z}_2$ and a linear map :

$$\lambda : V \to \mathbf{Z}_2$$

for which there exists $y \in V$ such that:

$$\lambda(x) = x \cdot y \ \forall \ x \in V$$

We get a $\mathbf{Z}_2$-quadratic space on V by adding the datum of a map defined by:

$$q(x + y) := q(x) + q(y) + x \cdot y$$

A geometric application is for the $\pmod 2$ reduced self linking of a knot.

Combining the duality and q, we easily obtain an hyperbolic decomposition of $(V, q) = \oplus_{i=1}^{n} H_i$,where $H_i = \langle x, y \rangle$ belongs to one of the following class:
i) either $q = 1$ on the three element $x, y, x + y$ of H_i
ii) or $q = 0$ on two of them.
We denote an element of the first (resp. second) class by H^1 (resp. H^0).
We have an isomorphism $H^0 \oplus H^0 \simeq H^1 \oplus H^1$ by sending x_i, y_i on the 4 maximal linear combinations of them; so our decomposition reduces to:
i) either $(V, q) = \oplus_{i=1}^{n-1} H^0 \oplus H^1$,
ii) or $(V, q) = \oplus_{i=1}^{n} H^0$.
In other words, there are $2^{2n-1} + 2n - 1$ terms on which q is zero in the first case and $2^{2n-1} - 2n - 1$ in the second case. We set:

$$Arf(q) := \{ \begin{array}{ll} 1 & \text{if i) is verified} \\ 0 & \text{otherwise} \end{array}$$

This furnishes our algebraic invariant,which is obviously additive.

A.2 Brown invariant

See [**Br**]

More than a simple extension of the Arf invariant, it will support most of our geometric background in being able to deal with the specificities of the intersection pairing on nonorientable surfaces. Just start as before, but chose a $\mathbf{Z}_4$-quadratic form $q : V \to \mathbf{Z}_4$ as follows:

$$q(x+y) = q(x) + q(y) + 2 \cdot x \cdot y$$

where we note $2 : \mathbf{Z}_2 \to \mathbf{Z}_4$ the homomorphism sending 1 to 2. Any $\mathbf{Z}_4$-quadratic space can be decomposed into a direct sum of indecomposable direct summands taken in the following list:

$$\begin{array}{ll} P_+ = (\mathbf{Z}_2 a, \cdot, q) & \text{where } a \cdot a = 1, q(a) = 1 \\ P_- = (\mathbf{Z}_2 a, \cdot, q) & \text{where } a \cdot a = 1, q(a) = -1 \\ T_0 = (\mathbf{Z}_2 b + \mathbf{Z}_2 c, \cdot, q) & \text{where } b \cdot b = 0, c \cdot c = 0, b \cdot c = 1, q(b) = q(c) = 0 \\ T_4 = (\mathbf{Z}_2 b + \mathbf{Z}_2 c, \cdot, q) & \text{where } b \cdot b = 0, c \cdot c = 0, b \cdot c = 1, q(b) = q(c) = 2 \end{array}$$

Now $H \subset (V, q)$ is said to be isotropic (and then (V, q) is split) if there exists $H \subset V$ such that $q(H) = 0, H \cdot H = 0$, and $dim(H) = 1/2 dim(V)$; for exemple $P_+ \oplus P_-$ and T_0 contain isotropic subspaces.
Let us define the Witt group of $\mathbf{Z}_4$-quadratic form on $\mathbf{Z}_2$-vector spaces by:

$$W := WQ(\mathbf{Z}_2, \mathbf{Z}_4) = \{\mathbf{Z}_4 - quadratic\ spaces\}/\{X \oplus S_1 \equiv X \oplus S_2\}$$

where S_i is split. Then $[T_0] \in W$ is zero and we easily see that:

$$\begin{cases} P_+ \oplus T_4 \simeq P_- \oplus P_- \oplus P_- \\ P_- \oplus T_4 \simeq P_+ \oplus P_+ \oplus P_+ \end{cases}$$

Finally $[T_4] = 4[P_-]$ and $[T_4] = 4[P_+]$, so W is generated by $[P_+]$ and is isomorphic to $\mathbf{Z}_8$ as follows from: $8[P_+] = 4([P_+] + [P_-]) = 0$.

In view to classify the isometric (i.e. equivalence) classes of $\mathbf{Z}_4$-quadratic form viewed as quadratic enhancements of a fixed bilinear pairing (for us : the intersection pairing),we are going to define an invariant of linear isomorphisms $T : (V_1, q_1) \to (V_2, q_2)$ preserving q_1 and q_2.

A Monski sum associated to a $\mathbf{Z}_4$-quadratic space $X = (V, \cdot, q)$ is written as: $\lambda(X) = \sum_{x \in V} i^{q(x)}$.
The operator λ clearly verifies $\lambda(X \oplus Y) = \lambda(X)\lambda(Y)$ and P_+ is sent onto $1 + i$. Consequently $\lambda(X) = \sqrt{2}^{dim(V)}(1 + i/\sqrt{2})^2$, where $m \in \mathbf{Z}$. The multiplicativity of the Monsky sum and this well defined integer $\pmod 8$ gives us our Brown invariant: $\beta(X) = m \in \mathbf{Z}_8$.
It is shown in [**Br**], th.1.20, that when X is split, it is sent to zero, and $\beta(P_+) = 1$ implies that β is an isomorphism between the Witt group W and $\mathbf{Z}_8$.

Here is a list of properties of β,extracted from **[Br]**, th.1.20:
1) If $(V, \cdot, q)$ is a $\mathbf{Z}_2$-quadratic space, then $(V, \cdot, 2q)$ is a $\mathbf{Z}_4$-quadratic space and $\beta(V, \cdot, 2q) = 4Arf(q)$, where we set $4 : 1 \in \mathbf{Z}_2 \to 4 \in \mathbf{Z}_8$.
2) For every $X = (V, \cdot, q)$, we have $\beta(X) \cong dim(V) \pmod 2$.
3) β is additive and multiplicative.
4) Suppose that U is a finitely generated abelian group, with $\Theta : U \otimes U \to \mathbf{Z}$ a unimodular symetric bilinear form; denote by $\Psi(u) = \Theta(u, u)$ and define $\varphi(u) : U/2U \to \mathbf{Z}$ by $\varphi(u) = \Psi(u) \pmod 4$. Then Φ is quadratic and $\beta(\Phi) = \sigma(\Psi) \pmod 8$.

An application of the last property is in considering the $\pmod 4$ reduction of the linking matrix of the simple closed curves generating $H^1(F^2, \mathbf{Z})$, for an orientable surface F^2: its Brown invariant is the $\pmod 8$ reduction of the index of the intersection pairing of F^2.

Appendix B

Pin groups

The main reference is [**KT**].

Given a real vector space V of dimension n with basis $\{e_i\}_{1\leq i\leq n}$ and a given inner product $\langle\ ,\ \rangle$, consider its universal algebra generated (as a vector space) by the free products $e_I = e_{i_1}e_{i_2}\dots e_{i_k},\ 1 \leq i_j \leq n, j \in \mathbf{N}, I = (i_1, \dots, i_k)$, the unity $e_\emptyset$ and either relations:

$$(+)\ \ vw + wv = 2\langle v,\ w\rangle$$

or

$$(-)\ \ vw + wv = -\ 2\langle v,\ w\rangle$$

This algebra will be called the Clifford algebra $Cliff^{\pm}(V)$, depending on the choice of the relation above.
Define $Pin^{\pm}(V)$ as the set of elements of $Cliff^{\pm}(V)$ which can be written in the form $v_1v_2\dots v_k$, where each v_i is a *unit* vector in V; this is a compact Lie group, and when k is even these elements generate the famous 2-sheeted covering $Spin(V)$ of $SO(V)$. This can be seen as follows.

We can represent geometrically the multiplication by a vector $v \in V$ in the algebra by the action of the reflexion r_v in $O(V)$ across the *non oriented* $(n-1)$-plane $v^\perp$ perpendicular to v. Since the reflection in $v^\perp$ has the form $x \mapsto x - 2\ \langle x,\ v\rangle\ v$, a product $(\pm v_1)(\pm v_2)\dots(\pm v_k)$ of basis vectors of V may be thought of as either the homomorphism $r_{v_1} \circ \dots \circ r_{v_k} \in O(V)$ or the image of v_k under $r_{v_1} \circ \dots \circ r_{v_{k-1}}$.
Now, the elements of $Pin^{\pm}(V)$ act on V by *choosing an orientation* on $v^\perp \in V$. Hence we consider arbitrary products of vectors of V, in the form $v_1 \dots v_k$. In particular, when we restrict to even numbered products of vectors, we have an action of $Spin(V)$ onto V which may be seen as a 2-sheeted covering of $SO(V)$ (generated by products of even numbers of reflexions in V). Moreover this is a non trivial double covering for $n \geq 2$, since for orthonormal vectors $v_1,\ v_2 \in V$ the invertible arc:

$$t \mapsto cos(t) + sin(t)v_1\ v_2,\ \ 0 \leq t \leq \pi$$

connects 1 and -1 *inside* $Spin(V)$. Considering the long exact sequence of homotopy groups for the principal bundle $SO(n) \to SO(n+1) \to S^n$ with $\pi_1(SO(3)) \cong \mathbb{Z}_2$, we see that for $n \geq 3$ the group $Spin(V)$ is in fact the universal covering of the group $SO(V)$.

How does this representation of $Spin(V)$ onto $SO(V)$ generalizes to the action of $Pin^{\pm}(V)$ onto V?

Define a transposition on $Cliff^{\pm}(V)$ by extending linearly over V the obvious transposition of vectors:

$$(v_{i_1} \dots v_{i_k})^t := v_{i_k} \dots v_{i_1}$$

In the same manner, define another algebra homomorphism by $\alpha(e_I) = (-1)^{Card(I)} e_I$;

There are representations $\rho^{\pm} : \; Pin^{\pm}(V) \to O(V)$, given by:

$$\rho^{-}(w)(v) = wvw^t$$

$$\rho^{+}(w)(v) = \alpha(w)vw^t$$

where the left and right action of elements of $Pin^{\pm}(V)$ correspond, as stated above, to composition with their images in $O(V)$.
Let us denote by $r, id \in O(V)$ the reflexion across an arbitrary $(n-1)$-plane $e_1^{\perp}$ in V and the identity element of $O(V)$; then it is easy to see that $(\rho^{\pm})^{-1}\{r, \; id\} = \{\pm e_1, \; \pm 1\}$. Furthermore we have $e_1^2 = \pm 1 \in Pin^{\pm}(V)$, hence:

$$(\star) \quad (\rho^{+})^{-1}\{r, \; id\} \simeq \mathbf{Z}_2 \oplus \mathbf{Z}_2$$
$$(\star\star) \quad (\rho^{-})^{-1}\{r, \; id\} \cong \mathbf{Z}_4$$

Clearly the elements $1, \; -1 \in Pin^{\pm}(V)$ are the whole center $\mathbf{Z}_2$ of $Pin^{\pm}(V)$ if $n > 1$ and the above constructions show that $Pin^{\pm}(V)/\{\pm 1\} = O(V)$. Notice that since $O(V)$ has a non trivial center, the groups $Pin^{\pm}(V)$ are non trivial central extensions of $O(V)$, which restrict to $Spin(V)$ over $SO(V)$.

The two groups $Pin^{\pm}(V)$ are $Spin(V) \amalg \; Spin(V)$ as spaces, but *the group structure is different in the two cases*, as is clear from the equations $(\star)$ and $(\star\star)$. We can think of $-1 \in \rho^{-1}(id)$ as a rotation of V by 2π about any axis, for the following reason: for $\theta \in [0, \; 1]$, the arc $\theta \to \pm e_1 \cdot (cos(\theta)e_1 + sin(\theta)e_2)$ in $Pin^{\pm}(V)$ goes from 1 to -1 and is sent by ρ onto a loop in $O(V)$ that generates $\pi_1(O(V))$. It is the group theoretic reason for which $Pin^{\pm}$ may be used to distinguish an odd from an even number of geometric full twists: for instance, a section s of the normal bundle ν to an embedded curve C in a 3-manifold describe a loop l in $O(2)$ under the identification of all the fibers to $\mathbf{R}^2$. The class of l in $\pi_1(O(2), \; id)$ is different from zero if the image curve of s in the total space of $\partial\nu$ (which is a torus or a Klein bottle) describes an odd number of twists about C.

Let us finally give the construction of a $Pin^{-}(2)$-structure on the tangeant bundle of $\mathbf{RP}^2$.
Take two vector fields e_1 and e_2 in the tangeant bundle to $\mathbf{RP}^2_{|Mb}$, where $\mathbf{RP}^2$ is considered as $B^2 \cup_{\partial B^2} \mathbf{RP}^1$ and Mb is the Mobius band neighborhood of $\mathbf{RP}^1 \subset \mathbf{RP}^2$, such that e_1 is parallel to the core circle $\mathbf{RP}^1$ and e_2 is normal to it. Consider the trivialization of $T\mathbf{RP}^2{}_{|Mb}$ over two coordinate charts U_1 and U_2, with transition

functions $U_1 \cap U_2 \to Pin^-(2)$ that send the two components of $U_1 \cap U_2$ to 1 and e_2 respectively.
The induced $\mathbf{R}^2$-bundle $T\mathbf{RP}^2{}_{|\partial Mb}$ over $S^1 = \partial Mb$ is trivialized by the transition functions 1 and $e_2^2 = -1$. This means that we add a rotation by 2π to the framing $(e_1,\ e_2)$, which is then turned into the opposite one, after a complete traverse of $\partial Mb = S^1$. But this trivialization of $T\mathbf{RP}^2{}_{|S^1}$ is exactly the one that extend over B^2. Notice that $\mathbf{RP}^2$ does not support a Pin^+ structure, as follows from the same argument (with $e_2^2 = 1$).

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