HOPF DIAGRAMS AND QUANTUM INVARIANTS

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ABSTRACT. The Reshetikhin-Turaev invariant, Turaev's TQFT, and many related constructions rely on the encoding of certain ribbon tangles (n-string links, or n-ribbon handles) as n-forms on the coend of a ribbon category. We introduce the monoidal category of Hopf diagrams, and describe a universal encoding of ribbon string links as Hopf diagrams. This universal encoding is an injective monoidal functor and admits a straightforward monoidal retraction. Any Hopf diagram with n legs yields a n-form on the coend of a ribbon category in a completely explicit way. Thus computing a quantum invariant of a 3-manifold reduces to the purely formal computation of the associated Hopf diagram, followed by the evaluation of this diagram in a given category (using in particular the so-called Kirby elements).

CONTENTS

INTRODUCTION

In 1991, Reshetikhin and Turaev [RT91] introduced a new 3-manifold invariant. The construction proceeds in two steps: representing a 3-manifold by surgery along a link and then *coloring* the link to obtain a scalar invariant. Here, *colors* are (linear combinations of) simple representations of a quantum group at a root of unity. Since then, this construction has been re-visited many times.

In particular, Turaev [Tur94] introduced the notion of a modular category, which is a semisimple ribbon category satisfying a finiteness and a non-degeneracy condition, and showed that such a category defines a 3-manifold invariant, and indeed a TQFT. In this approach, colors are simple objects of the category.

Following these ideas, a more general approach on quantum invariants of 3-manifolds has been subsequently developed, see [Lyu95] and more recently [KL01, Vir03]. It avoids in particular the semisimplicity condition. Let us briefly outline it: the

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initial data used to construct the invariants is a ribbon category $\mathcal{C}$ endowed with a coend $A = \int^{X \in \mathcal{C}} {}^{\vee}X \otimes X$. Let L be a framed n-link. We can always present L as the closure of some ribbon n-string link T. By using the universal property of the coend A, to such a string link T is associated a n-form on A, that is, a morphism $T_{\mathcal{C}} \colon A^{\otimes n} \to \mathbb{1}$ in $\mathcal{C}$. Given a morphism $\alpha \colon \mathbb{1} \to A$ (which plays here the role of the color), set:

$$\tau_{\mathcal{C}}(L; \alpha) = T_{\mathcal{C}} \circ \alpha^{\otimes n} \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}).$$

A *Kirby element* of $\mathcal{C}$, as defined in [Vir03], is a morphism $\alpha \colon \mathbb{1} \to A$ such that, for all framed link L, $\tau_{\mathcal{C}}(L; \alpha)$ is well-defined and invariant under isotopies of L and under 2-handle slides. In this case, by the Kirby theorem [Kir78] and under some invertibility condition, the invariant $\tau_{\mathcal{C}}(L; \alpha)$ can be normalized to an invariant of 3-manifolds.

At this stage, two main questions naturally arise. Firstly, how to recognize the Kirby elements of a ribbon category? And secondly, how to compute the forms $T_{\mathcal{C}}$ obtained via universal property?

Concerning the first question, recall that the coend A has a structure of a Hopf algebra in the category $\mathcal{C}$, see [Maj93, Lyu94]. This means in particular that A is endowed with a product $\mu_A \colon A \otimes A \to A$, a coproduct $\Delta_A \colon A \to A \otimes A$, and an antipode $S_A \colon A \to A$ which satisfy the same axioms as those of a Hopf algebra except one has to replace the usual flip map with the braiding of $\mathcal{C}$. If A admits a two-sided integral $\lambda \colon \mathbb{1} \to A$, then λ is a Kirby element and the corresponding 3-manifold invariant is that of Lyubashenko [Lyu95]. More generally, if a morphism $\alpha \colon \mathbb{1} \to A$ in $\mathcal{C}$ satisfies:

$$(\mathrm{id}_A \otimes \mu_A)(\Delta_A \otimes \mathrm{id}_A)(\alpha \otimes \alpha) = \alpha \otimes \alpha \quad \text{and} \quad S_A \alpha = \alpha,$$

then it is a Kirby element, see [Vir03]. In particular, any Reshetikhin-Turaev invariant computed from a semisimple sub-quotient of $\mathcal{C}$ can be defined directly via a Kirby element of $\mathcal{C}$ satisfying this last equation.

The main motivation of the present paper is to answer the second question. Given a ribbon string link T, we express $T_{\mathcal{C}}$ in terms of some structural morphisms of A (avoiding the product). This can be done by means of a universal formula, that is, independently of $\mathcal{C}$. To this end, we introduce the notion of *Hopf diagrams*. They are braided planar diagrams (with inputs but no output) obtained by stacking boxes $\Delta = \,$, $S = \,$, and $\omega = \,$. These are submitted to relations corresponding to those of a coproduct, an antipode, and a Hopf pairing. To each Hopf diagram D with n inputs is associated a ribbon n-string link $\phi(D)$:
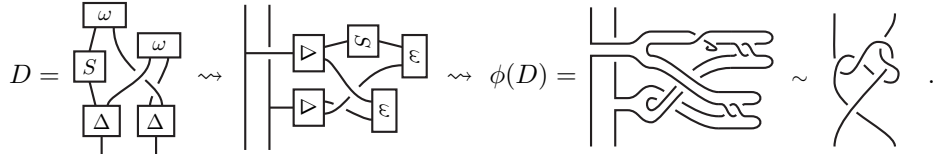
This gives a universal encoding for ribbon string links for which all the forms $?_{\mathcal{C}}$ are realizations.

In Section 1, we define the monoidal category of Hopf diagrams, as a convolution category. The category of Hopf diagrams comes in two versions: with, or without antipode. Both versions are isomorphic, however. In Section 2, we review the monoidal category of ribbon string links and the related monoidal category of ribbon handles. These categories are isomorphic. In Section 3, we construct a

monoidal functor ϕ from the category $\mathcal{D}iag^S$ of Hopf diagrams to the category RSL of ribbon string links. We introduce a quotient $\overline{\mathcal{D}iag}^S$ of $\mathcal{D}iag^S$ in such a way that the monoidal functor $\overline{\phi} \colon \overline{\mathcal{D}iag}^S \to \mathrm{RSL}$ induced by ϕ admits a right inverse. More precisely, in Section 4, we construct a monoidal functor $\Phi \colon \mathrm{RSL} \to \overline{\mathcal{D}iag}^S$ such that $\overline{\phi} \circ \Phi = 1_{\mathrm{RSL}}$. In particular, ϕ is surjective. Finally, in Section 5, given a ribbon category $\mathcal{C}$ endowed with a coend A, we explain how to represent the category of Hopf diagrams into $\mathcal{C}$ by using some structural morphisms of A. Moreover, we give a general criterion, using Hopf diagrams, for recognizing Kirby elements.

Unless otherwise specified, categories are assumed to be small and monoidal categories are assumed to be strict.

1. HOPF DIAGRAMS

1.1. Categorical (co)algebras. Recall that an *algebra* in a monoidal category is an object A endowed with morphisms $\mu \colon A \otimes A \to A$ (the *product*) and $\eta \colon A \to \mathbb{1}$ (the *unit*) which satisfy:

$$\mu(\mathrm{id}_A \otimes \mu) = \mu(\mu \otimes \mathrm{id}_A) \quad \text{and} \quad \mu(\eta \otimes \mathrm{id}_A) = \mathrm{id}_A = \mu(\mathrm{id}_A \otimes \eta).$$

Dually, a *coalgebra* in a tensor category is an object C endowed with morphisms $\Delta \colon C \to C \otimes C$ (the *coproduct*) and $\varepsilon \colon C \to \mathbb{1}$ (the *counit*) which satisfy:

$$(\mathrm{id}_C \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_C)\Delta \quad \text{and} \quad (\varepsilon \otimes \mathrm{id}_C)\Delta = \mathrm{id}_C = (\mathrm{id}_C \otimes \varepsilon)\Delta.$$

Note that the unit object $\mathbb{1}$ of a monoidal category $\mathcal{C}$ is an algebra and a coalgebra in $\mathcal{C}$ (with trivial structure morphisms).

When the category $\mathcal{C}$ is braided with braiding τ, the tensor product $A \otimes A'$ of two algebras A and A' in $\mathcal{C}$ is an algebra in $\mathcal{C}$ with product $(\mu \otimes \mu')(\mathrm{id}_A \otimes \tau_{A,A'} \otimes \mathrm{id}_{A'})$ and unit $\eta \otimes \eta'$. In particular, for any non-negative integer n, $A^{\otimes n}$ is an algebra in $\mathcal{C}$. Likewise, the tensor product $C \otimes C'$ of two coalgebras C and C' in a braided category $\mathcal{C}$ is a coalgebra in $\mathcal{C}$ with coproduct $(\mathrm{id}_C \otimes \tau_{C,C'} \otimes \mathrm{id}_{C'})(\Delta \otimes \Delta')$ and counit $\varepsilon \otimes \varepsilon'$. In particular, for any non-negative integer n, $C^{\otimes n}$ is a coalgebra in $\mathcal{C}$.

1.2. The convolution product. Let $\mathcal{C}$ be a tensor category, (A, μ, η) be an algebra in $\mathcal{C}$, and (C, Δ, ε) be a coalgebra in $\mathcal{C}$. The *convolution product* of two morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ is the morphism $f \star g = \mu(f \otimes g)\Delta \in \mathrm{Hom}_{\mathcal{C}}(C, A)$. This makes the set $\mathrm{Hom}_{\mathcal{C}}(C, A)$ a monoid with unit $\eta\varepsilon \colon C \to A$.

1.3. Convolution categories. Let $\mathcal{C}$ be a braided category, A be an algebra in $\mathcal{C}$, and C be a coalgebra in $\mathcal{C}$. Let us define the *convolution category* $\mathrm{Conv}_{\mathcal{C}}(C, A)$ as follows: the objects of $\mathrm{Conv}_{\mathcal{C}}(C, A)$ are the non-negative integers $\mathbb{N}$. For $m, n \in \mathbb{N}$, the set of morphisms from m to n is empty if $m \neq n$ and is the monoid $\mathrm{Hom}_{\mathcal{C}}(C^{\otimes n}, A)$ endowed with the convolution product if $m = n$, see Section 1.2 (recall indeed that $C^{\otimes n}$ is a coalgebra in $\mathcal{C}$). In particular, the identity of an object $n \in \mathbb{N}$ is:

$$\mathrm{id}_n = \eta\varepsilon^{\otimes n} \colon C^{\otimes n} \to A,$$

and the composition of two endomorphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(C^{\otimes n}, A)$ of an object $n \in \mathbb{N}$ is given by the convolution product:

$$f \circ g = f \star g = \mu(f \otimes g)\Delta_{C^{\otimes n}} \colon C^{\otimes n} \to A,$$

where $\Delta_{C^{\otimes n}}$ denotes the coproduct of the coalgebra $C^{\otimes n}$.

Note that the category $\mathrm{Conv}_{\mathcal{C}}(C, A)$ is a monoidal category: the monoidal product of two objects $m, n \in \mathbb{N}$ is given by $m \otimes n = m + n$, the unit object is $0 \in \mathbb{N}$, and the monoidal product of two morphisms $f \colon m \to m$ and $g \colon n \to n$ (where $m, n \in \mathbb{N}$) is the morphism $f \otimes g = \mu(f \otimes_{\mathcal{C}} g) \colon m + n \to m + n$.

1.4. The category $\mathcal{D}iag$. Let $\mathcal{D}$ be the braided category freely generated by one object $*$ and the following morphisms:

$$\Delta \colon * \to * \otimes *, \qquad \omega_+ \colon * \otimes * \to \mathbb{1}, \qquad \theta_+ \colon * \to \mathbb{1},$$
$$\varepsilon \colon * \to \mathbb{1}, \qquad \omega_- \colon * \otimes * \to \mathbb{1}, \qquad \theta_- \colon * \to \mathbb{1},$$

where $\mathbb{1}$ denotes the unit object of the tensor product. Let $\mathcal{D}_0$ be the quotient of the category $\mathcal{D}$ by the following relations:

(1.1) $$(\mathrm{id}_* \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_*)\Delta,$$

(1.2) $$(\mathrm{id}_* \otimes \varepsilon)\Delta = \mathrm{id}_* = (\varepsilon \otimes \mathrm{id}_*)\Delta.$$

The category $\mathcal{D}_0$ is still braided (with induced braiding) and $(*, \Delta, \varepsilon)$ is a coalgebra in $\mathcal{D}_0$. We define the category $\mathcal{D}iag$ to be the convolution category $\mathrm{Conv}_{\mathcal{D}_0}(*, \mathbb{1})$, see Section 1.3, where $\mathbb{1}$ is endowed with the trivial algebra structure.

1.5. The category $\mathcal{D}iag^S$. Let $\mathcal{D}^S$ be the braided category freely generated by one object $*$ and the following morphisms:

$$\Delta \colon * \to * \otimes *, \qquad \omega_+ \colon * \otimes * \to \mathbb{1}, \qquad \theta_+ \colon * \to \mathbb{1},$$
$$\varepsilon \colon * \to \mathbb{1}, \qquad \omega_- \colon * \otimes * \to \mathbb{1}, \qquad \theta_- \colon * \to \mathbb{1},$$
$$S \colon * \to *, \qquad S^{-1} \colon * \to *.$$

Let $\mathcal{D}_0^S$ be the quotient of the category $\mathcal{D}^S$ by the relations (1.1) and (1.2), and the following relations:

(1.3) $$SS^{-1} = \mathrm{id}_* = S^{-1}S,$$

(1.4) $$\Delta S = (S \otimes S)\tau_{*,*}\Delta,$$

(1.5) $$\varepsilon S = \varepsilon,$$

(1.6) $$\theta_{\pm} S = \theta_{\pm},$$

(1.7) $$\omega_+(S \otimes \mathrm{id}_*) = \omega_- = \omega_+(\mathrm{id}_* \otimes S),$$

(1.8) $$\omega_+(S^{-1} \otimes \mathrm{id}_*) = \omega_- \tau_{*,*} = \omega_+(\mathrm{id}_* \otimes S^{-1}),$$

where $\tau_{*,*} \colon * \otimes * \to * \otimes *$ denotes the braiding of the object $*$ with itself in $\mathcal{D}^S$.

The category $\mathcal{D}_0^S$ is still braided (with induced braiding) and $(*, \Delta, \varepsilon)$ is a coalgebra in $\mathcal{D}_0^S$. We define the category $\mathcal{D}iag^S$ to be the convolution category $\mathrm{Conv}_{\mathcal{D}_0^S}(*, \mathbb{1})$, see Section 1.3, where $\mathbb{1}$ is endowed with the trivial algebra structure.

1.6. Relations between $\mathcal{D}iag$ and $\mathcal{D}iag^S$. The inclusion functor $\mathcal{D} \hookrightarrow \mathcal{D}^S$ induces a functor $\mathcal{D}_0 \to \mathcal{D}_0^S$ and so a functor $\iota \colon \mathcal{D}iag \to \mathcal{D}iag^S$. Note that ι is the identity on the objects.

Theorem 1.1. $\iota \colon \mathcal{D}iag \to \mathcal{D}iag^S$ *is an isomorphism of categories.*

Proof. Let $\mathcal{C}$ be the quotient of the category $\mathcal{D}^S$ by the relations (1.4)-(1.8) together with the following additional relations (which can be shown to be satisfied in $\mathcal{D}_0^S$ by using (1.3)):

$$\Delta S^{-1} = (S^{-1} \otimes S^{-1})\tau_{*,*}^{-1}\Delta, \quad \varepsilon S^{-1} = \varepsilon, \quad \theta_{\pm} S^{-1} = \theta_{\pm},$$

$$\omega_-(S \otimes \mathrm{id}_*) = \omega_+ \tau_{*,*}^{-1} = \omega_-(\mathrm{id}_* \otimes S),$$

$$\omega_-(S^{-1} \otimes \mathrm{id}_*) = \omega_+ = \omega_-(\mathrm{id}_* \otimes S^{-1}).$$

From these rules, we can show that if m is a word in S and S^{-1}, then $\varepsilon m = \varepsilon$ and $\theta_{\pm} m = \theta_{\pm}$. Moreover if m_1 and m_2 are words in S and S^{-1}, then

$$\omega_+(m_1 \otimes m_2) = \omega_+ \tau_{*,*}^{-k} \qquad \text{if } e(m_1) + e(m_2) = 2k,$$

$$\omega_+(m_1 \otimes m_2) = \omega_- \tau_{*,*}^{-k} \qquad \text{if } e(m_1) + e(m_2) = 2k+1,$$

$$\omega_-(m_1 \otimes m_2) = \omega_+ \tau_{*,*}^{-k-1} \qquad \text{if } e(m_1) + e(m_2) = 2k+1,$$

$$\omega_-(m_1 \otimes m_2) = \omega_- \tau_{*,*}^{-k} \qquad \text{if } e(m_1) + e(m_2) = 2k,$$

where $e(S^{\alpha_1} S^{\alpha_2} \cdots S^{\alpha_r}) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ with $\alpha_i = \pm 1$.

Denote by $\rho \colon \mathcal{D} \to \mathcal{C}$ the composition of the inclusion functor $\mathcal{D} \hookrightarrow \mathcal{D}^S$ with the projection functor $\mathcal{D}^S \twoheadrightarrow \mathcal{C}$. Let n be a non negative integer and $f \in \mathrm{Hom}_{\mathcal{C}}(*^{\otimes n}, \mathbb{1})$. By using the above relations, one can remove all the generators S and S^{-1} from an expression of f in a unique way. Therefore there exists a unique morphism $\tilde{f} \in \mathrm{Hom}_{\mathcal{D}}(*^{\otimes n}, \mathbb{1})$ such that $\rho(\tilde{f}) = f$. This means that ρ induces a bijection between $\mathrm{Hom}_{\mathcal{D}}(*^{\otimes n}, \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(*^{\otimes n}, \mathbb{1})$.

Let $\mathcal{C}_0$ be the quotient of the category $\mathcal{C}$ by the relations (1.1) and (1.2). Note that the object $*$ is then a coalgebra in $\mathcal{C}_0$. By definition of convolution categories and by using the preceding, ρ induces an isomorphism $\bar{\rho} \colon \mathcal{D}iag = \mathrm{Conv}_{\mathcal{D}_0}(*, \mathbb{1}) \to \mathrm{Conv}_{\mathcal{C}_0}(*, \mathbb{1})$.

Now remark that the relation (1.3) holds in $\mathrm{Hom}_{\mathcal{C}}(*^{\otimes n}, \mathbb{1})$. Indeed let $f \in \mathrm{Hom}_{\mathcal{C}}(*^{\otimes n}, \mathbb{1})$. Suppose for example that $f = g(\mathrm{id}_*^{i-1} \otimes S^{-1}S \otimes \mathrm{id}_*^{m-i})h$ for some $g \in \mathrm{Hom}_{\mathcal{C}}(*^{\otimes m}, \mathbb{1})$ and $h \in \mathrm{Hom}_{\mathcal{C}}(*^{\otimes n}, *^{\otimes m})$. Recall that $g = \rho(\tilde{g})$ where $\tilde{g} \in \mathrm{Hom}_{\mathcal{D}}(*^{\otimes m}, \mathbb{1})$. Since $\rho(\tilde{g})$ is expressed without using S or S^{-1} and since the morphism $S^{-1}S$ satisfies:

$$\Delta S^{-1}S = (S^{-1}S \otimes S^{-1}S)\Delta, \quad \varepsilon S^{-1}S = \varepsilon, \quad \theta_{\pm} S^{-1}S = \theta_{\pm},$$

$$\omega_+(S^{-1}S \otimes \mathrm{id}_*) = \omega_+ = \omega_+(\mathrm{id}_* \otimes S^{-1}S),$$

$$\omega_-(S^{-1}S \otimes \mathrm{id}_*) = \omega_- = \omega_-(\mathrm{id}_* \otimes S^{-1}S),$$

we have that $\rho(\tilde{g})(\mathrm{id}_*^{i-1} \otimes S^{-1}S \otimes \mathrm{id}_*^{m-i}) = \rho(\tilde{g})$ and so $f = gh$.

Since (1.3) holds in $\mathrm{Hom}_{\mathcal{C}}(*^{\otimes n}, \mathbb{1})$, the projection functor $\mathcal{D}^S \to \mathcal{C}$ induces a isomorphism $\pi \colon \mathcal{D}iag^S \to \mathrm{Conv}_{\mathcal{C}_0}(*, \mathbb{1})$ such that $\bar{\rho} = \pi \circ \iota$. Hence ι is an isomorphism. $\qquad\square$

1.7. **Hopf diagrams.** By a *Hopf diagram*, we shall mean a morphism of $\mathcal{D}iag$ or $\mathcal{D}iag^S$. Hopf diagrams can be represented by plane diagrams: we draw one of their preimage in the braided category $\mathcal{D}_0$ or $\mathcal{D}_0^S$ by using Penrose graphical calculus with the ascending convention (diagrams are read from bottom to top) as for instance in [Tur94]. We depict the generators as in Figure 1 except Figure 1(e) which depicts $\Delta^{(n)} \colon * \to *^{\otimes(n+1)}$ defined inductively by:

$$\Delta^{(0)} = \mathrm{id}_*, \quad \Delta^{(1)} = \Delta \quad \text{and} \quad \Delta^{(n+1)} = (\Delta^{(n)} \otimes \mathrm{id}_*)\Delta.$$

The relations defining $\mathcal{D}iag^S$ (except those concerning the braiding) are depicted in Figure 2. Recall that the composition $D_1 \circ D_2 = D_1 \star D_2$ of two Hopf diagrams D_1 and D_2 is given by the convolution product, see Figure 3.
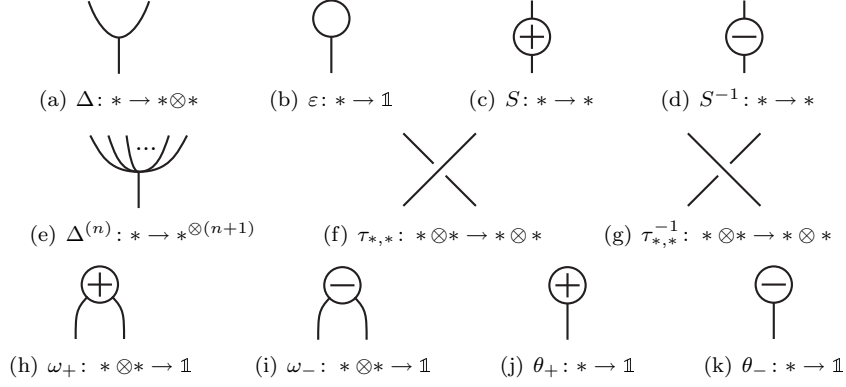
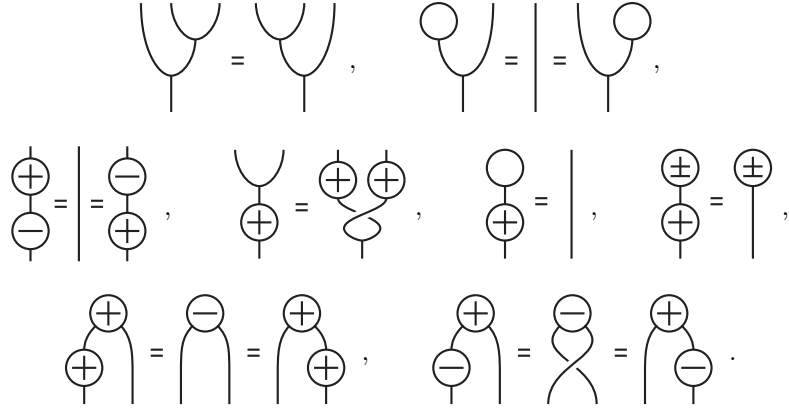(a) $\Delta \colon * \to * \otimes *$ (b) $\varepsilon \colon * \to \mathbb{1}$ (c) $S \colon * \to *$ (d) $S^{-1} \colon * \to *$

(e) $\Delta^{(n)} \colon * \to *^{\otimes(n+1)}$ (f) $\tau_{*,*} \colon * \otimes * \to * \otimes *$ (g) $\tau_{*,*}^{-1} \colon * \otimes * \to * \otimes *$

(h) $\omega_+ \colon * \otimes * \to \mathbb{1}$ (i) $\omega_- \colon * \otimes * \to \mathbb{1}$ (j) $\theta_+ \colon * \to \mathbb{1}$ (k) $\theta_- \colon * \to \mathbb{1}$

FIGURE 1. Generators of $\mathcal{D}iag^S$

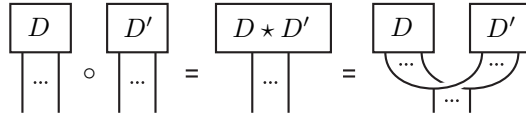FIGURE 2. Relations in $\mathcal{D}iag^S$

FIGURE 3. Composition in $\mathcal{D}iag^S$

2. Ribbon string links and ribbon handles

2.1. **Ribbon string links.** Let n be a positive integer. By a *ribbon n-string link* we shall mean a framed (n, n)-tangle $T \subset \mathbb{R}^2 \times [0, 1]$ consisting of n arc components, without any closed component, such that the kth arc $(1 \le k \le n)$ joins the

kth bottom endpoint to the kth top endpoint. Note that a ribbon string link is canonically oriented by orienting each component from bottom to top.

We will represent ribbon string links by plane diagrams. Recall that two such diagrams represent the same isotopy class of a ribbon string link if and only if one can be obtained from the other by deformation and a finite sequence of ribbon Reidemeister moves depicted in Figure 4.
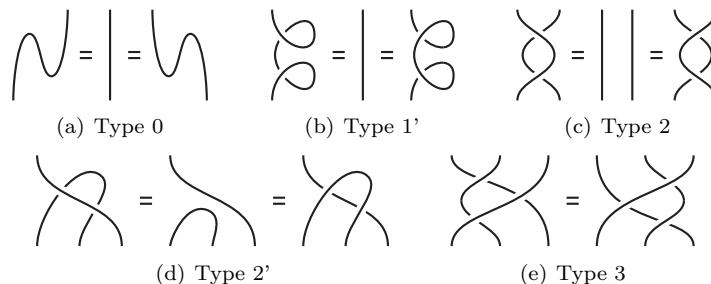

(a) Type 0   (b) Type 1'   (c) Type 2

(d) Type 2'   (e) Type 3

FIGURE 4. Ribbon Reidemeister moves

We denote by RSL the category of ribbon string links. The objects of RSL are the non-negative integers. For two non-negative integers m and n, the set of morphisms from m to n is

$$\mathrm{Hom}_{\mathrm{RSL}}(m, n) = \begin{cases} \emptyset & \text{if } m \neq n, \\ \mathrm{RSL}_n & \text{if } m = n, \end{cases}$$

where RSL_n denotes the set of (isotopy classes) of ribbon n-string links. The composition is given by the stacking product (with ascending convention) and identities are the trivial string links.

Note that the category RSL is a monoidal category: $m \otimes n = m + n$ on objects and the monoidal product $T \otimes T'$ of two ribbon string links T and T' is the ribbon string link obtained by stacking T on the left of T' (see, e.g., [Tur94]).

2.2. **Ribbon handles.** Let n be a positive integer. By a *ribbon n-handle* we shall mean a framed $(2n, 0)$-tangle $T \subset \mathbb{R}^2 \times [0, 1]$ consisting of n arc components, without any closed component, such that the kth arc $(1 \leq k \leq n)$ joins the $(2k-1)$-th bottom endpoint to the $2k$-th bottom endpoint. Note that a ribbon handle is canonically oriented by orienting each component upwards near its right bottom input.

As for ribbon string links, we will represent ribbon handless by plane diagrams. Two such diagrams represent the same isotopy class of a ribbon handle if and only if one can be obtained from the other by deformation and a finite sequence of ribbon Reidemeister moves depicted in Figure 4.

We denote by RHand the category of ribbon handles. The objects of RHand are the non-negative integers. For two non-negative integers m and n, the set of morphisms from m to n is

$$\mathrm{Hom}_{\mathrm{RHand}}(m, n) = \begin{cases} \emptyset & \text{if } m \neq n, \\ \mathrm{RHand}_n & \text{if } m = n, \end{cases}$$

where RHand_n denotes the set of (isotopy classes) of ribbon n-handles. The composition of two ribbon n-handles T and T' is the ribbon n-handle defined in Figure 5(c). The identity for this composition consists in n caps, see Figure 5(d).

(a) T (b) T' (c) $T \circ T'$ (d) id   (e) $T \otimes T'$
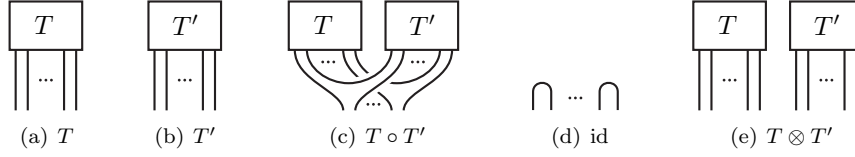
FIGURE 5. Composition, identity, and monoidal product in RHand

Note that the category RHand is a monoidal category: $m \otimes n = m + n$ on objects and the monoidal product $T \otimes T'$ of two ribbon handles T and T' is the ribbon handle obtained by stacking T on the left of T', see Figure 5(e)

2.3. **An isomorphism between ribbon handles and string links.** Let us construct functors $F \colon \mathrm{RSL} \to \mathrm{RHand}$ and $G \colon \mathrm{RHand} \to \mathrm{RSL}$ as follows. On objects, set $F(n) = n$ and $G(n) = n$ for any non-negative integer n. For any ribbon n-string link S, let $F(S)$ be the ribbon n-handle defined in Figure 6(b). For any ribbon n-handle T, let $G(T)$ be the ribbon n-string link defined in Figure 6(d).

(a) S (b) $F(S)$ (c) T (d) $G(T)$
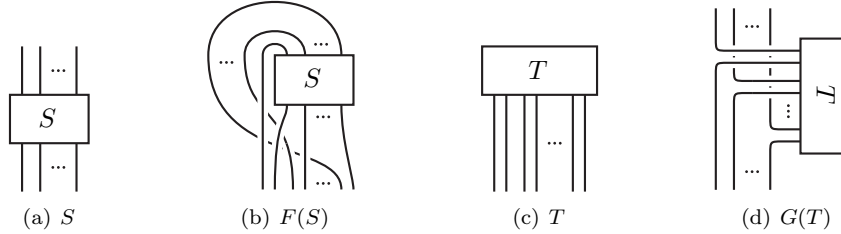
FIGURE 6. Definition of the functors F and G

Proposition 2.1. *The functors F and G are mutually inverse monoidal functors.*

In particular, the categories RSL and RHand are isomorphic. In the following, depending on the context, we will use indifferently the notions of ribbon string link and of ribbon handle as more convenient.

Proof. Straightforward!   □

3. FROM HOPF DIAGRAMS TO RIBBON STRING LINKS

3.1. **From Hopf diagrams to ribbon handles.** Let us define a functor ϕ from the category $\mathcal{D}iag^S$ of Hopf diagrams to the category RHand of ribbon handles. For any non-negative integer n, let $\phi(n) = n$. If D is a Hopf diagram, we construct a diagram of ribbon handle ϕ_D by using the rules of Figure 7 and the stacking product (with ascending convention). See Figure 8 for an example. Then let $\phi(D)$ be the isotopy class of the ribbon handles defined by ϕ_D.

Theorem 3.1. *The functor $\phi \colon \mathcal{D}iag^S \to \mathrm{RHand}$ is well-defined, monoidal and surjective.*

Proof. Let us first verify that ϕ is well-defined. We just have to verify both sides of the relations defining $\mathcal{D}iag^S$ are transformed by the rules of Figure 7 to isotopic tangles. Examples of such verifications are depicted in Figure 9. The fact that ϕ is a monoidal functor comes from the definitions of composition and monoidal
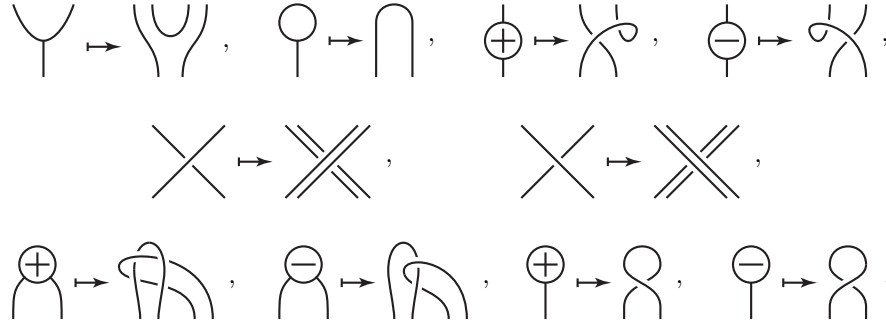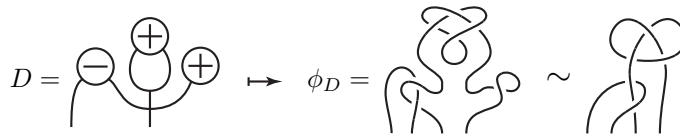
FIGURE 7. Rules for defining ϕ

FIGURE 8

product in $\mathcal{D}iag^S$ and RHand. The fact that ϕ is surjective is a direct consequence of Corollary 4.5 proved below. □
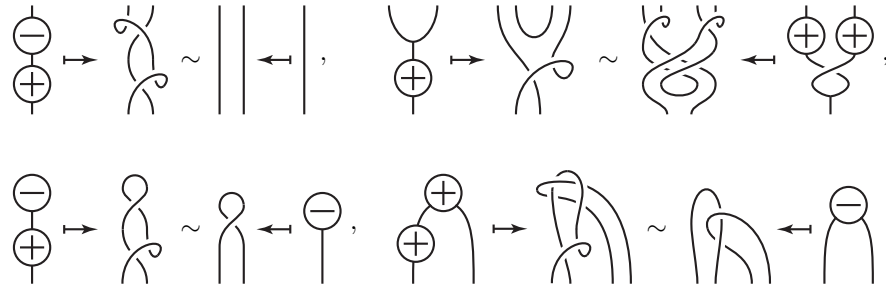

FIGURE 9

3.2. **From Hopf diagrams to ribbon string links.** Let us define a functor $\psi \colon \mathcal{D}iag^S \to \mathrm{RSL}$ as follows. For any non-negative integer n, let $\psi(n) = n$. If D is a Hopf diagram, we construct a diagram of ribbon string link ψ_D as in Figure 10(a), where ϕ_D is defined as above. For an example, see Figure 10(b). Then we let $\psi(D)$ be the isotopy class of the ribbon string link defined by ψ_D.
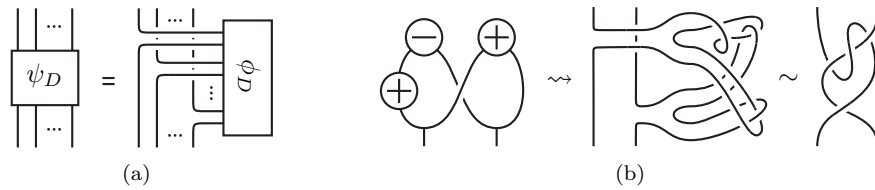

(a) (b)

FIGURE 10

Corrolary 3.2. *The functor $\psi\colon \mathcal{D}iag^S \to$ RSL is well-defined and surjective. Moreover $F \circ \psi = \phi$ and $G \circ \phi = \psi$, where F and G are the functors of Section 2.3.*

Proof. This is an immediate consequence of Theorem 3.1 and Proposition 2.1. $\quad\square$

3.3. **More relations on Hopf diagrams.** In this section, we quotient $\mathcal{D}iag^S$ by additional relations which will allow us to construct right inverses of the functors ϕ and ψ (see Section 4). Denoting by $\star$ the convolution product on Hopf diagrams, these relations are the following:

$$(3.1) \qquad (\mathrm{id}_* \otimes \theta_\pm)\Delta = (\theta_\pm \otimes \mathrm{id}_*)\Delta,$$

$$(3.2) \qquad (\theta_+ \otimes \theta_-)\Delta = \varepsilon,$$

$$(3.3) \qquad (\theta_+ \otimes \theta_+)\Delta = \omega_+ \tau_{*,*}^{-1}\Delta,$$

$$(3.4) \qquad (\theta_- \otimes \theta_-)\Delta = \omega_-\Delta,$$

$$(3.5) \qquad \omega_+ \star \omega_- = \varepsilon \otimes \varepsilon = \omega_- \star \omega_+,$$

$$(3.6) \qquad \omega_{+12} \star \omega_{+13} \star \omega_{+23} = \omega_{+13} \star \omega_{+23} \star \omega_{+12} = \omega_{+23} \star \omega_{+12} \star \omega_{+13},$$

$$(3.7) \qquad \omega_{+13} \star \omega_{+23} \star \omega_{+24} \star \omega_{-23} = \omega_{+23} \star \omega_{+24} \star \omega_{-23} \star \omega_{+13}.$$

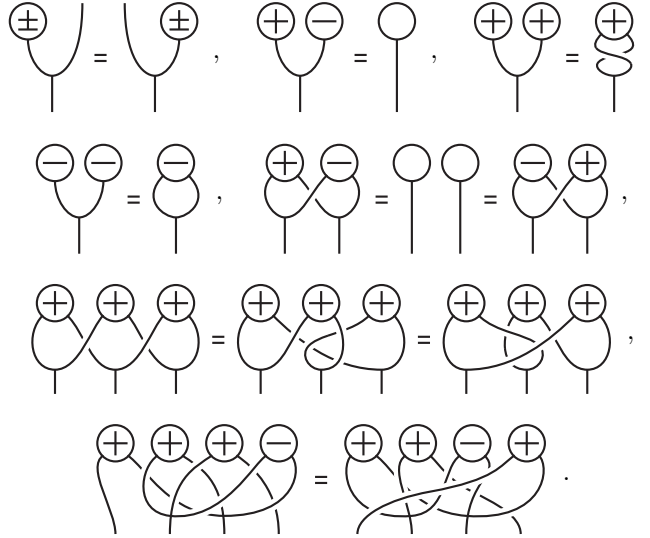Graphically, Relations (3.1)-(3.7) can be depicted as in Figure 11.

FIGURE 11

Let $\overline{\mathcal{D}iag}^S$ be the quotient of the category $\mathcal{D}iag^S$ by Relations (3.1)-(3.7). We will still call the morphisms of $\overline{\mathcal{D}iag}^S$ by *Hopf diagrams*. Denote by $\pi\colon \mathcal{D}iag^S \to \overline{\mathcal{D}iag}^S$ the projection functor.

Proposition 3.3. *The functors ϕ and ψ factorize through π to surjective functors $\overline{\phi}$ and $\overline{\psi}$ so that $\overline{\phi} = F \circ \overline{\psi}$.*

Proof. We have to verify both sides of Relations (3.1)-(3.7) are transformed by the rules of Figure 7 to isotopic tangles. Examples of such verifications are depicted in Figure 12. Note that in the first picture, we used the fact that the $(2i-1)$-th and $2i$-th inputs of a ribbon handle are connected. $\quad\square$
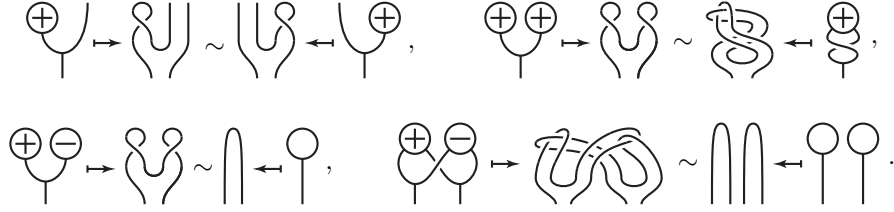
FIGURE 12

4. FROM RIBBON STRING LINKS TO HOPF DIAGRAMS

In this section, we construct right inverses of the functors $\overline{\phi}$ and $\overline{\psi}$, that is, monoidal functors $\Phi\colon \mathrm{RHand} \to \overline{\mathcal{D}iag}^S$ and $\Psi\colon \mathrm{RSL} \to \overline{\mathcal{D}iag}^S$ such that:

$$\overline{\phi} \circ \Phi = 1_{\mathrm{RHand}} \quad \text{and} \quad \overline{\psi} \circ \Psi = 1_{\mathrm{RSL}}.$$

4.1. **From ribbon pure braids to Hopf diagrams.** Let RPB the subcategory of RSL made of ribbon pure braids. The objects of RPB are the non-negative integers. For two non-negative integers m and n, the set of morphisms from m to n is

$$\mathrm{Hom}_{\mathrm{RPB}}(m, n) = \begin{cases} \emptyset & \text{if } m \neq n, \\ \mathrm{RPB}_n & \text{if } m = n, \end{cases}$$

where $\mathrm{RPB}_n \subset \mathrm{RSL}_n$ denotes the set of (isotopy classes) of ribbon pure n-braids. The composition is given by the stacking product (with ascending convention) and identities are the trivial braids. Note that RPB is a monoidal subcategory of RSL.

Recall we have a canonical group isomorphism:

$$(u, t_1, \ldots, t_n)\colon \mathrm{RPB}_n \xrightarrow{\sim} \mathrm{PB}_n \times \mathbb{Z}^n,$$

where PB_n denotes the group of pure n-braids, $u\colon \mathrm{RPB}_n \to \mathrm{PB}_n$ is the forgetful morphism, and t_i the self-linking number of the i-th component. Hence, using a presentation of PB_n by generators and relations due to Markov [Mar45], we get that RPB_n is generated by t_k ($1 \leq k \leq n$) and $\sigma_{i,j}$ ($1 \leq i < j \leq n$) subject to the following relations:

(4.1) $t_k t_l = t_l t_k$ for any k, l;

(4.2) $t_k \sigma_{i,j} = \sigma_{i,j} t_k$ for any $i < j$ and k;

(4.3) $\sigma_{i,j} \sigma_{k,l} = \sigma_{k,l} \sigma_{i,j}$ for any $i < j < k < l$ or any $i < k < j < l$;

(4.4) $\sigma_{i,j} \sigma_{i,k} \sigma_{j,k} = \sigma_{i,k} \sigma_{j,k} \sigma_{i,j} = \sigma_{j,k} \sigma_{i,j} \sigma_{i,k}$ for any $i < j < k$;

(4.5) $\sigma_{i,k} \sigma_{j,k} \sigma_{j,l} \sigma_{j,k}^{-1} = \sigma_{j,k} \sigma_{j,l} \sigma_{j,k}^{-1} \sigma_{i,k}$ for any $i < j < k < l$.

Graphically, the generators may be represented as:

$$\sigma_{i,j} = \left| \cdots \right| \begin{array}{c} \cdots \end{array} \left| \cdots \right| \quad \text{and} \quad t_k = \left| \cdots \right| \begin{array}{c} \end{array} \left| \cdots \right|.$$

Let us define a functor $\Psi_0 \colon \mathrm{RPB} \to \overline{\mathcal{D}iag}^S$ as follows: for any non-negative integer n set $\Psi(n) = n$. For $1 \le i < j \le n$ and $1 \le k \le n$, set

$$\Psi_0(\sigma_{i,j}^{\pm 1}) = \Sigma_{i,j}^{\pm 1} \quad \text{and} \quad \Psi_0(t_k^{\pm 1}) = \Omega_k^{\pm 1},$$

where

(4.6) $\qquad \Sigma_{i,j}^{\pm 1} = $ $\quad$ and $\qquad \Omega_k^{\pm 1} = $.

Lemma 4.1. *The functor $\Psi_0 \colon \mathrm{RPB} \to \overline{\mathcal{D}iag}^S$ is well defined, monoidal, and is such that $\overline{\psi} \circ \Psi_0(P) = P$ for all ribbon pure braid P.*

Proof. Firstly, from Relation (3.5) (resp. Relations (3.1) and (3.2)), we have that the Hopf diagrams $\Sigma_{i,j}^{-1}$ and $\Sigma_{i,j}$ (resp. Ω_k^{-1} and Ω_k) are inverse each other.

Secondly Relations (4.1)-(4.5) hold in $\overline{\mathcal{D}iag}_n^S$, where we replace $\sigma_{i,j}$ and t_k with $\Sigma_{i,j}$ and Ω_k respectively. Indeed Relations (4.1) and (4.2) follow from (3.1). Relation (4.3) follows from (1.2). Relations (4.4) and (4.5) correspond to (3.6) and (3.7) respectively.

Finally the isotopies depicted in Figure 13 show respectively that $\overline{\psi} \circ \Psi_0(\sigma_{i,j}) = \sigma_{i,j}$ and $\overline{\psi} \circ \Psi_0(t_k) = t_k$. Hence $\overline{\psi} \circ \Psi_0(P) = P$ for all ribbon pure braid P. $\qquad \square$
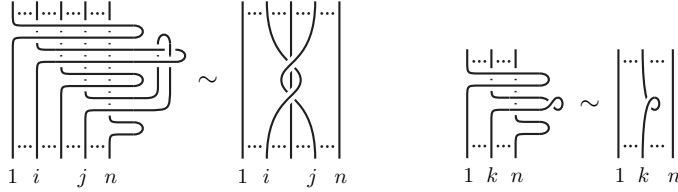

FIGURE 13

4.2. **Contractions.** Let $n \ge 3$ and $1 < i < n$. For a ribbon n-string link T, we define the *i-th left contraction* of T to be the ribbon $(n-2)$-string link $c_i(T)$ defined as in Figure 14(a). For a Hopf diagram D with n inputs, we define the *i-th contraction* of D to be the Hopf diagram with $(n-2)$ inputs $C_i(D)$ defined as in Figure 14(b).
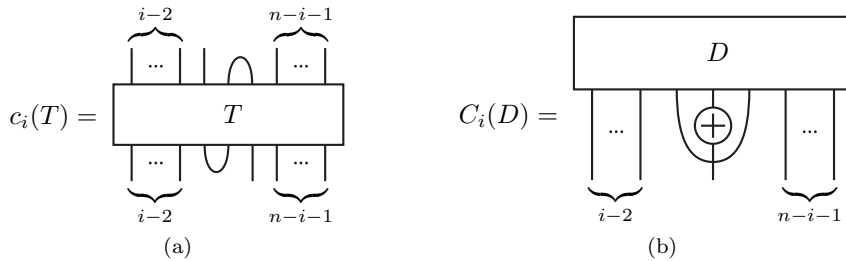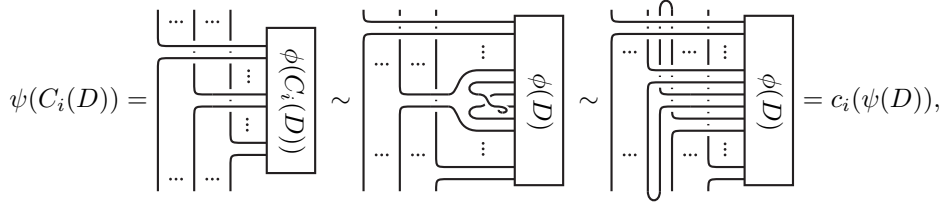

FIGURE 14

Lemma 4.2. *Let $n \geq 3$ and $1 < i < n$. For any Hopf diagram D with n inputs, we have that $c_i(\psi(D)) = \psi(C_i(D))$.*

Proof. This follows from the following equalities:

where we used that the $(2i - 1)$-th and $2i$-th inputs of $\phi(D)$ are connected (since it is a ribbon handle). $\square$

Lemma 4.3. $c_i c_j = c_j c_{i+2}$ *and* $C_i C_j = C_j C_{i+2}$ *for any* $i \geq j$.

Proof. This results directly from the definitions of the contraction operators. $\square$

4.3. **From ribbon string links to Hopf diagram.** In this section, we extend the functor $\Psi_0 \colon \mathrm{RPB} \to \overline{\mathcal{D}iag}^S$ to a functor $\Psi \colon \mathrm{RSL} \to \overline{\mathcal{D}iag}^S$. The construction of this extension follows, broadly speaking, the same pattern as the proof of [Bru04, Theorem 3]. The point is to see that a ribbon string link can be obtained from a ribbon pure braid by a sequence of contractions. This will at least show that Ψ extends uniquely and suggest a construction for it. We then must check the coherence of this construction, that is, its independence from the choices we made.

The main trick we use consists in "pulling a max to the top line". Let Γ be a tangle diagram with a local max m, with n outputs. We may write Γ as in Figure 15(a), where U, V are tangle diagrams. Let i be an integer, $1 \leq i \leq n + 1$. Let j be the number of strands to the left of m on the same horizontal line. Let $U \cup \ell$ be a tangle diagram obtained from U by inserting a new component ℓ going from a point between the j-th and $(j + 1)$-th inputs of U to a point between the $(i - 1)$-th and i-th outputs of U, see Figure 15(b). We assume also that ℓ has no local extremum. Let $U_\ell = \Delta_\ell(U \cup \ell)$ be the tangle diagram obtained from $U \cup \ell$ by doubling ℓ. Set $\Gamma_\ell = U_\ell V$, see Figure 15(c). We say that Γ_ℓ *is obtained from Γ by pulling m to the top in the i-th position (along the path ℓ)*. Likewise, one defines the action of *pulling a local min to the bottom*.

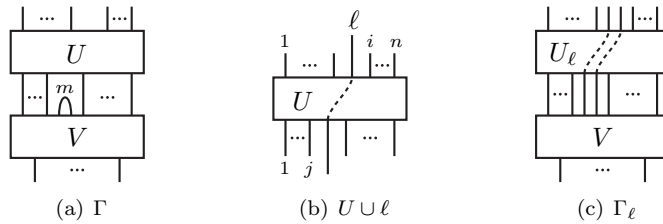(a) Γ (b) $U \cup \ell$ (c) Γ_ℓ

FIGURE 15. Pulling a max to the top line

Let us define $\Psi \colon \mathrm{RSL} \to \overline{\mathcal{D}iag}^S$ as follows: on objects $n \in \mathbb{N}$, set $\Psi(n) = n$. Let n be a positive integer and T be a ribbon n-string link. Consider a diagram

of T. For each local extremum pointing to the right (once the strands canonically oriented from bottom to top), modify the diagram using the following rule:

(4.7)

This leads to a diagram Γ which is *left handed*, that is, with all local extrema pointing to the left. Pulling all local max to the top and all local min to the bottom, we obtain a diagram of a pure braid. Here is an algorithm. Denote by m_i the number of local max (which is equal to the number of local min) on the i-th component of Γ. Let $m(\Gamma) = m_1 + \cdots + m_n$ be the number of local max of Γ. If $m(\Gamma) = 0$, we are already done. Otherwise, chose i maximal so that $m_i > 0$. Let m be the first max and m' be the first min you meet on the i-th component, going from bottom to top. Pull m to the top, in the $(i+1)$-th position, and m' to the bottom, in the i-th position. Let Γ' be the diagram so constructed. Then Γ' is a string link diagram, with $m(\Gamma') = m(\Gamma) - 1$. Let us denote by $\{\Gamma\}$ the ribbon string link defined by the diagram Γ. Then $\{\Gamma\} = c_{i+1}\{\Gamma'\}$, where c_{i+1} is the $(i+1)$-th left contraction as in Section 4.2. Repeating $m(\Gamma)$ times this transformation yields a pure braid diagram P with $n + 2m(\Gamma)$ strands, and we have $\{\Gamma\} = c_{j_{m(\Gamma)}} \cdots c_{j_1}\{P\}$ where $1 \leq j_1 \leq \cdots \leq j_{m(\Gamma)} \leq n$, and j_k takes m_i times the value $i+1$. Note that

$$T = (t_1^{\alpha_1} \cdots t_n^{\alpha_n}) \, c_{j_{m(\Gamma)}} \cdots c_{j_1}\{P\},$$

where α_i is the number of modifications (4.7) made on the i-th component and $t_i \in \mathrm{RPB}_n$ is as in Section 4.1. Finally, as suggested by Lemmas 4.1 and 4.2, set:

$$\Psi(T) = (\Omega_1^{\alpha_1} \cdots \Omega_n^{\alpha_n}) \, C_{j_{m(\Gamma)}} \cdots C_{j_1} \Psi_0(\{P\}),$$

where $\Psi_0 \colon \mathrm{RPB} \to \overline{\mathcal{D}iag}^S$ is the functor of Lemma 4.1, the $\Omega_j = \Psi_0(t_j)$ are the Hopf diagrams of Section 4.1, and the C_j are the contractions on Hopf diagrams defined in Section 4.2.

Theorem 4.4. *The functor* $\Psi \colon \mathrm{RSL} \to \overline{\mathcal{D}iag}^S$ *is well-defined, monoidal, and is the only functor from* RSL *to* $\overline{\mathcal{D}iag}^S$ *satisfying:*

(a) $\Psi(P) = \Psi_0(P)$ *for any ribbon pure braid* P;
(b) $\Psi(c_i(T)) = C_i \Psi(T)$ *for any ribbon n-string link T and $1 < i < n$.*

*Moreover* $\overline{\psi} \circ \Psi = 1_{\mathrm{RSL}}$.

Remark in particular that Ψ is injective and $\overline{\psi}$ (and so ψ) is surjective.
We prove the theorem in Section 4.5.

Set $\Phi = \Psi \circ G \colon \mathrm{RHand} \to \overline{\mathcal{D}iag}^S$, where $G \colon \mathrm{RHand} \to \mathrm{RSL}$ is the monoidal isomorphism defined in Section 2.3. From Proposition 2.1 and Theorem 4.4, we immediately deduce that:

Corrollary 4.5. *The functor* $\Phi \colon \mathrm{RHand} \to \overline{\mathcal{D}iag}^S$ *is monoidal and satisfies* $\overline{\phi} \circ \Phi = 1_{\mathrm{RHand}}$.

Note in particular that Φ is injective and that $\overline{\phi}$ (and so ϕ) is surjective.

4.4. **A summary and an open question.** The preceding results may be summarized in the commutativity of the following diagram:
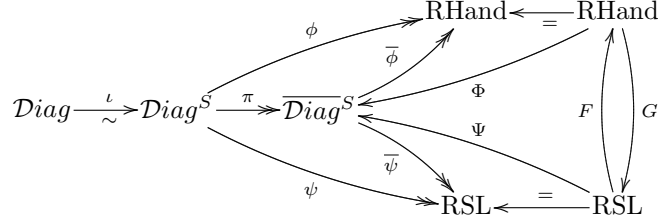
The main open question concerning Hopf diagram is: are the relations we put on them (that is, those defining $\overline{\mathcal{Diag}}^S$) sufficient to insure that the functors $\overline{\phi}$ and Φ (resp. $\overline{\psi}$ and Ψ) are isomorphisms (and so inverse each other)?

4.5. **Proof of Theorem 4.4.** Before proving Theorem 4.4, we first establish some lemmas.

Lemma 4.6. *Let P be a ribbon pure n-braid and $1 \leq i \leq n+1$. Insert a new component ℓ between the $(i-1)$-th and i-th strand of P so that $P \cup \ell$ is a ribbon pure $(n+1)$-braid. Let $P_\ell = \Delta_\ell(P \cup \ell)$ be the ribbon pure $(n+2)$-braid from $P \cup \ell$ by doubling ℓ. Then:*

(a) *The equalities of Figure 16 hold;*
(b) *$C_i\big(\Psi_0(P_\ell)D\big) = \Psi_0(P)C_i(D)$ and $C_{i+1}\big(D\Psi_0(P_\ell)\big) = C_{i+1}(D)\Psi_0(P)$ for any Hopf diagram D with $(n+2)$ inputs.*
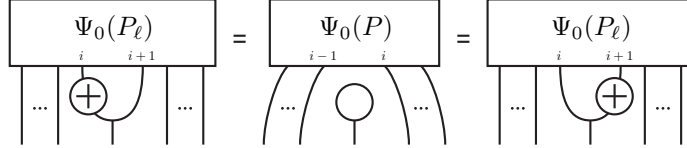

FIGURE 16

Proof. Let us prove Part (a) by induction on the length m of $P \cup l$ in the generators $\sigma_{k,l}^{\pm 1}$ and $t_k^{\pm 1}$ of RPB$_{n+1}$. If $m = 0$, then it is an immediate consequence of (1.2) and (1.5). Suppose that $m = 1$. Given a ribbon pure braid Q, denote by $\Delta_i(Q)$ (resp. $\delta_i(Q)$) the ribbon pure braid obtained from Q by doubling (resp. deleting) its i-th component. We have to verify that the statement is true for $P = \delta_i(\sigma_{k,l}^{\pm 1})$ and $P_\ell = \Delta_i(\sigma_{k,l}^{\pm 1})$, and for $P = \delta_i(t_k^{\pm 1})$ and $P_\ell = \Delta_i(t_k^{\pm 1})$. This can be done case by case by using the descriptions of Table 1. Examples of such verifications are depicted in Figure 17.

Let $m \geq 1$ and suppose the statement true for rank m. Let P and ℓ such that $P \cup \ell = w_1 \ldots w_{m+1}$ where the w_j are generators of RPB$_{n+1}$. Remark that $P_\ell = \Delta_i(w_1)\Delta_i(w_2 \cdots w_{m+1})$ and $P = \delta_i(w_1)\delta_i(w_2 \cdots w_{m+1})$. By using (1.4), (1.2), and the statement for ranks 1 and m, we get the equalities depicted in Figure 18. The left equality of Figure 16 is then true for P and ℓ. The right equality can be verified similarly by remarking that $P_\ell = \Delta_i(w_1 \cdots w_m)\Delta_i(w_{m+1})$. Hence the statement is true for rank $m + 1$.

	$\Delta_i(\sigma_{k,l}^{\pm1})$	$\delta_i(\sigma_{k,l}^{\pm1})$
$i < k$	$\sigma_{k+1,l+1}^{\pm1}$	$\sigma_{k-1,l-1}^{\pm1}$
$i = k$	$(\sigma_{i,l+1}\sigma_{i+1,l+1})^{\pm1}$	I_n
$k < i < l$	$\sigma_{k,l+1}^{\pm1}$	$\sigma_{k,l-1}^{\pm1}$
$i = l$	$(\sigma_{k,i}\sigma_{k,i+1})^{\pm1}$	I_n
$l < i$	$\sigma_{k,l}^{\pm1}$	$\sigma_{k,l}^{\pm1}$

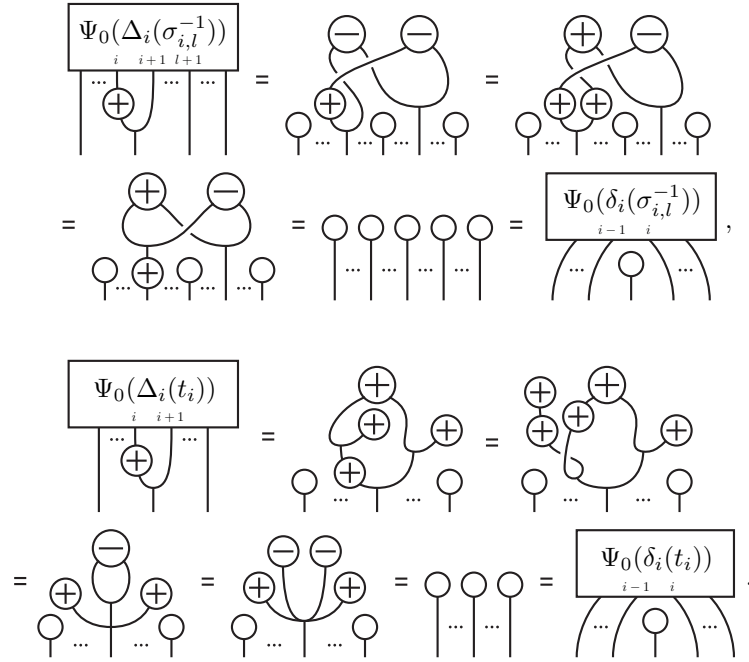|	$\Delta_i(t_k^{\pm1})$	$\delta_i(t_k)$
$i < k$	$t_{k+1}^{\pm1}$	$t_{k-1}^{\pm1}$
$i = k$	$\sigma_{i,i+1}^{\pm1}t_i^{\pm1}t_{i+1}^{\pm1}$	I_n
$k < i$	$t_k^{\pm1}$	$t_k^{\pm1}$

TABLE 1

FIGURE 17

Let us prove Part (b). Let D be a Hopf $(n+2)$-diagram. By using (1.1), (1.2), (1.4), and Part (a) of the lemma, we get the equalities of Figure 19, which mean that $C_i\big(\Psi_0(P_\ell)D\big) = \Psi_0(P)C_i(D)$. Likewise, one can show that $C_{i+1}\big(D\Psi_0(P_\ell)\big) = C_{i+1}(D)\Psi_0(P)$. $\square$

Lemma 4.7. *Let $n \geq 3$ and $1 < i < n$ (resp. $0 < i < n-1$). Let P and P' be two pure n-braids which have diagrams which differ only inside a disk. Inside this disk, the i-th and $(i+1)$-th strands pass respectively to the front and the back of another strand. Suppose also that above (resp. below) the disk, the i-th and $(i+1)$-th strands run parallel, see Figure 20. Then:*

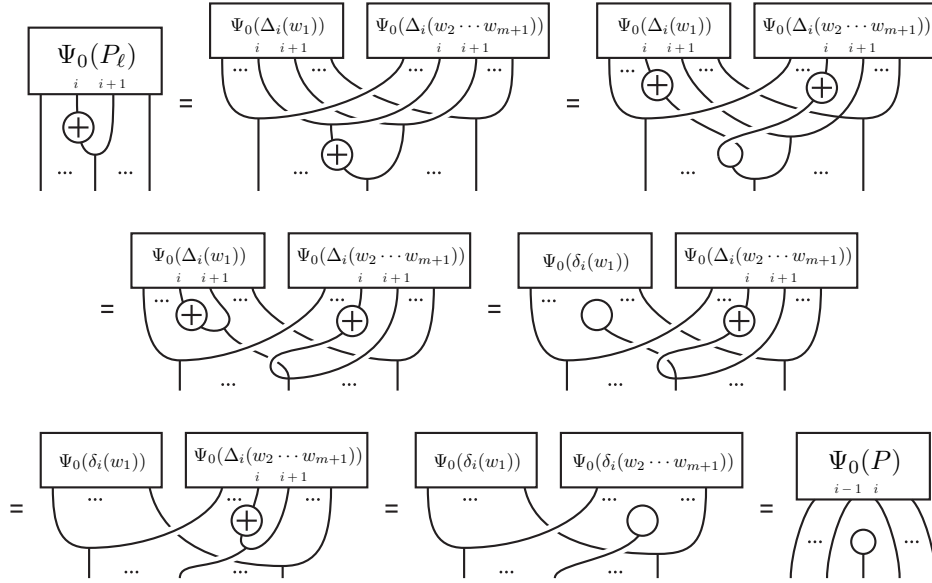$$C_i\Psi_0(P) = C_i\Psi_0(P') \ (resp. \ C_{i+1}\Psi_0(P) = C_{i+1}\Psi_0(P')).$$

FIGURE 18

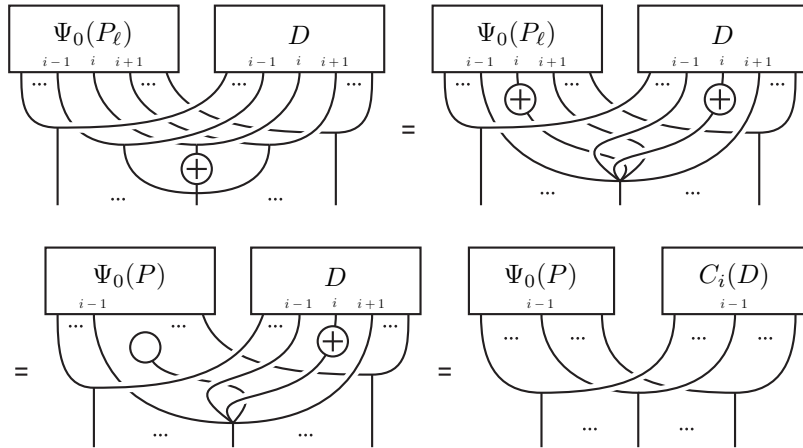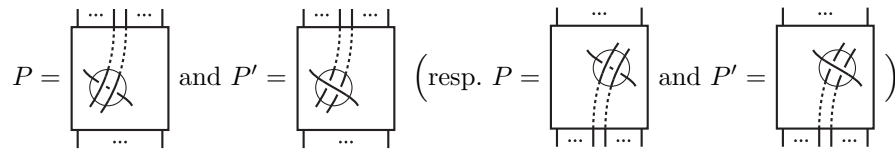FIGURE 19

FIGURE 20

Proof. Let us denote by k the other component. Suppose that above the disk, the i-th and $(i+1)$-th strands run parallel (the resp. case can be done similarly).

Assume first that $k < i$. We can write $P = A\sigma_{k,i}\sigma_{k,i+1}B$ and $P' = AB$ where A and B are ribbon pure n-braids. Moreover $A = \Delta_\ell(E \cup \ell)$ where E is ribbon pure braid with $n-2$ strands and ℓ is a new component inserted in E in i-th position. Now remark that $\sigma_{k,i}\sigma_{k,i+1} = \Delta_\gamma(I_{n-2} \cup \gamma)$ where I_{n-2} is the trivial braid with $n-2$ strands and γ is new component added to I_{n-2} in i-th position so that $I_{n-2} \cup \gamma = \sigma_{k,i}$ in RPB_{n-1}. Therefore, by using Lemma 4.6, we get:

$$C_i\Psi_0(P) = C_i\Psi_0(A\sigma_{k,i}\sigma_{k,i+1}B) = C_i\Psi_0\big(\Delta_\ell(E \cup \ell)\Delta_\gamma(I_{n-2} \cup \gamma)B\big)$$
$$= \Psi_0(E)\,\Psi_0(I_{n-2})\,C_i\Psi_0(B) = \Psi_0(E)\,C_i\Psi_0(B)$$
$$= C_i\Psi_0\big(\Delta_\ell(E \cup \ell)B\big) = C_i\Psi_0(AB) = C_i\Psi_0(P').$$

The case $k > i+1$ is done similarly by writing $P = A\sigma_{i,k}\sigma_{i+1,k}B$ and $P' = AB$. $\square$

Let us now prove Theorem 4.4. Let n be a positive integer and T be a ribbon n-string link. Consider a diagram Γ_T of T. Applying rules (4.7) changes (in a unique manner) Γ_T to a left handed diagram. Denote by α_i is the number of modifications (4.7) made on the i-th component of Γ_T. Then pulling maxima to the top and minima to the bottom as explained in Section 4.3 leads to a diagram P of a ribbon pure braid so that $T = (t_1^{\alpha_1} \cdots t_n^{\alpha_n})c_{j_m} \cdots c_{j_1}\{P\}$ for some $1 \leq j_1 \leq \cdots \leq j_m \leq n$. Pulling extrema in another way may lead to another diagram P' of a ribbon pure braid so that $T = (t_1^{\alpha_1} \cdots t_n^{\alpha_n})c_{j_m} \cdots c_{j_1}\{P'\}$. Now the ribbon pure braids $\{P\}$ and $\{P'\}$ are related by moves described in Figure 20. Therefore, by using Lemmas 4.3 and 4.7, we get that $C_{j_m} \cdots C_{j_1}\Psi_0(\{P\}) = C_{j_m} \cdots C_{j_1}\Psi_0(\{P'\})$. Hence

$$\Psi_{\Gamma_T} = (\Omega_1^{\alpha_1} \cdots \Omega_n^{\alpha_n})\,C_{j_m} \cdots C_{j_1}\Psi_0(\{P\})$$

only depends on the diagram Γ_T of T.

Let us verify that Ψ_{Γ_T} remains unchanged when applying to Γ_T a Reidemeister's move, see Figure 4. Invariance under moves of type 2 or type 3 is a consequence of the existence of the functor Ψ_0. Invariance under moves of type 2' is a consequence of Lemmas 4.3 and 4.7.

Suppose that a Reidemeister move of Type 0 is applied to Γ_T, and denote by Γ_T' the so-obtained diagram. There are four cases to consider, depending on the orientation of the considered strand and the direction (left or right handed) of the move. These cases, together with the way we apply the algorithm, are depicted in Figure 21. Let us for example verify invariance in the case depicted in Figure 21(d). Recall that applying the algorithm to Γ_T gives rise to a diagram P of a pure braid such that $\{\Gamma_T\} = (t_1^{\alpha_1} \cdots t_n^{\alpha_n})\,c_{j_m} \cdots c_{j_1}\{P\}$. Let i be the number of the component of Γ_T on which the move is performed. Since the orientation of the strand is downwards and T is a (canonically oriented) string link, we know that there exist a maximum just before and a minimum just after the place where the move is performed, when going through the component i from bottom to top (see the left picture of Figure 21(d)). Let c_{j_r} be the contraction corresponding to this pair of extrema when applying the algorithm to Γ_T (we have $j_r = i+1$). Denote by k the number of the component of P where the move is performed. Up to using invariance under Reidemeister's moves of type 2 and type 3, we can write $\{P\} = UV$, with U and V pure braids, so that the move is performed on the k-th component between U and V. Insert a new component ℓ in U in k-th position and a new component ℓ' in V in $(k+1)$-th position. Set $U_\ell = \Delta_\ell(U \cup \ell)$ and $V_{\ell'} = \Delta_{\ell'}(V \cup \ell')$. As depicted

$\Gamma_T =$ ... $\mapsto \Gamma'_T =$ ... $\rightsquigarrow$ ...

(a)

$\Gamma_T =$ ... $\mapsto \Gamma'_T =$ ... $\rightsquigarrow$ ... $\rightsquigarrow$ ...

(b)

$\Gamma_T =$ ... $\mapsto \Gamma'_T =$ ... $\rightsquigarrow$ ...

(c)

$\Gamma_T =$ ... $\mapsto \Gamma'_T =$ ... $\rightsquigarrow$ ... $\rightsquigarrow$ ...
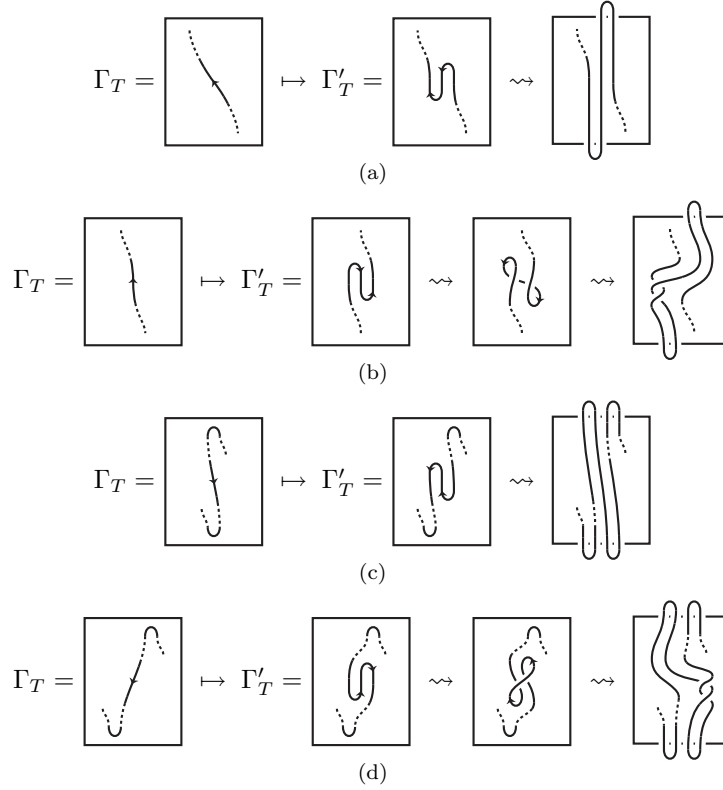
(d)

FIGURE 21

in Figures 21(d) and 22, we can apply the algorithm to Γ'_T in such a way that:

$$\{\Gamma'_T\} = (t_1^{\alpha_1} \cdots t_i^{\alpha_i+2} \cdots t_n^{\alpha_n})c_{j_m} \cdots c_{j_r}c_{i+1}c_{j_{r-1}} \cdots c_{j_1}(U_\ell \sigma_{k+1,k+2} V_{\ell'}).$$

Now, by using (1.4), (1.6) and (4.8), we have that, for any Hopf diagram D,

$$(4.8) \qquad C_j(\Omega_p D) = \begin{cases} \Omega_p C_j(D) & \text{for } p < j, \\ \Omega_{j-1}C_j(D) & \text{for } p = j, \\ \Omega_{p-2}C_j(D) & \text{for } p > j. \end{cases}$$

Moreover we have the equalities of Figure 23 where are used in particular (1.7), (3.1), (3.4) and Lemma 4.6(a). Then we get that:

$$\begin{aligned} C_k C_k \Psi_0(U_\ell \sigma_{k+1,k+2} V_{\ell'}) &= C_k C_{k+2} \Psi_0(U_\ell \sigma_{k+1,k+2} V_{\ell'}) \\ &= C_k C_{k+2}\big(\Psi_0(U_\ell)\Sigma_{k+1,k+2}\Psi_0(V_{\ell'})\big) \\ &= \Omega_{k-1}^{-2} C_k\big(\Psi_0(U)\Psi_0(V)\big) \\ &= \Omega_{k-1}^{-2} C_k \Psi_0(\{P\}). \end{aligned}$$
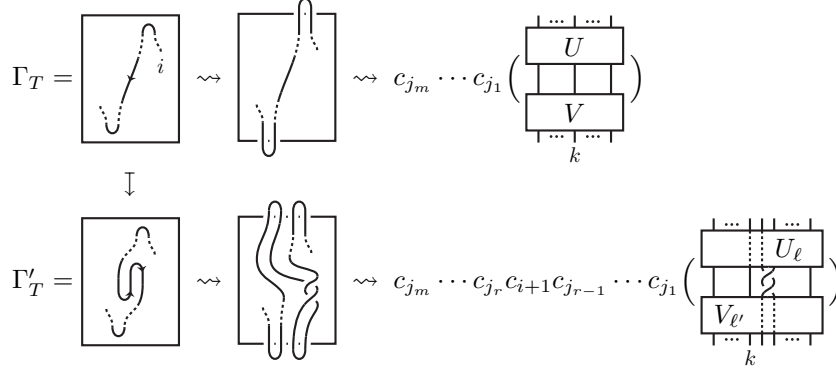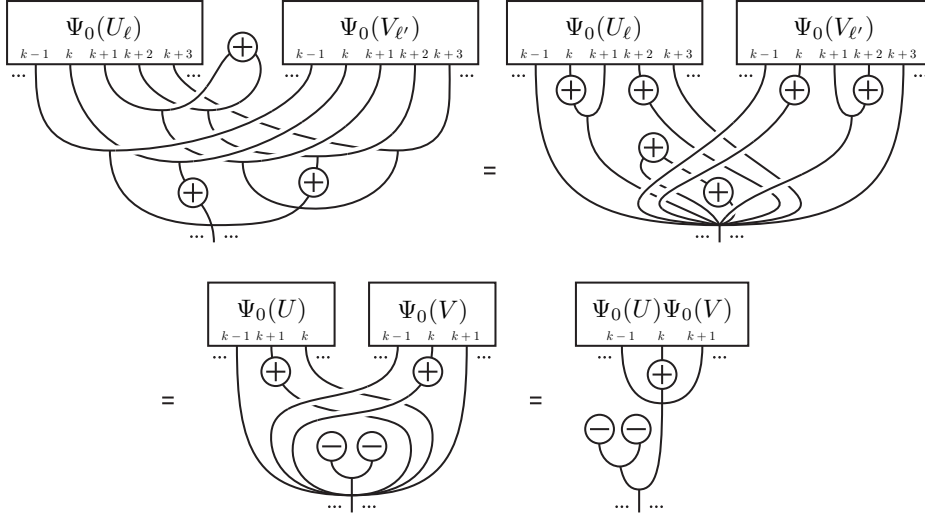
Therefore we can conclude that:

FIGURE 22

FIGURE 23

$$\psi_{\Gamma'_T} = (\Omega_1^{\alpha_1} \cdots \Omega_i^{\alpha_i+2} \cdots \Omega_n^{\alpha_n})\, C_{j_m} \cdots C_{j_r} C_{i+1} C_{j_{r-1}} \cdots C_{j_1} \Psi_0(U_\ell \sigma_{k+1,k+2} V_{\ell'})$$

$$= (\Omega_1^{\alpha_1} \cdots \Omega_i^{\alpha_i+2} \cdots \Omega_n^{\alpha_n})\, C_{j_m} \cdots \widehat{C_{j_r}} \cdots C_{j_1} C_k C_k \Psi_0(U_\ell \sigma_{k+1,k+2} V_{\ell'})$$

$$= (\Omega_1^{\alpha_1} \cdots \Omega_i^{\alpha_i+2} \cdots \Omega_n^{\alpha_n})\, C_{j_m} \cdots \widehat{C_{j_r}} \cdots C_{j_1} \big(\Omega_{k-1}^{-2} C_k \Psi_0(\{P\})\big)$$

$$= (\Omega_1^{\alpha_1} \cdots \Omega_i^{\alpha_i+2} \cdots \Omega_n^{\alpha_n})\Omega_i^{-2}\, C_{j_m} \cdots \widehat{C_{j_r}} \cdots C_{j_1} C_k \Psi_0(\{P\}) \quad \text{by (4.8)}$$

$$= (\Omega_1^{\alpha_1} \cdots \Omega_n^{\alpha_n})\, C_{j_m} \cdots C_{j_1} \Psi_0(\{P\}) = \psi_{\Gamma_T}.$$

Invariance in the cases depicted in Figures 21(a), 21(b), and 21(c) can be checked as above.

By using similar techniques, one can show the invariance of Ψ_{Γ_T} under Reidemeister's moves of type 1'. Hence we conclude that Ψ_{Γ_T} remains unchanged when applying to Γ_T a Reidemeister's move, and so $\Psi(T) = \Psi_{\Gamma_T}$ is well-defined.

We get directly from its construction that the functor Ψ is monoidal and satisfies Condition (a). Let us check that it satisfies Condition (b). Let T be a ribbon n-string link and $1 < i < n$. By applying the algorithm we have that

$T = (t_1^{\alpha_1} \cdots t_n^{\alpha_n}) c_{j_m} \cdots c_{j_1}(P)$ for some $1 \leq j_1 \leq \cdots \leq j_m \leq n$ and some pure braid P. We can apply the algorithm to a diagram of $c_i(T)$ in such a way that:

$$c_i(T) = (t_1^{\alpha_1} \cdots t_{i-2}^{\alpha_{i-2}} t_{i-1}^{\alpha_{i-1}+\alpha_i+\alpha_{i+1}} t_i^{\alpha_{i+2}} \cdots t_{n-2}^{\alpha_n}) c_{j_m-2} \cdots c_{j_{k+1}-2} c_i c_{j_k} \cdots c_{j_1}(P),$$

where k is such that $j_{k+1} > i + 1 \geq j_k$. Hence:

$$\begin{aligned}
C_i(\Psi(T)) &= C_i\big(\Omega_1^{\alpha_1} \cdots \Omega_n^{\alpha_n} C_{j_m} \cdots C_{j_1} \Psi_0(P)\big) \\
&= (\Omega_1^{\alpha_1} \cdots \Omega_{i-1}^{\alpha_{i-1}+\alpha_i+\alpha_{i+1}} \cdots \Omega_{n-2}^{\alpha_n}) C_i C_{j_m} \cdots C_{j_1}(P) \quad \text{by (4.8)} \\
&= (\Omega_1^{\alpha_1} \cdots \Omega_{i-1}^{\alpha_{i-1}+\alpha_i+\alpha_{i+1}} \cdots \Omega_{n-2}^{\alpha_n}) C_{j_m-2} \cdots C_{j_{k+1}-2} C_i C_{j_k} \cdots C_{j_1}(P) \\
&= \Psi(c_i(T)).
\end{aligned}$$

Uniqueness of a functor $\overline{\mathcal{D}iag}^S \to \mathrm{RSL}$ satisfying (a) and (b) comes from the fact that every ribbon string link can be realized as a sequence of contractions of a ribbon pure braid.

Finally, let T be a ribbon string link. We can always write $T = c_{j_m} \cdots c_{j_1}(P)$ for some ribbon pure braid P. Then we have:

$$\begin{aligned}
\overline{\psi}\Psi(T) &= \overline{\psi}\Psi(c_{j_m} \cdots c_{j_1}(P)) \\
&= \overline{\psi}\big(C_{j_m} \cdots C_{j_1} \Psi_0(P)\big) \quad \text{by Conditions (a) and (b)} \\
&= c_{j_m} \cdots c_{j_1}\big(\overline{\psi}\Psi_0(P)\big) \quad \text{by Lemma 4.2} \\
&= c_{j_m} \cdots c_{j_1}(P) = T \quad \text{by Lemma 4.1.}
\end{aligned}$$

Hence $\overline{\psi} \circ \Psi = 1_{\mathrm{RSL}}$. This completes the proof of Theorem 4.4.

5. Quantum invariants via Hopf diagrams and Kirby elements

A general method is given in [Vir03] for defining quantum invariants of 3-manifolds starting from a ribbon category (or a ribbon Hopf algebra). In this section, we explain the role played by Hopf diagrams in this theory. Note that this was the initial motivation of this work.

5.1. **Dinatural transformations and coends.** We give here definitions adapted to our purposes. For more general situations, we refer to [Mac98].

Let $\mathcal{C}$ be a category with left duals. By a *dinatural transformation of $\mathcal{C}$*, we shall mean a pair (Z, d) consisting in an object Z of $\mathcal{C}$ and a family d, indexed by $\mathrm{Ob}(\mathcal{C})$, of morphisms $d_X \colon {}^\vee X \otimes X \to Z$ in $\mathcal{C}$ satisfying $d_Y(\mathrm{id}_{{}^\vee Y} \otimes f) = d_X({}^\vee f \otimes \mathrm{id}_X)$ for any morphism $f \colon X \to Y$.

By a *coend of $\mathcal{C}$*, we shall mean a dinatural transformation (A, i) which is universal in the sense that, if (Z, d) is any dinatural transformation, then there exists a unique morphism $r \colon A \to Z$ such that $d_X = r \circ i_X$ for all object X in $\mathcal{C}$, see Figure 24. Note that a coend, if it exists, is unique (up to isomorphism).

Remarks. 1) In general, the coend of $\mathcal{C}$ always exists in a completion $\widehat{\mathcal{C}}$ of $\mathcal{C}$, namely the category of Ind-objects of $\mathcal{C}$ (see [Lyu94]). In the following, for simplicity, we will restrict to our definition, that is, to the case where the coend exists in $\mathcal{C}$ (although most of the material still remains true when working with $\widehat{\mathcal{C}}$).

2) The coend always exists (in $\mathcal{C}$) when $\mathcal{C}$ is the category of representations of a finite-dimensional Hopf algebra or is a premodular category, see [Vir03].

FIGURE 24

Assume that (A, i) is the coend of $\mathcal{C}$. The following dinatural transformations:

$$(i_X \otimes i_X)(1 \otimes \mathrm{coev}_X \otimes 1) \colon {}^\vee X \otimes X \to A \otimes A \quad \text{and} \quad \mathrm{ev}_X \colon {}^\vee X \otimes X \to \mathbb{1}$$

factorize respectively to morphisms $\Delta_A \colon A \to A \otimes A$ (the *coproduct*) and $\varepsilon_A \colon A \to \mathbb{1}$ (the *counit*). This makes A being a coalgebra in the category $\mathcal{C}$.

5.2. **The coend of a ribbon category.** Let $\mathcal{C}$ be a ribbon category (see [Tur94]) and assume that the coend (A, i) of $\mathcal{C}$ exists. We denote the braiding of $\mathcal{C}$ by $c_{X,Y} \colon X \otimes Y \to Y \otimes X$.

The coalgebra structure $(\Delta_A, \varepsilon_A)$ on the object A (see Section 5.1) extends to a structure of a Hopf algebra, see [Lyu94]. This means that there exist morphisms $\mu_A \colon A \otimes A \to A$ (the product), $\eta_A \colon \mathbb{1} \to A$ (the unit), and $S_A \colon A \to A$ (the *antipode*). They satisfy the same axioms as those of a Hopf algebra except the usual flip is replaced by the braiding $c_{A,A} \colon A \otimes A \to A \otimes A$. Namely, the unit morphism is $\eta_A = i_{\mathbb{1}} \colon \mathbb{1} = {}^\vee\mathbb{1} \otimes \mathbb{1} \to A$. By using the universal property of a coend[1], the product μ_A and the antipode S_A are defined as follows:

$$\mu_A(i_X \otimes i_Y) = i_{Y \otimes X}(1_{\vee X} \otimes c_{X, {}^\vee Y \otimes Y}) \colon {}^\vee X \otimes X \otimes {}^\vee Y \otimes Y \to A,$$

$$S_A i_X = i_{\vee X}(\theta_X \otimes \mathrm{id}_{\vee X}) c_{X,X} \colon {}^\vee X \otimes X \to A.$$

Here, the morphisms $\theta_X \colon X \to X$ denote the twist of $\mathcal{C}$. Note that S_A is invertible, with inverse $S_A^{-1} \colon A \to A$ coming from:

$$S_A^{-1} i_X = i_{\vee X}(\theta_X^{-1} \otimes \mathrm{id}_X) c_{X, \vee X}^{-1} \colon {}^\vee X \otimes X \to A.$$

Let us define morphisms $\omega_A \colon A \otimes A \to \mathbb{1}$ and $\theta_A^\pm \colon A \to \mathbb{1}$ as follows:

$$\theta_A^\pm i_X = \mathrm{ev}_X(\mathrm{id}_{\vee X} \otimes \theta_X^{\pm 1}) \colon {}^\vee X \otimes X \to \mathbb{1},$$

$$\omega_A(i_X \otimes i_Y) = (\mathrm{ev}_X \otimes \mathrm{ev}_Y)(\mathrm{id}_{\vee X} \otimes c_{\vee Y, X} c_{X, \vee Y} \otimes \mathrm{id}_{\vee Y}) \colon {}^\vee X \otimes X \otimes {}^\vee Y \otimes Y \to \mathbb{1}.$$

It can be shown that ω_A is a Hopf pairing (see [Vir03]). Finally, we set:

$$\omega_A^+ = \omega_A(S_A^{-1} \otimes \mathrm{id}_A) \quad \text{and} \quad \omega_A^- = \omega_A.$$

5.3. **Hopf diagrams and factorization.** Let $\mathcal{C}$ be a ribbon category. Assume that the coend (A, i) of $\mathcal{C}$ exists. Consider the Hopf algebra structure of A and the morphisms $\omega_A^\pm$ and $\theta_A^\pm$ as in Section 5.2.

Let T be a ribbon n-handle. Recall that for any objects $X_1, \dots, X_n$ of $\mathcal{C}$, the handle T defines a morphism $T_{X_1,\dots,X_n} \colon {}^\vee X_1 \otimes X_1 \otimes \cdots \otimes {}^\vee X_n \otimes X_n \to \mathbb{1}$ by canonically orienting T (see Section 2.2) and decorating the kth component of T

---

[1]For the product μ_A, use it twice.

by X_k. Hence it can be factorized thought the coend to a morphism $T_{\mathcal{C}} \colon A^{\otimes n} \to \mathbb{1}$ so that:

$$\text{(5.1)} \qquad\qquad T_{X_1, \ldots, X_n} = T_{\mathcal{C}} \circ (i_{X_1} \otimes \cdots \otimes i_{X_n})$$

for all objects $X_1, \ldots, X_n$ of $\mathcal{C}$.

Theorem 5.1. *There exists a unique braided functor $E_{\mathcal{C}} \colon \overline{\mathcal{D}iag}^S \to \mathcal{C}$ sending:*
- *the objects $n \in \mathbb{N}$ to $A^{\otimes n}$;*
- *the morphisms Δ, ε, $S^{\pm 1}$, $\omega_\pm$, $\theta_\pm$ to Δ_A, ε_A, $S_A^{\pm 1}$, $\omega_A^\pm$, $\theta_A^\pm$ respectively.*

Furthermore, for any ribbon n-handle T, the factorization morphism $T_{\mathcal{C}} \colon A^{\otimes n} \to \mathbb{1}$ defined by T is given by $T_{\mathcal{C}} = E_{\mathcal{C}}(D)$ where D is any Hopf diagram with $\overline{\phi}(D) = T$. In particular, $T_{\mathcal{C}} = E_{\mathcal{C}} \circ \Phi(T)$.

Remark 5.2. We expect to construct such functors $E_{\mathcal{C}} \colon \overline{\mathcal{D}iag}^S \to \mathcal{C}$ for categories $\mathcal{C}$ which are not ribbon but only turban, see [Bru04].

Proof. Uniqueness of the functor $E_{\mathcal{C}}$ is clear since we impose the image of all its generators. Its existence comes from the fact that the relations imposed on the generators Δ, ε, $S^{\pm 1}$, $\omega_\pm$, $\theta_\pm$ and $\tau_{*,*}^{\pm 1}$ in $\overline{\mathcal{D}iag}^S$ are still true in $\mathcal{C}$ when replacing them by Δ_A, ε_A, $S_A^{\pm 1}$, $\omega_A^\pm$, $\theta_A^\pm$ and $c_{A,A}^{\pm 1}$ respectively. This is because A is a Hopf algebra in $\mathcal{C}$, ω_A is a Hopf pairing, and θ_A is defined using the twist θ_X of $\mathcal{C}$.

Using the uniqueness of the factorization morphism via a coend, it remains to show that if D is a Hopf diagram with n inputs, then

$$\text{(5.2)} \qquad\qquad \overline{\phi}(D)_{X_1, \ldots, X_n} = E_{\mathcal{C}}(D) \circ (i_{X_1} \otimes \cdots \otimes i_{X_n})$$

for any objects $X_1, \ldots, X_n$ of $\mathcal{C}$. Clearly, we have that if (5.2) is true for two Hopf diagrams D and D', then it is also true for the Hopf diagram $D \otimes D'$. Moreover, by definition of ϕ (see Figure 7) and of ε_A, $\omega_A^\pm$ and $\theta_A^\pm$, we have that (5.2) is true for the trivial Hopf diagram $\varepsilon \otimes \cdots \otimes \varepsilon$ and for the Hopf diagrams $\Sigma_{i,j}^{\pm 1}$ and $\Omega_k^{\pm 1}$ depicted in (4.6). Now suppose that (5.2) is true for some Hopf diagram D. Then, by definition of ϕ (see Figure 7) and of $\underline{\Delta_A, S_A^{\pm 1}}$ and $c_{A,A}^{\pm 1}$, (5.2) remains true for the following diagrams:
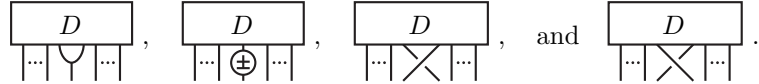
Hence we can deduce that (5.2) is always true. $\qquad\qquad\square$

5.4. **Kirby elements.** Let $\mathcal{C}$ and A be as in Section 5.3. Let L be a framed link in S^3 with n components. Choose an orientation for L. There always exists a ribbon n-handle T (not necessarily unique) such that L is isotopic $T \circ (\cup_- \otimes \cdots \otimes \cup_-)$, where $\cup_-$ denotes the cup with clockwise orientation and T is canonically oriented. For $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$, set:

$$\text{(5.3)} \qquad\qquad \tau_{\mathcal{C}}(L; \alpha) = T_{\mathcal{C}} \circ \alpha^{\otimes n} \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}),$$

where $T_{\mathcal{C}} \colon A^{\otimes n} \to \mathbb{1}$ is defined as in (5.1). Following [Vir03], by a *Kirby element* of $\mathcal{C}$, we shall mean a morphism $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ such that, for any framed link L, $\tau_{\mathcal{C}}(L; \alpha)$ is well-defined and invariant under isotopies and 2-handle slides of L.

In general, determining the set of morphisms $T_{\mathcal{C}}$ when T runs over ribbon handles is quite difficult. Nevertheless, by Theorem 5.1, the $T_{\mathcal{C}}$ belong to the set of morphisms given by evaluations of Hopf diagrams. Hence we have a manner of verifying that a morphism is a Kirby element:

Corrolary 5.3. *Let* $\alpha\colon \mathbb{1} \to A$ *in* $\mathcal{C}$*. Suppose that the two following conditions are satisfied:*

(a) *for any integer* $n \geq 1$ *and any Hopf diagram* D *with* n *entries, we have:*

(b) *for any integer* $n \geq 2$ *and any Hopf diagram* D *with* n *entries, we have:*

where $\curlywedge$ *and* $\curlyvee$ *denote* $\mu_A\colon A \otimes A \to A$ *and* $\Delta_A\colon A \to A \otimes A$ *respectively.*

Then α *is a Kirby element of* $\mathcal{C}$*.*

Remark 5.4. One recovers the fact (see [Vir03, Theorem 2.5]) that a morphism $\alpha\colon \mathbb{1} \to A$ is a Kirby element if it satisfies:

$$\begin{cases} S_A\alpha = \alpha, \\ (\mu_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta_A)(\alpha \otimes \alpha) = \alpha \otimes \alpha, \end{cases}$$

or, in case $\mathcal{C}$ is linear, if the morphisms

$$\begin{cases} S_A\alpha - \alpha, \\ (\mu_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta_A)(\alpha \otimes \alpha) - \alpha \otimes \alpha, \end{cases}$$

are negligible.

Proof. We just have to adapt the proof of [Vir03, Theorem 2.5] to our more general situation. Let $L = L_1 \cup \cdots \cup L_n$ be a framed link. Choose an orientation of L and a ribbon n-handle T such that L is isotopic $T \circ (\cup_- \otimes \cdots \otimes \cup_-)$.

Suppose firstly that we reverse the orientation of L_i. As in the proof of [Vir03, Theorem 2.5], we can choose a ribbon n-handle T' such that new oriented framed link L' is isotopic to $T' \circ (\cup_- \otimes \cdots \otimes \cup_-)$ and $T'_{\mathcal{C}} = T_{\mathcal{C}} \circ (\mathrm{id}_{A^{\otimes(i-1)}} \otimes S_A \otimes \mathrm{id}_{A^{\otimes(n-i)}})$. Now let D be a Hopf diagram such that $\overline{\phi}(D) = T$. Then $T_{\mathcal{C}} = E_{\mathcal{C}}(D)$ and so

$$\begin{aligned} \tau_{\mathcal{C}}(L';\alpha) &= E_{\mathcal{C}}(D)(\mathrm{id}_{A^{\otimes(i-1)}} \otimes S_A \otimes \mathrm{id}_{A^{\otimes(n-i)}})\alpha^{\otimes n} \\ &= E_{\mathcal{C}}(D)(c_{A,A^{\otimes(i-1)}} \otimes \mathrm{id}_{A^{(n-i)}})(S_A\alpha \otimes \alpha^{\otimes n-1}) \\ &= E_{\mathcal{C}}(D')(S_A\alpha \otimes \alpha^{\otimes n-1}) \\ &= E_{\mathcal{C}}(D')\alpha^{\otimes n} \quad \text{by Condition (a)} \\ &= E_{\mathcal{C}}(D)(c_{A,A^{\otimes(i-1)}} \otimes \mathrm{id}_{A^{(n-i)}})\alpha^{\otimes n} \\ &= E_{\mathcal{C}}(D)\alpha^{\otimes n} = \tau_{\mathcal{C}}(L;\alpha). \end{aligned}$$

where

Hence $\tau_{\mathcal{C}}(L;\alpha)$ does not depend on the choice of an orientation for L.

Suppose now that the component L_2 slides over the component L_1. Let L_1' be a parallel copy of L_1 and set $L' = L_1 \cup (L_1' \# L_2) \cup L_3 \cup \cdots \cup L_n$. As in the proof of [Vir03, Theorem 2.5], we can choose a ribbon n-handle T' such L' is isotopic to $T' \circ (\cup_- \otimes \cdots \otimes \cup_-)$ and $T_{\mathcal{C}}' = T_{\mathcal{C}} \circ \big((\mu_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta_A) \otimes \mathrm{id}_{A^{\otimes(n-2)}}\big)$. Hence, choosing a Hopf diagram D such that $\overline{\phi}(D) = T$, we get that $T_{\mathcal{C}} = E_{\mathcal{C}}(D)$ and so

$$
\begin{aligned}
\tau_{\mathcal{C}}(L'; \alpha) &= E_{\mathcal{C}}(D)\big((\mu_A \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \Delta_A) \otimes \mathrm{id}_{A^{\otimes(n-2)}}\big)\alpha^{\otimes n} \\
&= E_{\mathcal{C}}(D)\alpha^{\otimes n} \quad \text{by Condition (b)} \\
&= \tau_{\mathcal{C}}(L; \alpha).
\end{aligned}
$$

Likewise, using the braiding, we can show that $\tau_{\mathcal{C}}(L; \alpha)$ is invariant under the other handle slides. Hence α is a Kirby element of $\mathcal{C}$. $\qquad\square$

5.5. Quantum invariants of 3-manifolds.

Recall (see [Lic97]) that every closed, connected and oriented 3-dimensional manifold can be obtained from S^3 by surgery along a framed link $L \subset S^3$.

For any framed link L in S^3, we will denote by S_L^3 the 3-manifold obtained from S^3 by surgery along L, by n_L the number of components of L, and by $b_-(L)$ the number of negative eigenvalues of the linking matrix of L.

A Kirby element α of $\mathcal{C}$ is said to be *normalized* if $\theta_A^+ \alpha$ and $\theta_A^- \alpha$ are invertible in the semigroup $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$. To each normalized Kirby element α is associated an invariant $\tau_{\mathcal{C}}(M; \alpha)$ of (closed, connected, and oriented) 3-manifolds M with values in $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$, see [Vir03, Proposition 2.3]. From Theorem 5.1, we immediately deduce that:

Corollary 5.5. *For a normalized Kirby element α and any framed link L, we have*

$$
\tau_{\mathcal{C}}(S_L^3; \alpha) = (\theta_A^+ \alpha)^{b_-(L) - n_L} \, (\theta_A^- \alpha)^{-b_-(L)} \, E_{\mathcal{C}}(D)\alpha^{\otimes n},
$$

where D is any Hopf diagram such that L is isotopic to $\phi(D) \circ (\cup \otimes \cdots \otimes \cup)$.

Remarks. 1) Corollary 5.5 gives an intrinsic description, in terms of Hopf algebraic structures, of quantum invariants of 3-manifolds.

2) In Corollary 5.5, we can in particular take $D = \Psi(T)$ where Ψ is as in Theorem 4.4 and T is a ribbon sting link such that L is isotopic to closure of T. Recall also that the constructive proof of Theorem 4.4 provides us with an algorithm for computing $D = \Psi(T)$ starting from a diagram of T.

Let us conclude by giving an example. Recall that the Poincaré sphere $\mathbb{P}$ can be presented as $\mathbb{P} \simeq S_K^3$ where K is the right-handed trefoil with framing $+1$. Now this trefoil can be represented as $K = \phi(D) \circ \cup$ where $D = (\omega_+ \Delta \otimes \theta_-)\Delta$. Indeed, we have that:

Hence, if $\alpha \colon \mathbb{1} \to A$ is a normalized Kirby element of $\mathcal{C}$, then:

$$\tau_{\mathcal{C}}(\mathbb{P};\alpha) = \left(\theta_A^+\right)^{-1} \begin{array}{c} \omega_A^+ \\ \Delta_A \quad \theta_A^- \\ \Delta_A \\ \alpha \end{array} .$$

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