CATEGORICAL CENTERS AND RESHETIKHIN-TURAEV INVARIANTS

ALAIN BRUGUIÈRES AND ALEXIS VIRELIZIER

Abstract. A theorem of Müger asserts that the center $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$ over $\Bbbk$ is a modular fusion category if $\Bbbk$ is an algebraically closed field and the dimension of $\mathcal{C}$ is invertible. We generalize this result to the case where $\Bbbk$ is an arbitrary commutative ring, without restriction on the dimension of the category. Moreover we construct a variant of the Reshetikhin-Turaev invariant associated to $\mathcal{Z}(\mathcal{C})$, still defined when $\dim \mathcal{C}$ is not invertible, and give an algorithm for computing this invariant in terms of certain explicit morphisms in the category $\mathcal{C}$. Our approach is based on (a) Lyubashenko's construction of the Reshetikhin-Turaev invariant in terms of the coend of a ribbon category; (b) an explicit algorithm for computing this invariant via Hopf diagrams; (c) an algebraic interpretation of the center of $\mathcal{C}$ as the category of modules over a canonical Hopf monad on $\mathcal{C}$; (d) a generalization of the Drinfeld double construction to Hopf monads which, applied to the canonical Hopf monad of $\mathcal{C}$, provides an explicit description of the coend of $\mathcal{Z}(\mathcal{C})$ in terms of the category $\mathcal{C}$.

Contents

Introduction

In the early 90's, two new 'quantum' invariants of 3-manifolds were introduced: the *Reshetikhin-Turaev invariant*, and the *Turaev-Viro invariant*. The definition of the Reshetikhin-Turaev invariant $\mathrm{RT}_\mathcal{B}$ [RT91, Tur94] involves a *modular category* $\mathcal{B}$, that is, a ribbon fusion category with invertible S-matrix (hence invertible dimension). The algorithm for computing its value on a 3-manifold consists in presenting the manifold by surgery along a ribbon link, coloring this link by simple objects of $\mathcal{B}$ to obtain scalars, and then forming a linear combination of these scalars.

Similarly, the definition of the Turaev-Viro invariant $\mathrm{TV}_\mathcal{C}$ [TV92], as revisited by Barrett and Westbury [BW96], involves a *spherical fusion category* (that is, a sovereign fusion category such that left and right traces coincide) with invertible dimension. The algorithm for computing $\mathrm{TV}_\mathcal{C}(M)$ consists in presenting the

Date: February 22, 2008.
2000 *Mathematics Subject Classification.* 57M27,16W30,18C20.
The second author thanks the organizers of the *International Conference on Quantum Topology* held at the Institute of Mathematics of Hanoi (Vietnam) in August 2007. This paper is an enhanced version of the talk he gave there.

3-manifold M by a triangulation, coloring the edges of the triangulation with simple objects of $\mathcal{C}$, and then evaluating the colored tetrahedra by means of the $6j$-symbols of $\mathcal{C}$.

A modular category $\mathcal{B}$ is also a spherical category and, in this case, the Reshetikhin-Turaev and Turaev-Viro invariants are related [Tur94, Ro95] by:

$$\mathrm{TV}_{\mathcal{B}}(M) = \mathrm{RT}_{\mathcal{B}}(M)\mathrm{RT}_{\mathcal{B}}(-M)$$

for any 3- manifold M, where $-M$ is the 3-manifold M with opposite orientation.

But in general a spherical category need not to be braided and so cannot be used as input to define the Reshetikhin-Turaev invariant. However, spherical and modular categories are related by a theorem of Müger [Mü03]: if $\mathcal{C}$ is a spherical fusion category over an algebraically closed field $\Bbbk$ and has invertible dimension, then its center $\mathcal{Z}(\mathcal{C})$ is a modular fusion category of dimension $\dim \mathcal{Z}(\mathcal{C}) = (\dim \mathcal{C})^2$. In this setting, Turaev conjectured that, for any 3-manifold M,

$$\mathrm{TV}_{\mathcal{C}}(M) = \mathrm{RT}_{\mathcal{Z}(\mathcal{C})}(M).$$

This conjecture was shown to be true for some spherical categories $\mathcal{C}$ arising from subfactors, see [KSW05]. The general case is still open.

In this context, a natural question is: how can we compute $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}(M)$? Using the algorithm given by Reshetikhin and Turaev is not a practicable approach here, as that would require a description of the simple objects of $\mathcal{Z}(\mathcal{C})$ in terms of those of $\mathcal{C}$, and no such description is available in general. What we need is a different algorithm for computing $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}(M)$, which one should be able to perform inside $\mathcal{C}$, without reference to the simple objects of $\mathcal{Z}(\mathcal{C})$. This is the primary objective of this paper.

In order to fulfill this objective, it will be convenient to adopt an alternative approach for constructing RT-like quantum invariants of 3-manifolds, due to Lyubashenko [Lyu95] and later developed in [KL01, Vir06], where the input data is a (non-necessarily linear neither semisimple) ribbon category $\mathcal{B}$ which admits a coend $C = \int^{X \in \mathcal{B}} {}^{\vee}X \otimes X$. This coend C is naturally endowed with a very rich algebraic structure. In particular, it is a Hopf algebra in the braided category $\mathcal{B}$ and comes equipped with a Hopf pairing $\omega\colon C \otimes C \to \mathbb{1}$. Such a category $\mathcal{B}$ is modular if the pairing ω is non-degenerate (this is the natural way of formulating the invertibility of the S-matrix in this setting).

The construction of the Lyubashenko invariant consists in presenting the 3-manifold by surgery along a ribbon link L, using the universal property of the coend C to associate a form ϕ_L to the link, and then evaluating this form on an integral Λ of the Hopf algebra C. Note that, more generally, one can evaluate the form ϕ_L by a 'Kirby element' α of $\mathcal{B}$ to get other invariants $\tau_{\mathcal{B}}(M; \alpha)$ of 3-manifold invariants, see [Vir06]. In particular, up to normalization, $\tau_{\mathcal{B}}(M; \Lambda)$ is the Lyubashenko invariant and, in the special case where $\mathcal{B}$ is a modular fusion category, $\tau_{\mathcal{B}}(M; \Lambda)$ is the Reshetikhin-Turaev invariant.

In order to make this construction effective, we need an algorithm for computing the forms ϕ_L which are defined by universal property. Such an algorithm, based on an encoding of certain tangles by means of *Hopf diagrams*, is given in [BV05]. Thus the invariants $\tau_{\mathcal{B}}(M; \alpha)$ can be expressed in terms of certain structural morphisms of the coend C. Section 2 is devoted to these quantum invariants and their computation.

Hence, when $\mathcal{C}$ is a spherical fusion category, we may compute $\tau_{\mathcal{Z}(\mathcal{C})}(M; \Lambda)$ provided we can describe explicitly the structural morphisms of the coend of $\mathcal{Z}(\mathcal{C})$. In other words, we need an algebraic interpretation of the center construction. If $\mathcal{C}$ is braided and has a coend A (which is a Hopf algebra), then the category $\mathcal{Z}(\mathcal{C})$

coincides with the category of A-modules in $\mathcal{C}$. However the difficulty here is that we do not want to assume $\mathcal{C}$ is braided. To bypass this difficulty, we use the notion of *Hopf monad* introduced in [BV07].

Hopf monads generalize Hopf algebras in a non-braided setting. In particular, finite-dimensional Hopf algebras and their different generalizations (Hopf algebras in braided autonomous categories, quantum bialgebroids, etc...) provide examples of Hopf monads. If fact, any monoidal adjunction between autonomous categories gives rise to a Hopf monad. It turns out that much of the theory of finite-dimensional Hopf algebras extends to Hopf monads, see [BV07]. In Section 3, we recall a few results on Hopf monads.

The whole point of introducing Hopf monads here is that they provide an algebraic interpretation of the center construction [BV08a]. If $\mathcal{C}$ is a *centralizable* autonomous category, meaning that the coend $Z(X) = \int^{Y \in \mathcal{C}} {}^{\vee}Y \otimes X \otimes Y$ exists for any object X of $\mathcal{C}$, then Z is a quasitriangular Hopf monad on $\mathcal{C}$ and the center $\mathcal{Z}(\mathcal{C})$ coincides, as a braided category, with the category of Z-modules in $\mathcal{C}$. In addition, Drinfeld's double construction extends naturally to Hopf monads. This theory provides a description of the coend of $\mathcal{Z}(\mathcal{C})$. In Section 4, we review a few facts on the double of Hopf monads.

In Section 5, we apply the above results to spherical fusion categories. Firstly, we obtain a generalization of Müger's theorem on the modularity of the center of a spherical fusion category $\mathcal{C}$ to the case where $\dim \mathcal{C}$ is not necessarily invertible and $\Bbbk$ is any commutative ring. Denoting by $\{V_i\}_{i \in I}$ a (finite) representative family of scalar[1] objects of $\mathcal{C}$, we get:

$$Z(X) = \bigoplus_{i \in I} {}^{\vee}V_i \otimes X \otimes V_i,$$

Moreover $\mathcal{Z}(\mathcal{C})$ is centralizable and $\dim \mathcal{Z}(\mathcal{C}) = (\dim \mathcal{C})^2$. The underlying object of the coend of $\mathcal{Z}(\mathcal{C})$ is:

$$C = \bigoplus_{i,j \in I} {}^{\vee}V_i \otimes {}^{\vee}V_j \otimes {}^{\vee\vee}V_i \otimes V_j \,,$$

and all structural morphisms of C (including an integral $\Lambda \colon \mathbb{1} \to C$) can be written down explicitly in $\mathcal{C}$. Furthermore, $\mathcal{Z}(\mathcal{C})$ is always modular. When $\Bbbk$ is an algebraically closed field and $\dim \mathcal{C}$ is invertible, $\mathcal{Z}(\mathcal{C})$ is a fusion category and so we recover Müger's theorem. However, when $\dim \mathcal{C}$ is not invertible, $\mathcal{Z}(\mathcal{C})$ is a non-semisimple ribbon category. Nevertheless, in this case, the version $\tau_{\mathcal{Z}(\mathcal{C})}(M; \Lambda)$ of the Lyubashenko invariant is still defined and computable in terms of $\mathcal{C}$.

1. Conventions and notations

1.1. **Autonomous categories.** Monoidal categories are assumed to be strict.

Recall that a *duality* in a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ is a quadruple (X, Y, e, c), where X, Y are objects of $\mathcal{C}$, $e \colon X \otimes Y \to \mathbb{1}$ (the *evaluation*) and $c \colon \mathbb{1} \to Y \otimes X$ (the *coevaluation*) are morphisms in $\mathcal{C}$, such that:

$$(e \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes c) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{id}_Y \otimes e)(c \otimes \mathrm{id}_Y) = \mathrm{id}_Y.$$

Then (X, e, c) is a *left dual* of Y, and (Y, e, c) is a *right dual* of X.

A *left autonomous* category is a monoidal category for which every object X admits a left dual $({}^{\vee}X, \mathrm{ev}_X, \mathrm{coev}_X)$. Likewise, a *right autonomous* category is a monoidal category for which every object X admits a right dual $(X^{\vee}, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$.

[1] An object X of $\mathcal{C}$ is *scalar* if $\mathrm{End}_{\mathcal{C}}(X) \cong \Bbbk$.

An *autonomous category* is a monoidal category which is left and right autonomous. Note that in an autonomous category, there are canonical isomorphisms:

$$\begin{aligned} &{}^\vee(X^\vee) \cong X, && {}^\vee(X \otimes Y) \cong {}^\vee Y \otimes {}^\vee X, && {}^\vee\mathbb{1} \cong \mathbb{1}, \\ &({}^\vee X)^\vee \cong X, && (X \otimes Y)^\vee \cong Y^\vee \otimes X^\vee, && \mathbb{1}^\vee \cong \mathbb{1}. \end{aligned}$$

We will often abstain from writing down these isomorphisms.

1.2. **Sovereign categories.** A *sovereign category* is a left autonomous category endowed with a strong monoidal natural transformation $\phi_X \colon X \to {}^{\vee\vee}X$. Such a transformation is then an isomorphism. A sovereign category is actually autonomous. Furthermore, in a sovereign category $\mathcal{C}$, one can define the *left and right traces* of an endomorphism $f \colon X \to X$ as:

$$\begin{aligned} \mathrm{tr}_l(f) &= \mathrm{ev}_X(\mathrm{id}_{{}^\vee X} \otimes f\phi_X^{-1})\mathrm{coev}_{{}^\vee X} \in \mathrm{End}_\mathcal{C}(\mathbb{1}), \\ \mathrm{tr}_r(f) &= \widetilde{\mathrm{ev}}_X(f\phi_{X^{\vee\vee}} \otimes \mathrm{id}_{X^\vee})\widetilde{\mathrm{coev}}_{X^\vee} \in \mathrm{End}_\mathcal{C}(\mathbb{1}), \end{aligned}$$

and the *left and right dimensions* of an object X as $\dim_l(X) = \mathrm{tr}_l(\mathrm{id}_X)$ and $\dim_r(X) = \mathrm{tr}_r(\mathrm{id}_X)$. We have $\dim_r({}^\vee X) = \dim_l(X)$.

1.3. **Braided categories.** A *braided* category is a monoidal category endowed with a *braiding*, that is, a natural isomorphism $\tau_{X,Y} \colon X \otimes Y \to Y \otimes X$ satisfying: $\tau_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes \mathrm{id}_Z)$ and $\tau_{X\otimes Y,Z} = (\tau_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \tau_{Y,Z})$.

1.4. **Ribbon categories.** A *twist* on a braided category $\mathcal{B}$ is a natural isomorphism $\theta_X \colon X \to X$ satisfying: $\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y)\tau_{Y,X}\tau_{X,Y}$. If $\mathcal{B}$ is braided and autonomous, a twist θ on $\mathcal{B}$ is *self-dual* if ${}^\vee(\theta_X) = \theta_{{}^\vee X}$ (or, equivalently, $(\theta_X)^\vee = \theta_{X^\vee}$).

A *ribbon category* is a braided autonomous category endowed with a self-dual twist. A ribbon category is naturally equipped with a sovereign structure such that the left and right traces coincide.

1.5. **Coends.** Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor.

A *dinatural transformation* from the functor F to an object D of $\mathcal{D}$ is a family $d = \{d_X \colon F(X,X) \to D\}_{X\in\mathrm{Ob}(\mathcal{C})}$ of morphisms in $\mathcal{D}$ satisfying:

$$d_X F(f, \mathrm{id}_X) = d_Y F(\mathrm{id}_Y, f)$$

for every morphism $f \colon X \to Y$ in $\mathcal{C}$.

A *coend* of F consists of an object C of $\mathcal{D}$ and a dinatural transformation i from F to C which is *universal*, that is, for every dinatural transformation d from F to an object D of $\mathcal{D}$, there exists a unique morphism $\phi \colon C \to D$ such that $d_X = \phi \circ i_X$.

If F admits a coend (C, i), then it is unique (up to unique isomorphism) and is denoted $C = \int^{X\in\mathcal{C}} F(X,X)$. See [Mac98] for details.

1.6. **Coends of autonomous categories.** By *coend* of an autonomous category $\mathcal{C}$, we mean a coend $C = \int^{X\in\mathcal{C}} {}^\vee X \otimes X$ of the functor $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ defined by $F(X,Y) = {}^\vee X \otimes Y$. The object C is then a coalgebra in $\mathcal{C}$ which coacts universally on the objects of $\mathcal{C}$ via the (right) coaction:

$$\delta_X = (\mathrm{id}_X \otimes i_X)(\mathrm{coev}_X \otimes \mathrm{id}_X) \colon X \to X \otimes C.$$

Furthermore, when $\mathcal{C}$ is braided, C is a Hopf algebra in $\mathcal{C}$ (see [Maj93, Lyu94]).

1.7. **Dimension of sovereign categories.** Let $\mathcal{C}$ be a sovereign category which admits a coend. The *left* and *right dimensions of* $\mathcal{C}$ are defined respectively as the left and right dimensions of its coend. These dimensions are actually independent of the choice of sovereign structure on $\mathcal{C}$. If they coincide (for instance when $\mathcal{C}$ is a ribbon category or $\mathcal{C}$ is a fusion category), they are called the *dimension of* $\mathcal{C}$ and denoted $\dim \mathcal{C}$.

1.8. **Fusion categories.** A *fusion category* over a commutative ring $\Bbbk$ is a $\Bbbk$-linear autonomous category $\mathcal{C}$ endowed with a finite family $\{V_i\}_{i\in I}$ of objects of $\mathcal{C}$ satisfying:

- $\mathrm{Hom}_{\mathcal{C}}(V_i, V_j) = \delta_{i,j}\,\Bbbk$ for all $i, j \in I$;
- each object of $\mathcal{C}$ is a finite direct sum of objects of $\{V_i\}_{i\in I}$;
- $\mathbb{1}$ is isomorphic to some V_0 with $0 \in I$.

An object X of $\mathcal{C}$ is *scalar*[2] if $\mathrm{End}_{\mathcal{C}}(X) \cong \Bbbk$. The family $\{V_i\}_{i\in I}$ is a representative family of scalar objects of $\mathcal{C}$. The left or right dual of a scalar object is still scalar. Moreover, in a fusion category, the left and right duals of a scalar object are isomorphic.

Let $\mathcal{C}$ be a fusion category. The Hom spaces in $\mathcal{C}$ are free $\Bbbk$-modules of finite type. The *multiplicity* of $i \in I$ in an objet X of $\mathcal{C}$ is defined as:

$$N_X^i = \mathrm{rank}_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(V_i, X) = \mathrm{rank}_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(X, V_i).$$

Note there exist morphisms $(p_X^{i,\alpha} \colon X \to V_i)_{1\leq\alpha\leq N_X^i}$ and $(q_X^{i,\alpha} \colon V_i \to X)_{1\leq\alpha\leq N_X^i}$ such that:

$$\mathrm{id}_X = \sum_{\substack{i\in I\\ 1\leq\alpha\leq N_X^i}} q_X^{i,\alpha} p_X^{i,\alpha} \quad \text{and} \quad p_X^{i,\alpha} q_X^{j,\beta} = \delta_{i,j}\delta_{\alpha,\beta}\,\mathrm{id}_{V_i}.$$

A fusion category $\mathcal{C}$ admits a coend $C = \bigoplus_{i\in I} {}^{\vee}V_i \otimes V_i$ with universal dinatural transformation given by:

$$i_X = \sum_{\substack{i\in I\\ 1\leq\alpha\leq N_X^i}} {}^{\vee}q_X^{i,\alpha} \otimes p_X^{i,\alpha}.$$

Since $\dim_l(C) = \dim_r(C)$, the dimension of a sovereign fusion category $\mathcal{C}$ is:

$$\dim \mathcal{C} = \sum_{i\in I} \dim_l(V_i) \dim_r(V_i) \in \Bbbk.$$

In a sovereign fusion category $\mathcal{C}$, the dimensions $\dim_l(V_i)$ and $\dim_r(V_i)$ of the scalar objects are invertible. However $\dim \mathcal{C}$ may be not invertible.

A fusion category $\mathcal{C}$ is *spherical* if it is sovereign and the left and right traces of endomorphisms in $\mathcal{C}$ coincide. This last condition is equivalent to the equality of left and right dimensions of the scalar objects V_i for $i \in I$. In a spherical category, the left (and right) dimension of an object X is denoted $\dim(X)$.

2. Quantum invariants and Hopf diagrams

In this section, we review a general construction of quantum invariants (of Reshetikhin-Turaev type) and a method for computing them via Hopf diagrams.

2.1. **Constructing quantum invariants.** Let $\mathcal{B}$ be a ribbon autonomous category ($\mathcal{B}$ is not necessarily linear). Assume that $\mathcal{B}$ admits a coend $C = \int^{Y\in\mathcal{B}} {}^{\vee}Y \otimes Y$. Let $\delta_Y \colon Y \to Y \otimes C$ its associated universal coaction.

Let T be a ribbon n-string link with n a non-negative integer. Recall T is a ribbon (n,n)-tangle consisting of n arc components, without any closed component, such that the kth arc ($1 \leq k \leq n$) joins the kth bottom endpoint to the kth top endpoint. We orient (each component of) T from bottom to top. Let $Y_1, \dots, Y_n$ be objects of $\mathcal{B}$. Denote by:

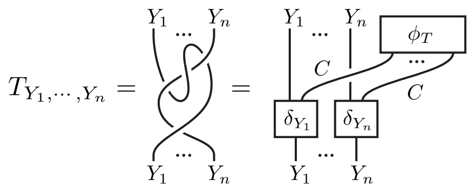
$$T_{Y_1,\cdots,Y_n} \colon Y_1 \otimes \cdots Y_n \to Y_1 \otimes \cdots Y_n$$

[2] Scalar objects coincide with simple objects if $\mathcal{C}$ is abelian and $\Bbbk$ is an algebraically closed field.

the morphism in $\mathcal{B}$ obtained by coloring the kth component of T with the object Y_k. Then $T_{Y_1,\cdots,Y_n}$ is natural in each variable Y_k and so, by universality of the coend C, there exists a unique morphism:

$$\phi_T \colon C^{\otimes n} \to \mathbb{1}$$

such that:



Two natural questions arise in this context:

- How to evaluate the forms ϕ_T to get invariants of ribbon links[3] and, further, of 3-manifolds?
- How to compute the forms ϕ_T which are defined by universal property?

We address the first question in Section 2.2 and the second one in Section 2.3.

2.2. **Kirby elements and quantum invariants.** As in the previous section, let $\mathcal{B}$ be a ribbon autonomous category admitting a coend C. In this setting, $\Bbbk = \mathrm{End}_{\mathcal{C}}(\mathbb{1})$ is a commutative monoid.

Let L be a ribbon link in S^3 with n components. There always exists a (non-unique) ribbon n-string link T such that L is isotopic to the closure of T. For $\alpha \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, C)$, set

$$\tau_{\mathcal{B}}(L;\alpha) = \phi_T \circ \alpha^{\otimes n} \in \Bbbk,$$

where $\phi_T \colon C^{\otimes n} \to \mathbb{1}$ is defined as above.

A *Kirby element* [Vir06] of $\mathcal{B}$ is a morphism $\alpha \in \mathrm{Hom}_{\mathcal{B}}(\mathbb{1}, C)$ such that, for any ribbon link L, $\tau_{\mathcal{B}}(L;\alpha)$ is well-defined and invariant under isotopies and 2-handle slides of L. A Kirby element α of $\mathcal{B}$ is said to be *normalizable* if $\tau_{\mathcal{B}}(\bigcirc^{+1};\alpha)$ and $\tau_{\mathcal{B}}(\bigcirc^{-1};\alpha)$ are invertible in $\Bbbk$, where $\bigcirc^{\pm 1}$ denotes the unknot with framing ± 1.

By universality of the coend C, the twist $\theta_Y \colon Y \to Y$ of $\mathcal{B}$ and its inverse lead to morphisms $\theta_C^{\pm} \colon C \to \mathbb{1}$ such that:

$$\theta_Y^{\pm 1} = (\mathrm{id}_Y \otimes \theta_C^{\pm})\delta_Y.$$

If α is a Kirby element of $\mathcal{B}$, we have: $\tau_{\mathcal{B}}(\bigcirc^{\pm 1};\alpha) = \theta_C^{\pm}\alpha$, so that α is normalizable if and only if $\theta_C^{\pm}\alpha$ are invertible in $\Bbbk$.

Recall (see [Lic97]) that every (closed, connected, oriented) 3-manifold can be obtained from S^3 by surgery along a ribbon link $L \subset S^3$. For any ribbon link L in S^3, we will denote by M_L the 3-manifold obtained from S^3 by surgery along L, by n_L the number of components of L, and by $b_-(L)$ the number of negative eigenvalues of the linking matrix of L.

An immediate consequence of the Kirby theorem [Kir78] is that if α is a normalizable Kirby element of $\mathcal{B}$, then:

$$\tau_{\mathcal{B}}(M_L;\alpha) = (\theta_C^{+}\alpha)^{b_-(L)-n_L}\,(\theta_C^{-}\alpha)^{-b_-(L)}\,\tau_{\mathcal{B}}(L;\alpha)$$

is an invariant of 3-manifolds. Furthermore these invariants are multiplicative under the connected sum of 3- manifolds: $\tau_{\mathcal{B}}(M\#M';\alpha) = \tau_{\mathcal{B}}(M;\alpha)\,\tau_{\mathcal{B}}(M';\alpha)$.

Note that if α is a normalizable Kirby element and k is an automorphism of $\mathbb{1}$, then $k\alpha$ is also a normalizable Kirby element. The normalization of the invariant $\tau_{\mathcal{B}}(M;\alpha)$ has been chosen so that $\tau_{\mathcal{B}}(M;k\alpha) = \tau_{\mathcal{B}}(M;\alpha)$.

[3] A ribbon link with n components is always the closure of some ribbon n-string link.

How to determine the (normalizable) Kirby element of $\mathcal{B}$? A partial answer was given in [Vir06]. Denoting by m_C, Δ_C, and S_C respectively the product, coproduct, and antipode of the Hopf algebra C, we have:

Theorem 2.1 ([Vir06, Theorem 2.5])**.** *Any morphism $\alpha\colon \mathbb{1} \to C$ in $\mathcal{B}$ such that:*

$$S_C\alpha = \alpha \quad \textit{and} \quad (m_C \otimes \mathrm{id}_C)(\mathrm{id}_C \otimes \Delta_C)(\alpha \otimes \alpha) = \alpha \otimes \alpha$$

is a Kirby element of $\mathcal{B}$.

For instance, the unit u_C of C is a normalizable Kirby element (its associated invariant is the trivial one).

A more interesting example of a Kirby element is an S_C-invariant integral Λ of C, that is, a morphism $\Lambda\colon \mathbb{1} \to C$ such that $S_C(\Lambda) = \Lambda$ and $m_C(\Lambda \otimes \mathrm{id}_C) = \Lambda\, \varepsilon_C = m_C(\mathrm{id}_C \otimes \Lambda)$, where ε_C is the counit of C. For the existence of such integrals, we refer to [BKLT00]. If Λ is normalizable, then the associated invariant is the Lyubashenko's one [Lyu95], up to a different normalization.

Note that other Kirby elements exist in general (see [Vir06]).

Remark 2.2. Assume $\mathcal{B}$ is a modular category in the sense of [Tur94], that is, a ribbon fusion category with invertible S-matrix. Let $\{V_i\}_{i\in I}$ be a representative family of simple objects of $\mathcal{B}$. Then $\mathcal{B}$ admits a coend $C = \bigoplus_{i\in I} {}^\vee V_i \otimes V_i$. Let $\phi_X\colon X \to {}^{\vee\vee}X$ be the sovereign structure of $\mathcal{B}$ and set:

$$\Lambda = \sum_{i\in I} \dim(V_i)\, (\mathrm{id}_{{}^\vee V_i} \otimes \phi_i^{-1})\mathrm{coev}_{V_i} \colon \mathbb{1} \to C,$$

Then Λ is a S_C-invariant integral of C. Furthermore it is normalizable and its associated invariant is the Reshetikhin-Turaev one [Tur94], up to a different normalization. More precisely, assuming $\dim \mathcal{B} = \sum_{i\in I} \dim(V_i)^2$ has a square root D in $\Bbbk$ (which is then invertible in this context), setting $\Delta_- = \theta_C^-\Lambda$, and denoting by $b_1(M)$ the first Betti number of M, we have:

$$\mathrm{RT}_{\mathcal{B}}(M) = D^{-1}\Big(\frac{D}{\Delta_-}\Big)^{b_1(M)}\, \tau_{\mathcal{B}}(M;\Lambda).$$

We will see in Section 5 that, unlike $\mathrm{RT}_{\mathcal{B}}(M)$, $\tau_{\mathcal{B}}(M;\Lambda)$ may be still defined for ribbon categories $\mathcal{B}$ with $\dim \mathcal{B} = 0$.

2.3. **Hopf diagrams.** For a precise treatment of the theory of Hopf diagrams, we refer to [BV05]. Note that Habiro had comparable results in [Hab06].

A *Hopf diagram* is a planar diagram, with inputs but no output, obtained by stacking the generators of Figure 1 (diagrams are read from bottom to top). Examples of Hopf diagrams with one and two inputs are depicted in Figure 2. Hopf diagrams are submitted to the relations of Figure 3 (plus relations expressing that τ is an invertible QYBE solution which is natural with respect to the other generators). In particular, the relations of Figure 3 say that Δ behaves as a coproduct with counit ε, S behaves as an antipode, $\omega_\pm$ behaves as a Hopf pairing, and $\theta_\pm$ behaves as a twist form. The last two relations of Figure 3 are nothing but the Markov relations for pure braids.

Hopf diagrams with the same number of inputs can be composed as in Figure 4. This leads to the category $\mathcal{D}iag$ of Hopf diagrams. Objects of $\mathcal{D}iag$ are the non-negative integers. For two non-negative integers m and n, the set $\mathrm{Hom}_{\mathcal{D}iag}(m,n)$ of morphisms from m to n is the empty set if $m \neq n$ and is the set of Hopf diagrams with m inputs (up to their relations) if $m = n$. The identity of n is the juxtaposition of n copies of ε.

The category $\mathcal{D}iag$ is a monoidal category: $m \otimes n = m + n$ on objects and the monoidal product $D \otimes D'$ of two Hopf diagrams D and D' is the Hopf diagram obtained by juxtaposing D on the left of D'.

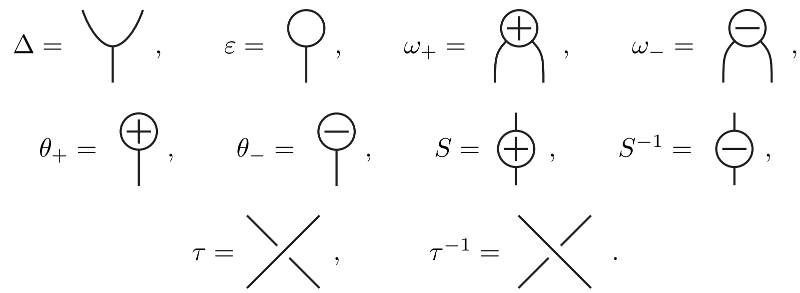


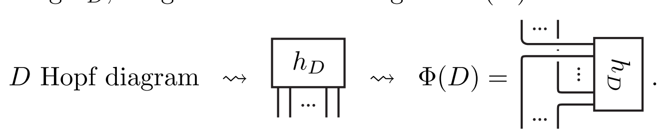
FIGURE 1. Generators of the Hopf diagrams

(a) A Hopf diagram with 1 input (b) A Hopf diagram with 2 inputs

FIGURE 2. Examples of Hopf diagrams

Let us denote by RSL the category of ribbon string links. The objects of RSL are the non-negative integers. For two non-negative integers m and n, the set of morphisms from m to n is the empty set if $m \neq n$ and is the set of (isotopy classes) of ribbon m-string links if $m = n$. The composition $T' \circ T$ of two ribbon n-string links is given by stacking T' on the top of T (i.e., with ascending convention). Identities are the trivial string links. Note that the category RSL is a monoidal category: $m \otimes n = m + n$ on objects and the monoidal product $T \otimes T'$ of two ribbon string links T and T' is the ribbon string link obtained by juxtaposing T on the left of T'.

Hopf diagrams give a 'Hopf algebraic' description of ribbon string links. Indeed, any Hopf diagram D with n inputs gives rise to a ribbon n-string link $\Phi(D)$ in the following way: using the rules of Figure 5, we obtain a ribbon n-handle[4] h_D, that is, a ribbon $(2n, 0)$-tangle consisting of n arc components, without any closed component, such that the k-th arc joins the $(2k-1)$-to the $2k$-th bottom endpoints. Then, by rotating h_D, we get a ribbon n-string link $\Phi(D)$:



An example of this procedure is depicted in Figure 6.

This leads to a functor $\Phi\colon \mathcal{D}iag \to \mathrm{RSL}$ defined on objects by $n \mapsto \Phi(n) = n$ and on morphisms by $D \mapsto \Phi(D)$.

Theorem 2.3 ([BV05, Theorem 4.5])**.** $\Phi\colon \mathcal{D}iag \to \mathrm{RSL}$ *is a well-defined monoidal functor and there exists (constructive proof) a monoidal functor* $\Psi\colon \mathrm{RSL} \to \mathcal{D}iag$ *which satisfies* $\Phi \circ \Psi = 1_{\mathrm{RSL}}$.

Note that by 'constructing proof' we mean there is an explicit algorithm that associates to a ribbon string T a Hopf diagram $\Psi(T)$ such that $\Phi\big(\Psi(T)\big) = T$ (see [BV05]). The key point is that such a functor Ψ exists thanks to the relations we put on Hopf diagrams.

[4]Ribbon handles are called bottom tangles in [Hab06].

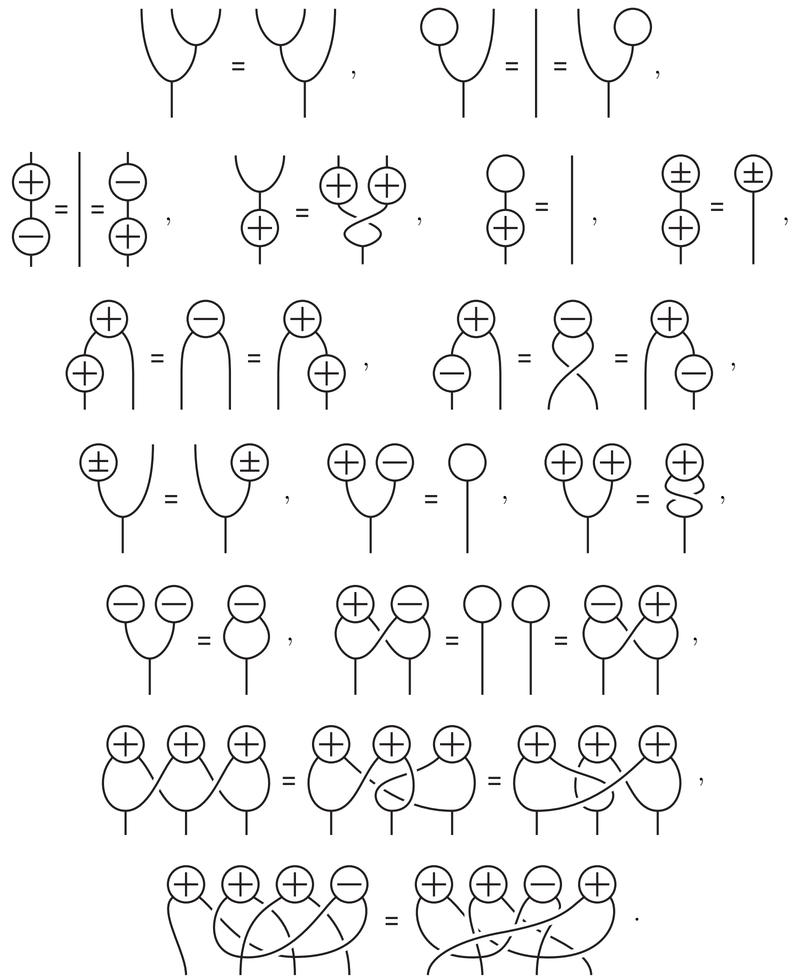


FIGURE 3. Relations on Hopf diagrams

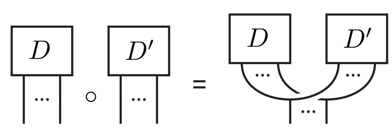


FIGURE 4. Composition of Hopf diagrams

We can now answer to the second question of Section 2.1. Let $\mathcal{B}$ be a ribbon autonomous category which admits a coend C. Given a ribbon n-string link T, how to compute the morphism $\phi_T\colon C^{\otimes n} \to \mathbb{1}$ which is defined by universal property? Recall C is a Hopf algebra in $\mathcal{B}$ and denote its coproduct, counit, and antipode by Δ_C, ε_C, and S_C respectively. The twist (and its inverse) of $\mathcal{B}$ is encoded by morphisms $\theta_C^{\pm} \to C \to \mathbb{1}$ (see Section 2.2). Furthermore, we can defined a Hopf pairing $\omega_C\colon C \otimes C \to \mathbb{1}$ via:

$$\omega_C(i_X \otimes i_Y) = (\mathrm{ev}_X \otimes \mathrm{ev}_Y)(\mathrm{id}_{^\vee X} \otimes \tau_{^\vee Y, X}\tau_{X, ^\vee Y} \otimes \mathrm{id}_{^\vee Y}),$$

where τ is the braiding of $\mathcal{B}$ and $i_Y\colon {^\vee Y} \otimes Y \to C$ is the universal dinatural transformation of the coend C. Finally, we set $\omega_C^+ = \omega_C(S_C^{-1} \otimes \mathrm{id}_C)$ and $\omega_C^- = \omega_C$.

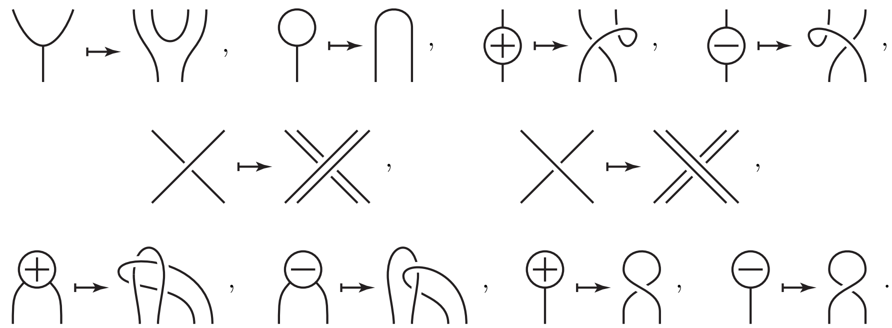


FIGURE 5. Rules for transforming Hopf diagrams to tangles

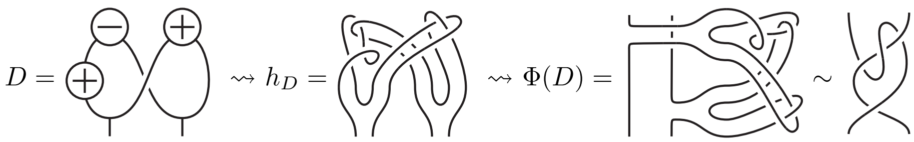
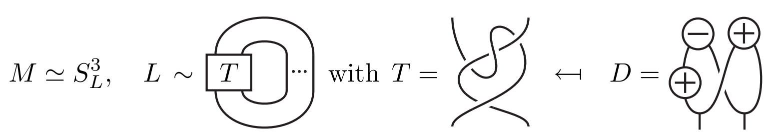


FIGURE 6. From Hopf diagrams to ribbon string links

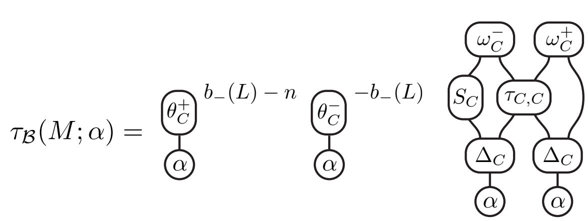
Theorem 2.4 ([BV05, Theorem 5.1]). *Let T be ribbon n-string link. Let D be any Hopf diagram (with n entries) which encodes T, that is, such that $\Phi(D) = T$ (recall there is an algorithm producing such a Hopf diagram). Then the morphism $\phi_T : C^{\otimes n} \to \mathbb{1}$ defined by T is given by replacing the generators Δ, ε, $\omega_\pm$, $\theta_\pm$, $S^{\pm 1}$, and $\tau^{\pm 1}$ by Δ_C, ε_C, $\omega_C^\pm$, $\theta_C^\pm$, $S_C^{\pm 1}$, and $\tau_{C,C}^{\pm 1}$ respectively.*

Remark that the product and unit of the coend C are not needed to represent Hopf diagrams.

Let us summarize the above universal construction of quantum invariants, starting from a ribbon category $\mathcal{B}$ which admits a coend C. Pick a normalizable Kirby element α of $\mathcal{B}$ (for example as in Theorem 2.1). Recall it gives rise to an invariant $\tau_\mathcal{B}(M, \alpha)$ of 3-manifolds. Let M be a 3-manifold. Present M by surgery along a ribbon link L, which can be viewed as the closure of a ribbon n-string link T where n is the number of components of L. Encode the string link T by a Hopf diagram D:



The morphism $\phi_T : C^{\otimes n} \to \mathbb{1}$ associated with T can be computed by replacing the generators of D by the corresponding structural morphisms of the coend C. Then evaluate ϕ_T with the Kirby element α and normalize to get the invariant:



In particular, to compute such quantum invariants defined from the center $\mathcal{Z}(\mathcal{C})$ of a autonomous category $\mathcal{C}$, one needs to give an explicit description of the structural morphism of the coend of $\mathcal{Z}(\mathcal{C})$ in terms of the category $\mathcal{C}$. In Section 4, we give such a description by using Hopf monads. This was our original motivation for introducing Hopf monads.

3. Hopf monads

In this section, we give the definition and first properties of Hopf monads [BV07].

3.1. **Monads.** Let $\mathcal{C}$ be a category. Recall that the category $\mathrm{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$ is strict monoidal with composition for monoidal product and identity functor $1_\mathcal{C}$ for unit object.

A *monad* on $\mathcal{C}$ is an algebra in $\mathrm{End}(\mathcal{C})$, that is, a triple (T, μ, η), where $T\colon \mathcal{C} \to \mathcal{C}$ is a functor, $\mu\colon T^2 \to T$ and $\eta\colon 1_\mathcal{C} \to T$ are natural transformations, such that:

$$\mu_X T(\mu_X) = \mu_X \mu_{T(X)} \quad \text{and} \quad \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X)$$

for any object X of $\mathcal{C}$.

3.2. **Modules over a monad.** Let T be a monad on a category $\mathcal{C}$. An *action* of T on an object M of $\mathcal{C}$ is a morphism $r\colon T(M) \to M$ in $\mathcal{C}$ such that:

$$rT(r) = r\mu_M \quad \text{and} \quad r\eta_M = \mathrm{id}_M.$$

The pair (M, r) is then called a *T-module in $\mathcal{C}$*, or just a *T-module*.

Given two T-modules (M, r) and (N, s) in $\mathcal{C}$, a morphism $f \in \mathrm{Hom}_\mathcal{C}(M, N)$ is said to be *T-linear* if $fr = sT(f)$. This gives rise to the *category T-$\mathcal{C}$ of T-modules*, with composition inherited from $\mathcal{C}$.

We will denote by $U_T\colon T\text{-}\mathcal{C} \to \mathcal{C}$ the *forgetful functor of T* defined by $U_T(M, r) = M$ for any T-module (M, r) and $U_T(f) = f$ for any T-linear morphism f.

3.3. **The philosophy.** A monad T on an autonomous category $\mathcal{C}$ is a Hopf monad, a quasitriangular Hopf monad, or a ribbon Hopf monad if the category T-$\mathcal{C}$ of T-modules is respectively autonomous, braided, or ribbon (in such a way that the forgetful functor $U_T\colon T\text{-}\mathcal{C} \to \mathcal{C}$ is strict monoidal).

The cornerstone of the theory is that these categorical properties of T-$\mathcal{C}$ can be encoded as structural morphisms of T. In the next sections, we give the definitions of these structural morphisms. Their relations with the category T-$\mathcal{C}$ is summarized in Theorem 3.1. For a complete treatment, we refer to [BV07].

3.4. **Bimonads.** A *bimonad*[5] on a monoidal category $\mathcal{C}$ is a monad T on $\mathcal{C}$ endowed with a natural transformation $T_2(X, Y)\colon T(X \otimes Y) \to T(X) \otimes T(Y)$ and a morphism $T_0\colon T(\mathbb{1}) \to \mathbb{1}$ satisfying:

$$(\mathrm{id}_{T(X)} \otimes T_2(Y, Z)) T_2(X, Y \otimes Z) = (T_2(X, Y) \otimes \mathrm{id}_{T(Z)}) T_2(X \otimes Y, Z);$$
$$(\mathrm{id}_{T(X)} \otimes T_0) T_2(X, \mathbb{1}) = \mathrm{id}_{T(X)} = (T_0 \otimes \mathrm{id}_{T(X)}) T_2(\mathbb{1}, X);$$
$$T_2(X, Y) \mu_{X \otimes Y} = (\mu_X \otimes \mu_Y) T_2(T(X), T(Y)) T(T_2(X, Y));$$
$$T_0 \mu_\mathbb{1} = T_0 T(T_0); \qquad T_2(X, Y) \eta_{X \otimes Y} = \eta_X \otimes \eta_Y; \qquad T_0 \eta_\mathbb{1} = \mathrm{id}_\mathbb{1};$$

for all objects X, Y, Z of $\mathcal{C}$.

[5]This notion of bimonad coincides exactly with the notion of 'Hopf monad' introduced in [Moe02]. However, by analogy with the notions of bialgebra and Hopf algebra, we prefer to reserve the term 'Hopf monad' for bimonads with antipodes (see Section 3.6)

3.5. **Antipodes.** Let T be a bimonad on a monoidal category $\mathcal{C}$.

If $\mathcal{C}$ is left autonomous, then a *left antipode for* T is a natural transformation $s^l = \{s^l_X : T({}^\vee T(X)) \to {}^\vee X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfying:

$$T_0 T(\mathrm{ev}_X) T({}^\vee\eta_X \otimes \mathrm{id}_X) = \mathrm{ev}_{T(X)}(s^l_{T(X)} T({}^\vee\mu_X) \otimes \mathrm{id}_{T(X)}) T_2({}^\vee T(X), X);$$
$$(\eta_X \otimes \mathrm{id}_{{}^\vee X})\mathrm{coev}_X T_0 = (\mu_X \otimes s^l_X) T_2(T(X), {}^\vee T(X)) T(\mathrm{coev}_{T(X)}).$$

Likewise, if $\mathcal{C}$ is right autonomous, then a *right antipode for* T is a natural transformation $s^r = \{s^r_X : T(T(X)^\vee) \to X^\vee\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfying:

$$T_0 T(\widetilde{\mathrm{ev}}_X) T(\mathrm{id}_X \otimes \eta_X^\vee) = \widetilde{\mathrm{ev}}_{T(X)}(\mathrm{id}_{T(X)} \otimes s^r_{T(X)} T(\mu_X^\vee)) T_2(X, T(X)^\vee);$$
$$(\mathrm{id}_{X^\vee} \otimes \eta_X)\widetilde{\mathrm{coev}}_X T_0 = (s^r_X \otimes \mu_X) T_2(T(X)^\vee, T(X)) T(\widetilde{\mathrm{coev}}_{T(X)}).$$

If a left (resp. right) antipode exists, then it is unique. When both exist, the left antipode s^l and the right antipode s^r are 'inverse' to each other in the sense that $\mathrm{id}_{T(X)} = s^r_{{}^\vee T(X)} T((s^l_X)^\vee) = s^l_{T(X)^\vee} T({}^\vee(s^r_X))$ for any object X of $\mathcal{C}$.

Moreover, as in the classical case, left and right antipodes are 'anti-(co)multiplicative' (see [BV07, Theorem 3.7]).

3.6. **Hopf monads.** A *Hopf monad* is a bimonad on an autonomous category which has a left antipode and a right antipode.

Hopf monads generalize Hopf algebras to a non-braided (and non-linear) setting. Furthermore: if $\mathcal{C}, \mathcal{D}$ are two autonomous categories and $U : \mathcal{D} \to \mathcal{C}$ is a strong monoidal functor which admits a left adjoint $F : \mathcal{C} \to \mathcal{D}$, then $T = UF$ is a Hopf monad on $\mathcal{C}$ (see [BV07, Corollary 3.15]).

Many fundamental results of the theory of Hopf algebras (such as the decomposition of Hopf modules, the existence of integrals, Maschke's criterium of semisimplicity, etc...) can be generalized to Hopf monads (see [BV07]).

3.7. **Quasitriangular Hopf monads.** Let T be a Hopf monad on an autonomous category $\mathcal{C}$. An R-*matrix* for T is a natural transformation $R_X, Y : X \otimes Y \to T(Y) \otimes T(X)$ satisfying:

$$(\mu_Y \otimes \mu_X) R_{T(X),T(Y)} T_2(X, Y) = (\mu_Y \otimes \mu_X) T_2(T(Y), T(X)) T(R_{X,Y});$$
$$(\mathrm{id}_{T(Z)} \otimes T_2(X, Y)) R_{X \otimes Y, Z}$$
$$= (\mu_Z \otimes \mathrm{id}_{T(X) \otimes T(Y)})(R_{X,T(Z)} \otimes \mathrm{id}_{T(Y)})(\mathrm{id}_X \otimes R_{Y,Z});$$
$$(T_2(Y, Z) \otimes \mathrm{id}_{T(X)}) R_{X, Y \otimes Z}$$
$$= (\mathrm{id}_{T(Y) \otimes T(Z)} \otimes \mu_X)(\mathrm{id}_{T(Y)} \otimes R_{T(X),Z})(R_{X,Y} \otimes \mathrm{id}_Z).$$

An R-matrix satisfies a QYB equation and is invertible for a convolution product (see [BV07, corollary 8.7]).

A *quasitriangular Hopf monad* is a Hopf monad equipped with an R-matrix.

3.8. **Ribbon Hopf monads.** Let T be a quasitriangular Hopf monad T on an autonomous category $\mathcal{C}$. A *twist* for T is a central and $*$-invertible natural transformation $\theta_X : X \to T(X)$ satisfying:

$$T_2(X, Y)\theta_{X \otimes Y} = (\mu_X \theta_{T(X)} \mu_X \otimes \mu_Y \theta_{T(Y)} \mu_Y) R_{T(Y),T(X)} R_{X,Y}.$$

Here central and $*$-invertible means central and invertible in the monoid $\mathrm{H{\scriptsize OM}}(1_\mathcal{C}, T)$ of natural transformations from $1_\mathcal{C}$ to T. This monoid is endowed with the convolution product defined by: $(\phi * \psi)_X = \mu_X \phi_{T(X)} \psi_X = \mu_X T(\psi_X) \phi_X : X \to T(X)$ and with the unit η.

A twist of a quasitriangular Hopf monad on an autonomous category is said to be *self-dual* if it satisfies:

$${}^\vee\theta_X = s^l_X \theta_{{}^\vee T(X)} \quad \text{(or, equivalently, } \theta_X^\vee = s^r_X \theta_{T(X)^\vee}).$$

A *ribbon Hopf monad* is a quasitriangular Hopf monad on an autonomous category endowed with a self-dual twist.

3.9. **Relations with modules.** Bimonads, Hopf monads, quasitriangular Hopf monads, and ribbon Hopf monads encode the expected structures on their category of modules:

Theorem 3.1 ([BV07]). (a) *Let T be a monad on a monoidal category $\mathcal{C}$. If T is a bimonad, then the category T-$\mathcal{C}$ of T-modules is monoidal by setting:*

$$(M,r) \otimes_{T\text{-}\mathcal{C}} (N,s) = (M \otimes N, (r \otimes s)T_2(M,N)) \quad \textit{and} \quad \mathbb{1}_{T\text{-}\mathcal{C}} = (\mathbb{1}, T_0).$$

Moreover this gives a bijective correspondence between bimonad structures for the monad T and monoidal structures of T-$\mathcal{C}$ such that the forgetful functor $U_T \colon T\text{-}\mathcal{C} \to \mathcal{C}$ is strict monoidal.

(b) *Let T be a bimonad on a left autonomous $\mathcal{C}$. Then T has a left antipode s^l if and only if the category T-$\mathcal{C}$ of T-modules is left autonomous. In terms of a left antipode s^l, left duals in T-$\mathcal{C}$ are given by:*

$$^\vee(M,r) = (^\vee M, s^l_M T(^\vee r)), \quad \mathrm{ev}_{(M,r)} = \mathrm{ev}_M, \quad \mathrm{coev}_{(M,r)} = \mathrm{coev}_M.$$

(c) *Let T be a bimonad on a right autonomous $\mathcal{C}$. Then T has a right antipode s^l if and only if the category T-$\mathcal{C}$ of T-modules is right autonomous. In terms of a right antipode s^r, right duals in T-$\mathcal{C}$ are given by:*

$$(M,r)^\vee = (M^\vee, s^r_M T(r^\vee)), \quad \widetilde{\mathrm{ev}}_{(M,r)} = \widetilde{\mathrm{ev}}_M, \quad \widetilde{\mathrm{coev}}_{(M,r)} = \widetilde{\mathrm{coev}}_M.$$

(d) *Let T be a bimonad on an autonomous $\mathcal{C}$. Then T is a Hopf monad if and only if the category T-$\mathcal{C}$ of T-modules is autonomous.*

(e) *Let T be a bimonad on a monoidal category $\mathcal{C}$. Any* R*-matrix R for T yields a braiding τ on T-$\mathcal{C}$ as follows:*

$$\tau_{(M,r),(N,s)} = (s \otimes t)R_{M,N} \colon (M,r) \otimes (N,s) \to (N,s) \otimes (M,r).$$

This assignment gives a bijection between R*-matrices for T and braidings on T-$\mathcal{C}$.*

(f) *Let T be a quasitriangular Hopf monad on an autonomous category $\mathcal{C}$. Any twist θ for T yields a twist Θ on T-$\mathcal{C}$ as follows:*

$$\Theta_{(M,r)} = r\theta_M \colon (M,r) \to (M,r).$$

This assignment gives a bijection between twists for T and twists on T-$\mathcal{C}$. Moreover, in this correspondence, θ is self-dual (and so T is ribbon) if and only if Θ is self-dual (and so T-$\mathcal{C}$ is ribbon).

4. Quantum double of Hopf monads

In this section, we review the construction of the double of a Hopf monad and its relations with the center construction (see [BV08a] for details).

4.1. **The center of a monoidal category.** Let $\mathcal{C}$ be a braided category. Recall that the *center of* $\mathcal{C}$ is the category $\mathcal{Z}(\mathcal{C})$ defined as follows: the objects are pairs (M,σ), where M is an object of $\mathcal{C}$ and $\sigma_Y \colon M \otimes Y \to Y \otimes M$ is a natural isomorphism verifying $\sigma_{Y\otimes Z} = (\mathrm{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \mathrm{id}_Z)$. A morphism $f\colon (M,\sigma) \to (M',\sigma')$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f \colon M \to M'$ in $\mathcal{C}$ which satisfies $(\mathrm{id}_Y \otimes f)\sigma_Y = \sigma'_Y(f \otimes \mathrm{id}_Y)$. The composition and identities are inherited from that of $\mathcal{C}$.

The category $\mathcal{Z}(\mathcal{C})$ is monoidal with unit object $(\mathbb{1}, \mathrm{id}_M)$ and monoidal product defined by $(M,\sigma) \otimes (N,\gamma) = \big(M \otimes N, (\sigma \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes \gamma)\big)$. Furthermore, if $\mathcal{C}$ is autonomous, so is $\mathcal{Z}(\mathcal{C})$.

4.2. **The centralizer and the double of a Hopf monad.** An endofunctor T of an autonomous category $\mathcal{C}$ is *centralizable* if the coend:

$$Z_T(X) = \int^{Y \in \mathcal{C}} {}^{\vee}T(Y) \otimes X \otimes Y$$

exists for every object X of $\mathcal{C}$. Denote by $i_{X,Y} : {}^{\vee}T(Y) \otimes X \otimes Y \to Z_T(X)$ the associated universal dinatural transformation. By the parameter theorem for coends, Z_T is an endofunctor of $\mathcal{C}$, called the *centralizer* of T, and $i_{X,Y}$ is natural in X and dinatural in Y.

Let T be a Hopf monad on an autonomous category $\mathcal{C}$. In [BV08a], we construct an explicit Hopf monad structure on Z_T, inherited from that of T. Since T preserves colimits and so coends (see [BV07, Remark 3.13]), $T(i)$ is a universal dinatural transformation. Therefore we can define a natural transformation $\Omega : TZ_T \to Z_T T$ by:

$$\Omega_X T(i_{X,Y}) = i_{T(X),T(Y)}\big({}^{\vee}\mu_Y s^l_{T(Y)} T({}^{\vee}\mu_Y) \otimes T_2(X,Y)\big) T_2({}^{\vee}T(Y), X \otimes Y),$$

where μ and s^l denote the product and left antipode of T.

Theorem 4.1 ([BV08a]). *The natural transformation $\Omega : TZ_T \to Z_T T$ is a comonoidal distributive law*[6].

The distributive law Ω is called the *canonical distributive law* of T over Z_T. Since Ω is a comonoidal distributive law, $D_T = Z_T \circ_\Omega T$ is a Hopf monad on $\mathcal{C}$. We call D_T the *double* of T.

Theorem 4.2 ([BV08a]). *Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$. Then the forgetful functor $\mathcal{U} : \mathcal{Z}(T\text{-}\mathcal{C}) \to \mathcal{C}$, given by $\big((M,r),\sigma\big) \mapsto M$, is monadic with monad the double D_T of T. Hence an equivalence:*

$$\mathcal{Z}(T\text{-}\mathcal{C}) \cong D_T\text{-}\mathcal{C}.$$

This is an equivalence of braided categories when the Hopf monad D_T is equipped with the R*-matrix:*

$$R_{X,Y} = \big(u_{T(Y)} \otimes Z_T(\eta_X)\big)(\mathrm{id}_{T(Y)} \otimes i_{X,Y})(\mathrm{coev}_Y \otimes \mathrm{id}_X),$$

where η and u the units of T and Z_T respectively.

Remark 4.3. Let $\mathcal{C}$ be an autonomous category which is *centralizable*, that is, such that the trivial Hopf monad $1_\mathcal{C}$ is centralizable. In that case, the centralizer $Z = Z_{1_\mathcal{C}}$ and the double $D_{1_\mathcal{C}}$ of $1_\mathcal{C}$ coincide. Then, by Theorem 4.2, Z is a quasitriangular Hopf monad on $\mathcal{C}$ such that $\mathcal{Z}(\mathcal{C}) \cong Z\text{-}\mathcal{C}$ as braided categories. In Section 5.1, we explicitly describe Z in terms of $\mathcal{C}$ when $\mathcal{C}$ is a fusion category.

Example 4.4. Let H be a finite-dimensional Hopf algebra over a field $\Bbbk$. Then the Hopf monad $T = ? \otimes_\Bbbk H$ on $\mathrm{vect}_\Bbbk$ is centralizable. We have: $Z_T = ? \otimes_\Bbbk H^*$ and $D_T = ? \otimes_\Bbbk H \otimes_\Bbbk H^*$. From Theorem 4.2, the vector space $D(H) = H \otimes_\Bbbk H^*$ inherits a quasitriangular Hopf algebra structure from the quasitriangular Hopf monad D_T. In particular the algebra structure on $D(H)$ is a twist of that of $H \otimes H^*$ by an isomorphism $H^* \otimes H \to H \otimes H^*$ coming from the distributive law $\Omega : TZ_T \to Z_T T$. This quasitriangular Hopf algebra $D(H)$ is precisely the *Drinfeld double* of H. Furthermore, since $T\text{-}\mathrm{vect}_\Bbbk = \mathrm{rep}H$ and $D_T\text{-}\mathrm{vect}_\Bbbk = \mathrm{rep}D(H)$, one recovers that $\mathcal{Z}(\mathrm{rep}H) \cong \mathrm{rep}D(H)$ as braided categories.

---

[6] A comonoidal distributive law $\Omega : TP \to PT$ between two Hopf monads P and T makes their composition PT a Hopf monad denoted by $P \circ_\Omega T$.

The previous example may be generalized to Hopf algebras in braided categories. Let $\mathcal{B}$ be a braided autonomous category which admits a coend:

$$C=\int^{Y\in\mathcal{B}} {}^\vee Y\otimes Y.$$

Recall C is then a Hopf algebra in $\mathcal{B}$ (see Section 1.6). Let A be a Hopf algebra in $\mathcal{B}$. Then the Hopf monad $?\otimes A$ on $\mathcal{B}$ is centralizable and we have:

$$Z_{?\otimes A}=?\otimes {}^\vee A\otimes C, \qquad D_{?\otimes A}=?\otimes A\otimes {}^\vee A\otimes C.$$

From Theorem 4.2, we get that the object $D(A)=A\otimes {}^\vee A\otimes C$ is a quasitriangular Hopf algebra in $\mathcal{B}$, whose structure is inherited from the quasitriangular Hopf monad $D_{?\otimes A}$. Here $D(A)$ quasitriangular means that there exists a R-matrix:

$$R\colon C\otimes C\to D(A)\otimes D(A)$$

verifying axioms generalizing the usual ones (when $\mathcal{B}=\mathrm{vect}_{\Bbbk}$, we have $C=\Bbbk$). This R-matrix defines a braiding on the category $\mathrm{rep}D(A)$ of $D(A)$- modules in $\mathcal{B}$ and $\mathcal{Z}(\mathrm{rep}A)\cong\mathrm{rep}D(A)$ as braided categories.

In particular $\mathcal{Z}(\mathcal{B})\cong\mathrm{rep}\,C$ (since $D(\mathbb{1})=C$), that is, the center of a braided category is the category of modules over its coend.

4.3. **The coend of a category of modules over a Hopf monad.** Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$. Denote by Z_T its centralizer, $i_{X,Y}\colon {}^\vee T(Y)\otimes X\otimes Y\to Z_T(X)$ its universal dinatural transformation, and $\Omega\colon TZ_T\to Z_TT$ the canonical distributive law of T over Z_T. Then:

Theorem 4.5 ([BV08a])**.** *The category T-$\mathcal{C}$ of T-modules admits a coend:*

$$\big(Z_T(\mathbb{1}),Z_T(T_0)\Omega_{\mathbb{1}}\big)=\int^{(M,r)\in T\text{-}\mathcal{C}} {}^\vee(M,r)\otimes(M,r),$$

with $I_{(M,r)}=i_{\mathbb{1},M}({}^\vee r\otimes\mathrm{id}_M)\colon {}^\vee(M,r)\otimes(M,r)\to\big(Z_T(\mathbb{1}),Z_T(T_0)\Omega_{\mathbb{1}}\big)$ as universal dinatural transformation.

If, in addition, T is quasitriangular, the coend $\big(Z_T(\mathbb{1}),Z_T(T_0)\Omega_{\mathbb{1}}\big)$ of T-$\mathcal{C}$ is a Hopf algebra in the braided category T-$\mathcal{C}$ (see Section 1.6).

Remark 4.6. Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$. Then $\mathcal{Z}(T\text{-}\mathcal{C})$ admits a coend if and only if the double D_T of T is centralizable. If such is the case, we obtain via Theorems 4.2 and 4.5 an explicit description of the coend of the braided category $\mathcal{Z}(T\text{-}\mathcal{C})\cong D_T\text{-}\mathcal{C}$ in terms of the monad T.

5. Reshetikhin-Turaev invariants from categorical centers

In this section, we treat in details the case of the center $\mathcal{Z}(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$. This leads to an explicit algorithm for computing invariants of Reshetikhin-Turaev type defined using $\mathcal{Z}(\mathcal{C})$ as input.

5.1. **On the center of a fusion category.** Fix a commutative ring $\Bbbk$. Let $\mathcal{C}$ be a fusion category over $\Bbbk$ (see Section 1.8). Remark that any $\Bbbk$-linear endofunctor T of $\mathcal{C}$ is centralizable with centralizer:

$$Z_T(X)=\bigoplus_{i\in I} {}^\vee T(V_i)\otimes X\otimes V_i$$

and universal dinatural transformation $i_{X,Y}\colon {}^\vee Y\otimes X\otimes Y\to Z(X)$ given by:

$$i_{X,Y}=\sum_{\substack{i\in I\\ 1\leq\alpha\leq N_Y^i}} {}^\vee T\big(q_Y^{i,\alpha}\big)\otimes\mathrm{id}_X\otimes p_Y^{i,\alpha}.$$

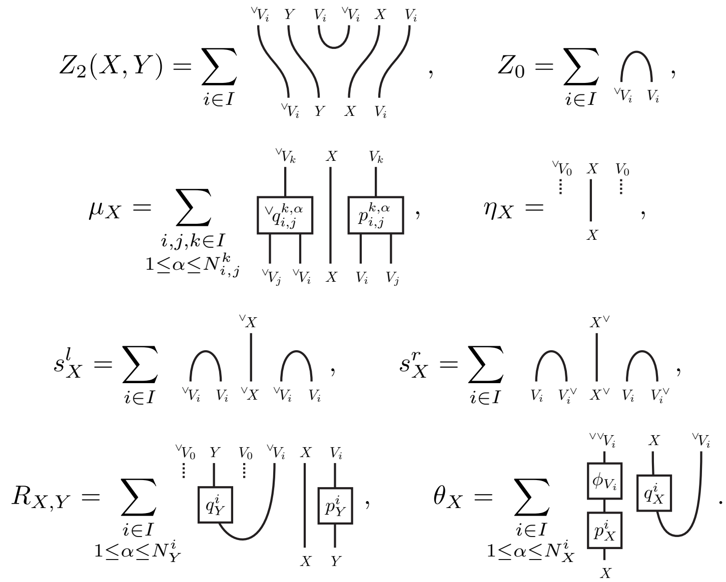


FIGURE 7. Structural morphisms of Z

In particular, the trivial Hopf monad $1_\mathcal{C}$ is centralizable with centralizer $Z = Z_{1_\mathcal{C}}$ given by:

$$Z(X) = \bigoplus_{i \in I} {}^\vee V_i \otimes X \otimes V_i.$$

By Remark 4.3, Z is a quasitriangular Hopf monad and $\mathcal{Z}(\mathcal{C}) \cong Z\text{-}\mathcal{C}$ as braided categories. Furthermore, if $\mathcal{C}$ is spherical, then Z is a ribbon Hopf monad and so $\mathcal{Z}(\mathcal{C})$ is a ribbon category.

The structural morphisms of Z can be described in terms of the category $\mathcal{C}$, that is, only using the p, q's (see Section 1.8), the duality morphisms, and the sovereign structure $\phi_X \colon X \to {}^{\vee\vee}X$. They are depicted in Figure 7. The dotted lines in the figures represent $\mathrm{id}_{V_0} = \mathrm{id}_\mathbb{1}$ and can be removed without changing the morphisms. We depicted them in order to remember which factor of $Z(X)$ is concerned. In the pictures, we simplify the notations as follows: $p^{i,\alpha}_{k^\vee, l, {}^{\vee\vee}m, n}$ denotes $p^{i,\alpha}_{V_k^\vee \otimes V_l \otimes {}^{\vee\vee}V_m \otimes V_n}$.

Using the Maschke theorem for Hopf monads which characterize semisimplicity (see [BV07, Theorem 6.5]), we have:

Proposition 5.1. [BV08b] *Let $\mathcal{C}$ be a spherical fusion category. Then the (ribbon) Hopf monad Z is semisimple if and only if* $\dim \mathcal{C}$ *is invertible.*

Since $\mathcal{Z}(\mathcal{C}) \cong Z\text{-}\mathcal{C}$, a consequence of Proposition 5.1 is:

Corollary 5.2. *Let $\mathcal{C}$ be a spherical fusion category over an algebraic closed field. Assume* $\dim \mathcal{C}$ *is invertible. Then $\mathcal{Z}(\mathcal{C})$ is a ribbon fusion category.*

5.2. **The coend of the center of a fusion category.** Let $\mathcal{C}$ be a fusion category over a commutative ring $\Bbbk$. Let Z be the centralizer of $1_\mathcal{C}$ as in Section 5.1. Since Z is $\Bbbk$-linear and $\mathcal{C}$ is fusion, it is centralizable, and so $\mathcal{Z}(\mathcal{C}) \cong Z\text{-}\mathcal{C}$ admits a coend. By Theorem 4.5, the underlying object of this coend is:

$$C = \bigoplus_{j \in I} {}^\vee Z(V_j) \otimes V_j = \bigoplus_{i,j \in I} {}^\vee V_i \otimes {}^\vee V_j \otimes {}^{\vee\vee} V_i \otimes V_j.$$

An immediate consequence of this is: $\dim \mathcal{Z}(\mathcal{C}) = (\dim \mathcal{C})^2$.

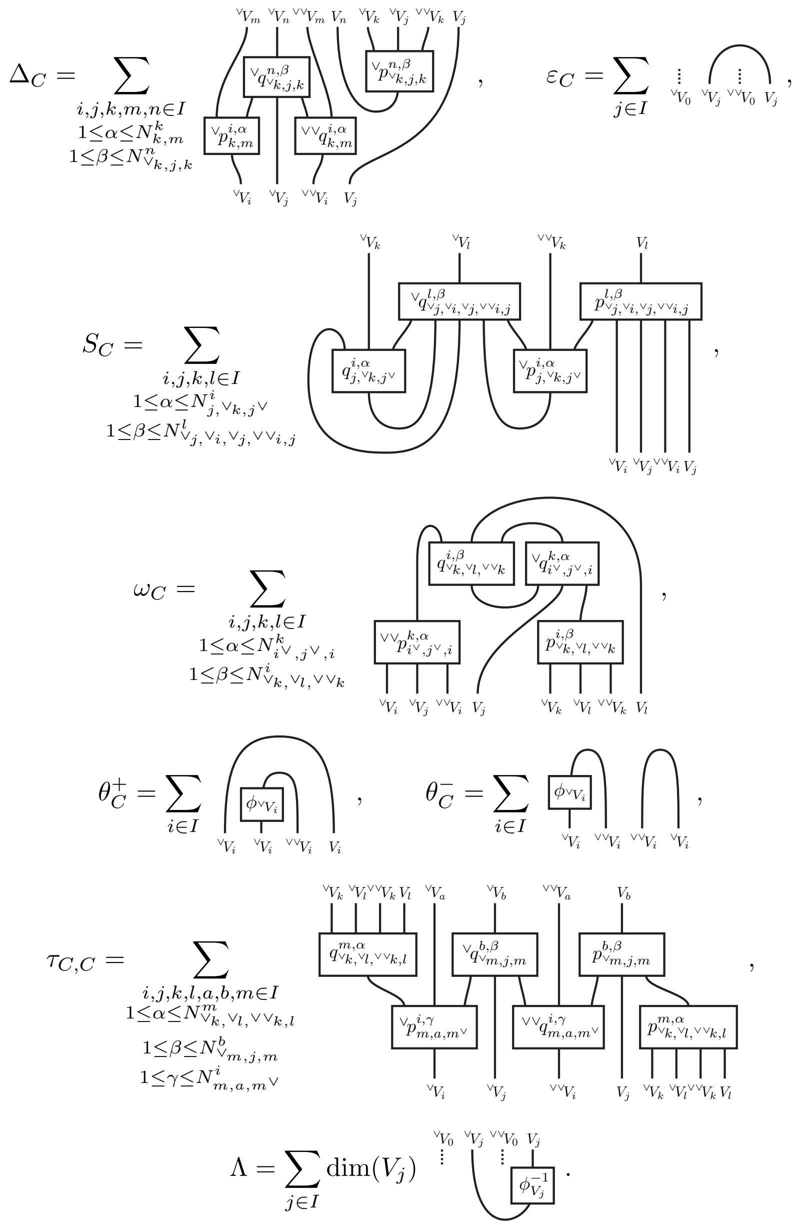


FIGURE 8. Structural morphisms of the coend of $\mathcal{Z}(\mathcal{C})$

The structural morphisms of C can be expressed using only the category $\mathcal{C}$. Those needed to represent Hopf diagrams are depicted in Figure 8.

Theorem 5.3 ([BV08b])**.** *The morphism $\Lambda\colon \mathbb{1} \to C$ of Figure 8 is a S_C-invariant integral of the coend of $\mathcal{Z}(\mathcal{C}) \cong Z\text{-}\mathcal{C}$.*

Following [Lyu95], a braided autonomous category $\mathcal{B}$ is said to be *modular* if it admits a coend C whose Hopf pairing $\omega_C\colon C \otimes C \to \mathbb{1}$ is non-degenerate. Note that this extends the usual notion of modularity to the non-semisimple case (when $\mathcal{B}$ is a ribbon fusion category, $\mathcal{B}$ is modular in the above sense if and only if the S- matrix is invertible).

Corollary 5.4 ([BV08b]). *The center of a spherical fusion category is modular.*

Remark 5.5. Let $\mathcal{C}$ a spherical fusion category over an algebraic closed field with $\dim \mathcal{C}$ invertible. By Corollaries 5.2 and 5.4, the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ is a modular ribbon fusion category. This last result was first proved in [Mü03] using a different method.

5.3. **Computing** $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}(M^3)$ **from** $\mathcal{C}$**.** Let $\mathcal{C}$ be a spherical fusion category over a commutative ring $\Bbbk$. As explained in Sections 5.1 and 5.2, the center $Z(\mathcal{C})$ of $\mathcal{C}$ is a ribbon category and admits a coend C whose structural morphisms are explicit (see Figure 8).

By Theorem 5.3, the morphism $\Lambda\colon \mathbb{1} \to C$ of Figure 8 is a Kirby element. Moreover we have: $\theta_C^+\Lambda = 1$ and $\theta_C^-\Lambda = 1$. Therefore Λ is a normalizable Kirby element , hence the 3-manifold invariant $\tau_{\mathcal{Z}(\mathcal{C})}(M, \Lambda)$ (see Section 2.2). Note that the normalization coefficient in the expression of $\tau_{\mathcal{Z}(\mathcal{C})}(M; \Lambda)$ is equal to 1.

For example, we have:

$$\tau_{\mathcal{Z}(\mathcal{C})}(S^3; \Lambda) = 1 \quad \text{and} \quad \tau_{\mathcal{Z}(\mathcal{C})}(S^2 \times S^1; \Lambda) = \dim \mathcal{C}.$$

Since we have an explicit description of the structural morphisms of the coend C (see Figures 8), we have a way to compute this invariant by using Hopf diagrams (see Section 2.3).

Note that the invariant $\tau_{\mathcal{Z}(\mathcal{C})}(M, \Lambda)$ is defined even if $\dim \mathcal{C}$ is not invertible. When $\dim \mathcal{C}$ is invertible and $\Bbbk$ is an algebraic closed field, so that $\mathcal{Z}(\mathcal{C})$ is a modular fusion category (see Remark 5.5), the invariant $\tau_{\mathcal{Z}(\mathcal{C})}(M, \Lambda)$ coincides with the Reshetikhin-Turaev invariant $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}(M)$ (up to normalization, see Remark 2.2). Hence a way to compute $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}(M)$ in terms of the structural morphisms of $\mathcal{C}$ (recall one cannot use the original algorithm of Reshetikhin-Turaev since the simple objects of $\mathcal{Z}(\mathcal{C})$ are not known in general).

References

[BKLT00] Y. Bespalov, T. Kerler, V. Lyubashenko, and V. Turaev, *Integrals for braided hopf algebras*, J. Pure Appl. Algebra **148** (2000), no. 2, 113–164.

[BV05] A. Bruguières and A. Virelizier, *Hopf diagrams and quantum invariants*, Algebr. Geom. Topol. **5** (2005), 1677–1710 (electronic).

[BV07] ———, *Hopf monads*, Adv. Math. **215** (2007), 679–733.

[BV08a] ———, *The double of a Hopf monad*, 2008, preprint.

[BV08b] ———, *On the center of fusion category*, 2008, in preparation.

[BW96] J. Barrett and B. Westbury, *Invariants of piecewise-linear* 3*-manifolds*, Trans. Amer. Math. Soc. **348** (1996), no. 10, 3997–4022.

[Hab06] K. Habiro, *Bottom tangles and universal invariants*, Algebr. Geom. Topol. **6** (2006), 1113–1214 (electronic).

[KSW05] Y. Kawahigashi, N. Sato, and M. Wakui, *(2+1)-dimensional topological quantum field theory from subfactors and Dehn surgery formula for 3-manifold invariants*, Adv. Math. **195** (2005), 165–204.

[Kir78] R. Kirby, *A calculus of framed links in S^3*, Invent. Math. **45** (1978), no. 3, 35–56.

[KL01] T. Kerler and V. Lyubashenko, *Non-semisimple topological quantum field theories for 3-manifolds with corners*, Lecture Notes in Mathematics, vol. 1765, Springer-Verlag, Berlin, 2001.

[Lic97] W. B. R. Lickorish, *An introduction to knot theory*, Springer-Verlag, New York, 1997.

[Lyu94] V. Lyubashenko, *Tangles and Hopf algebras in braided categories*, J. Pure Appl. Algebra **98** (1994), no. 3, 245–278.

[Lyu95] ———, *Invariants of* 3*-manifolds and projective representations of mapping class groups via quantum groups at roots of unity*, Comm. Math. Phys. **172** (1995), no. 3, 467–516.

[Mac98] S. Mac Lane, *Categories for the working mathematician*, second ed., Springer-Verlag, New York, 1998.

[Maj93] S. Majid, *Braided groups*, J. Pure Appl. Algebra **86** (1993), no. 2, 187–221.

[Moe02] I. Moerdijk, *Monads on tensor categories*, J. Pure Appl. Algebra **168** (2002), no. 2-3, 189–208.

[Mü03] M. Müger, *From subfactors to categories and topology ii. The quantum double of tensor categories and subfactors.*, J. Pure Appl. Alg. **180** (2003), 159–219.
[RT91] N. Reshetikhin and V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), no. 3, 547–597.
[Ro95] J. Roberts, *Skein theory and Turaev-Viro invariants*, Topology **34** (1995), no. 4, 771–787.
[Tur94] V. Turaev, *Quantum invariants of knots and* 3*-manifolds*, Walter de Gruyter & Co., Berlin, 1994.
[TV92] V. Turaev and O. Viro, *State sum invariants of* 3*-manifolds and quantum* $6j$*-symbols*, Topology **31** (1992), no. 4, 865–902.
[Vir06] A. Virelizier, *Kirby elements and quantum invariants*, Proc. London Math. Soc. **93** (2006), 474–514.

E-mail address: `bruguier@math.univ-montp2.fr` and `virelizi@math.univ-montp2.fr`