ON THE DOUBLE OF A BRAIDED HOPF ALGEBRA

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ABSTRACT. In 2011, S. Lack and the authors defined the double $D(A)$ of a Hopf algebra A in a braided category $\mathcal{B}$ with duals and a coend; $D(A)$ is a quasitriangular Hopf algebra in $\mathcal{B}$ whose category of modules is isomorphic to the center of the category of A- modules as a braided category. Here, quasitriangular means endowed with a R-matrix; however this involves a notion of R-matrix different from that introduced by Majid in 1993. If A is a Hopf algebra in a braided category $\mathcal{B}$, we show that the center of the category of A- modules can be described as the category of Yetter-Drinfeld modules for A viewed as a Hopf algebra in the center $\mathcal{Z}(\mathcal{B})$ of $\mathcal{B}$, but it cannot be described, in general, as a the category of modules of a Hopf algebra in (the center of) $\mathcal{Z}(\mathcal{B})$. However, we show that the center of the category of A- modules is the category of modules of a quasitriangular Hopf monad d_A on $\mathcal{Z}(\mathcal{B})$, which we describe explicitly. As an endofunctor of $\mathcal{Z}(\mathcal{B})$, d_A it is given by $X \mapsto X \otimes A \otimes {}^\vee A$, but, in general, d_A is not the Hopf monad associated with a Hopf algebra in (the center of) $\mathcal{Z}(\mathcal{B})$. If $\mathcal{B}$ has a coend C, then $D(A)$ is the cross product of the Hopf monad d_A by the coend C. Equivalently, d_A is the cross quotient of $D(A)$ by C. This illustrates the fact that the cross-quotient of two Hopf algebras is not representable by a Hopf algebra in general.

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INTRODUCTION

The notion of Hopf algebra extends naturally to the context of braided categories. Hopf algebras in braided categories, also called braided Hopf algebras, have been studied by many authors and it has been shown that several aspects of the theory of Hopf algebras can be extended to this setting. This paper is devoted to the study of the double of a braided Hopf algebra. In [BV2], the authors defined the double $D(A)$ of a Hopf algebra in a rigid braided category admitting a coend C. As an object, $D(A) = A \otimes {}^\vee A \otimes C$; it is a quasitriangular Hopf algebra in $\mathcal{B}$, whose category of modules is isomorphic to the categorical center of the category of A- modules in $\mathcal{B}$ as braided categories. A quasitriangular Hopf algebra is a Hopf algebra A endowed

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with an R-matrix, that is a morphism $\mathfrak{r}\colon C \otimes C \to A \otimes A$ satisfying certain axioms. R-matrixes are in one-to-one correspondance with braidings on the category of A-modules. This notion of R-matrix is different from that previously introduced by Majid in [Ma3], which did not involve the coend, and did not have this property.

If A is a Hopf algebra in a braided category $\mathcal{B}$, there is a canonical strict monoidal functor $\mathcal{U}$ from the center of the category of A-modules to the center of $\mathcal{B}$. It would seem reasonable to try to describe the center of the category of A-modules as the category of modules of a Hopf algebra in $\mathcal{Z}(\mathcal{B})$, whose underlying object would be $A \otimes {}^{\vee}\!A$. However, in general this is not possible.

We do show that the center of the category of A-modules can be described as a category of Yetter-Drinfeld modules for A, viewed as a Hopf algebra in the center of $\mathcal{B}$. (Actually for this result $\mathcal{B}$ needs not be braided, if we assume that A is a Hopf algebra in the center of $\mathcal{B}$). We also show that, in general, the center of the category of A-modules cannot be described as the category of modules over a Hopf algebra in the center of $\mathcal{B}$.

In fact, the center of the category of A-modules is the category of modules of a quasitriangular Hopf *monad* d_A on $\mathcal{Z}(\mathcal{B})$ which we describe explicitly. As an endofunctor of $\mathcal{Z}(\mathcal{B})$, d_A is defined by $X \mapsto X \otimes A \otimes {}^{\vee}\!A$.

Hopf monads, which were introduced in [BV1] and further studied in [BLV], are algebraic objects which generalize Hopf algebras in braided categories to the setting of monoidal categories. A Hopf algebra in the center of a monoidal category $\mathcal{C}$ defines a Hopf monad on $\mathcal{C}$; a Hopf monad T on $\mathcal{C}$ can be obtained from a Hopf algebra of the center of $\mathcal{C}$ if and only if T admits an augmentation, that is a Hopf monad morphism $T \to \mathrm{id}_{\mathcal{C}}$. In [BLV], the notions of cross-product and cross-quotient of Hopf monads were introduced. It turns out that the cross-quotient of two Hopf monads representable by Hopf algebras is not always representable by a Hopf algebra. The present situation is an illustration of this fact: the Hopf monad d_A is the cross-quotient of the double of A, $D(A)$ by the coend C (both viewed as Hopf monads); but this cross-quotient is not representable by a Hopf algebra in general.

1. Preliminaries on categories

1.1. Monoidal categories and (co-)monoidal functors. Given an object X of a monoidal category $\mathcal{C}$, we denote by $X\otimes ?$ the endofunctor of $\mathcal{C}$ defined on objects by $Y \mapsto X \otimes Y$ and on morphisms by $f \mapsto X \otimes f = \mathrm{id}_X \otimes f$. Similarly one defines the endofunctor $? \otimes X$ of $\mathcal{C}$.

Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_2, F_0), where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, $F_2\colon F \otimes F \to F\otimes$ is a natural transformation, and $F_0\colon \mathbb{1} \to F(\mathbb{1})$ is a morphism in $\mathcal{D}$, such that:

$$F_2(X, Y \otimes Z)(\mathrm{id}_{F(X)} \otimes F_2(Y,Z)) = F_2(X \otimes Y, Z)(F_2(X,Y) \otimes \mathrm{id}_{F(Z)});$$
$$F_2(X, \mathbb{1})(\mathrm{id}_{F(X)} \otimes F_0) = \mathrm{id}_{F(X)} = F_2(\mathbb{1}, X)(F_0 \otimes \mathrm{id}_{F(X)});$$

for all objects X, Y, Z of $\mathcal{C}$.

A monoidal functor (F, F_2, F_0) is said to be *strong* (resp. *strict*) if F_2 and F_0 are isomorphisms (resp. identities).

A natural transformation $\varphi\colon F \to G$ between comonoidal functors is *monoidal* if it satisfies:

$$\varphi_{X\otimes Y} F_2(X,Y) = G_2(X,Y)(\varphi_X \otimes \varphi_Y) \quad \text{and} \quad G_0 = \varphi_{\mathbb{1}} F_0.$$

1.2. **Comonoidal functors.** Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ be two monoidal categories. A *comonoidal functor* (also called *opmonoidal functor*) from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_2, F_0), where $F \colon \mathcal{C} \to \mathcal{D}$ is a functor, $F_2 \colon F\otimes \to F \otimes F$ is a natural transformation, and $F_0 \colon F(\mathbb{1}) \to \mathbb{1}$ is a morphism in $\mathcal{D}$, such that:

$$\big(\mathrm{id}_{F(X)} \otimes F_2(Y,Z)\big) F_2(X, Y \otimes Z) = \big(F_2(X,Y) \otimes \mathrm{id}_{F(Z)}\big) F_2(X \otimes Y, Z);$$
$$(\mathrm{id}_{F(X)} \otimes F_0) F_2(X, \mathbb{1}) = \mathrm{id}_{F(X)} = (F_0 \otimes \mathrm{id}_{F(X)}) F_2(\mathbb{1}, X);$$

for all objects X, Y, Z of $\mathcal{C}$.

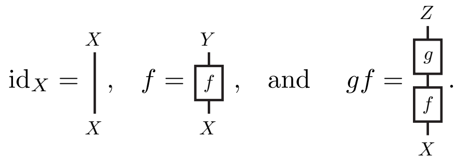
A comonoidal functor (F, F_2, F_0) is said to be *strong* (resp. *strict*) if F_2 and F_0 are isomorphisms (resp. identities). In that case, (F, F_2^{-1}, F_0^{-1}) is a strong (resp. strict) monoidal functor.

A natural transformation $\varphi \colon F \to G$ between monoidal functors is *comonoidal* if it satisfies:

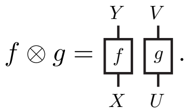
$$G_2(X,Y)\varphi_{X \otimes Y} = (\varphi_X \otimes \varphi_Y) F_2(X,Y) \quad \text{and} \quad G_0 \varphi_{\mathbb{1}} = F_0.$$

Note that the notions of comonoidal functor and comonoidal natural transformation are dual to the notions of monoidal functor and monoidal natural transformation.

1.3. **Graphical conventions.** We represent morphisms in a category by diagrams to be read from bottom to top. Thus we draw the identity id_X of an object X, a morphism $f \colon X \to Y$, and its composition with a morphism $g \colon Y \to Z$ as follows:



In a monoidal category, we represent the monoidal product of two morphisms $f \colon X \to Y$ and $g \colon U \to V$ by juxtaposition:



1.4. **Duals and rigid categories.** Let $\mathcal{C}$ be a monoidal category. Recall that a *left dual* of an object X of $\mathcal{C}$ is an object ${}^\vee X$ of $\mathcal{C}$ endowed with morphisms $\mathrm{ev}_X \colon {}^\vee X \otimes X \to \mathbb{1}$ (the *left evaluation*) and $\mathrm{coev}_X \colon \mathbb{1} \to X \otimes {}^\vee X$ (the *left coevaluation*) such that

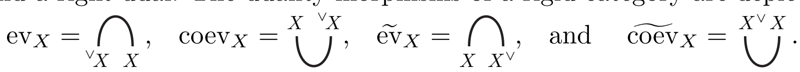
$$(\mathrm{ev}_X \otimes \mathrm{id}_{^\vee X})(\mathrm{id}_{^\vee X} \otimes \mathrm{coev}_X) = \mathrm{id}_{^\vee X} \quad \text{and} \quad (\mathrm{id}_X \otimes \mathrm{ev}_X)(\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X.$$

Likewise, a *right dual* of an object X of $\mathcal{C}$ is an object $X^\vee$ of $\mathcal{C}$ endowed with morphisms $\widetilde{\mathrm{ev}}_X \colon X \otimes X^\vee \to \mathbb{1}$ (the *right evaluation*) and $\widetilde{\mathrm{coev}}_X \colon \mathbb{1} \to X^\vee \otimes X$ (the *right coevaluation*) such that

$$(\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \widetilde{\mathrm{coev}}_X) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{id}_{X^\vee} \otimes \widetilde{\mathrm{ev}}_X)(\widetilde{\mathrm{coev}}_X \otimes \mathrm{id}_{X^\vee}) = \mathrm{id}_{X^\vee}.$$

Left and right duals, if they exist, are unique up to unique isomorphisms preserving the evaluation and coevaluation morphisms.

A *rigid category* is a monoidal category where every objects admits both a left dual and a right dual. The duality morphisms of a rigid category are depicted as:



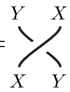
1.5. **Braided categories.** A *braiding* of a monoidal category $\mathcal{B}$ is a natural isomorphism $\tau = \{\tau_{X,Y} \colon X \otimes Y \to Y \otimes X\}_{X,Y \in \mathcal{B}}$ such that

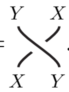
$$\tau_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes \mathrm{id}_Z) \quad \text{and} \quad \tau_{X\otimes Y,Z} = (\tau_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \tau_{Y,Z})$$

for all $X, Y, Z \in \mathrm{Ob}(\mathcal{B})$. These conditions imply that $\tau_{X,\mathbb{1}} = \tau_{\mathbb{1},X} = \mathrm{id}_X$.

A *braided category* is a monoidal category endowed with a braiding.

The braiding τ of a braided category, and its inverse, are depicted as

$\tau_{X,Y} =$ 

and $\tau^{-1}_{Y,X} =$ 

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If $\mathcal{B}$ is a braided category, then its braiding τ defines a fully faithful braided functor

$$\begin{cases} \mathcal{B} & \to \quad \mathcal{Z}(\mathcal{B}) \\ X & \mapsto \quad (X, \tau_{X,-}) \end{cases}$$

which is a monoidal section of the forgetful functor $\mathcal{Z}(\mathcal{B}) \to \mathcal{B}$.

If $\mathcal{B}$ is a braided category with braiding τ, then the *mirror of* $\mathcal{B}$ is the braided category which coincides with $\mathcal{B}$ as a monoidal category and equipped with the braiding $\overline{\tau}$ defined by $\overline{\tau}_{X,Y} = \tau^{-1}_{Y,X}$.

1.6. **The center of a monoidal category.** Let $\mathcal{C}$ be a monoidal category. A *half braiding* of $\mathcal{C}$ is a pair (A, σ), where A is an object of $\mathcal{C}$ and

$$\sigma = \{\sigma_X \colon A \otimes X \to X \otimes A\}_{X \in \mathcal{C}}$$

is a natural isomorphism such that

$$\sigma_{X\otimes Y} = (\mathrm{id}_X \otimes \sigma_Y)(\sigma_X \otimes \mathrm{id}_Y) \tag{1}$$

for all $X, Y \in \mathrm{Ob}(\mathcal{C})$. This implies that $\sigma_{\mathbb{1}} = \mathrm{id}_A$.

The *center of* $\mathcal{C}$ is the braided category $\mathcal{Z}(\mathcal{C})$ defined as follows. The objects of $\mathcal{Z}(\mathcal{C})$ are half braidings of $\mathcal{C}$. A morphism $(A, \sigma) \to (A', \sigma')$ in $\mathcal{Z}(\mathcal{C})$ is a morphism $f \colon A \to A'$ in $\mathcal{C}$ such that $(\mathrm{id}_X \otimes f)\sigma_X = \sigma'_X(f \otimes \mathrm{id}_X)$ for any object X of $\mathcal{C}$. The unit object of $\mathcal{Z}(\mathcal{C})$ is $\mathbb{1}_{\mathcal{Z}(\mathcal{C})} = (\mathbb{1}, \{\mathrm{id}_X\}_{X \in \mathcal{C}})$ and the monoidal product is

$$(A, \sigma) \otimes (B, \rho) = \big(A \otimes B, (\sigma \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \rho)\big).$$

The braiding τ in $\mathcal{Z}(\mathcal{C})$ is defined by

$$\tau_{(A,\sigma),(B,\rho)} = \sigma_B \colon (A, \sigma) \otimes (B, \rho) \to (B, \rho) \otimes (A, \sigma).$$

If $\mathcal{C}$ is rigid, then so is $\mathcal{Z}(\mathcal{C})$.

The *forgetful functor* $\mathcal{U} \colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$, $(A, \sigma) \mapsto A$, is strict monoidal.

1.7. **Coends.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *dinatural transformation* from a functor $F \colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{C}$ to an object A of $\mathcal{C}$ is a family of morphisms in $\mathcal{C}$

$$d = \{d_Y \colon F(Y, Y) \to A\}_{Y \in \mathcal{D}}$$

such that for every morphism $f \colon X \to Y$ in $\mathcal{D}$, we have

$$d_X F(f, \mathrm{id}_X) = d_Y F(\mathrm{id}_Y, f) \colon F(Y, X) \to A.$$

A *coend* of F is a pair (C, ρ) consisting in an object C of $\mathcal{C}$ and a dinatural transformation ρ from F to C satisfying the following universality condition: for each dinatural transformation d from F to an object c of $\mathcal{C}$, there exists a unique morphism $\tilde{d} : C \to c$ such that $d_Y = \tilde{d}\rho_Y$ for any Y in $\mathcal{D}$. Thus if F has a coend (C, ρ), then it is unique up to unique isomorphism. One writes $C = \int^{Y \in \mathcal{D}} F(Y,Y)$. For more details on coends, see [Mac].

2. Hopf algebras in braided categories

2.1. **Hopf algebras.** Let $\mathcal{C}$ be a monoidal category. Recall that an *algebra in* $\mathcal{C}$ is an object A of $\mathcal{C}$ endowed with morphisms $m\colon A \otimes A \to A$ (the product) and $u\colon \mathbb{1} \to A$ (the unit) such that

$$m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m) \quad \text{and} \quad m(\mathrm{id}_A \otimes u) = \mathrm{id}_A = m(u \otimes \mathrm{id}_A).$$

A *coalgebra in* $\mathcal{C}$ is an object C of $\mathcal{C}$ endowed with morphisms $\Delta\colon C \to C \otimes C$ (the coproduct) and $\varepsilon\colon C \to \mathbb{1}$ (the counit) such that

$$(\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta \quad \text{and} \quad (\mathrm{id}_C \otimes \varepsilon)\Delta = \mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C)\Delta.$$

Let $\mathcal{B}$ be a braided category, with braiding τ. A *bialgebra in* $\mathcal{B}$ is an object A of $\mathcal{B}$ endowed with an algebra structure (in $\mathcal{B}$) and a coalgebra structure (in $\mathcal{B}$) such that its coproduct Δ and counit ε are algebra morphisms (or equivalently, such that it product m and unit u are coalgebra morphisms), that is,

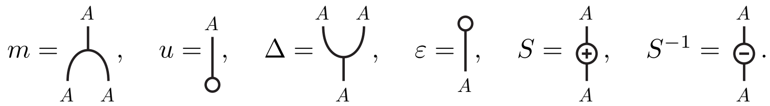
$$\begin{aligned} \Delta m &= (m \otimes m)(\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_A)(\Delta \otimes \Delta), & \Delta u &= u \otimes u, \\ \varepsilon m &= \varepsilon \otimes \varepsilon, & \varepsilon u &= \mathrm{id}_{\mathbb{1}}. \end{aligned}$$

An *antipode* for a bialgebra A is a morphism $S\colon A \to A$ in $\mathcal{B}$ such that

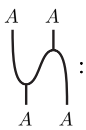
$$m(S \otimes \mathrm{id}_A)\Delta = u\varepsilon = m(\mathrm{id}_A \otimes S)\Delta.$$

If it exists, an antipode is unique. A *Hopf algebra in* $\mathcal{B}$ is a bialgebra in $\mathcal{B}$ which admits an invertible antipode.

Given a Hopf algebra A in a braided category, we depict its product m, unit u, coproduct Δ, counit ε, antipode S, and S^{-1} as follows:



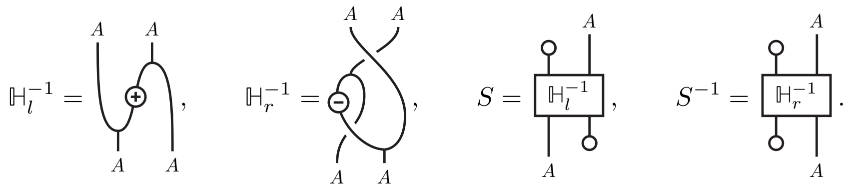
Remark 2.1. Let A be a bialgebra in a braided category $\mathcal{B}$. Then A is a Hopf algebra if and only if its left *left fusion operator*

$$\mathbb{H}_l = (\mathrm{id}_A \otimes m)(\Delta \otimes \mathrm{id}_A) =$$  $$: A \otimes A \to A \otimes A$$

and its *right fusion operator*

$$\mathbb{H}_r = (m \otimes \mathrm{id}_A)(\mathrm{id}_A \otimes \tau_{A,A})(\Delta \otimes \mathrm{id}_A) =$$  $$: A \otimes A \to A \otimes A$$

are invertible. If such is the case, the inverses of the fusion operators and the antipode S of A and its inverse are related by



2.2. **Modules in categories.** Let (A, m, u) be an algebra in a monoidal category $\mathcal{C}$. A *left A-module* (in $\mathcal{C}$) is a pair (M, r), where M is an object of $\mathcal{C}$ and $r\colon A \otimes M \to M$ is a morphism in $\mathcal{C}$, such that

$$r(m \otimes \mathrm{id}_M) = r(\mathrm{id}_A \otimes r) \quad \text{and} \quad r(u \otimes \mathrm{id}_M) = \mathrm{id}_M.$$

An *A-linear morphism* between two left A-modules (M, r) and (N, s) is a morphism $f\colon M \to N$ such that $fr = s(\mathrm{id}_A \otimes f)$. Hence the category ${}_A\mathcal{C}$ of left A-modules. Likewise, one defines the category $\mathcal{C}_A$ of right A-modules.

Let A be a bialgebra in a braided category $\mathcal{B}$. Then the category ${}_A\mathcal{B}$ is monoidal, with unit object $(\mathbb{1}, \varepsilon)$ and monoidal product

$$(M, r) \otimes (N, s) = (r \otimes s)(\mathrm{id}_A \otimes \tau_{A,M} \otimes \mathrm{id}_N)(\Delta \otimes \mathrm{id}_{M \otimes N}),$$

where Δ and ϵ are the coproduct and counit of A, and τ is the braiding of $\mathcal{B}$. Likewise the category $\mathcal{B}_A$ is monoidal, with unit object $(\mathbb{1}, \varepsilon)$ and monoidal product:

$$(M, r) \otimes (N, s) = (r \otimes s)(\mathrm{id}_M \otimes \tau_{N,A} \otimes \mathrm{id}_A)(\Delta \otimes \mathrm{id}_{M \otimes N}).$$

Assume $\mathcal{B}$ is rigid. Then ${}_A\mathcal{B}$ is rigid if and only if $\mathcal{B}_A$ is rigid, if and only if A is a Hopf algebra. If A is a Hopf algebra, with antipode S, then the duals of a left A-module (M, r) are:

$$\begin{aligned}
{}^\vee(M, r) &= \big({}^\vee M, (\mathrm{ev}_M \otimes \mathrm{id}_{{}^\vee M})(\mathrm{id}_{{}^\vee M} \otimes r(S \otimes \mathrm{id}_M) \otimes \mathrm{id}_{{}^\vee M})(\tau_{A,{}^\vee M} \otimes \mathrm{coev}_M)\big),\\
(M, r)^\vee &= \big(M^\vee, (\mathrm{id}_{M^\vee} \otimes \widetilde{\mathrm{ev}}_M)(\mathrm{id}_{M^\vee} \otimes r\tau^{-1}_{A,M} \otimes \mathrm{id}_{M^\vee})(\widetilde{\mathrm{coev}}_M \otimes S^{-1} \otimes \mathrm{id}_{M^\vee})\big),
\end{aligned}$$

and the duals of a right A-module (M, r) are:

$$\begin{aligned}
{}^\vee(M, r) &= \big({}^\vee M, (\mathrm{ev}_M \otimes \mathrm{id}_{{}^\vee M})(\mathrm{id}_{{}^\vee M} \otimes r\tau^{-1}_{M,A} \otimes \mathrm{id}_{{}^\vee M})(\mathrm{id}_{{}^\vee M} \otimes S^{-1} \otimes \mathrm{coev}_M)\big),\\
(M, r)^\vee &= \big(M^\vee, (\mathrm{id}_{M^\vee} \otimes \widetilde{\mathrm{ev}}_M)(\mathrm{id}_{M^\vee} \otimes r(\mathrm{id}_M \otimes S) \otimes \mathrm{id}_{M^\vee})(\widetilde{\mathrm{coev}}_M \otimes \tau_{M^\vee,A})\big).
\end{aligned}$$

Remark 2.2. Let A be a Hopf algebra in a braided category $\mathcal{B}$, with braiding τ. The functor $F_A\colon {}_A\mathcal{B} \to \mathcal{B}_A$, defined by $F_A(M, r) = \big(M, r\tau_{M,A}(\mathrm{id}_M \otimes S)\big)$ and $F_A(f) = f$, gives rise to a monoidal isomorphism of categories:

$$F_A = (F_A, \tau, \mathbb{1})\colon ({}_A\mathcal{B})^{\otimes \mathrm{op}} \to \mathcal{B}_A.$$

Therefore braidings on ${}_A\mathcal{B}$ are in bijection with braidings on $\mathcal{B}_A$. More precisely, if c is a braiding on $\mathcal{B}_A$, then:

$$c'_{(M,r),(N,s)} = \tau_{M,N}\, c_{F_A(N,s),F_A(M,r)}\, \tau^{-1}_{N,M}$$

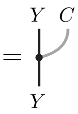
is a braiding on ${}_A\mathcal{B}$ (making F_A braided), and the correspondence $c \mapsto c'$ is bijective.

2.3. **The coend of a braided rigid category.** Let $\mathcal{B}$ be braided rigid category. The coend

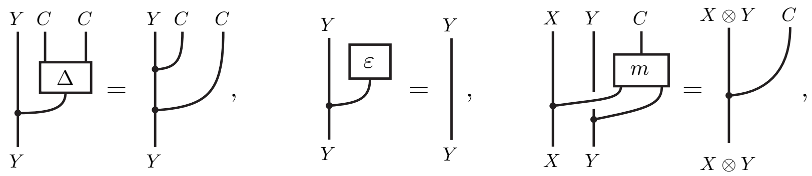
$$C = \int^{Y \in \mathcal{B}} {}^{\vee}Y \otimes Y,$$

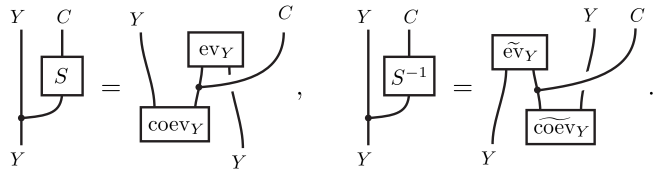
if it exists, is called the *coend of* $\mathcal{B}$.

Assume $\mathcal{B}$ has a coend C and denote by $i_Y : {}^{\vee}Y \otimes Y \to C$ the corresponding universal dinatural transformation. The *universal coaction* of C on the objects of $\mathcal{B}$ is the natural transformation δ defined by:

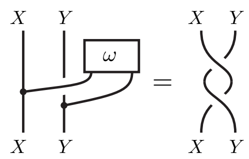
(2) $\quad \delta_Y = (\mathrm{id}_Y \otimes i_Y)(\mathrm{coev}_Y \otimes \mathrm{id}_Y) : Y \to Y \otimes C, \quad$ depicted as $\quad \delta_Y =$  .

As shown by Majid [Ma2], C is a Hopf algebra in $\mathcal{B}$. Its coproduct Δ, product m, counit ε, unit u, and antipode S with inverse S^{-1} are characterized by the following equalities, where $X, Y \in \mathcal{B}$:



$u = \delta_{\mathbb{1}},$ 

Furthermore, the morphism $\omega : C \otimes C \to \mathbb{1}$ defined by

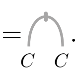


is a Hopf pairing for C, called the *canonical pairing*. This means that

$$\omega(m \otimes \mathrm{id}_C) = \omega(\mathrm{id}_C \otimes \omega \otimes \mathrm{id}_C)(\mathrm{id}_{C^{\otimes 2}} \otimes \Delta), \qquad \omega(u \otimes \mathrm{id}_C) = \varepsilon,$$
$$\omega(\mathrm{id}_C \otimes m) = \omega(\mathrm{id}_C \otimes \omega \otimes \mathrm{id}_C)(\Delta \otimes \mathrm{id}_{C^{\otimes 2}}), \qquad \omega(\mathrm{id}_C \otimes u) = \varepsilon.$$

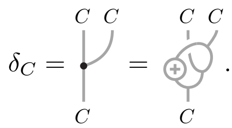
These axioms imply: $\omega(S \otimes \mathrm{id}_C) = \omega(\mathrm{id}_C \otimes S)$. Moreover the canonical pairing ω satisfies the self-duality condition: $\omega \tau_{C,C}(S \otimes S) = \omega$.

In this section, the structural morphisms of C are drawn in grey and the Hopf pairing $w : C \otimes C \to \mathbb{1}$ is depicted as:

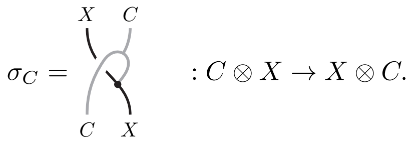
$\omega =$  .

Remark 2.3. The category $\mathcal{B}$ is symmetric if and only if $\omega = \epsilon \otimes \epsilon$. Such is the case if $C = \mathbb{1}$.

Remark 2.4. The universal coaction of the coend on itself can be expressed in terms of its Hopf algebra structure as follows:

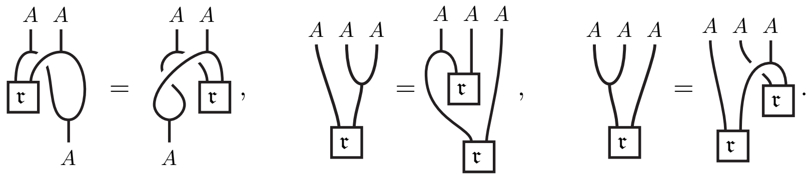


Remark 2.5. The coend C has a canonical half braiding $\sigma = \{\sigma_X\}_{X \in \mathcal{C}}$ defined by

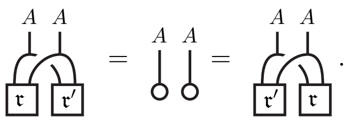


Then (C, σ), endowed with the coproduct and counit of C, is a coalgebra in $\mathcal{Z}(\mathcal{C})$ which is cocomutative. Indeed its coproduct Δ_C satisfies $\sigma_C \Delta_C = \Delta_C$.

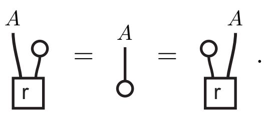
2.4. R**-matrices: Majid's approach.** Let A be a Hopf algebra in braided category $\mathcal{B}$, with braiding τ. By extending Drinfeld's axioms, Majid ([Ma3]) defines an R-matrix for A to be as a convolution-invertible morphism $\mathfrak{r} \colon \mathbb{1} \to A \otimes A$ satisfying



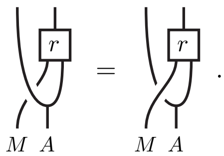
Here $\mathfrak{r}$ convolution-invertible means that there exists a (necessarily unique) morphism $\mathfrak{r}' \colon \mathbb{1} \to A \otimes A$ such that



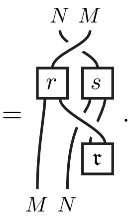
Note that if $\mathcal{B}$ is rigid, then $\mathfrak{r}$ is convolution-invertible if and only if it satisfies



Majid remarks that such a morphism $\mathfrak{r}$ does not define a braiding on the category $\mathcal{B}_A$ of A-modules braided, but only on the full subcategory $\mathcal{O}_A$ of $\mathcal{B}_A$ whose objects are right A-modules (M, r) such that



This braiding on $\mathcal{O}_A$ is given by

$c_{(M,r),(N,s)} =$  .

Note that in general $\mathcal{O}_A \neq \mathcal{B}_A$; equality occurs only when the Hopf algebra A is transparent, that is, $\tau_{A,X} = \tau^{-1}_{X,A}$ for any object X of $\mathcal{B}$.

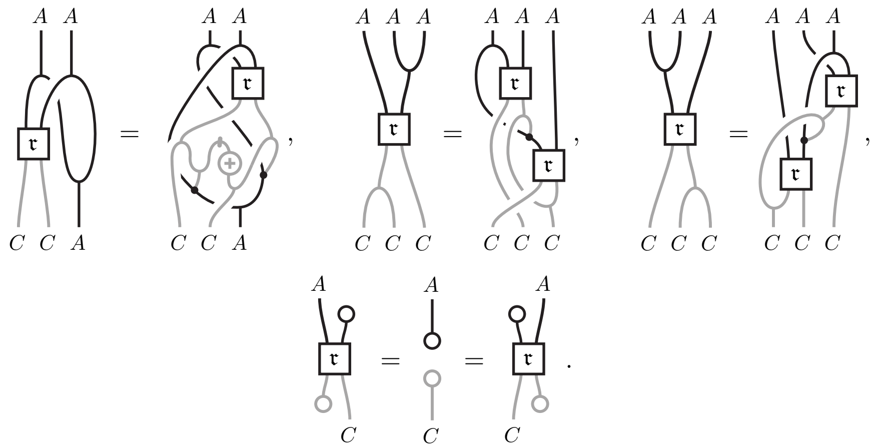
2.5. **R-matrices revisited.** Recall that a key feature of R-matrices for Hopf algebras over a field is the following (see [Dri90]): if H is finite-dimensional Hopf algebra H over a field $\Bbbk$, then R-matrices for H are in natural bijection with braidings on the category of finite-dimensional H-modules. As noted above, this bijective correspondance is lost with Majid's definition.

In [BV2], using the theories of Hopf monads and coends, we extended the notion of an R-matrix to a Hopf algebra A in braided rigid category admitting a coend, so as to preserve this bijective correspondence.

Let $\mathcal{B}$ be a braided rigid category admitting a coend C, and A be a Hopf algebra in $\mathcal{B}$. An R-*matrix* for A is a morphism

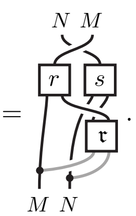
$$\mathfrak{r}\colon C \otimes C \to A \otimes A$$

in $\mathcal{B}$, which satisfies



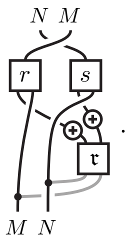
Note that for finite-dimensional Hopf algebras over a field $\Bbbk$, our definition of an R-matrix coincides with Drinfeld's definition, as in that case $C = \Bbbk$.

Theorem 2.6 ([BV2, Section 8.6])**.** *Any* R-*matrix* $\mathfrak{r}$ *for* A *defines a braiding* c *on* $\mathcal{B}_A$ *as follows: for right* A*-modules* $(M, r), (N, s)$*,*

$c_{(M,r),(N,s)} =$  .

Furthermore, the map $\mathfrak{r} \mapsto c$ *is a bijection between* R*-matrices for* A *and braidings on* $\mathcal{B}_A$

Remark 2.7. R-matrices also encode braiding on the category ${}_A\mathcal{B}$ of left A-modules. Indeed, since braidings on ${}_A\mathcal{B}$ are in bijective correspondence with braidings on $\mathcal{B}_A$ (see Remark 2.2), an R-matrix $\mathfrak{r}$ for A defines a braiding c' on ${}_A\mathcal{B}$ as follows: for left A-modules $(M, r), (N, s)$,

$c'_{(M,r),(N,s)} =$  .

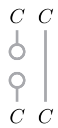
Furthermore, the map $\mathfrak{r} \mapsto c'$ is a bijection between R-matrices for A and braidings on ${}_A\mathcal{B}$.

2.6. **Quasitriangular Hopf algebras.** Let $\mathcal{B}$ be a braided rigid category admitting a coend. A *quasitriangular Hopf algebra* in $\mathcal{B}$ is a Hopf algebra in $\mathcal{B}$ endowed with an R-matrix.

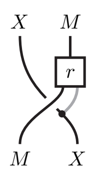
By Theorem 2.6 and Remark 2.7, if A is a quasitriangular Hopf algebra in $\mathcal{B}$, then the rigid categories $\mathcal{B}_A$ and ${}_A\mathcal{B}$ are braided.

Remark 2.8. If A is a quasitriangular Hopf algebra in $\mathcal{B}$, then ${}_A\mathcal{B}$ and $\mathcal{B}_A$ are isomorphic as braided categories. Indeed, the monoidal functor $(1_{{}_A\mathcal{B}}, c', \mathrm{id}_{\mathbb{1}})\colon {}_A\mathcal{B}^{\otimes\mathrm{op}} \to {}_A\mathcal{B}$ is a braided isomorphism (where c' is the braiding of ${}_A\mathcal{B}$), and by construction the monoidal isomorphism $F_A\colon ({}_A\mathcal{B})^{\otimes\mathrm{op}} \to \mathcal{B}_A$ of Remark 2.2 is braided.

Example 2.9. The coend C of $\mathcal{B}$ is a quasitriangular Hopf algebra in $\mathcal{B}$ with R-matrix

$\mathfrak{r} =$  ,

and so the category $\mathcal{B}_C$ of right C-modules is braided. For any right C-module (M, r) and any C-linear morphism f, set $I(M, r) = (M, \sigma)$ and $I(f) = f$ with

$\sigma_X =$  .

This defines a functor $I\colon \mathcal{B}_C \to \mathcal{Z}(\mathcal{B})$ which is a braided strict monoidal isomorphism.

2.7. **The double of a Hopf algebra.** Let $\mathcal{B}$ be a braided rigid category admitting a coend C, and let A be a Hopf algebra in $\mathcal{B}$. Set

$$D(A) = A \otimes {}^{\vee}\!A \otimes C$$

and define the product $m_{D(A)}$, the unit $u_{D(A)}$, the coproduct $\Delta_{D(A)}$, the counit $\varepsilon_{D(A)}$, the antipode $S_{D(A)}$, and the R-matrix $\mathfrak{r}_{D(A)}$ as in Figure 1.

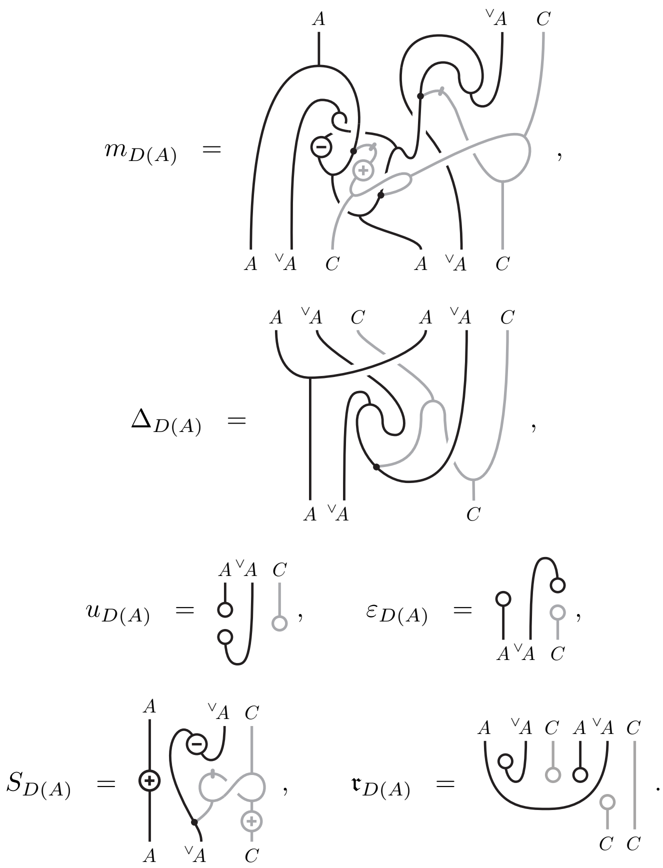


FIGURE 1. Structural morphisms of the double $D(A)$ of A

Theorem 2.10 ([BV2, Theorem 8.13])**.** *In the above notation, $D(A)$ is a quasitriangular Hopf algebra in $\mathcal{B}$, and we have isomorphisms of braided categories*

$$\mathcal{Z}(\mathcal{B}_A) \simeq \mathcal{B}_{D(A)} \simeq {}_{D(A)}\mathcal{B} \simeq \overline{\mathcal{Z}({}_A\mathcal{B})}.$$

Remark 2.11. When $\mathcal{B} = \mathrm{vect}_{\Bbbk}$ is the category of finite-dimensional vector spaces over a field $\Bbbk$, whose coend is $\Bbbk$, we recover the usual Drinfeld double and the interpretation of its category of modules in terms of the center. More precisely, let H be a finite-dimensional Hopf algebra over $\Bbbk$ and (e_i) be a basis of H with dual basis (e^i). Then $D(H) = H \otimes (H^*)^{\mathrm{cop}}$ is a quasitriangular Hopf algebra over $\Bbbk$, with R-matrix $\mathfrak{r} = \sum_i e_i \otimes \varepsilon \otimes 1_H \otimes e_i$, such that

$$\mathcal{Z}\big((\mathrm{vect}_{\Bbbk})_H\big) \simeq (\mathrm{vect}_{\Bbbk})_{D(H)} \simeq {}_{D(H)}(\mathrm{vect}_{\Bbbk}) \simeq \overline{\mathcal{Z}\big({}_H(\mathrm{vect}_{\Bbbk})\big)}$$

as braided categories.

Remark 2.12. The coend C is nothing but the quasitriangular Hopf algebra $D(\mathbb{1})$ (see Example 2.9).

2.8. **Yetter-Drinfeld modules.** In the previous section, we have seen that, given a Hopf algebra A in a braided category $\mathcal{B}$, the center of the category of A-modules in $\mathcal{B}$ can be described as the category of modules of a Hopf algebra in $\mathcal{B}$ under certain assumptions (*viz.* $\mathcal{B}$ is rigid and has a coend).

However we also have a natural monoidal functor

$$\mathcal{U} : \mathcal{Z}({}_A\mathcal{B}) \to \mathcal{Z}(\mathcal{B}),$$

defined by $\mathcal{U}((M,r),\sigma) = (M,s)$, where s is the half-braiding of $\mathcal{B}$ given by $s_Y = \sigma_{(Y,\varepsilon\otimes Y)}$, ε being the counit of A.

Is there an algebraic description of the centrer of the category of A- modules in terms of the center of $\mathcal{B}$? Is the center of the category of A- modules the category of modules of a Hopf algebra in the center of $\mathcal{B}$?

We will show in this section that the answer to the first question is positive: $\mathcal{Z}({}_A\mathcal{B})$ can be described as the category of Yetter-Drinfeld modules of A viewed as a Hopf algebra of $\mathcal{B}$, but the answer to the second question is negative. However, we will see in Section 4 that the center of the category of A- modules is the category of modules of a quasitriangular Hopf monad on the center of $\mathcal{B}$.

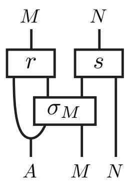
Let A be a bialgebra in a braided category $\mathcal{B}$ with braiding τ. A *Yetter-Drinfeld* of A in $\mathcal{B}$ is a an object M of $\mathcal{B}$, endowed with a left A-action $r\colon A\otimes X \to M$ and a left A-coaction $\delta\colon A\otimes X \to M$, such that

$$(m\otimes M)(A\otimes\tau_{M,A})(\delta r\otimes M)(A\otimes\tau_{A,M})(\Delta\otimes M) = (m\otimes r)(A\otimes\tau_{A,A}\otimes M)(\Delta\otimes\delta),$$

where m is the product of A and Δ its coproduct.

These Yetter-Drinfeld modules are the objects of a category ${}_A\mathcal{YD}(\mathcal{B})$, whose morphisms are morphisms in $\mathcal{B}$ which are A- linear and A- colinear. This category is monoidal, the tensor product of two Yetter-Drinfeld modules being the usual tensor product as left A-modules and as left A-comodules.

Now let $\mathcal{C}$ be a monoidal category, and let $\mathbb{A} = (A,\sigma)$ be a Hopf algebra in the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$. The category ${}_A\mathcal{C}$ of left A-modules in $\mathcal{C}$ has a monoidal structure, denoted by ${}_{\mathbb{A}}\mathcal{C}$, with monoidal product $(M,r)\otimes(N,s) = (M\otimes N,\omega)$, where

$\omega =$ ,

and monoidal unit $(\mathbb{1},\varepsilon)$, where ε is the counit of $\mathbb{A}$.

Theorem 2.13. *Let $\mathcal{C}$ be a monoidal category and let $\mathbb{A}$ be a Hopf algebra in the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$. There exists an isomorphism of monoidal categories*

$$\phi\colon {}_{\mathbb{A}}\mathcal{YD}(\mathcal{Z}(\mathcal{C})) \to \mathcal{Z}({}_{\mathbb{A}}\mathcal{C}),$$

defined by $((X,s),r,\delta) \mapsto ((X,r),\sigma)$, where σ is a half-braiding of ${}_{\mathbb{A}}\mathcal{C}$ given by $\sigma_{N,t} = (t\otimes M)(A\otimes s_N)(\delta\otimes N)$.

Proof. By straightforward computation. The inverse of ϕ is given by $((M,r),\sigma) \mapsto ((M,s),r,\delta)$, with $s_Y = \sigma_{Y,\varepsilon\otimes Y}$ for Y object of $\mathcal{C}$, and $\delta = \sigma_{(A,m)}(M\otimes u)$, where u is the unit, and ε the counit, of $\mathbb{A}$. □

In particular, let $\mathcal{B}$ be a braided category and A be a Hopf algebra in $\mathcal{B}$. Via the canonical braided embedding $\mathcal{B} \to \mathcal{Z}(\mathcal{B})$, $x \mapsto (x,\tau_{x,-})$, we may view A as a Hopf algebra in $\mathcal{Z}(\mathcal{B})$. Applying Theorem 2.13, we see that $\mathcal{Z}({}_A\mathcal{B})$ is isomorphic to ${}_A\mathcal{YD}(\mathcal{Z}(\mathcal{B}))$.

Proposition 2.14. *Let G be a non-trivial finite commutative group, let $\mathcal{B}$ be the category of G-graded vector spaces over a field $\Bbbk$, endowed with its canonical structure of monoidal symmetric category, and let $A = \Bbbk[G]$, viewed as a Hopf algebra in $\mathcal{B}$ with the obvious G-graduation. Then the natural monoidal functor $\mathcal{U} : \mathcal{Z}({}_A\mathcal{B}) \to \mathcal{Z}(\mathcal{B})$ is not essentially surjective. In particular, it is impossible to find a Hopf algebra B in $\mathcal{Z}(\mathcal{B})$, or its center, and a monoidal equivalence $F\colon {}_B\mathcal{Z}(\mathcal{B}) \to \mathcal{Z}({}_A\mathcal{B})$ such that the diagram*

$$\begin{array}{ccc} {}_B\mathcal{Z}(\mathcal{B}) & \xrightarrow{\quad F \quad} & \mathcal{Z}({}_A\mathcal{B}) \\ & {\scriptstyle f.\,f.}\searrow \quad \swarrow{\scriptstyle \mathcal{U}} & \\ & \mathcal{Z}(\mathcal{B}) & \end{array}$$

commutes up to monoidal isomorphism.

Proof. The 'in particular' assertion follows from the observation that if $\mathbb{A}$ is a Hopf algebra in the center of a monoidal category $\mathcal{C}$, then the forgetful functor ${}_{\mathbb{A}}\mathcal{C} \to \mathcal{C}$ admits a strict monoidal section, given by $c \mapsto (c, \varepsilon \otimes c)$, where ε is the counit of $\mathbb{A}$, and it is therefore essentially surjective.

For the main assertion, by Theorem 2.13 it is enough to show that the forgetful functor ${}_A\mathcal{YD}(\mathcal{Z}(\mathcal{B})) \to \mathcal{Z}(\mathcal{B})$ is not essentially surjective. Let g be an element of G, and let (g, s) be a half-braiding of $\mathcal{B}$. Such half-braidings are in one-to-one correspondance with characters of G. Let $((g', s'), r, \delta)$ be a Yetter-Drinfeld module on A above (g, s). We may assume $(g', s') = (g, s)$. Note that all A- actions or coactions on g are trival, that is, we have $\delta = u \otimes g$ and $r = \varepsilon \otimes g$, where u is the unit, and ε the counit of A. In that case, the Yetter-Drinfeld compatibility axiom implies that s is induced by the symmetry. Thus, if the character defining s is not trivial (g, s) doesn't belong to the essential image of the forgetful functor. □

3. Hopf monads

In this section we recall the definition of Hopf monads, and we list several results. See [BLV] for more a more detailed discussion, and for the proofs.

3.1. **Hopf monads and their modules.** Let $\mathcal{C}$ be a category. A *monad* on $\mathcal{C}$ is a monoid in the category of endofunctors of $\mathcal{C}$, that is, a triple (T, μ, η) consisting of a functor $T\colon \mathcal{C} \to \mathcal{C}$ and two natural transformations

$$\mu = \{\mu_X \colon T^2(X) \to T(X)\}_{X \in \mathcal{C}} \quad \text{and} \quad \eta = \{\eta_X \colon X \to T(X)\}_{X \in \mathcal{C}}$$

called the *product* and the *unit* of T, such that for any object X of $\mathcal{C}$,

$$\mu_X T(\mu_X) = \mu_X \mu_{T(X)} \quad \text{and} \quad \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X).$$

Given a monad $T = (T, \mu, \eta)$ on $\mathcal{C}$, a T- module in $\mathcal{C}$ is a pair (M, r) where M is an object of $\mathcal{C}$ and $r\colon T(M) \to M$ is a morphism in $\mathcal{C}$ such that $rT(r) = r\mu_M$ and $r\eta_M = \mathrm{id}_M$. A morphism from a T- module (M, r) to a T- module (N, s) is a morphism $f\colon M \to N$ in $\mathcal{C}$ such that $fr = sT(f)$. This defines the *category $\mathcal{C}^T$ of T-modules in $\mathcal{C}$* with composition induced by that in $\mathcal{C}$. We define a forgetful functor $U_T\colon \mathcal{C}^T \to \mathcal{C}$ by $U_T(M, r) = M$ and $U_T(f) = f$. The forgetful functor U_T has a left adjoint $F_T\colon \mathcal{C} \to \mathcal{C}^T$, called the free module functor, defined by $F_T(X) = (T(X), \mu_X)$ and $F_T(f) = T(f)$. Note that if $\mathcal{C}$ is $\Bbbk$-additive and T is

$\Bbbk$-linear (that is, T induces $\Bbbk$-linear maps on Hom spaces), then the category $\mathcal{C}^T$ is $\Bbbk$-additive and the functors U_T and F_T are $\Bbbk$-linear.

Let $\mathcal{C}$ be a monoidal category. A *bimonad* on $\mathcal{C}$ is a monoid in the category of comonoidal endofunctors of $\mathcal{C}$. In other words, a bimonad on $\mathcal{C}$ is a monad (T, μ, η) on $\mathcal{C}$ such that the functor $T\colon \mathcal{C} \to \mathcal{C}$ and the natural transformations μ and η are comonoidal, that is, T comes equipped with a natural transformation $T_2 = \{T_2(X,Y)\colon T(X \otimes Y) \to T(X) \otimes T(Y)\}_{X,Y \in \mathcal{C}}$ and a morphism $T_0\colon T(\mathbb{1}) \to \mathbb{1}$ such that

$$\begin{aligned}
&\big(\mathrm{id}_{T(X)} \otimes T_2(Y,Z)\big)T_2(X, Y \otimes Z) = \big(T_2(X,Y) \otimes \mathrm{id}_{T(Z)}\big)T_2(X \otimes Y, Z);\\
&(\mathrm{id}_{T(X)} \otimes T_0)T_2(X, \mathbb{1}) = \mathrm{id}_{T(X)} = (T_0 \otimes \mathrm{id}_{T(X)})T_2(\mathbb{1}, X);\\
&T_2(X,Y)\mu_{X \otimes Y} = (\mu_X \otimes \mu_Y)T_2(T(X), T(Y))T(T_2(X,Y));\\
&T_2(X,Y)\eta_{X \otimes Y} = \eta_X \otimes \eta_Y.
\end{aligned}$$

For any bimonad T on $\mathcal{C}$, the category of T-modules $\mathcal{C}^T$ has a monoidal structure with unit object $(\mathbb{1}, T_0)$ and with tensor product

$$(M,r) \otimes (N,s) = \big(M \otimes N, (r \otimes s)\, T_2(M,N)\big).$$

Note that the forgetful functor $U_T\colon \mathcal{C}^T \to \mathcal{C}$ is strict monoidal.

A *quasitriangular bimonad* on $\mathcal{C}$ is a bimonad T on $\mathcal{C}$ equipped with an R-matrix, that is, a natural transformation

$$R = \{R_{X,Y}\colon X \otimes Y \to T(Y) \otimes T(X)\}_{X,Y \in \mathcal{C}}$$

satisfying appropriate axioms which ensure that the natural transformation $\tau = \big\{\tau_{(M,r),(N,s)}\big\}_{(M,r),(N,s) \in \mathcal{C}^T}$ defined by

$$\tau_{(M,r),(N,s)} = (s \otimes r)R_{M,N}\colon (M,r) \otimes (N,s) \to (N,s) \otimes (M,r)$$

form a braiding in the category $\mathcal{C}^T$ of T-modules, see [BV1].

Given a bimonad (T, μ, η) on $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, one defines the *left fusion operator*

$$H^l_{X,Y} = (T(X) \otimes \mu_Y)T_2(X, T(Y))\colon T(X \otimes T(Y)) \to T(X) \otimes T(Y)$$

and the *right fusion operator*

$$H^r_{X,Y} = (\mu_X \otimes T(Y))T_2(T(X), Y)\colon T(T(X) \otimes Y) \to T(X) \otimes T(Y).$$

A *Hopf monad* on $\mathcal{C}$ is a bimonad on $\mathcal{C}$ whose left and right fusion operators are isomorphisms for all objects X, Y of $\mathcal{C}$. When $\mathcal{C}$ is a rigid category, a bimonad T on $\mathcal{C}$ is a Hopf monad if and only if the category $\mathcal{C}^T$ is rigid. The structure of a rigid category in $\mathcal{C}^T$ can then be encoded in terms of natural transformations

$$s^l = \{s^l_X\colon T({}^\vee T(X)) \to {}^\vee X\}_{X \in \mathcal{C}} \quad \text{and} \quad s^r = \{s^r_X\colon T(T(X)^\vee) \to X^\vee\}_{X \in \mathcal{C}}$$

called the *left and right antipodes.* They are computed from the fusion operators:

$$\begin{aligned}
s^l_X &= \big(T_0 T(\mathrm{ev}_{T(X)})(H^l_{{}^\vee T(X),X})^{-1} \otimes {}^\vee\eta_X\big)\big(\mathrm{id}_{T({}^\vee T(X))} \otimes \mathrm{coev}_{T(X)}\big);\\
s^r_X &= \big(\eta^\vee_X \otimes T_0 T(\widetilde{\mathrm{ev}}_{T(X)})(H^r_{X,T(X)^\vee})^{-1}\big)\big(\widetilde{\mathrm{coev}}_{T(X)} \otimes \mathrm{id}_{T(T(X)^\vee)}\big).
\end{aligned}$$

The left and right duals of any T-module (M,r) are then defined by

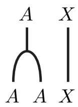
$${}^\vee(M,r) = ({}^\vee M, s^l_M T({}^\vee r) \quad \text{and} \quad (M,r)^\vee = (M^\vee, s^r_M T(r^\vee)).$$

3.2. **Centralizers.** Let T be a Hopf monad on a rigid category $\mathcal{C}$. We say that T is *centralizable* if, for any object X of $\mathcal{C}$, the coend

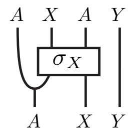
$$Z_T(X) = \int^{Y\in\mathcal{C}} {}^{\vee}T(Y) \otimes X \otimes Y$$

exists (see [BV2]). In that case, the assignment $X \mapsto Z_T(X)$ is a Hopf monad on $\mathcal{C}$, called the centralizer of T and denoted by Z_T. In particular, we say that $\mathcal{C}$ is *centralizable* if the identity functor $\mathrm{id}_\mathcal{C}$ is centralizable. In that case, its centralizer $Z = Z_{\mathrm{id}\mathcal{C}}$ is a quasi-triangular Hopf monad on $\mathcal{C}$, called the *centralizer of* $\mathcal{C}$. Moreover, there is a canonical isomorphism of braided categories $\mathcal{C}^Z \xrightarrow{\sim} \mathcal{Z}(\mathcal{C})$, see [?].

3.3. **Hopf algebras of the center define Hopf monads.** Let $\mathcal{C}$ be a monoidal category. Any bialgebra $\mathbb{A} = (A, \sigma)$ of the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ gives rise to a bimonad on $\mathcal{C}$, denoted by $A\otimes_\sigma ?$. It is defined by $A\otimes ?$ as a functor, with the monad structure defined by

$\mu_X = m \otimes X =$  and $\eta_X = u \otimes X =$  ,

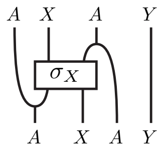
where m and u are the product and unit of A, and endowed with the comonoidal structure:

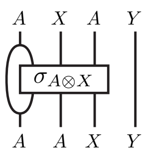
$(A\otimes_\sigma ?)_2(X,Y) = (A \otimes \sigma_X)(\Delta \otimes X) \otimes Y =$  , $(A\otimes_\sigma ?)_0 = \varepsilon =$  ,

where Δ and ε denote the coproduct and counit of (A, σ).

The monoidal category $\mathcal{C}^{A\otimes_\sigma ?}$ is the monoidal category ${}_\mathbb{A}\mathcal{C}$ encountered previously.

Given a bialgebra (A, σ) of $\mathcal{Z}(\mathcal{C})$, The left and right fusion operators of the monad $A\otimes_\sigma ?$ are:

$H^l_{X,Y} = (A \otimes X \otimes m)(A \otimes \sigma_X \otimes A)(\Delta \otimes X \otimes A) \otimes Y =$  ,

$H^r_{X,Y} = (m \otimes X \otimes A)(A \otimes \sigma_{A\otimes X})(\Delta \otimes A \otimes X) \otimes Y =$  .

Proposition 3.1. *Let (A, σ) be a bialgebra in $\mathcal{Z}(\mathcal{C})$. Then the corresponding bimonad $A\otimes_\sigma ?$ is a Hopf monad if and only if (A, σ) is a Hopf algebra.*

3.4. **Characterization of representable Hopf monads.** Let $\mathcal{C}$ be a monoidal category. A bimonad T on $\mathcal{C}$ is *augmented* if it is endowed with an *augmentation*, that is, a bimonad morphism $e\colon T \to 1_\mathcal{C}$.

Augmented bimonads on $\mathcal{C}$ form a category $\mathrm{BiMon}(\mathcal{C})/1_\mathcal{C}$, whose objects are augmented bimonads on $\mathcal{C}$, and morphisms between two augmented bimonads (T, e) and (T', e') are morphisms of bimonads $f\colon T \to T'$ such that $e'f = e$.

If (A,σ) is a bialgebra of the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$, the bimonad $A\otimes_\sigma ?$ is augmented, with augmentation $e=\varepsilon\otimes ?\colon A\otimes_\sigma ? \to 1_\mathcal{C}$, where ε is the counit of (A,σ). Hence a functor $\mathrm{BiAlg}(\mathcal{Z}^{\mathrm{lax}}(\mathcal{C})) \to \mathrm{BiMon}(\mathcal{C})/1_\mathcal{C}$ which, according to Proposition 3.1, induces by restriction a functor

$$\mathfrak{R}\colon \left\{ \begin{array}{ccc} \mathrm{HopfAlg}(\mathcal{Z}(\mathcal{C})) & \to & \mathrm{HopfMon}(\mathcal{C})/1_\mathcal{C} \\ (A,\sigma) & \mapsto & (A\otimes_\sigma ?, \varepsilon\otimes ?) \end{array} \right.$$

where $\mathrm{HopfMon}(\mathcal{C})/1_\mathcal{C}$ denotes the category of augmented Hopf monads on $\mathcal{C}$.

Theorem 3.2. *The functor $\mathfrak{R}$ is an equivalence of categories.*

In other words, Hopf monads representable by Hopf algebras of the center are nothing but augmented Hopf monads. Not all Hopf monads are of this kind:

Remark 3.3. Let $\mathcal{C}$ be a centralizable rigid category, and let $Z=Z_{\mathrm{id}_\mathcal{C}}$ be its centralizer, which is a quasitriangular Hopf monad on $\mathcal{C}$. Then augmentations of Z are in one-to-one correspondance with braidings on $\mathcal{C}$. In particular if $\mathcal{C}$ is not braided, then Z is not representable by a Hopf algebra of the center of $\mathcal{C}$. For example, let $\mathcal{C}=G\text{-}\mathrm{vect}$ be the category of finite-dimensional G-graded vector spaces over a field $\Bbbk$ for some finite group G. It is centralisable, and its centralizer is representable by a Hopf algebra of the center of $\mathcal{C}$ if and only if G is abelian (see [BV2, Remark 9.2]).

3.5. **Functoriality of categories of modules.** Let $\mathcal{C}$ be a category. If T is a monad on $\mathcal{C}$, then $(\mathcal{C}^T, U_T)$ is a category over $\mathcal{C}$, that is, an object of $\mathrm{Cat}/\mathcal{C}$. Any morphism $f\colon T\to P$ of monads on $\mathcal{C}$ induces a functor

$$f^*\colon \left\{ \begin{array}{ccc} \mathcal{C}^P & \to & \mathcal{C}^T \\ (M,r) & \mapsto & (M, rf_M) \end{array} \right.$$

over $\mathcal{C}$, that is, $U_T f^* = U_P$. Moreover, any functor $F\colon \mathcal{C}^P \to \mathcal{C}^T$ over $\mathcal{C}$ is of this form. This construction defines a fully faithful functor

$$\left\{ \begin{array}{ccc} \mathrm{Mon}(\mathcal{C})^{\mathrm{op}} & \to & \mathrm{Cat}/\mathcal{C} \\ T & \mapsto & (\mathcal{C}^T, U_T) \end{array} \right.$$

If $f\colon T\to P$ is a morphism of bimonads on a monoidal category $\mathcal{C}$, then $f^*\colon \mathcal{C}^P \to \mathcal{C}^T$ is a strict monoidal functor over $\mathcal{C}$, and any strong monoidal functor $F\colon \mathcal{C}^P\to\mathcal{C}^T$ over $\mathcal{C}$ (that is, such that $U_T F = U_P$ as monoidal functors) is of this form (see [BV1, Lemma 2.9]). Hence a fully faithful functor

$$\mathrm{BiMon}(\mathcal{C})^{\mathrm{op}} \to \mathrm{MonCat}/\mathcal{C}.$$

3.6. **Cross products.** Let T be a monad on a category $\mathcal{C}$. If Q is a monad on the category $\mathcal{C}^T$ of T-modules, the monad of the composite adjunction

$$(\mathcal{C}^T)^Q \underset{F_Q}{\overset{U_Q}{\rightleftarrows}} \mathcal{C}^T \underset{F_T}{\overset{U_T}{\rightleftarrows}} \mathcal{C}$$

is called the *cross product of T by Q* and denoted by $Q\rtimes T$ (see [BV2, Section 3.7]). As an endofunctor of $\mathcal{C}$, $Q\rtimes T = U_T Q F_T$. The product p and unit e of $Q\rtimes T$ are:

$$p = q_{F_T} Q(\varepsilon_{QF_T}) \quad \text{and} \quad e = v_{F_T}\eta,$$

where q and v are the product and the unit of Q, and η and ε are the unit and counit of the adjunction (F_T, U_T).

Note that the composition of two monadic functors is not monadic in general. However:

Proposition 3.4. *If T is a monad on a category $\mathcal{C}$ and Q is a monad on $\mathcal{C}^T$ which preserves reflexive coequalizers, then the forgetful functor $U_T U_Q$ is monadic with monad $Q \rtimes T$. Moreover the comparison functor*

$$K\colon (\mathcal{C}^T)^Q \to \mathcal{C}^{Q \rtimes T}$$

is an isomorphism of categories.

If T is a bimonad on a monoidal category $\mathcal{C}$ and Q is a bimonad on $\mathcal{C}^T$, then $Q \rtimes T = U_T Q F_T$ is a bimonad on $\mathcal{C}$ (since a composition of comonoidal adjunctions is a comonoidal adjunction), with comonoidal structure given by:

$$\begin{aligned} (Q \rtimes T)_2(X, Y) &= Q_2\big(F_T(X), F_T(Y)\big)\, Q\big((F_T)_2(X, Y)\big), \\ (Q \rtimes T)_0 &= Q_0\, Q\big((F_T)_0\big). \end{aligned}$$

In that case the comparison functor $K\colon (\mathcal{C}^T)^Q \to \mathcal{C}^{Q \rtimes T}$ is strict monoidal.

The cross product is functorial in Q: the assignment $Q \mapsto Q \rtimes T$ defines a functor $? \rtimes T\colon \mathrm{BiMon}(\mathcal{C}^T) \to \mathrm{BiMon}(\mathcal{C})$.

Proposition 3.5. *The cross product of two Hopf monads is a Hopf monad.*

Example 3.6. Let H be a bialgebra over a field $\Bbbk$ and A be a H-module algebra, that is, an algebra in the monoidal category ${}_H\mathrm{Mod}$ of left H-modules. In this situation, we may form the cross product $A \rtimes H$, which is a $\Bbbk$-algebra (see [Ma2]). Recall that $H \otimes ?$ is a monad on $\mathrm{Vect}_\Bbbk$ and $A \otimes ?$ is a monad on ${}_H\mathrm{Mod}$. Then:

$$(A \otimes ?) \rtimes (H \otimes ?) = (A \rtimes H) \otimes ?$$

as monads. Moreover, if H is a quasitriangular Hopf algebra and A is a H-module Hopf algebra, that is, a Hopf algebra in the braided category ${}_H\mathrm{Mod}$, then $A \rtimes H$ is a Hopf algebra over $\Bbbk$, and $(A \otimes ?) \rtimes (H \otimes ?) = (A \rtimes H) \otimes ?$ as Hopf monads.

3.7. **Cross quotients.** Let $f\colon T \to P$ be a morphism of monads on a category $\mathcal{C}$. We say that f is *cross quotientable* if the functor $f^*\colon \mathcal{C}^P \to \mathcal{C}^T$ is monadic. In that case, the monad of f^* (on $\mathcal{C}^T$) is called the *cross quotient* of f and is denoted by $P \mathbin{\div\!\!\!|}_f T$ or simply $P \mathbin{\div\!\!\!|} T$. Note that the comparison functor

$$\begin{array}{ccc} \mathcal{C}^P & \xrightarrow{\ K\ } & (\mathcal{C}^T)^{P \mathbin{\div\!\!\!|} T} \\ & {\scriptstyle f^*}\searrow \qquad \swarrow{\scriptstyle U_{P \mathbin{\div\!\!\!|} T}} & \\ & \mathcal{C}^T & \end{array}$$

is then an isomorphism of categories.

A morphism $f\colon T \to P$ of monads on $\mathcal{C}$ is cross quotientable whenever $\mathcal{C}$ admits coequalizers of reflexive pairs and P preserve them.

A cross quotient of bimonads is a bimonad: let $f\colon T \to P$ be a cross quotientable morphism of bimonads on a monoidal category $\mathcal{C}$. Since f^* is strong monoidal, $P \mathbin{\div\!\!\!|}_f T$ is a bimonad on $\mathcal{C}^T$ and the comparison functor $K\colon \mathcal{C}^P \to (\mathcal{C}^T)^{P \mathbin{\div\!\!\!|}_f T}$ is an isomorphism of monoidal categories.

The cross quotient is inverse to the cross product in the following sense:

Proposition 3.7. *Let T be a (bi)monad on a (monoidal) category $\mathcal{C}$.*

(a) *If $T \to P$ is a cross quotientable morphism of (bi)monads on $\mathcal{C}$, then*

$$(P \div\!\!\dashv T) \rtimes T \simeq P$$

as (bi)monads.

(b) *Let Q be a (bi)monad on $\mathcal{C}^T$ such that $U_T U_Q$ is monadic. Then the unit of Q defines a cross quotientable morphism of (bi)monads $T \to Q \rtimes T$ and*

$$(Q \rtimes T) \div\!\!\dashv T \simeq Q$$

as (bi)monads.

Remark 3.8. Let T be a bimonad on a monoidal category $\mathcal{C}$. Let $\mathrm{BiMon}(\mathcal{C}^T)_m$ be the full subcategory of $\mathrm{BiMon}(\mathcal{C}^T)$ whose objects are monads Q on $\mathcal{C}^T$ such that $U_T U_Q$ is monadic. Let $T\backslash\mathrm{BiMon}(\mathcal{C})_q$ be the full subcategory of $T\backslash\mathrm{BiMon}(\mathcal{C})$ whose objects are quotientable morphisms of bimonads from T. Then the functor

$$\begin{cases} \mathrm{BiMon}(\mathcal{C}^T) & \to \quad T\backslash\mathrm{BiMon}(\mathcal{C}) \\ \qquad Q & \mapsto \quad (Q, T \to Q \rtimes T) \end{cases}$$

induces an equivalence of categories $\mathrm{BiMon}(\mathcal{C}^T)_m \simeq T\backslash\mathrm{BiMon}(\mathcal{C})_q$, with quasi-inverse given by $(T \to P) \mapsto (P \div\!\!\dashv T)$.

Proposition 3.9. *Let $\mathcal{C}$ be a monoidal category admitting reflexive coequalizers, and whose monoidal product preserves reflexive coequalizers. Let T and P be two Hopf monads on $\mathcal{C}$ which preserve reflexive coequalizers. Then any morphism of bimonads $T \to P$ is cross quotientable and $P \div\!\!\dashv T$ is a Hopf monad.*

Example 3.10. Let $f\colon L \to H$ be a morphism of Hopf algebras over a field $\Bbbk$, so that H becomes a L-bimodule by setting $\ell \cdot h \cdot \ell' = f(\ell) h f(\ell')$. The morphism f induces a morphism of Hopf monads on $\mathrm{Vect}_\Bbbk$:

$$f \otimes_\Bbbk ?\colon L \otimes_\Bbbk ? \to H \otimes_\Bbbk ?$$

which is cross quotientable, and $(H\otimes ?) \div\!\!\dashv (L \otimes ?)$ is a $\Bbbk$-linear Hopf monad on the monoidal category ${}_L\mathrm{Mod}$ given by $N \mapsto H \otimes_L N$. (Note that in general this cross quotient is not representable by a Hopf algebra of the center of the category of left L-modules, see Remark 3.12). This construction defines an equivalence of categories

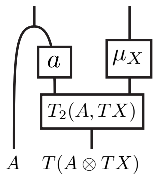
$$L\backslash\mathrm{HopfAlg}_\Bbbk \to \mathrm{HopfMon}_\Bbbk({}_L\mathrm{Mod}),$$

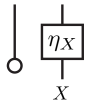
where $L\backslash\mathrm{HopfAlg}_\Bbbk$ is the category of Hopf $\Bbbk$-algebras under L and $\mathrm{HopfMon}_\Bbbk({}_L\mathrm{Mod})$ is the category of $\Bbbk$-linear Hopf monads on ${}_L\mathrm{Mod}$.

3.8. **Bosonization.** Let $\mathcal{C}$ be a monoidal category. Given a Hopf monad (T, μ, η) on $\mathcal{C}$ and a central Hopf algebra $(\mathbb{A}, \sigma)$ of $\mathcal{C}^T$ (that is, a Hopf algebra in the center $\mathcal{Z}(\mathcal{C}^T)$ of $\mathcal{C}^T$), set:

$$\mathbb{A} \rtimes_\sigma T = (\mathbb{A} \otimes_\sigma ?) \rtimes T.$$

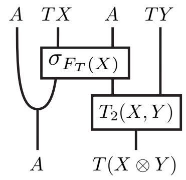
As a cross product of Hopf monads, $\mathbb{A} \rtimes_\sigma T$ is a Hopf monad on $\mathcal{C}$ (see Proposition 3.5). Set $\mathbb{A} = (A, a)$, where $A = U_T(\mathbb{A})$ and a is the T-action on A. As an endofunctor of $\mathcal{C}$, $\mathbb{A} \rtimes_\sigma T = A \otimes T$. The product p and unit v of $\mathbb{A} \rtimes_\sigma T$ are:

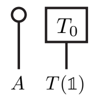
$p_X =$ 

and $v_X =$ 

.

The comonoidal structure of $\mathbb{A} \rtimes_\sigma T$ is given by:

$(\mathbb{A} \rtimes_\sigma T)_2(X,Y) =$ 

and $(\mathbb{A} \rtimes_\sigma T)_0 =$ 

.

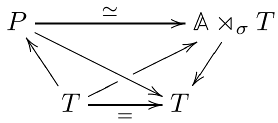
Denoting by u and ε the unit and counit of $(\mathbb{A}, \sigma)$, the morphisms

$$\iota = u \otimes T \colon T \to \mathbb{A} \rtimes_\sigma T \quad \text{and} \quad \pi = \varepsilon \otimes T \colon \mathbb{A} \rtimes_\sigma T \to T$$

are Hopf monads morphisms such that $\pi\iota = \mathrm{id}_T$. Hence T is a retract of $\mathbb{A} \rtimes_\sigma T$ in the category $\mathrm{HopfMon}(\mathcal{C})$ of Hopf monads on $\mathcal{C}$.

Conversely, under exactness assumptions, a Hopf monad which admits T as a retract is of the form $\mathbb{A} \rtimes_\sigma T$ for some central Hopf algebra $(\mathbb{A}, \sigma)$ of $\mathcal{C}^T$.

Theorem 3.11. *Let P and T be Hopf monads on a monoidal category $\mathcal{C}$ such that T is a retract of P. Assume that reflexive coequalizers exist in $\mathcal{C}$ and are preserved by P and the monoidal product of $\mathcal{C}$. Then there exists a central Hopf algebra $(\mathbb{A}, \sigma)$ of $\mathcal{C}^T$ and an isomorphism of Hopf monads $P \simeq \mathbb{A} \rtimes_\sigma T$ such that we have a commutative diagram of Hopf monads:*



Remark 3.12. Let H be a Hopf algebra over a field $\Bbbk$, and A a Hopf algebra in the braided category of Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$. In that situation, Radford constructed a Hopf algebra $A\#H$, known as *Radford's biproduct*, or *Radford-Majid bosonization*. Radford [Rad85] (see also [Ma3]) showed that if K is a Hopf algebra on a field $\Bbbk$ and p is a projection of K, that is, an idempotent endomorphism of the Hopf algebra K, then K may be described as a biproduct as follows. Denote by H the image of p, which is a Hopf subalgebra of K. Then there exists a Hopf algebra A in ${}^H_H\mathcal{YD}$ such that $K = A\#H$. Corollary 3.11 generalizes Radford's theorem. Indeed, in the situation of the theorem, the Hopf monad $H\otimes ?$ is a retract of the Hopf monad $K\otimes ?$ on $\mathrm{Vect}_\Bbbk$. Hence, by Corollary 3.11, there exists a Hopf algebra $(\mathbb{A}, \sigma)$ in $\mathcal{Z}({}_H\mathrm{Mod})$ such that $K\otimes = \mathbb{A} \rtimes_\sigma (H\otimes ?)$. Identifying the center of ${}_H\mathrm{Mod}$ with the category of Yetter-Drinfeld modules, we view $(\mathbb{A}, \sigma)$ as a Hopf algebra A in ${}^H_H\mathcal{YD}$. Then $K\otimes ? = \mathbb{A} \rtimes_\sigma (H\otimes ?) = A\#H\otimes ?$ as Hopf monad, and so $K = A\#H$.

4. Monadicity of the canonical functor $\mathcal{Z}(\mathcal{B}_A) \to \mathcal{Z}(\mathcal{B})$

Let A be a Hopf algebra in a braided rigid category $\mathcal{B}$. Remark that any object X of $\mathcal{B}$ has a trivial right A-action given by $\mathrm{id}_X \otimes \varepsilon \colon X \otimes A \to X$, where ε is the counit of A. This defines a functor

$$\mathcal{U} \colon \mathcal{Z}(\mathcal{B}_A) \to \mathcal{Z}(\mathcal{B}),$$

by setting $\mathcal{U}((M,r),\gamma) = (M, \sigma = \{\sigma_X = \gamma_{(X, \mathrm{id}_X \otimes \varepsilon_A)}\}_{X \in \mathrm{Ob}(\mathcal{B})})$ on objects and $\mathcal{U}(f) = f$ on morphisms. Then $\mathcal{U}$ is a strict monoidal functor. In this section, we prove that $\mathcal{U}$ is monadic and we explicit its associated quasitriangular Hopf monad.

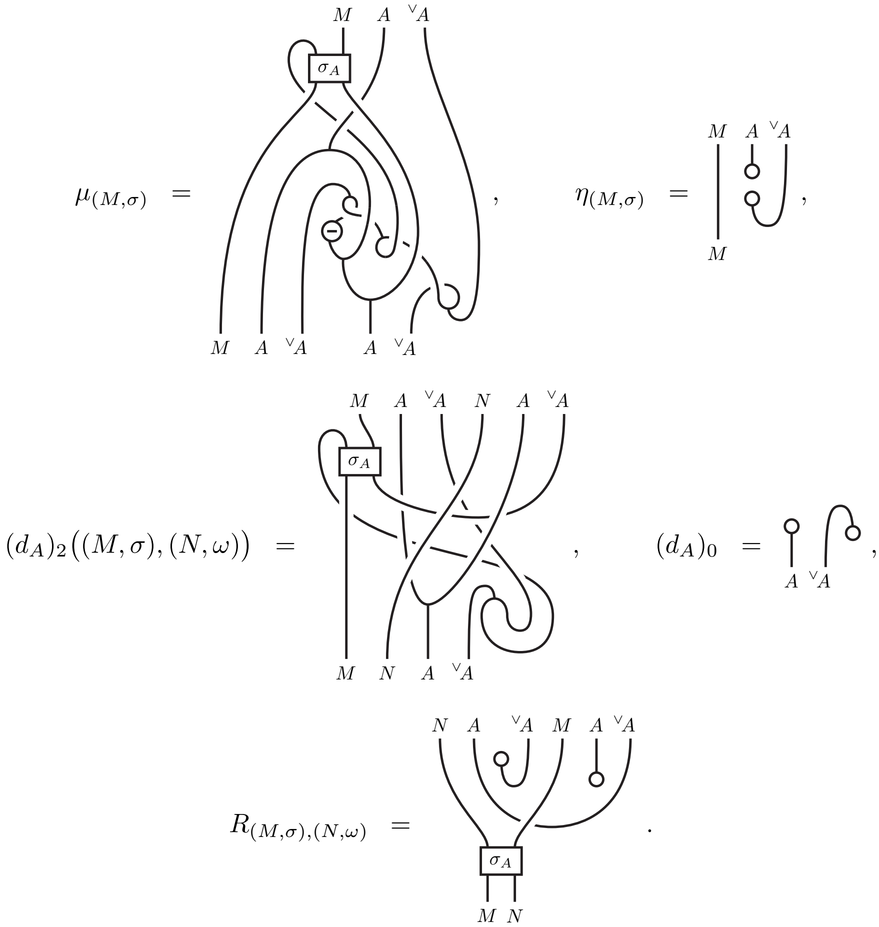
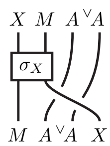


FIGURE 2. Structural morphisms of d_A

For any object (M,σ) of $\mathcal{Z}(\mathcal{B})$, set

$$d_A(M,\sigma) = \big(M \otimes A \otimes {}^\vee A, \varsigma = \{\varsigma_X\}_{X \in \mathrm{Ob}(\mathcal{B})}\big) \quad \text{with} \quad \varsigma_X =$$



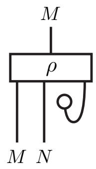
.

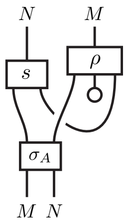
For any morphism f in $\mathcal{Z}(\mathcal{B})$, set $d_A(f) = f \otimes \mathrm{id}_{A\otimes {}^\vee A}$. Then d_A is clearly an endofunctor of $\mathcal{Z}(\mathcal{B})$.

Theorem 4.1. *The endofunctor d_A is a quasitriangular Hopf monad on $\mathcal{Z}(\mathcal{B})$, with product μ, unit η, comonoidal structure, and* R*-matrix R given in Figure 2. Furthermore the functor*

$$\left\{ \begin{array}{ccc} \mathcal{Z}(\mathcal{B})^{d_A} & \to & \mathcal{Z}(\mathcal{B}_A) \\ \big((M,\sigma),\rho\big) & \mapsto & \big((M,r),\gamma\big) \\ f & \mapsto & f \end{array} \right.$$

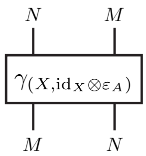
where

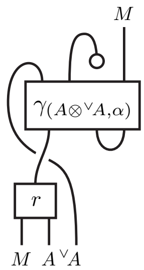
$r =$ 

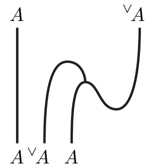
and $\gamma_{(N,s)} =$ 

,

is a braided strict monoidal isomorphism, with inverse given by

$\sigma_X =$ 

and $\rho =$ 

where $\alpha =$ 

.

Proof. By direct computation; one constructs d_A as the cross-quotient of the Hopf monad $? \otimes D(A)$ by the Hopf monad $? \otimes C$. □

Remark 4.2. From Proposition 2.14, we see that the Hopf monad d_A is not representable by a Hopf algebra in general. This can also be verified by hand, showing that, with the same category $\mathcal{B}$ and algebra A as in the Proposition, the Hopf monad d_A admits no augmentation.

Corollary 4.3. *If $\mathcal{B}$ has a coend C, then*

$$? \otimes D(A) = d_A \rtimes (? \otimes C)$$

as quasi-triangular Hopf monads. In particular, $d_A = D(A) \div C$.

Proof. The Hopf monads $? \otimes C$, $? \otimes D(A)$, and d_A are respectively the monads of the forgetful functor $U\colon \mathcal{Z}(\mathcal{B}) \to \mathcal{B}$, the forgetful functor $U'\colon \mathcal{Z}(\mathcal{B}_A) \to \mathcal{B}$, and the functor $\mathcal{U} : \mathcal{Z}(\mathcal{B}_A) \to \mathcal{Z}(\mathcal{B})$. Since $U' = U \circ \mathcal{U}$, $? \otimes D(A)$ is the cross-product of d_A by $? \otimes C$. This can be restated by saying that d_A is the cross-quotient of $? \otimes D(A)$ by $? \otimes C$, or in short, $d_A = D(A) \div C$. □

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