Hopf (co)monads, tensor functors and exact sequences of tensor categories

Alain Bruguières

(Université Montpellier II)

based on joint works
with Alexis Virelizier and Steve Lack [BLV]
and with Sonia Natale [BN]

Conference ‘Quantum Groups’

Clermont-Ferrand August 30- September 3 2010

Conference of the ANR project *GALOISINT Quantum Groups : Galois and integration techniques*

Motivation : Tannaka theory

Motivation : Tannaka theory

Over $\mathbb{k}$ field:

Motivation : Tannaka theory

Over $\Bbbk$ field:

H Hopf algebra $\longrightarrow$ a *tensor category* $\mathcal{C} = \mathrm{comod} H$
+ a *fiber functor* $\mathcal{C} \to \mathrm{vect}$

Motivation : Tannaka theory

Over $\mathbb{k}$ field:

H Hopf algebra $\longrightarrow$ a *tensor category* $\mathcal{C} = \mathrm{comod} H$
+ a *fiber functor* $\mathcal{C} \to \mathrm{vect}$

Reconstruction: given $\mathcal{C}$ tensor category + $\omega : \mathcal{C} \to \mathrm{vect}$ fiber functor

Motivation : Tannaka theory

Over $\mathbb{k}$ field:

H Hopf algebra $\longrightarrow$ a *tensor category* $\mathcal{C} = \mathrm{comod} H$
+ a *fiber functor* $\mathcal{C} \to \mathrm{vect}$

Reconstruction: given $\mathcal{C}$ tensor category + $\omega : \mathcal{C} \to \mathrm{vect}$ fiber functor

$$\rightsquigarrow H = \mathrm{Coend}(\omega) = \int^{X \in \mathcal{C}} \omega(X) \otimes \omega(X)^* \qquad \text{Hopf algebra}$$

Motivation : Tannaka theory

Over $\mathbb{k}$ field:

H Hopf algebra $\longrightarrow$ a *tensor category* $\mathcal{C} = \mathrm{comod} H$
\+ a *fiber functor* $\mathcal{C} \to \mathrm{vect}$

Reconstruction: given $\mathcal{C}$ tensor category + $\omega : \mathcal{C} \to \mathrm{vect}$ fiber functor

$$\rightsquigarrow H = \mathrm{Coend}(\omega) = \int^{X \in \mathcal{C}} \omega(X) \otimes \omega(X)^* \qquad \text{Hopf algebra}$$

with commutative diagram:

Motivation : Tannaka theory

Over $\mathbb{k}$ field:

H Hopf algebra $\longrightarrow$ a *tensor category* $\mathcal{C} = \mathrm{comod} H$
+ a *fiber functor* $\mathcal{C} \to \mathrm{vect}$

Reconstruction: given $\mathcal{C}$ tensor category + $\omega : \mathcal{C} \to \mathrm{vect}$ fiber functor

$$\rightsquigarrow H = \mathrm{Coend}(\omega) = \int^{X \in \mathcal{C}} \omega(X) \otimes \omega(X)^* \qquad \text{Hopf algebra}$$

with commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\omega"] \arrow[dr, "\simeq_\otimes"'] & & \mathrm{vect} \\ & \mathrm{comod} H \arrow[ur] & \end{tikzcd}$$

Motivation : Tannaka theory

Over $\Bbbk$ field:

H Hopf algebra $\longrightarrow$ a *tensor category* $\mathcal{C} = \mathrm{comod} H$
+ a *fiber functor* $\mathcal{C} \to \mathrm{vect}$

Reconstruction: given $\mathcal{C}$ tensor category + $\omega : \mathcal{C} \to \mathrm{vect}$ fiber functor

$$\rightsquigarrow H = \mathrm{Coend}(\omega) = \int^{X \in \mathcal{C}} \omega(X) \otimes \omega(X)^* \qquad \text{Hopf algebra}$$

with commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\omega"] \arrow[dr, "\simeq_{\otimes}"'] & & \mathrm{vect} \\ & \mathrm{comod} H \arrow[ur] & \end{tikzcd}$$

A fiber functor is encoded by a Hopf algebra (in Vect)

G affine group scheme/$\mathbb{k}$ = commutative Hopf algebra $H = \mathcal{O}(G)$.

G affine group scheme/$\mathbb{k}$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

G affine group scheme/$\Bbbk$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

Converse: $\mathcal{C}$ symmetric tensor category $+\ \omega$ symmetric fiber functor

G affine group scheme/$\mathbb{k}$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

Converse: $\mathcal{C}$ symmetric tensor category $+\ \omega$ symmetric fiber functor $\rightsquigarrow H = \mathrm{Coend}(\omega)$ commutative Hopf algebra,

G affine group scheme/$\Bbbk$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

Converse: $\mathcal{C}$ symmetric tensor category $+\ \omega$ symmetric fiber functor $\rightsquigarrow H = \mathrm{Coend}(\omega)$ commutative Hopf algebra, $G = \mathrm{Spec} H$ affine group scheme

G affine group scheme/$\Bbbk$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

Converse: $\mathcal{C}$ symmetric tensor category $+ \ \omega$ symmetric fiber functor $\rightsquigarrow H = \mathrm{Coend}(\omega)$ commutative Hopf algebra, $G = \mathrm{Spec} H$ affine group scheme and $\mathcal{C} \simeq \mathrm{rep} G$ as symmetric tensor categories.

G affine group scheme/$\mathbb{k}$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

Converse: $\mathcal{C}$ symmetric tensor category $+\ \omega$ symmetric fiber functor $\rightsquigarrow H = \mathrm{Coend}(\omega)$ commutative Hopf algebra, $G = \mathrm{Spec} H$ affine group scheme and $\mathcal{C} \simeq \mathrm{rep} G$ as symmetric tensor categories.
Then there exists a commutative algebra A in $\mathcal{C}$ (or its Ind-completion) satisfying

- $\forall X$ in $\mathcal{C}$, $A \otimes X \xrightarrow{\sim} A^n$ as left A- modules
- $\mathrm{Hom}(\mathbb{1}, A) = \mathbb{k}$

and we have

$$\omega(X) = \mathrm{Hom}(\mathbb{1}, A \otimes X).$$

G affine group scheme/$\Bbbk$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

Converse: $\mathcal{C}$ symmetric tensor category $+ \, \omega$ symmetric fiber functor $\rightsquigarrow H = \mathrm{Coend}(\omega)$ commutative Hopf algebra, $G = \mathrm{Spec} H$ affine group scheme and $\mathcal{C} \simeq \mathrm{rep} G$ as symmetric tensor categories.
Then there exists a commutative algebra A in $\mathcal{C}$ (or its Ind-completion) satisfying

- $\forall X$ in $\mathcal{C}$, $A \otimes X \xrightarrow{\sim} A^n$ as left A-modules
- $\mathrm{Hom}(\mathbb{1}, A) = \Bbbk$

and we have

$$\omega(X) = \mathrm{Hom}(\mathbb{1}, A \otimes X).$$

The proof of Deligne's internal characterization of tannaka categories consists in constructing such a *trivializing algebra*.

G affine group scheme/$\Bbbk$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

Converse: $\mathcal{C}$ symmetric tensor category $+\ \omega$ symmetric fiber functor $\rightsquigarrow H = \mathrm{Coend}(\omega)$ commutative Hopf algebra, $G = \mathrm{Spec} H$ affine group scheme and $\mathcal{C} \simeq \mathrm{rep} G$ as symmetric tensor categories.
Then there exists a commutative algebra A in $\mathcal{C}$ (or its Ind-completion) satisfying

- $\forall X$ in $\mathcal{C}$, $A \otimes X \xrightarrow{\sim} A^n$ as left A- modules
- $\mathrm{Hom}(\mathbb{1}, A) = \Bbbk$

and we have

$$\omega(X) = \mathrm{Hom}(\mathbb{1}, A \otimes X).$$

The proof of Deligne's internal characterization of tannaka categories consists in constructing such a *trivializing algebra*.

A symmetric fiber functor is encoded by a certain commutative algebra in $\mathcal{C}$ (or $\mathrm{Ind}\mathcal{C}$)

G affine group scheme/$\Bbbk$ = commutative Hopf algebra $H = \mathcal{O}(G)$. Then $\mathcal{C} = \mathrm{comod} H = \mathrm{rep} G$ and the fiber functor $\mathcal{C} \to \mathrm{vect}$ are both symmetric.

Converse: $\mathcal{C}$ symmetric tensor category $+\ \omega$ symmetric fiber functor $\rightsquigarrow H = \mathrm{Coend}(\omega)$ commutative Hopf algebra, $G = \mathrm{Spec} H$ affine group scheme and $\mathcal{C} \simeq \mathrm{rep} G$ as symmetric tensor categories.
Then there exists a commutative algebra A in $\mathcal{C}$ (or its Ind-completion) satisfying

- $\forall X$ in $\mathcal{C}$, $A \otimes X \xrightarrow{\sim} A^n$ as left A-modules
- $\mathrm{Hom}(\mathbb{1}, A) = \Bbbk$

and we have

$$\omega(X) = \mathrm{Hom}(\mathbb{1}, A \otimes X).$$

The proof of Deligne's internal characterization of tannaka categories consists in constructing such a *trivializing algebra*.

A symmetric fiber functor is encoded by a certain commutative algebra in $\mathcal{C}$ (or $\mathrm{Ind}\mathcal{C}$)

Can we give similar encodings for arbitrary tensor functors?

Tensor categories and tensor functors

Let $\mathbb{k}$ be a field.

Definition

In this talk a *tensor category* is a $\mathbb{k}$- linear abelian category with a structure of rigid category (=monoidal with duals) such that:

- $\mathcal{C}$ is locally finite (Hom's are finite dim'l and objects have finite length)
- $\otimes$ is $\mathbb{k}$- bilinear and $\mathrm{End}(\mathbb{1}) = \mathbb{k}$

$\mathcal{C}$ is *finite* if $\mathcal{C} \overset{\mathbb{k}}{\simeq}_R$ *mod* for some finite dimensional $\mathbb{k}$- algebra R.

Definition

A *tensor functor* $F : \mathcal{C} \to \mathcal{D}$ is a $\mathbb{k}$- linear exact strong monoidal functor between tensor categories.

A tensor functor F is faithful. It has a right adjoint iff it has a left adjoint; in that case we say that F is *finite*.

Examples

1. vect is the initial tensor category

Examples

1. vect is the initial tensor category
2. A fiber functor for C is a tensor functor $C \to \mathrm{vect}$

Examples

1. vect is the initial tensor category
2. A fiber functor for $\mathcal{C}$ is a tensor functor $\mathcal{C} \to \mathrm{vect}$
3. A Hopf algebra morphism $f : H \to H'$ induces a tensor functor

$$f_* : \mathrm{comod}H \to \mathrm{comod}H'$$

Examples

1. vect is the initial tensor category
2. A fiber functor for $\mathcal{C}$ is a tensor functor $\mathcal{C} \to \mathrm{vect}$
3. A Hopf algebra morphism $f : H \to H'$ induces a tensor functor

$$f_* : \mathrm{comod}H \to \mathrm{comod}H'$$

Tannaka duality asserts that we have an equivalence of categories

$$\{\{\text{Hopf Algebras}\}\} \simeq \{\{\text{Tensor categories}\}\} \,/\, \mathrm{vect}$$

Examples

1. vect is the initial tensor category
2. A fiber functor for $\mathcal{C}$ is a tensor functor $\mathcal{C} \to \mathrm{vect}$
3. A Hopf algebra morphism $f : H \to H'$ induces a tensor functor

$$f_* : \mathrm{comod} H \to \mathrm{comod} H'$$

Tannaka duality asserts that we have an equivalence of categories

$$\{\{\text{Hopf Algebras}\}\} \simeq \{\{\text{Tensor categories}\}\} \,/\, \mathrm{vect}$$

But many tensor categories do not come from Hopf algebras!

Examples

1. vect is the initial tensor category
2. A fiber functor for C is a tensor functor $C \to \mathrm{vect}$
3. A Hopf algebra morphism $f : H \to H'$ induces a tensor functor

$$f_* : \mathrm{comod}H \to \mathrm{comod}H'$$

Tannaka duality asserts that we have an equivalence of categories

$$\{\{\text{Hopf Algebras}\}\} \simeq \{\{\text{Tensor categories}\}\} / \mathrm{vect}$$

But many tensor categories do not come from Hopf algebras!

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor.

Question 1

Can one encode F by algebraic data in $\mathcal{D}$ (or $\mathrm{Ind}\mathcal{D}$)?

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor.

Question 1

Can one encode F by algebraic data in $\mathcal{D}$ (or $\mathrm{Ind}\mathcal{D}$)?

Yes. But this data cannot be a Hopf algebra, as $\mathcal{D}$ is not braided.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor.

Question 1

Can one encode F by algebraic data in $\mathcal{D}$ (or $\mathrm{Ind}\mathcal{D}$)?

Yes. But this data cannot be a Hopf algebra, as $\mathcal{D}$ is not braided. It is a Hopf (co)monad.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor.

Question 1

Can one encode F by algebraic data in $\mathcal{D}$ (or $\mathrm{Ind}\mathcal{D}$)?

Yes. But this data cannot be a Hopf algebra, as $\mathcal{D}$ is not braided. It is a Hopf (co)monad.

Question 2

Can one encode F by an algebraic data in $\mathcal{C}$ (or $\mathrm{Ind}\mathcal{C}$)?

Yes, if F is *dominant.*

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor.

Question 1

Can one encode F by algebraic data in $\mathcal{D}$ (or $\mathrm{Ind}\mathcal{D}$)?

Yes. But this data cannot be a Hopf algebra, as $\mathcal{D}$ is not braided. It is a Hopf (co)monad.

Question 2

Can one encode F by an algebraic data in $\mathcal{C}$ (or $\mathrm{Ind}\mathcal{C}$)?

Yes, if F is *dominant.*
This data is a commutative algebra

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor.

Question 1

Can one encode F by algebraic data in $\mathcal{D}$ (or $\mathrm{Ind}\mathcal{D}$)?

Yes. But this data cannot be a Hopf algebra, as $\mathcal{D}$ is not braided. It is a Hopf (co)monad.

Question 2

Can one encode F by an algebraic data in $\mathcal{C}$ (or $\mathrm{Ind}\mathcal{C}$)?

Yes, if F is *dominant.*
This data is a commutative algebra in the center of $\mathcal{C}$ (or $\mathrm{Ind}\mathcal{C}$).

Outline of the talk

1. Introduction

Outline of the talk

Outline of the talk

Outline of the talk

Monads

Let C be a category. The category $\mathrm{EndoFun}(C)$ is strict monoidal ($\otimes$=composition, $\mathbb{1} = 1_C$)

Monads

Let $\mathcal{C}$ be a category. The category $\mathrm{EndoFun}(\mathcal{C})$ is strict monoidal ($\otimes$=composition, $\mathbb{1} = 1_{\mathcal{C}}$)

A monad on $\mathcal{C}$ is an algebra (=monoid) in $\mathrm{EndoFun}(\mathcal{C})$:

$$T\colon \mathcal{C} \to \mathcal{C}, \quad \mu\colon T^2 \to T \ \text{(product)}, \quad \eta\colon 1_{\mathcal{C}} \to T \ \text{(unit)}$$

Monads

Let $\mathcal{C}$ be a category. The category $\mathrm{EndoFun}(\mathcal{C})$ is strict monoidal ($\otimes$=composition, $\mathbb{1} = 1_{\mathcal{C}}$)

A monad on $\mathcal{C}$ is an algebra (=monoid) in $\mathrm{EndoFun}(\mathcal{C})$:

$$T\colon \mathcal{C} \to \mathcal{C}, \quad \mu\colon T^2 \to T \text{ (product)}, \quad \eta\colon 1_{\mathcal{C}} \to T \text{ (unit)}$$

A T-module is a pair (M, r), $M \in \mathrm{Ob}(\mathcal{C})$, $r\colon T(M) \to M$ s. t.

$$r\mu_M = rT(r) \quad \text{and} \quad r\eta_M = \mathrm{id}_M.$$

Monads

Let $\mathcal{C}$ be a category. The category $\mathrm{EndoFun}(\mathcal{C})$ is strict monoidal ($\otimes$=composition, $\mathbb{1} = 1_{\mathcal{C}}$)

A monad on $\mathcal{C}$ is an algebra (=monoid) in $\mathrm{EndoFun}(\mathcal{C})$:

$$T\colon \mathcal{C} \to \mathcal{C}, \quad \mu\colon T^2 \to T \text{ (product)}, \quad \eta\colon 1_{\mathcal{C}} \to T \text{ (unit)}$$

A T-module is a pair (M, r), $M \in \mathrm{Ob}(\mathcal{C})$, $r\colon T(M) \to M$ s. t.

$$r\mu_M = rT(r) \quad \text{and} \quad r\eta_M = \mathrm{id}_M.$$

$\rightsquigarrow$ $\mathcal{C}^T$ category of T-modules.

Monads

Let $\mathcal{C}$ be a category. The category $\mathrm{EndoFun}(\mathcal{C})$ is strict monoidal ($\otimes$=composition, $\mathbb{1} = 1_{\mathcal{C}}$)

A monad on $\mathcal{C}$ is an algebra (=monoid) in $\mathrm{EndoFun}(\mathcal{C})$:

$$T\colon \mathcal{C} \to \mathcal{C}, \quad \mu\colon T^2 \to T \text{ (product)}, \quad \eta\colon 1_{\mathcal{C}} \to T \text{ (unit)}$$

A T-module is a pair (M, r), $M \in \mathrm{Ob}(\mathcal{C})$, $r\colon T(M) \to M$ s. t.

$$r\mu_M = rT(r) \quad \text{and} \quad r\eta_M = \mathrm{id}_M.$$

$\rightsquigarrow$ $\mathcal{C}^T$ category of T-modules.

Example

A algebra in a monoidal category $\mathcal{C}$
$\rightsquigarrow$ $T = ? \otimes A\colon X \mapsto X \otimes A$ is a monad on $\mathcal{C}$ and $\mathcal{C}^T = \mathrm{Mod}\text{-}A$

Monads

Let $\mathcal{C}$ be a category. The category $\mathrm{EndoFun}(\mathcal{C})$ is strict monoidal (⊗=composition, $\mathbb{1} = 1_{\mathcal{C}}$)

A monad on $\mathcal{C}$ is an algebra (=monoid) in $\mathrm{EndoFun}(\mathcal{C})$:

$$T\colon \mathcal{C} \to \mathcal{C}, \quad \mu\colon T^2 \to T \text{ (product)}, \quad \eta\colon 1_{\mathcal{C}} \to T \text{ (unit)}$$

A T-module is a pair (M, r), $M \in \mathrm{Ob}(\mathcal{C})$, $r\colon T(M) \to M$ s. t.

$$r\mu_M = rT(r) \quad \text{and} \quad r\eta_M = \mathrm{id}_M.$$

$\rightsquigarrow$ $\mathcal{C}^T$ category of T-modules.

Example

A algebra in a monoidal category $\mathcal{C}$

$\rightsquigarrow$ $T = ? \otimes A : X \mapsto X \otimes A$ is a monad on $\mathcal{C}$ and $\mathcal{C}^T = \mathrm{Mod}\text{-}A$

$T' = A \otimes ?$ is a monad on $\mathcal{C}$ and $\mathcal{C}^{T'} = A\text{-}\mathrm{Mod}$

Monads and adjunctions

A monad T on a category $\mathcal{C}$ $\rightsquigarrow$ an adjunction
$$\begin{tikzcd} \mathcal{C}^T \arrow[d, bend left, "U^T"] \\ \mathcal{C} \arrow[u, bend left, "F^T"] \end{tikzcd}$$
where $U^T(M,r) = M$ and $F^T(X) = (T(X), \mu_X)$.

Monads and adjunctions

A monad T on a category $\mathcal{C} \rightsquigarrow$ an adjunction $\begin{tikzcd} \mathcal{C}^T \arrow[d, bend left, "U^T"] \\ \mathcal{C} \arrow[u, bend left, "F^T"] \end{tikzcd}$

where $U^T(M,r) = M$ and $F^T(X) = (T(X), \mu_X)$.

An adjunction $\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$ $\rightsquigarrow$ a monad $T = (UF, \mu := U(\varepsilon_F), \eta)$ on $\mathcal{C}$

where $\eta : 1_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to 1_{\mathcal{D}}$ are the adjunction morphisms

$\rightsquigarrow$

Monads and adjunctions

A monad T on a category $\mathcal{C}$ $\rightsquigarrow$ an adjunction
$$\begin{tikzcd} \mathcal{C}^T \arrow[d, bend left, "U^T"] \\ \mathcal{C} \arrow[u, bend left, "F^T"] \end{tikzcd}$$

where $U^T(M,r) = M$ and $F^T(X) = (T(X), \mu_X)$.

An adjunction
$$\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$$
$\rightsquigarrow$ a monad $T = (UF, \mu := U(\varepsilon_F), \eta)$ on $\mathcal{C}$

where $\eta : 1_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to 1_{\mathcal{D}}$ are the adjunction morphisms

$\rightsquigarrow$
$$\begin{tikzcd} \mathcal{D} \arrow[dr, bend left, "U"] & & \mathcal{C}^T \arrow[dl, bend left, "U^T"] \\ & \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] & \end{tikzcd}$$

Monads and adjunctions

A monad T on a category $\mathcal{C}$ $\rightsquigarrow$ an adjunction $\begin{tikzcd} \mathcal{C}^T \arrow[d, bend left, "U^T"] \\ \mathcal{C} \arrow[u, bend left, "F^T"] \end{tikzcd}$

where $U^T(M, r) = M$ and $F^T(X) = (T(X), \mu_X)$.

An adjunction $\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$ $\rightsquigarrow$ a monad $T = (UF, \mu := U(\varepsilon_F), \eta)$ on $\mathcal{C}$

where $\eta : 1_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to 1_{\mathcal{D}}$ are the adjunction morphisms

$$\rightsquigarrow \begin{tikzcd} \mathcal{D} \arrow[rr, bend left, "{\color{red}K}"] \arrow[dr, bend left, "U"] & & \mathcal{C}^T \arrow[dl, bend left, "U^T"] \\ & \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] & \end{tikzcd}$$

Monads and adjunctions

A monad T on a category $\mathcal{C}$ $\rightsquigarrow$ an adjunction $\begin{tikzcd} \mathcal{C}^T \arrow[d, bend left, "U^T"] \\ \mathcal{C} \arrow[u, bend left, "F^T"] \end{tikzcd}$

where $U^T(M, r) = M$ and $F^T(X) = (T(X), \mu_X)$.

An adjunction $\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$ $\rightsquigarrow$ a monad $T = (UF, \mu := U(\varepsilon_F), \eta)$ on $\mathcal{C}$

where $\eta : 1_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to 1_{\mathcal{D}}$ are the adjunction morphisms

$\rightsquigarrow$ $\begin{tikzcd} \mathcal{D} \arrow[rr, bend left, "K"] \arrow[dr, bend left, "U"] & & \mathcal{C}^T \arrow[dl, bend left, "U^T"] \\ & \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] & \end{tikzcd}$

$K\colon D \mapsto (U(D), U(\varepsilon_D))$
(the *comparison functor*)

The adjunction (F, U) is *monadic* if K equivalence.

Monads and adjunctions

A monad T on a category $\mathcal{C}$ $\rightsquigarrow$ an adjunction
$$\begin{tikzcd} \mathcal{C}^T \arrow[d, bend left, "U^T"] \\ \mathcal{C} \arrow[u, bend left, "F^T"] \end{tikzcd}$$
where $U^T(M,r) = M$ and $F^T(X) = (T(X), \mu_X)$.

An adjunction
$$\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$$
$\rightsquigarrow$ a monad $T = (UF, \mu := U(\varepsilon_F), \eta)$ on $\mathcal{C}$

where $\eta : 1_{\mathcal{C}} \to UF$ and $\varepsilon : FU \to 1_{\mathcal{D}}$ are the adjunction morphisms

$\rightsquigarrow$
$$\begin{tikzcd} \mathcal{D} \arrow[rr, bend left, "K"] \arrow[dr, bend left, "U"] & & \mathcal{C}^T \arrow[dl, bend left, "U^T"] \\ & \mathcal{C} \arrow[ul, bend left, "F"] \arrow[ur, bend left, "F^T"] & \end{tikzcd}$$

$K\colon D \mapsto (U(D), U(\varepsilon_D))$
(the *comparison functor*)

The adjunction (F,U) is *monadic* if K equivalence.

Bimonads [Moerdijk]

$\mathcal{C}$ monoidal category, (T, μ, η) monad on $\mathcal{C}$

Bimonads [Moerdijk]

$\mathcal{C}$ monoidal category, (T, μ, η) monad on $\mathcal{C} \rightsquigarrow \mathcal{C}^T$, $U^T : \mathcal{C}^T \to \mathcal{C}$

Bimonads [Moerdijk]

$\mathcal{C}$ monoidal category, (T, μ, η) monad on $\mathcal{C} \rightsquigarrow \mathcal{C}^T$, $U^T : \mathcal{C}^T \to \mathcal{C}$

T is a *bimonad* if and only if $\mathcal{C}^T$ is monoidal and U^T is strict monoidal.

Bimonads [Moerdijk]

$\mathcal{C}$ monoidal category, (T, μ, η) monad on $\mathcal{C} \rightsquigarrow \mathcal{C}^T$, $U^T : \mathcal{C}^T \to \mathcal{C}$

T is a *bimonad* if and only if $\mathcal{C}^T$ is monoidal and U^T is strict monoidal. This is equivalent to:

- T is comonoidal endofunctor
  (with $\Delta_{X,Y} : T(X \otimes Y) \to TX \otimes TY$ and $\varepsilon : T\mathbb{1} \to \mathbb{1}$)
- μ and η are comonoidal natural transformations.

Bimonads [Moerdijk]

$\mathcal{C}$ monoidal category, (T, μ, η) monad on $\mathcal{C} \rightsquigarrow \mathcal{C}^T$, $U^T : \mathcal{C}^T \to \mathcal{C}$

T is a *bimonad* if and only if $\mathcal{C}^T$ is monoidal and U^T is strict monoidal. This is equivalent to:

- T is comonoidal endofunctor
  (with $\Delta_{X,Y} : T(X \otimes Y) \to TX \otimes TY$ and $\varepsilon : T\mathbb{1} \to \mathbb{1}$)
- μ and η are comonoidal natural transformations.

Axioms similar to those of a bialgebra except the compatibility between μ and Δ:

Bimonads [Moerdijk]

$\mathcal{C}$ monoidal category, (T, μ, η) monad on $\mathcal{C} \rightsquigarrow \mathcal{C}^T$, $U^T : \mathcal{C}^T \to \mathcal{C}$

T is a *bimonad* if and only if $\mathcal{C}^T$ is monoidal and U^T is strict monoidal. This is equivalent to:

- T is comonoidal endofunctor
  (with $\Delta_{X,Y} : T(X \otimes Y) \to TX \otimes TY$ and $\varepsilon : T\mathbb{1} \to \mathbb{1}$)
- μ and η are comonoidal natural transformations.

Axioms similar to those of a bialgebra except the compatibility between μ and Δ:

$$\begin{tikzcd} T^2(X \otimes Y) \arrow[r, "T\Delta_{X,Y}"] \arrow[d, "\mu_{X\otimes Y}"'] & T(TX \otimes TY) \arrow[r, "\Delta_{TX,TY}"] & T^2X \otimes T^2Y \arrow[d, "\mu_X \otimes \mu_Y"] \\ T(X \otimes Y) \arrow[rr, "\Delta_{X,Y}"'] & & TX \otimes TY \end{tikzcd}$$

Bimonads [Moerdijk]

$\mathcal{C}$ monoidal category, (T, μ, η) monad on $\mathcal{C} \rightsquigarrow \mathcal{C}^T$, $U^T : \mathcal{C}^T \to \mathcal{C}$

T is a *bimonad* if and only if $\mathcal{C}^T$ is monoidal and U^T is strict monoidal. This is equivalent to:

- T is comonoidal endofunctor
  (with $\Delta_{X,Y} : T(X \otimes Y) \to TX \otimes TY$ and $\varepsilon : T\mathbb{1} \to \mathbb{1}$)
- μ and η are comonoidal natural transformations.

Axioms similar to those of a bialgebra except the compatibility between μ and Δ:

$$\begin{tikzcd}
T^2(X \otimes Y) \arrow[r, "T\Delta_{X,Y}"] \arrow[d, "\mu_{X\otimes Y}"'] & T(TX \otimes TY) \arrow[r, "\Delta_{TX,TY}"] & T^2X \otimes T^2Y \arrow[d, "\mu_X \otimes \mu_Y"] \\
T(X \otimes Y) \arrow[rr, "\Delta_{X,Y}"'] & & TX \otimes TY
\end{tikzcd}$$

No braiding involved!

Hopf monads

For a bimonad T define the (left and right) *fusion morphisms*

- $H^l(X,Y) = (\mathrm{id}_{TX} \otimes \mu_Y)\Delta_{X,TY} \colon T(X \otimes TY) \to TX \otimes TY$,
- $H^r(X,Y) = (\mu_X \otimes \mathrm{id}_{TY})\Delta_{TX,Y} \colon T(TX \otimes Y) \to TX \otimes TY$.

Hopf monads

For a bimonad T define the (left and right) *fusion morphisms*

- $H^l(X,Y) = (\mathrm{id}_{TX} \otimes \mu_Y)\Delta_{X,TY} \colon T(X \otimes TY) \to TX \otimes TY$,
- $H^r(X,Y) = (\mu_X \otimes \mathrm{id}_{TY})\Delta_{TX,Y} \colon T(TX \otimes Y) \to TX \otimes TY$.

A bimonad T is a *Hopf monad* if the fusion morphisms are isomorphisms.

Hopf monads

For a bimonad T define the (left and right) *fusion morphisms*

- $H^l(X, Y) = (\mathrm{id}_{TX} \otimes \mu_Y)\Delta_{X,TY} \colon T(X \otimes TY) \to TX \otimes TY$,
- $H^r(X, Y) = (\mu_X \otimes \mathrm{id}_{TY})\Delta_{TX,Y} \colon T(TX \otimes Y) \to TX \otimes TY$.

A bimonad T is a *Hopf monad* if the fusion morphisms are isomorphisms.

Proposition

For T bimonad on $\mathcal{C}$ rigid, equivalence:

(i) $\mathcal{C}^T$ is rigid;

(ii) T is a Hopf monad;

(iii) (older definition) T admits a left and a right (unary) antipode $s^l_X \colon T({}^\vee TX) \to {}^\vee X$ and $s^r \colon T(TX^\vee) \to X^\vee$.

Hopf monads

For a bimonad T define the (left and right) *fusion morphisms*

- $H^l(X,Y) = (\mathrm{id}_{TX} \otimes \mu_Y)\Delta_{X,TY} \colon T(X \otimes TY) \to TX \otimes TY$,
- $H^r(X,Y) = (\mu_X \otimes \mathrm{id}_{TY})\Delta_{TX,Y} \colon T(TX \otimes Y) \to TX \otimes TY$.

A bimonad T is a *Hopf monad* if the fusion morphisms are isomorphisms.

Proposition

For T bimonad on $\mathcal{C}$ rigid, equivalence:

(i) $\mathcal{C}^T$ is rigid;

(ii) T is a Hopf monad;

(iii) (older definition) T admits a left and a right (unary) antipode $s^l_X \colon T({}^\vee TX) \to {}^\vee X$ and $s^r \colon T(TX^\vee) \to X^\vee$.

There is a similar result for closed categories (monoidal categories with internal Homs).

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$

$$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N})$$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)		

$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N})$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y} \colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	

$$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N})$$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y} \colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X \colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X \colon T(T(X)^\vee) \to X^\vee$

$$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N})$$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X\colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X\colon T(T(X)^\vee) \to X^\vee$

$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N})$ ${}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X\colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X\colon T(T(X)^\vee) \to X^\vee$
quasitriangular		

$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N})$ ${}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y} \colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X \colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X \colon T(T(X)^\vee) \to X^\vee$
quasitriangular	braided	

$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N})$

${}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y} \colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X \colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X \colon T(T(X)^\vee) \to X^\vee$
quasitriangular	braided	$R_{X,Y} \colon X \otimes Y \to T(Y) \otimes T(X)$

$(M,r) \otimes (N,s) = (M \otimes N, (r \otimes s)\Delta_{M,N})$

${}^\vee(M,r) = ({}^\vee M, s^l_M T({}^\vee r))$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X\colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X\colon T(T(X)^\vee) \to X^\vee$
quasitriangular	braided	$R_{X,Y}\colon X \otimes Y \to T(Y) \otimes T(X)$

$$(M,r) \otimes (N,s) = (M \otimes N, (r \otimes s)\Delta_{M,N}) \qquad {}^\vee(M,r) = ({}^\vee M, s^l_M T({}^\vee r))$$

$$\tau_{(M,r),(N,s)} = (s \otimes r)R_{M,N}$$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X\colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X\colon T(T(X)^\vee) \to X^\vee$
quasitriangular	braided	$R_{X,Y}\colon X \otimes Y \to T(Y) \otimes T(X)$
ribbon		

$$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N}) \qquad {}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$$

$$\tau_{(M,r),(N,s)} = (s \otimes r) R_{M,N}$$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X\colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X\colon T(T(X)^\vee) \to X^\vee$
quasitriangular	braided	$R_{X,Y}\colon X \otimes Y \to T(Y) \otimes T(X)$
ribbon	ribbon	

$$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N}) \qquad {}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$$

$$\tau_{(M,r),(N,s)} = (s \otimes r)R_{M,N}$$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y} \colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X \colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X \colon T(T(X)^\vee) \to X^\vee$
quasitriangular	braided	$R_{X,Y} \colon X \otimes Y \to T(Y) \otimes T(X)$
ribbon	ribbon	$\theta_X \colon X \to T(X)$

$$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N}) \qquad {}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$$

$$\tau_{(M,r),(N,s)} = (s \otimes r) R_{M,N}$$

Tannaka dictionary

There is a Tannaka dictionary relating properties of a monad T on a monoidal category $\mathcal{C}$ and properties of its category of modules $\mathcal{C}^T$.

T	$\mathcal{C}^T$	*Structural morphism*
bimonad	monoidal	$\Delta_{X,Y}\colon T(X \otimes Y) \to T(X) \otimes T(Y)$
Hopf monad ($\mathcal{C}$ rigid)	rigid	$s^l_X\colon T({}^\vee T(X)) \to {}^\vee X$ $s^r_X\colon T(T(X)^\vee) \to X^\vee$
quasitriangular	braided	$R_{X,Y}\colon X \otimes Y \to T(Y) \otimes T(X)$
ribbon	ribbon	$\theta_X\colon X \to T(X)$

$$(M, r) \otimes (N, s) = (M \otimes N, (r \otimes s)\Delta_{M,N}) \qquad {}^\vee(M, r) = ({}^\vee M, s^l_M T({}^\vee r))$$

$$\tau_{(M,r),(N,s)} = (s \otimes r) R_{M,N} \qquad \Theta_{(M,r)} = r\theta_M$$

Hopf comonads

The notion of a Hopf monad is not self-dual, unlike that of a Hopf algebra: if you reverse the arrows in the definition, you obtain the notion of a *Hopf comonad.* A Hopf comonad is a monoidal comonad such that the cofusion operators are invertible.

Hopf comonads

The notion of a Hopf monad is not self-dual, unlike that of a Hopf algebra: if you reverse the arrows in the definition, you obtain the notion of a *Hopf comonad.* A Hopf comonad is a monoidal comonad such that the cofusion operators are invertible.

All results about Hopf monads translate into results about Hopf comonads.

Hopf comonads

The notion of a Hopf monad is not self-dual, unlike that of a Hopf algebra: if you reverse the arrows in the definition, you obtain the notion of a *Hopf comonad.* A Hopf comonad is a monoidal comonad such that the cofusion operators are invertible.

All results about Hopf monads translate into results about Hopf comonads. In particular, if *T* is a Hopf comonad on $\mathcal{C}$,

Hopf comonads

The notion of a Hopf monad is not self-dual, unlike that of a Hopf algebra: if you reverse the arrows in the definition, you obtain the notion of a *Hopf comonad.* A Hopf comonad is a monoidal comonad such that the cofusion operators are invertible.
All results about Hopf monads translate into results about Hopf comonads.
In particular, if T is a Hopf comonad on $\mathcal{C}$,

1. the category $\mathcal{C}_T$ of comodules over T is monoidal,

Hopf comonads

The notion of a Hopf monad is not self-dual, unlike that of a Hopf algebra: if you reverse the arrows in the definition, you obtain the notion of a *Hopf comonad.* A Hopf comonad is a monoidal comonad such that the cofusion operators are invertible.

All results about Hopf monads translate into results about Hopf comonads. In particular, if T is a Hopf comonad on $\mathcal{C}$,

1. the category $\mathcal{C}_T$ of comodules over T is monoidal,
2. we have a Hopf monoidal adjunction: $\begin{tikzcd} \mathcal{D} \arrow[r, bend right, "U_T"'] & \mathcal{C} \arrow[l, bend right, "F_T"'] \end{tikzcd}$

where U_T is the forgetful functor and F_T is its right adjoint, the cofree comodule functor.

Hopf monads from adjunctions

Let $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "U"] & \mathcal{C} \arrow[l, bend left, "F"] \end{tikzcd}$ be a comonoidal adjunction (meaning $\mathcal{C}$, $\mathcal{D}$ are monoidal and U is strong monoidal)

Hopf monads from adjunctions

Let $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "U"] & \mathcal{C} \arrow[l, bend left, "F"] \end{tikzcd}$ be a comonoidal adjunction (meaning $\mathcal{C}$, $\mathcal{D}$ are monoidal and U is strong monoidal)

Then F is comonoidal and $T = UF$ is a bimonad on $\mathcal{C}$.

Hopf monads from adjunctions

Let $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "U"] & \mathcal{C} \arrow[l, bend left, "F"] \end{tikzcd}$ be a comonoidal adjunction (meaning $\mathcal{C}$, $\mathcal{D}$ are monoidal and U is strong monoidal)

Then F is comonoidal and $T = UF$ is a bimonad on $\mathcal{C}$.

Hopf monads from adjunctions

Let $\begin{tikzcd} \mathcal{D} \arrow[r, bend right, "F"', leftarrow] \arrow[r, bend left, "U"] & \mathcal{C} \end{tikzcd}$ be a comonoidal adjunction (meaning $\mathcal{C}$, $\mathcal{D}$ are monoidal and U is strong monoidal)

Then F is comonoidal and $T = UF$ is a bimonad on $\mathcal{C}$.

There are canonical morphisms:

- $F(c \otimes Ud) \to Fc \otimes d$
- $F(Ud \otimes c) \to d \otimes Fc$

and (F, U) is a *Hopf adjunction* if these morphisms are isos.

Hopf monads from adjunctions

Let $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "U"] & \mathcal{C} \arrow[l, bend left, "F"] \end{tikzcd}$ be a comonoidal adjunction (meaning $\mathcal{C}$, $\mathcal{D}$ are monoidal and U is strong monoidal)

Then F is comonoidal and $T = UF$ is a bimonad on $\mathcal{C}$.

There are canonical morphisms:

- $F(c \otimes Ud) \to Fc \otimes d$
- $F(Ud \otimes c) \to d \otimes Fc$

and (F, U) is a *Hopf adjunction* if these morphisms are isos.

Proposition

If the adjunction is *Hopf*, T is a Hopf monad. Such is the case if either of the following hold:

- $\mathcal{C}$, $\mathcal{D}$ are rigid;
- $\mathcal{C}$, $\mathcal{D}$ and U are closed.

Hopf monads from adjunctions

Let $\begin{tikzcd} \mathcal{D} \arrow[r, bend left, "U"] & \mathcal{C} \arrow[l, bend left, "F"] \end{tikzcd}$ be a comonoidal adjunction (meaning $\mathcal{C}$, $\mathcal{D}$ are monoidal and U is strong monoidal)

Then F is comonoidal and $T = UF$ is a bimonad on $\mathcal{C}$.

There are canonical morphisms:

- $F(c \otimes Ud) \to Fc \otimes d$
- $F(Ud \otimes c) \to d \otimes Fc$

and (F, U) is a *Hopf adjunction* if these morphisms are isos.

Proposition

If the adjunction is *Hopf*, T is a Hopf monad. Such is the case if either of the following hold:

- $\mathcal{C}$, $\mathcal{D}$ are rigid;
- $\mathcal{C}$, $\mathcal{D}$ and U are closed.

A bimonad is Hopf iff its adjunction is Hopf!

Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories

Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories

H Hopf algebra in $\mathcal{B}$ braided category with braiding τ

Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories

H Hopf algebra in $\mathcal{B}$ braided category with braiding τ
$\rightsquigarrow T = H \otimes ?$ is a Hopf monad on $\mathcal{B}$

Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories

H Hopf algebra in $\mathcal{B}$ braided category with braiding τ
$\rightsquigarrow T = H \otimes ?$ is a Hopf monad on $\mathcal{B}$
The monad structure of T comes from the algebra structure of H

Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories

H Hopf algebra in $\mathcal{B}$ braided category with braiding τ
$\rightsquigarrow T = H \otimes ?$ is a Hopf monad on $\mathcal{B}$
The monad structure of T comes from the algebra structure of H
The comonoidal structure of T is

$$\Delta_{X,Y} = (H \otimes \tau_{H,X} \otimes Y)(\Delta \otimes X \otimes Y)\colon H \otimes X \otimes Y \to H \otimes X \otimes H \otimes Y$$
$$\varepsilon = \text{counit of } H\colon H \to \mathbb{1}$$

Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories

H Hopf algebra in $\mathcal{B}$ braided category with braiding τ
$\rightsquigarrow T = H \otimes ?$ is a Hopf monad on $\mathcal{B}$
The monad structure of T comes from the algebra structure of H
The comonoidal structure of T is

$$\Delta_{X,Y} = (H \otimes \tau_{H,X} \otimes Y)(\Delta \otimes X \otimes Y) \colon H \otimes X \otimes Y \to H \otimes X \otimes H \otimes Y$$
$$\varepsilon = \text{counit of } H \colon H \to \mathbb{1}$$

We have $\mathcal{B}^T =_H \mathrm{Mod}$ as monoidal categories.

Hopf monads from Hopf algebras

Hopf monads generalize Hopf algebras in braided categories

H Hopf algebra in $\mathcal{B}$ braided category with braiding τ
$\rightsquigarrow T = H \otimes ?$ is a Hopf monad on $\mathcal{B}$
The monad structure of T comes from the algebra structure of H
The comonoidal structure of T is

$$\Delta_{X,Y} = (H \otimes \tau_{H,X} \otimes Y)(\Delta \otimes X \otimes Y) \colon H \otimes X \otimes Y \to H \otimes X \otimes H \otimes Y$$
$$\varepsilon = \text{counit of } H \colon H \to \mathbb{1}$$

We have $\mathcal{B}^T =_H \mathrm{Mod}$ as monoidal categories.

Can we extend this construction to non-braided categories?

The Joyal-Street Center

The Joyal-Street Center

$\mathcal{C}$ monoidal category $\xrightarrow[\textit{Center}]{\textit{Joyal-Street}}$ $\mathcal{Z}(\mathcal{C})$ braided category

The Joyal-Street Center

$\mathcal{C}$ monoidal category $\xrightarrow[\text{Center}]{\text{Joyal-Street}}$ $\mathcal{Z}(\mathcal{C})$ braided category

- Objects of $\mathcal{Z}(\mathcal{C})$ = *half-braidings* of $\mathcal{C}$:

  pair (X, σ) with $\sigma_Y \colon X \otimes Y \xrightarrow{\sim} Y \otimes X$ natural in Y s. t.

$$\sigma_{Y \otimes Z} = (\mathrm{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \mathrm{id}_Z)$$

The Joyal-Street Center

$\mathcal{C}$ monoidal category $\xrightarrow[\textit{Center}]{\textit{Joyal-Street}}$ $\mathcal{Z}(\mathcal{C})$ braided category

- Objects of $\mathcal{Z}(\mathcal{C})$ = *half-braidings* of $\mathcal{C}$:
  pair (X, σ) with $\sigma_Y : X \otimes Y \xrightarrow{\sim} Y \otimes X$ natural in Y s. t.

$$\sigma_{Y \otimes Z} = (\mathrm{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \mathrm{id}_Z)$$

- Morphisms $f : (X, \sigma) \to (X', \sigma')$ in $\mathcal{Z}(\mathcal{C})$ are morphisms $f : X \to X'$ in $\mathcal{C}$ s. t. $\sigma'(f \otimes \mathrm{id}) = (\mathrm{id} \otimes f)\sigma$

The Joyal-Street Center

$\mathcal{C}$ monoidal category $\xrightarrow[\textit{Center}]{\textit{Joyal-Street}}$ $\mathcal{Z}(\mathcal{C})$ braided category

- Objects of $\mathcal{Z}(\mathcal{C})$ = *half-braidings* of $\mathcal{C}$:

  pair (X, σ) with $\sigma_Y \colon X \otimes Y \xrightarrow{\sim} Y \otimes X$ natural in Y s. t.

$$\sigma_{Y \otimes Z} = (\mathrm{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \mathrm{id}_Z)$$

- Morphisms $f \colon (X, \sigma) \to (X', \sigma')$ in $\mathcal{Z}(\mathcal{C})$ are morphisms $f \colon X \to X'$ in $\mathcal{C}$ s. t. $\sigma'(f \otimes \mathrm{id}) = (\mathrm{id} \otimes f)\sigma$
- $(X, \sigma) \otimes_{\mathcal{Z}(\mathcal{C})} (X', \sigma') = \big(X \otimes X', (\sigma_Y \otimes \mathrm{id})(\mathrm{id} \otimes \sigma')\big)$

The Joyal-Street Center

$\mathcal{C}$ monoidal category $\xrightarrow[\textit{Center}]{\textit{Joyal-Street}}$ $\mathcal{Z}(\mathcal{C})$ braided category

- Objects of $\mathcal{Z}(\mathcal{C})$ = *half-braidings* of $\mathcal{C}$:
  pair (X, σ) with $\sigma_Y \colon X \otimes Y \xrightarrow{\sim} Y \otimes X$ natural in Y s. t.
  $$\sigma_{Y \otimes Z} = (\mathrm{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \mathrm{id}_Z)$$
- Morphisms $f \colon (X, \sigma) \to (X', \sigma')$ in $\mathcal{Z}(\mathcal{C})$ are morphisms $f \colon X \to X'$ in $\mathcal{C}$ s. t. $\sigma'(f \otimes \mathrm{id}) = (\mathrm{id} \otimes f)\sigma$
- $(X, \sigma) \otimes_{\mathcal{Z}(\mathcal{C})} (X', \sigma') = \big(X \otimes X', (\sigma_Y \otimes \mathrm{id})(\mathrm{id} \otimes \sigma')\big)$
- Braiding: $c_{(X,\sigma),(X',\sigma')} = \sigma_{X'}$

The Joyal-Street Center

$\mathcal{C}$ monoidal category $\xrightarrow[\textit{Center}]{\textit{Joyal-Street}}$ $\mathcal{Z}(\mathcal{C})$ braided category

- Objects of $\mathcal{Z}(\mathcal{C})$ = *half-braidings* of $\mathcal{C}$:
  pair (X, σ) with $\sigma_Y \colon X \otimes Y \xrightarrow{\sim} Y \otimes X$ natural in Y s. t.
  $$\sigma_{Y \otimes Z} = (\mathrm{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \mathrm{id}_Z)$$
- Morphisms $f \colon (X, \sigma) \to (X', \sigma')$ in $\mathcal{Z}(\mathcal{C})$ are morphisms $f \colon X \to X'$ in $\mathcal{C}$ s. t. $\sigma'(f \otimes \mathrm{id}) = (\mathrm{id} \otimes f)\sigma$
- $(X, \sigma) \otimes_{\mathcal{Z}(\mathcal{C})} (X', \sigma') = \big(X \otimes X', (\sigma_Y \otimes \mathrm{id})(\mathrm{id} \otimes \sigma')\big)$
- Braiding: $c_{(X,\sigma),(X',\sigma')} = \sigma_{X'}$

Representable Hopf monads

Representable Hopf monads

$\mathcal{C}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{C})$ (which is braided)
$\rightsquigarrow$ a Hopf monad $T = H \otimes_\sigma ?$ on $\mathcal{C}$, defined by $X \mapsto H \otimes X$.

Representable Hopf monads

$\mathcal{C}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{C})$ (which is braided) $\rightsquigarrow$ a Hopf monad $T = H \otimes_\sigma ?$ on $\mathcal{C}$, defined by $X \mapsto H \otimes X$. The comonoidal structure of T is

$$\Delta_{X,Y} = (H \otimes \sigma_X \otimes Y)(\Delta \otimes X \otimes Y)$$
$$\varepsilon = \text{counit of } H$$

Representable Hopf monads

$\mathcal{C}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{C})$ (which is braided) $\rightsquigarrow$ a Hopf monad $T = H \otimes_\sigma ?$ on $\mathcal{C}$, defined by $X \mapsto H \otimes X$. The comonoidal structure of T is

$$\Delta_{X,Y} = (H \otimes \sigma_X \otimes Y)(\Delta \otimes X \otimes Y)$$
$$\varepsilon = \text{counit of } H$$

Moreover T is equipped with a Hopf monad morphism

$$e = (\varepsilon \otimes ?) : T \to \mathrm{id}_\mathcal{C}$$

Representable Hopf monads

$\mathcal{C}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{C})$ (which is braided) $\rightsquigarrow$ a Hopf monad $T = H \otimes_\sigma ?$ on $\mathcal{C}$, defined by $X \mapsto H \otimes X$. The comonoidal structure of T is

$$\Delta_{X,Y} = (H \otimes \sigma_X \otimes Y)(\Delta \otimes X \otimes Y)$$
$$\varepsilon = \text{counit of } H$$

Moreover T is equipped with a Hopf monad morphism

$$e = (\varepsilon \otimes ?) : T \to \mathrm{id}_\mathcal{C}$$

Theorem (BVL)

This construction defines an equivalence of categories

$$\{\{\text{Hopf algebras in } \mathcal{Z}(\mathcal{C})\}\} \xrightarrow{\simeq} \{\{\text{Hopf monads on } \mathcal{C}\}\} / \mathrm{id}_\mathcal{C}$$

If H is a Hopf algebra and $T = H\otimes$ we recover Sweedler's Theorem.

Monadicity of the center

Let $\mathcal{C}$ be a rigid category, with center $\mathcal{Z}(\mathcal{C})$.

Monadicity of the center

Let $\mathcal{C}$ be a rigid category, with center $\mathcal{Z}(\mathcal{C})$.
Using duality, interpret a half-braiding $\sigma_Y : X \otimes Y \to Y \otimes X$ as a dinatural transformation ${}^\vee Y \otimes X \otimes Y \to X$

Monadicity of the center

Let $\mathcal{C}$ be a rigid category, with center $\mathcal{Z}(\mathcal{C})$.
Using duality, interpret a half-braiding $\sigma_Y : X \otimes Y \to Y \otimes X$ as a dinatural transformation ${}^\vee Y \otimes X \otimes Y \to X$
We say that $\mathcal{C}$ is *centralizable* if $Z(X) = \int^{Y \in \mathcal{C}} {}^\vee Y \otimes X \otimes Y$ exists for all $X \in \mathcal{C}$

Monadicity of the center

Let $\mathcal{C}$ be a rigid category, with center $\mathcal{Z}(\mathcal{C})$.
Using duality, interpret a half-braiding $\sigma_Y : X \otimes Y \to Y \otimes X$ as a dinatural transformation ${}^\vee Y \otimes X \otimes Y \to X$
We say that $\mathcal{C}$ is *centralizable* if $Z(X) = \int^{Y \in \mathcal{C}} {}^\vee Y \otimes X \otimes Y$ exists for all $X \in \mathcal{C}$ (note that $Z(\mathbb{1})$ is the coend of $\mathcal{C}$). Then a half braiding σ corresponds with $\tilde{\sigma} \colon Z(X) \to X$

Monadicity of the center

Let $\mathcal{C}$ be a rigid category, with center $\mathcal{Z}(\mathcal{C})$.
Using duality, interpret a half-braiding $\sigma_Y : X \otimes Y \to Y \otimes X$ as a dinatural transformation ${}^\vee Y \otimes X \otimes Y \to X$
We say that $\mathcal{C}$ is *centralizable* if $Z(X) = \int^{Y \in \mathcal{C}} {}^\vee Y \otimes X \otimes Y$ exists for all $X \in \mathcal{C}$ (note that $Z(\mathbb{1})$ is the coend of $\mathcal{C}$). Then a half braiding σ corresponds with $\tilde{\sigma} : Z(X) \to X$

Theorem (BV)

If $\mathcal{C}$ is centralizable, then $Z : X \mapsto Z(X)$ is a **quasitriangular Hopf monad on** $\mathcal{C}$ and we have a braided isomorphism of categories

$$\begin{aligned} \mathcal{Z}(\mathcal{C}) &\to \mathcal{C}^Z \\ (X, \sigma) &\mapsto (X, \tilde{\sigma}) \end{aligned}$$

Monadicity of the center

Let $\mathcal{C}$ be a rigid category, with center $\mathcal{Z}(\mathcal{C})$.
Using duality, interpret a half-braiding $\sigma_Y : X \otimes Y \to Y \otimes X$ as a dinatural transformation ${}^\vee Y \otimes X \otimes Y \to X$
We say that $\mathcal{C}$ is *centralizable* if $Z(X) = \int^{Y \in \mathcal{C}} {}^\vee Y \otimes X \otimes Y$ exists for all $X \in \mathcal{C}$ (note that $Z(\mathbb{1})$ is the coend of $\mathcal{C}$). Then a half braiding σ corresponds with $\tilde{\sigma} : Z(X) \to X$

Theorem (BV)

If $\mathcal{C}$ is centralizable, then $Z : X \mapsto Z(X)$ is a **quasitriangular Hopf monad on** $\mathcal{C}$ and we have a braided isomorphism of categories

$$\mathcal{Z}(\mathcal{C}) \to \mathcal{C}^Z$$
$$(X, \sigma) \mapsto (X, \tilde{\sigma})$$

Remark: In general the Hopf monad Z is not augmented, i e. not representable by a Hopf algebra: *e. g.* $\mathcal{C} = \{\{G\text{-graded vector spaces}\}\}$, for G non abelian finite group.

The centralizer of a Hopf monad

Let C be a monoidal rigid category

The centralizer of a Hopf monad

Let $\mathcal{C}$ be a monoidal rigid category
A Hopf monad $T: \mathcal{C} \to \mathcal{C}$ is *centralizable* if

The centralizer of a Hopf monad

Let $\mathcal{C}$ be a monoidal rigid category
A Hopf monad $T: \mathcal{C} \to \mathcal{C}$ is *centralizable* if

$$Z_T(X) = \int^{Y \in \mathcal{C}} {}^{\vee}T(Y) \otimes X \otimes Y \quad \text{exists for all } X \in \mathrm{Ob}(X)$$

Proposition (BV)

If T is a centralizable Hopf monad, $Z_T: X \mapsto Z_T(X)$ is a Hopf monad called the centralizer of T.

The centralizer of a Hopf monad

Let $\mathcal{C}$ be a monoidal rigid category
A Hopf monad $T : \mathcal{C} \to \mathcal{C}$ is *centralizable* if

$$Z_T(X) = \int^{Y \in \mathcal{C}} {}^{\vee}T(Y) \otimes X \otimes Y \quad \text{exists for all } X \in \mathrm{Ob}(X)$$

Proposition (BV)

If T is a centralizable Hopf monad, $Z_T : X \mapsto Z_T(X)$ is a Hopf monad called the centralizer of T.

In particular the monad Z of the previous slide is the centralizer of $1_{\mathcal{C}}$.

The centralizer of a Hopf monad

Let $\mathcal{C}$ be a monoidal rigid category
A Hopf monad $T\colon \mathcal{C} \to \mathcal{C}$ is *centralizable* if

$$Z_T(X) = \int^{Y \in \mathcal{C}} {}^{\vee}T(Y) \otimes X \otimes Y \quad \text{exists for all } X \in \mathrm{Ob}(X)$$

Proposition (BV)

If T is a centralizable Hopf monad, $Z_T\colon X \mapsto Z_T(X)$ is a Hopf monad called the centralizer of T.

In particular the monad Z of the previous slide is the centralizer of $1_{\mathcal{C}}$.
In a sense the centralizer plays the role of the dual of the Hopf monad T.

Hopf monads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Hopf monads as 'quantum groupoids'

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear bimonads on ${}_R Mod_R$ with a right adjoint is are bialgebroids in the sense of Takeuchi [Szlachányi]

Hopf monads as 'quantum groupoids'

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_RMod_R, \otimes_R, {}_RR_R)$.

Facts

- linear bimonads on ${}_RMod_R$ with a right adjoint is are bialgebroids in the sense of Takeuchi [Szlacháni]
- linear Hopf monads on ${}_RMod_R$ with a right adjoints are a Hopf algebroids in the sense of Schauenburg.

Hopf monads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear bimonads on ${}_R Mod_R$ with a right adjoint is are bialgebroids in the sense of Takeuchi [Szlachányi]
- linear Hopf monads on ${}_R Mod_R$ with a right adjoints are a Hopf algebroids in the sense of Schauenburg.

Hopf algebroids are non-commutative avatars of groupoids.

Hopf monads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear bimonads on ${}_R Mod_R$ with a right adjoint is are bialgebroids in the sense of Takeuchi [Szlacháni]
- linear Hopf monads on ${}_R Mod_R$ with a right adjoints are a Hopf algebroids in the sense of Schauenburg.

Hopf algebroids are non-commutative avatars of groupoids. Complicated axioms $\rightsquigarrow$ a Hopf adjunction $\rightsquigarrow$ a Hopf monad (much easier to manipulate).

Hopf monads as 'quantum groupoids'

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear bimonads on ${}_R Mod_R$ with a right adjoint is are bialgebroids in the sense of Takeuchi [Szlachányi]
- linear Hopf monads on ${}_R Mod_R$ with a right adjoints are a Hopf algebroids in the sense of Schauenburg.

Hopf algebroids are non-commutative avatars of groupoids. Complicated axioms $\rightsquigarrow$ a Hopf adjunction $\rightsquigarrow$ a Hopf monad (much easier to manipulate). Using Hopf monads one shows:

Theorem (BVL)

A finite tensor category $\mathcal{C}$ over a field $\Bbbk$ is tensor equivalent to the category of A-modules for some bialgebroid A.

Hopf monads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear bimonads on ${}_R Mod_R$ with a right adjoint is are bialgebroids in the sense of Takeuchi [Szlacháni]
- linear Hopf monads on ${}_R Mod_R$ with a right adjoints are a Hopf algebroids in the sense of Schauenburg.

Hopf algebroids are non-commutative avatars of groupoids. Complicated axioms $\rightsquigarrow$ a Hopf adjunction $\rightsquigarrow$ a Hopf monad (much easier to manipulate). Using Hopf monads one shows:

Theorem (BVL)

A finite tensor category $\mathcal{C}$ over a field $\Bbbk$ is tensor equivalent to the category of A-modules for some bialgebroid A.

Given a $\Bbbk$- equivalence $\mathcal{C} \overset{\Bbbk}{\simeq}_R$ *mod* for some finite dimensional $\Bbbk$- algebra R, one constructs a canonical Hopf algebroid A over R.

Outlook of General Theory of Hopf monads

- Tannaka dictionary

Outlook of General Theory of Hopf monads

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem

Outlook of General Theory of Hopf monads

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals (with values in a certain autoequivalence of $\mathcal{C}$)

Outlook of General Theory of Hopf monads

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals (with values in a certain autoequivalence of $\mathcal{C}$)
- Semisimplicity, Maschke criterion

Outlook of General Theory of Hopf monads

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals (with values in a certain autoequivalence of $\mathcal{C}$)
- Semisimplicity, Maschke criterion
- The drinfeld double of a Hopf monad

Outlook of General Theory of Hopf monads

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals (with values in a certain autoequivalence of $\mathcal{C}$)
- Semisimplicity, Maschke criterion
- The drinfeld double of a Hopf monad
- Cross-products

Outlook of General Theory of Hopf monads

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals (with values in a certain autoequivalence of $\mathcal{C}$)
- Semisimplicity, Maschke criterion
- The drinfeld double of a Hopf monad
- Cross-products
- Bosonization for Hopf monads

Outlook of General Theory of Hopf monads

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals (with values in a certain autoequivalence of $\mathcal{C}$)
- Semisimplicity, Maschke criterion
- The drinfeld double of a Hopf monad
- Cross-products
- Bosonization for Hopf monads
- Applications to construction and comparison of quantum invariants (non-braided setting)

Hopf modules and Sweedler's Theorem for Hopf Monads

T Hopf monad on $\mathcal{C} \rightsquigarrow T\mathbb{1}$ is a coalgebra in $\mathcal{C}$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit ε)

Hopf modules and Sweedler's Theorem for Hopf Monads

T Hopf monad on $\mathcal{C} \rightsquigarrow T\mathbb{1}$ is a coalgebra in $\mathcal{C}$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit ε) $\rightsquigarrow$ lifts to a coalgebra $\hat{C} = F^T(\mathbb{1})$ in $\mathcal{C}^T$. Moreover we have a natural isomorphism

$$\sigma : \hat{C} \otimes ? \to ? \otimes \hat{C}.$$

Hopf modules and Sweedler's Theorem for Hopf Monads

T Hopf monad on $\mathcal{C} \rightsquigarrow T\mathbb{1}$ is a coalgebra in $\mathcal{C}$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit ε) $\rightsquigarrow$ lifts to a coalgebra $\hat{C} = F^T(\mathbb{1})$ in $\mathcal{C}^T$. Moreover we have a natural isomorphism

$$\sigma : \hat{C} \otimes ? \to ? \otimes \hat{C}.$$

Proposition (BVL)

σ is a half-braiding

Hopf modules and Sweedler's Theorem for Hopf Monads

T Hopf monad on $\mathcal{C} \rightsquigarrow T\mathbb{1}$ is a coalgebra in $\mathcal{C}$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit ε) $\rightsquigarrow$ lifts to a coalgebra $\hat{C} = F^T(\mathbb{1})$ in $\mathcal{C}^T$. Moreover we have a natural isomorphism

$$\sigma : \hat{C} \otimes ? \to ? \otimes \hat{C}.$$

Proposition (BVL)

σ is a half-braiding and $(\hat{C}, \sigma)$ is a cocommutative coalgebra in $\mathcal{Z}(\mathcal{C}^T)$ called the *induced central coalgebra* of T.

Hopf modules and Sweedler's Theorem for Hopf Monads

T Hopf monad on $\mathcal{C} \rightsquigarrow T\mathbb{1}$ is a coalgebra in $\mathcal{C}$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit ε) $\rightsquigarrow$ lifts to a coalgebra $\hat{C} = F^T(\mathbb{1})$ in $\mathcal{C}^T$. Moreover we have a natural isomorphism

$$\sigma : \hat{C} \otimes ? \to ? \otimes \hat{C}.$$

Proposition (BVL)

σ is a half-braiding and $(\hat{C}, \sigma)$ is a cocommutative coalgebra in $\mathcal{Z}(\mathcal{C}^T)$ called the *induced central coalgebra* of T.

A (right) T-Hopf module is a (right) $\hat{C}$-comodule in $\mathcal{C}^T$

Hopf modules and Sweedler's Theorem for Hopf Monads

T Hopf monad on $\mathcal{C} \rightsquigarrow T\mathbb{1}$ is a coalgebra in $\mathcal{C}$ (coproduct $\Delta_{\mathbb{1},\mathbb{1}}$, counit ε) $\rightsquigarrow$ lifts to a coalgebra $\hat{C} = F^T(\mathbb{1})$ in $\mathcal{C}^T$. Moreover we have a natural isomorphism

$$\sigma : \hat{C} \otimes ? \to ? \otimes \hat{C}.$$

Proposition (BVL)

σ is a half-braiding and $(\hat{C}, \sigma)$ is a cocommutative coalgebra in $\mathcal{Z}(\mathcal{C}^T)$ called the *induced central coalgebra* of T.

A (right) T-Hopf module is a (right) $\hat{C}$-comodule in $\mathcal{C}^T$, *i. e.* a data (M, r, ∂) with (M, r) a T-module, (M, ∂) a $T\mathbb{1}$-comodule + T-linearity of ∂.

Under suitable exactness conditions (T is conservative, $\mathcal{C}$ has coequalizers and T preserves them):

Under suitable exactness conditions (T is conservative, $\mathcal{C}$ has coequalizers and T preserves them):

Theorem (BVL)

The assignment $X \mapsto (TX, \mu_X, \Delta_{X,\mathbb{1}})$ is an equivalence of categories

$$Q : \mathcal{C} \xrightarrow{\simeq} \{\{T\text{-}\,\mathrm{Hopf\ modules}\}\}$$

with quasi-inverse the functor *coinvariant part*.

Under suitable exactness conditions (T is conservative, $\mathcal{C}$ has coequalizers and T preserves them):

Theorem (BVL)

The assignment $X \mapsto (TX, \mu_X, \Delta_{X,\mathbb{1}})$ is an equivalence of categories

$$Q : \mathcal{C} \xrightarrow{\simeq} \{\{T\text{- Hopf modules}\}\}$$

with quasi-inverse the functor *coinvariant part*.
Moreover if $\mathcal{C}$ has equalizers and T preserves them, Q is a monoidal equivalence, the category of Hopf modules (i.e. $\hat{C}$- comodules) being endowed with the cotensor product over $\hat{C}$.

Proof of Sweedler's theorem for Hopf monads

An adjunction $\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$ $\rightsquigarrow$ a comonad $\hat{T} = (FU, F(\eta_U), \varepsilon)$ on $\mathcal{D}$.

Proof of Sweedler's theorem for Hopf monads

An adjunction $\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$ $\rightsquigarrow$ a comonad $\hat{T} = (FU, F(\eta_U), \varepsilon)$ on $\mathcal{D}$.

Denoting $\mathcal{D}_{\hat{T}}$ the category of $\hat{T}$-comodules we have a cocomparison

functor $\hat{K}$:

Proof of Sweedler's theorem for Hopf monads

An adjunction $\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$ $\rightsquigarrow$ a comonad $\hat{T} = (FU, F(\eta_U), \varepsilon)$ on $\mathcal{D}$.

Denoting $\mathcal{D}_{\hat{T}}$ the category of $\hat{T}$-comodules we have a cocomparison functor $\hat{K}$:

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, bend left] \arrow[dr, bend left] & \\ \mathcal{C} \arrow[ur, bend left] \arrow[rr, "\hat{K}"'] & & \mathcal{D}_{\hat{T}} \arrow[ul, bend left] \end{tikzcd}$$

The adjunction (F, U) is *comonadic* if $\hat{K}$ equivalence.

Proof of Sweedler's theorem for Hopf monads

An adjunction
$$\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$$
$\rightsquigarrow$ a comonad $\hat{T} = (FU, F(\eta_U), \varepsilon)$ on $\mathcal{D}$.

Denoting $\mathcal{D}_{\hat{T}}$ the category of $\hat{T}$-comodules we have a cocomparison functor $\hat{K}$:

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, bend right] \arrow[dr, bend left] & \\ \mathcal{C} \arrow[ur, bend left] \arrow[rr, "\hat{K}"'] & & \mathcal{D}_{\hat{T}} \arrow[ul, bend left] \end{tikzcd}$$

The adjunction (F, U) is *comonadic* if $\hat{K}$ equivalence.

If T is a monad on $\mathcal{C}$, its adjunction is comonadic under *suitable exactness assumptions* (descent), *i. e.* $\hat{K}: \mathcal{C} \to (\mathcal{C}^T)_{\hat{T}}$ is an equivalence.

Proof of Sweedler's theorem for Hopf monads

An adjunction $\begin{tikzcd} \mathcal{D} \arrow[d, bend left, "U"] \\ \mathcal{C} \arrow[u, bend left, "F"] \end{tikzcd}$ $\rightsquigarrow$ a comonad $\hat{T} = (FU, F(\eta_U), \varepsilon)$ on $\mathcal{D}$.

Denoting $\mathcal{D}_{\hat{T}}$ the category of $\hat{T}$-comodules we have a cocomparison functor $\hat{K}$:

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, bend left] \arrow[dr, bend left] & \\ \mathcal{C} \arrow[ur, bend left] \arrow[rr, "\hat{K}"'] & & \mathcal{D}_{\hat{T}} \arrow[ul, bend left] \end{tikzcd}$$

The adjunction (F, U) is *comonadic* if $\hat{K}$ equivalence.

If T is a monad on $\mathcal{C}$, its adjunction is comonadic under *suitable exactness assumptions* (descent), *i. e.* $\hat{K} : \mathcal{C} \to (\mathcal{C}^T)_{\hat{T}}$ is an equivalence.
For T Hopf monad, we have an isomorphism of comonads on $\mathcal{C}^T$

$$\phi : \hat{T} \xrightarrow{\sim} ? \otimes \hat{C}$$

defined by $\phi_{(M,r)} = (r \otimes \mathrm{id}_{T(\mathbb{1})}) T_{M,\mathbb{1}} : TM \to M \otimes T\mathbb{1}$.
Hence ${\mathcal{C}^T}_{\hat{T}} \xrightarrow{\sim} \{\{\text{right } T\text{-Hopf modules}\}\}$ $\square$

We now consider tensor categories over a field $\Bbbk$.

We now consider tensor categories over a field $\Bbbk$.

If $\mathcal{C}$ is a tensor category, its Ind-completion $\mathrm{Ind}\mathcal{C}$ is a monoidal abelian category containing $\mathcal{C}$ as a full subcategory and whose objects are formal filtering colimits of objects of $\mathcal{C}$.

We now consider tensor categories over a field $\Bbbk$.

If $\mathcal{C}$ is a tensor category, its Ind-completion $\mathrm{Ind}\mathcal{C}$ is a monoidal abelian category containing $\mathcal{C}$ as a full subcategory and whose objects are formal filtering colimits of objects of $\mathcal{C}$. For instance $\mathrm{Ind}\,\mathrm{vect} = \mathrm{Vect}$, and $\mathrm{Ind}\,\mathrm{comod}H = \mathrm{Comod}H$.

We now consider tensor categories over a field $\Bbbk$.

If $\mathcal{C}$ is a tensor category, its Ind-completion $\mathrm{Ind}\mathcal{C}$ is a monoidal abelian category containing $\mathcal{C}$ as a full subcategory and whose objects are formal filtering colimits of objects of $\mathcal{C}$. For instance $\mathrm{Ind}\,\mathrm{vect} = \mathrm{Vect}$, and $\mathrm{Ind}\,\mathrm{comod}H = \mathrm{Comod}H$. Note that these are no longer rigid.

We now consider tensor categories over a field $\mathbb{k}$.

If $\mathcal{C}$ is a tensor category, its Ind-completion $\mathrm{Ind}\mathcal{C}$ is a monoidal abelian category containing $\mathcal{C}$ as a full subcategory and whose objects are formal filtering colimits of objects of $\mathcal{C}$. For instance $\mathrm{Ind}\,\mathrm{vect} = \mathrm{Vect}$, and $\mathrm{Ind}\,\mathrm{comod}H = \mathrm{Comod}H$. Note that these are no longer rigid.

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. There exists a $\mathbb{k}$- linear left exact comonad on $\mathrm{Ind}\mathcal{C}$ such that we have a commutative diagram:

We now consider tensor categories over a field $\mathbb{k}$.

If $\mathcal{C}$ is a tensor category, its Ind-completion $\mathrm{Ind}\mathcal{C}$ is a monoidal abelian category containing $\mathcal{C}$ as a full subcategory and whose objects are formal filtering colimits of objects of $\mathcal{C}$. For instance $\mathrm{Ind}\,\mathrm{vect} = \mathrm{Vect}$, and $\mathrm{Ind}\,\mathrm{comod}H = \mathrm{Comod}H$. Note that these are no longer rigid.

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. There exists a $\mathbb{k}$- linear left exact comonad on $\mathrm{Ind}\mathcal{C}$ such that we have a commutative diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\simeq_{\otimes}"'] & & \mathrm{vect} \\ & \mathcal{D}_T \arrow[ur] & \end{tikzcd}$$

where $\mathcal{C}_T$ is the category of T-comodule whose underlying object is in $\mathcal{C}$.

Proof

The functor $F : \mathcal{C} \to \mathcal{D}$ extends to a linear faithful exact functor $\mathrm{Ind}F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{D}$ which preserves colimits and is strong monoidal.

Proof

The functor $F : \mathcal{C} \to \mathcal{D}$ extends to a linear faithful exact functor $\mathrm{Ind}F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{D}$ which preserves colimits and is strong monoidal. $\mathrm{Ind}F$ has a right adjoint, denoted by R.

Proof

The functor $F : \mathcal{C} \to \mathcal{D}$ extends to a linear faithful exact functor $\mathrm{Ind}F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{D}$ which preserves colimits and is strong monoidal.
$\mathrm{Ind}F$ has a right adjoint, denoted by R.
It is also a monoidal adjunction, which is Hopf.

Proof

The functor $F : \mathcal{C} \to \mathcal{D}$ extends to a linear faithful exact functor $\mathrm{Ind}F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{D}$ which preserves colimits and is strong monoidal.
$\mathrm{Ind}F$ has a right adjoint, denoted by R.
It is also a monoidal adjunction, which is Hopf. Its comonad $T = \mathrm{Ind}FR$ is a Hopf comonad on $\mathrm{Ind}\mathcal{C}$.

Proof

The functor $F : \mathcal{C} \to \mathcal{D}$ extends to a linear faithful exact functor $\mathrm{Ind}F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{D}$ which preserves colimits and is strong monoidal.
$\mathrm{Ind}F$ has a right adjoint, denoted by R.
It is also a monoidal adjunction, which is Hopf. Its comonad $T = \mathrm{Ind}FR$ is a Hopf comonad on $\mathrm{Ind}\mathcal{C}$.
$\mathrm{Ind}F$ being faithful exact, the adjunction $(\mathrm{Ind}F, R)$ is comonadic by Beck, hence the theorem.

Proof

The functor $F : \mathcal{C} \to \mathcal{D}$ extends to a linear faithful exact functor $\mathrm{Ind}F : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{D}$ which preserves colimits and is strong monoidal.
$\mathrm{Ind}F$ has a right adjoint, denoted by R.
It is also a monoidal adjunction, which is Hopf. Its comonad $T = \mathrm{Ind}FR$ is a Hopf comonad on $\mathrm{Ind}\mathcal{C}$.
$\mathrm{Ind}F$ being faithful exact, the adjunction $(\mathrm{Ind}F, R)$ is comonadic by Beck, hence the theorem.

Example

If $\mathcal{D} = \mathrm{vect}$, a linear Hopf comonad on Vect is of the form $H \otimes ?$ for some Hopf algebra H, so we recover the classical tannakian result.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. We say that F is *dominant* if the right adjoint R of $\mathrm{Ind}F$ is faithful exact.

Then applying the classification theorem for Hopf modules in its dual form we obtain:

Theorem

If F is dominant, there exists a commutative algebra (A, σ) in $\mathcal{Z}(\mathrm{Ind}\mathcal{C})$ - the induced central algebra of T - such that we have a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F_A"] \arrow[dr, "F"'] & & A\text{-}\operatorname{mod}_{\mathcal{C}} \\ & \mathcal{D} \arrow[ur, "\simeq_{\otimes}"'] & \end{tikzcd}$$

where A-mod is the category of 'finite type' A-modules in $\mathrm{Ind}\mathcal{C}$ (=quotients of $A \otimes X$, $X \in \mathcal{C}$), with tensor product $\otimes_{A,\sigma}$, and F_A is the tensor functor $X \mapsto A \otimes X$.

If $\mathcal{D} = \mathrm{vect}\Bbbk$ and $\mathcal{C}$, F are symmetric, then A is Deligne's trivializing algebra.

An exact sequence of Hopf algebras in the sense of Schneider is a sequence

$$K \xrightarrow{i} H \xrightarrow{p} H'$$

of Hopf algebras such that

1. $p^{-1}(0)$ is a normal Hopf ideal of H;
2. H is right faithfully coflat over H';
3. i is a categorical kernel of p.

An exact sequence of Hopf algebras in the sense of Schneider is a sequence

$$K \xrightarrow{i} H \xrightarrow{p} H'$$

of Hopf algebras such that

1. $p^{-1}(0)$ is a normal Hopf ideal of H;
2. H is right faithfully coflat over H';
3. i is a categorical kernel of p.

We extend this notion to tensor categories.
Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. We denote by $\Bbbk_F$ the full tensor subcategory of $\mathcal{C}$

$$\Bbbk_F = \{X \in \mathcal{C} \mid F(X) \text{is trivial}\}$$

An exact sequence of Hopf algebras in the sense of Schneider is a sequence

$$K \xrightarrow{i} H \xrightarrow{p} H'$$

of Hopf algebras such that

1. $p^{-1}(0)$ is a normal Hopf ideal of H;
2. H is right faithfully coflat over H';
3. i is a categorical kernel of p.

We extend this notion to tensor categories.
Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. We denote by $\Bbbk_F$ the full tensor subcategory of $\mathcal{C}$

$$\Bbbk_F = \{X \in \mathcal{C} \mid F(X) \text{is trivial}\}$$

Note that F induces a fiber functor $\mathcal{K}_F \to \mathrm{vect}$, $X \mapsto \mathrm{Hom}(\mathbb{1}, F(X))$.
We say that F is *normal* if the right adjoint R of $\mathrm{Ind}F$ satisfies $R(\mathbb{1}) \in \mathrm{Ind}(\mathcal{K}_F)$.
This means that the subcategory $< \mathbb{1} >$ of $\mathrm{Ind}\mathcal{C}$ generated by $\mathbb{1}$ is stable under the Hopf comonad $T = UR$ which encodes F.

An exact sequence of tensor categories is a sequence

$$\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

of tensor categories such that:

1. F is normal and dominant;
2. f induces a tensor equivalence $\mathcal{C}' \to \mathcal{K}_F$.

An exact sequence of tensor categories is a sequence

$$\mathcal{C}' \xrightarrow{f} \mathcal{C} \xrightarrow{F} \mathcal{C}''$$

of tensor categories such that:

1. F is normal and dominant;
2. f induces a tensor equivalence $\mathcal{C}' \to \mathcal{K}_F$.

If $H' \to H \to H''$ is an exact sequence of Hopf algebras, then

$$\mathrm{comod} H' \to \mathrm{comod} H \to \mathrm{comod} H''$$

is an exact sequence of tensor categories, and, if H is finite dimensional,

$$\text{mod } H'' \to \quad \text{mod } H \to \quad \text{mod } H'$$

is also an exact sequence of tensor categories.

Exact sequences of tensor categories are classified by certain Hopf (co)-monads.

Exact sequences of tensor categories are classified by certain Hopf (co)-monads.
A linear exact Hopf comonad T on tensor category $\mathcal{C}$ is normal if $T(\mathbb{1}) \in < \mathbb{1} >$. We have $< \mathbb{1} > \simeq \mathrm{Vect}$, so if T is normal it restricts to a Hopf algebra H on Vect. If in addition T is faithful, we have an exact sequence of tensor categories

$$\mathrm{comod} H \to \mathcal{C}_T \to \mathcal{C}$$

and 'all extensions of $\mathcal{C}$ by $\mathrm{comod} H$' are of this form up to tensor equivalence [one has to be more precise].

Examples

Equivariantization

Examples

Equivariantization

Let G be a finite group acting on a tensor category $\mathcal{C}$ by tensor automorphisms $(T_g)_{g \in G}$. Then we have an exact sequence

$$\mathrm{rep} G \to \mathcal{C}^G \to \mathcal{C}$$

where $\mathcal{C}^G \to \mathcal{C}$ is the equivariantization functor.

Examples

Equivariantization

Let G be a finite group acting on a tensor category $\mathcal{C}$ by tensor automorphisms $(T_g)_{g\in G}$. Then we have an exact sequence

$$\mathrm{rep}G \to \mathcal{C}^G \to \mathcal{C}$$

where $\mathcal{C}^G \to \mathcal{C}$ is the equivariantization functor.
The endofunctor $T = \bigoplus T_g$ admits a structure of Hopf comonad T^G (it admits also a structure of Hopf monad), and $\mathcal{C}^G$ is just $\mathbb{C}^{T^G}$. The Hopf comonad T^G is normal faithful exact, and its associated Hopf algebra is k^G. It has a certain commutativity property. These conditions characterize Hopf comonads corresponding with equivariantizations (at least over $\mathbb{C}$).

24. More on Hopf monads

BV1. *Hopf Diagrams and Quantum Invariants,* AGT **5** (2005) 1677-1710.

Where Hopf diagram are introduced as a means for computing the Reshetikhin-Turaev invariant in terms of the coend of a ribbon category and its structural morphisms.

BV2. *Hopf Monads,* Advances in Math. **215** (2007), 679-733.

Where the notion of Hopf monad is introduced, and several fundamental results of the theory of finite dimensional Hopf algebras are extended thereto.

BV3. *Categorical Centers and Reshetikhin-Turaev Invariants,* Acta Mathematica Vietnamica **33** 3, 255-279

Where the coend of the center of a fusion spherical category over a ring is described, the modularity of the center, proven, and the corresponding Reshetikhin-Turaev invariant, constructed.

BV4. *Quantum Double of Hopf monads and Categorical Centers,* arXiv:0812.2443, to appear in Transactions of the American Mathematical Society (2010)

Where the general theory of centralizers and doubles of Hopf monads is expounded.

BLV. *Hopf Monads on Monoidal Categories,* arXiv:1003.1920.

Where Hopf monads are defined anew in the monoidal world

BN. *Exact sequences of tensor categories*, arXiv:1006.0569.

See also: `http://www.math.univ-montp2.fr/~bruguieres/recherche.html`