Dualité de Galois-Grothendieck et dualité tannakienne

Alain Bruguières

(Université Montpellier II)

Journées en l'honneur de Georges Maltsiniotis

Université Paris Diderot - Paris 7

17 - 18 Juin 2013

In SGA1 (1960) Grothendieck defines the *étale fundamental group* $\pi(X, p)$ of a scheme S at a geometric point p.

In SGA1 (1960) Grothendieck defines the *étale fundamental group* $\pi(X, p)$ of a scheme S at a geometric point p.
Consider the *fibre functor* at p :

$$\omega_p : \begin{cases} \{\text{finite étale coverings of } S\} & \rightarrow & \text{set} \\ E & \mapsto & E_p \end{cases}$$

In SGA1 (1960) Grothendieck defines the *étale fundamental group* $\pi(X,p)$ of a scheme S at a geometric point p.
Consider the *fibre functor* at p :

$$\omega_p : \begin{cases} \{\text{finite étale coverings of } S\} & \rightarrow & \text{set} \\ E & \mapsto & E_p \end{cases}$$

Set $\pi = \pi(X,p) =$ 'profinite group of automorphisms' of ω_p. Then

In SGA1 (1960) Grothendieck defines the *étale fundamental group* $\pi(X,p)$ of a scheme S at a geometric point p.
Consider the *fibre functor* at p :

$$\omega_p : \begin{cases} \{\text{finite étale coverings of } S\} & \rightarrow & \mathrm{set} \\ E & \mapsto & E_p \end{cases}$$

Set $\pi = \pi(X,p) =$ 'profinite group of automorphisms' of ω_p. Then

$$\{\text{finite étale coverings of } S\} \cong \{\text{continuous finite } \pi\text{-sets}\}$$

In SGA1 (1960) Grothendieck defines the *étale fundamental group* $\pi(X, p)$ of a scheme S at a geometric point p.
Consider the *fibre functor* at p :

$$\omega_p : \begin{cases} \{\text{finite étale coverings of } S\} & \to & \mathrm{set} \\ E & \mapsto & E_p \end{cases}$$

Set $\pi = \pi(X, p) =$ 'profinite group of automorphisms' of ω_p. Then

$$\{\text{finite étale coverings of } S\} \cong \{\text{continuous finite } \pi\text{-sets}\}$$

This follows from

Theorem

Let $\mathcal{C}$ be a category and $\omega : \mathcal{C} \to \mathrm{set}$ be a functor, and assume :

1. *$\mathcal{C}$ has finite limits and colimits, and ω is exact ;*
2. *in $\mathcal{C}$ any morphism factorizes as mono $\circ$ epi, and epis are strict ;*
3. *ω is conservative ;*
4. *in $\mathcal{C}$ monos are summands.*

Then $\mathcal{C} \cong G - \mathrm{set}$, where G is the profinite group $\mathrm{Aut}(\omega)$.

Tannaka theory in its algebraic form (Saavedra Rivano's thesis in 1972, Deligne-Milne, Deligne...).

Theorem

Neutral case. *Let $\mathcal{C}$ be a symmetric tensor category over a field $\Bbbk$, $\omega : \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ a symmetric fibre functor, that is, a strong monoidal symmetric $\Bbbk$-linear exact functor.*

Tannaka theory in its algebraic form (Saavedra Rivano's thesis in 1972, Deligne-Milne, Deligne...).

Theorem

Neutral case. *Let* $\mathcal{C}$ *be a symmetric tensor category over a field* $\mathbb{k}$*,* $\omega : \mathcal{C} \to \mathrm{Vect}_{\mathbb{k}}$ *a symmetric fibre functor, that is, a strong monoidal symmetric* $\mathbb{k}$*-linear exact functor. Then* $G = \mathrm{Aut}(\omega)$ *is an affine algebraic group over* $\mathbb{k}$ *and* $\mathcal{C} \cong \mathrm{rep}G$*.*

Tannaka theory in its algebraic form (Saavedra Rivano's thesis in 1972, Deligne-Milne, Deligne...).

Theorem

Neutral case. *Let $\mathcal{C}$ be a symmetric tensor category over a field $\Bbbk$, $\omega : \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ a symmetric fibre functor, that is, a strong monoidal symmetric $\Bbbk$-linear exact functor. Then $G = \mathrm{Aut}(\omega)$ is an affine algebraic group over $\Bbbk$ and $\mathcal{C} \cong \mathrm{rep}G$.*

2) **General case.** *Still works if you replace $\mathrm{Vect}_{\Bbbk}$ with Mod_B, with B commutative $\Bbbk$-algebra $\neq 0$. Then G is an affine groupoid with base $\mathrm{Spec}B$.*

Tannaka theory in its algebraic form (Saavedra Rivano's thesis in 1972, Deligne-Milne, Deligne...).

Theorem

Neutral case. *Let $\mathcal{C}$ be a symmetric tensor category over a field $\Bbbk$, $\omega : \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ a symmetric fibre functor, that is, a strong monoidal symmetric $\Bbbk$-linear exact functor. Then $G = \mathrm{Aut}(\omega)$ is an affine algebraic group over $\Bbbk$ and $\mathcal{C} \cong \mathrm{rep} G$.*
2) **General case.** *Still works if you replace $\mathrm{Vect}_{\Bbbk}$ with Mod_B, with B commutative $\Bbbk$-algebra $\neq 0$. Then G is an affine groupoid with base $\mathrm{Spec} B$.*

This theorem has several variants, notably :

- Nori's approach, where $\Bbbk$ is a coherent commutative ring and G is a Hopf algebra in the pro objects of mod $_{\Bbbk}$

Tannaka theory in its algebraic form (Saavedra Rivano's thesis in 1972, Deligne-Milne, Deligne...).

Theorem

Neutral case. *Let* $\mathcal{C}$ *be a symmetric tensor category over a field* $\Bbbk$*,* $\omega : \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ *a symmetric fibre functor, that is, a strong monoidal symmetric* $\Bbbk$*-linear exact functor. Then* $G = \mathrm{Aut}(\omega)$ *is an affine algebraic group over* $\Bbbk$ *and* $\mathcal{C} \cong \mathrm{rep} G$*.*
2) **General case.** *Still works if you replace* $\mathrm{Vect}_{\Bbbk}$ *with* Mod_B*, with B commutative* $\Bbbk$*-algebra* $\neq 0$*. Then G is an affine groupoid with base* $\mathrm{Spec} B$*.*

This theorem has several variants, notably :

- Nori's approach, where $\Bbbk$ is a coherent commutative ring and G is a Hopf algebra in the pro objects of mod $_{\Bbbk}$
- Non-commutative case, where $\mathcal{C}$ is no longer symmetric $\rightsquigarrow$ G is a 'quantum' Hopf algebroid [Maltsiniotis, B]

Tannaka theory in its algebraic form (Saavedra Rivano's thesis in 1972, Deligne-Milne, Deligne...).

Theorem

Neutral case. *Let* $\mathcal{C}$ *be a symmetric tensor category over a field* $\Bbbk$*,* $\omega : \mathcal{C} \to \mathrm{Vect}_{\Bbbk}$ *a symmetric fibre functor, that is, a strong monoidal symmetric* $\Bbbk$*-linear exact functor. Then* $G = \mathrm{Aut}(\omega)$ *is an affine algebraic group over* $\Bbbk$ *and* $\mathcal{C} \cong \mathrm{rep}G$*.*
2) **General case.** *Still works if you replace* $\mathrm{Vect}_{\Bbbk}$ *with* Mod_B*, with B commutative* $\Bbbk$*-algebra* $\neq 0$*. Then G is an affine groupoid with base* $\mathrm{Spec}B$*.*

This theorem has several variants, notably :

- Nori's approach, where $\Bbbk$ is a coherent commutative ring and G is a Hopf algebra in the pro objects of mod $_{\Bbbk}$
- Non-commutative case, where $\mathcal{C}$ is no longer symmetric $\rightsquigarrow$ G is a 'quantum' Hopf algebroid [Maltsiniotis, B]
- Fully non-commutative case, B non-commutative, $\omega : \mathcal{C} \to_B \mathrm{Bimod}_B \rightsquigarrow G$ is a Hopf algebroid in the sense of Takeuchi/Schauenburg

Aim : unify Galois-Grothendieck duality and Tannaka duality (neutral or not, commutative or not).

Aim : unify Galois-Grothendieck duality and Tannaka duality (neutral or not, commutative or not).

Question

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. Describe $\mathcal{C}$ as the category of representations of a **Hopf structure** living at the level of $\mathcal{B}$?

Aim : unify Galois-Grothendieck duality and Tannaka duality (neutral or not, commutative or not).

Question

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. Describe $\mathcal{C}$ as the category of representations of a **Hopf structure** living at the level of $\mathcal{B}$?

In the neutral case, a partial answer has been given by Xavier Rochard in his unpublished thesis (1998).

Aim : unify Galois-Grothendieck duality and Tannaka duality (neutral or not, commutative or not).

Question

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. Describe $\mathcal{C}$ as the category of representations of a **Hopf structure** living at the level of $\mathcal{B}$?

In the neutral case, a partial answer has been given by Xavier Rochard in his unpublished thesis (1998).
More recently, work by Daniel Schäppi (*Tannaka duality for comonoids in cosmoi*).

Aim : unify Galois-Grothendieck duality and Tannaka duality (neutral or not, commutative or not).

Question

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. Describe $\mathcal{C}$ as the category of representations of a **Hopf structure** living at the level of $\mathcal{B}$?

In the neutral case, a partial answer has been given by Xavier Rochard in his unpublished thesis (1998).

More recently, work by Daniel Schäppi (*Tannaka duality for comonoids in cosmoi*).

Our first step to address this question will be to study (co)monoidal adjunction and introduce Hopf (co)monads which serve exactly that kind of purpose.

A **monoidal adjunction** is an adjunction $\omega : \mathcal{C} \rightleftarrows \mathcal{B} : R$ where $\mathcal{C}$, $\mathcal{B}$ are monoidal and ω is strong monoidal.

A **monoidal adjunction** is an adjunction $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$ where $\mathcal{C}$, $\mathcal{B}$ are monoidal and ω is strong monoidal. Then R, and the adjunction morphisms $\eta : 1_{\mathcal{C}} \to R\omega$ and $\varepsilon : \omega R \to 1_{\mathcal{B}}$ are monoidal.

A **monoidal adjunction** is an adjunction $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$ where $\mathcal{C}$, $\mathcal{B}$ are monoidal and ω is strong monoidal. Then R, and the adjunction morphisms $\eta : 1_{\mathcal{C}} \to R\omega$ and $\varepsilon : \omega R \to 1_{\mathcal{B}}$ are monoidal.

It is **Hopf** if for all $b \in \mathcal{B}$, $c \in \mathcal{C}$ the **fusion morphisms** are isomorphisms :

$$H^l_{b,c} : c \otimes Rb \to R(\omega c \otimes b) \quad \text{and} \quad H^r_{b,c} : Rb \otimes c \to R(b \otimes \omega c)$$

A **monoidal adjunction** is an adjunction $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$ where $\mathcal{C}$, $\mathcal{B}$ are monoidal and ω is strong monoidal. Then R, and the adjunction morphisms $\eta : 1_{\mathcal{C}} \to R\omega$ and $\varepsilon : \omega R \to 1_{\mathcal{B}}$ are monoidal.
It is **Hopf** if for all $b \in \mathcal{B}$, $c \in \mathcal{C}$ the **fusion morphisms** are isomorphisms :

$$H^l_{b,c} : c \otimes Rb \to R(\omega c \otimes b) \quad \text{and} \quad H^r_{b,c} : Rb \otimes c \to R(b \otimes \omega c)$$

Such is the case if $\mathcal{C}$ has duals or if $\mathcal{C}$ and ω are coclosed.

A **monoidal adjunction** is an adjunction $\omega : \mathcal{C} \rightleftarrows \mathcal{B} : R$ where $\mathcal{C}$, $\mathcal{B}$ are monoidal and ω is strong monoidal. Then R, and the adjunction morphisms $\eta : 1_{\mathcal{C}} \to R\omega$ and $\varepsilon : \omega R \to 1_{\mathcal{B}}$ are monoidal.
It is **Hopf** if for all $b \in \mathcal{B}$, $c \in \mathcal{C}$ the **fusion morphisms** are isomorphisms :

$$H^l_{b,c} : c \otimes Rb \to R(\omega c \otimes b) \quad \text{and} \quad H^r_{b,c} : Rb \otimes c \to R(b \otimes \omega c)$$

Such is the case if $\mathcal{C}$ has duals or if $\mathcal{C}$ and ω are coclosed.

Example

B bialgebra over $\Bbbk$ $\rightsquigarrow$ a monoidal adjunction $U : \mathrm{Comod}B \rightleftarrows \mathrm{Vect}_{\Bbbk} : \mathit{cofree}$

A **monoidal adjunction** is an adjunction $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$ where $\mathcal{C}$, $\mathcal{B}$ are monoidal and ω is strong monoidal. Then R, and the adjunction morphisms $\eta : 1_{\mathcal{C}} \to R\omega$ and $\varepsilon : \omega R \to 1_{\mathcal{B}}$ are monoidal.
It is **Hopf** if for all $b \in \mathcal{B}$, $c \in \mathcal{C}$ the **fusion morphisms** are isomorphisms :

$$H^l_{b,c} : c \otimes Rb \to R(\omega c \otimes b) \quad \text{and} \quad H^r_{b,c} : Rb \otimes c \to R(b \otimes \omega c)$$

Such is the case if $\mathcal{C}$ has duals or if $\mathcal{C}$ and ω are coclosed.

Example

B bialgebra over $\Bbbk$ $\rightsquigarrow$ a monoidal adjunction $\mathrm{Comod}B \underset{cofree}{\overset{U}{\rightleftarrows}} \mathrm{Vect}_{\Bbbk}$

It is Hopf iff B is a Hopf algebra (w. bijective antipode).

Hopf comonads

A **monoidal comonad** [I. Moerdijk] on a monoidal category $\mathscr{B}$ is a a comonad (T, Δ, ε) such that T, Δ and ε are monoidal, that is :

Hopf comonads

A **monoidal comonad** [I. Moerdijk] on a monoidal category $\mathcal{B}$ is a a comonad (T, Δ, ε) such that T, Δ and ε are monoidal, that is :

$$T\colon \mathcal{B} \to \mathcal{B}, \quad \Delta\colon T \to T^2 \text{ (coproduct)}, \qquad \varepsilon\colon T \to 1_{\mathcal{B}} \text{ (counit)}$$

$$\mu_{a,b}\colon Ta \otimes Tb \to T(a \otimes b), \qquad \eta\colon \mathbb{1} \to T\mathbb{1}$$

Hopf comonads

A **monoidal comonad** [I. Moerdijk] on a monoidal category $\mathcal{B}$ is a a comonad (T, Δ, ε) such that T, Δ and ε are monoidal, that is :

$$T\colon \mathcal{B} \to \mathcal{B}, \quad \Delta\colon T \to T^2 \text{ (coproduct)}, \qquad \varepsilon\colon T \to 1_{\mathcal{B}} \text{ (counit)}$$

$$\mu_{a,b}\colon Ta \otimes Tb \to T(a \otimes b), \qquad \eta\colon \mathbb{1} \to T\mathbb{1}$$

The $\mu - \Delta$ compatibility doesn't require a braiding :

$$\begin{array}{ccccc} Ta \otimes Tb & \xrightarrow{\Delta_a \otimes \Delta_b} & T^2 a \otimes T^2 b & \xrightarrow{\Delta_{Ta,Tb}} & T(Ta \otimes Tb) \\ \downarrow{\scriptstyle \mu_{a,b}} & & & & \downarrow{\scriptstyle T\mu_{a,b}} \\ T(a \otimes b) & & \xrightarrow[\Delta_{a\otimes b}]{} & & T^2(a \otimes b) \end{array}$$

Define the (left and right) *fusion morphisms*

- $H^l_{a,b} = \mu_{a,Tb}(Ta \otimes \Delta_b)\colon Ta \otimes Tb \to T(a \otimes Tb)$,
- $H^r_{a,b} = \mu_{Ta,b}(\Delta_a \otimes Tb)\colon Ta \otimes Tb \to T(Ta \otimes b)$.

Hopf comonads

A **monoidal comonad** [I. Moerdijk] on a monoidal category $\mathcal{B}$ is a a comonad (T, Δ, ε) such that T, Δ and ε are monoidal, that is :

$$T\colon \mathcal{B} \to \mathcal{B}, \quad \Delta\colon T \to T^2 \text{ (coproduct)}, \qquad \varepsilon\colon T \to 1_{\mathcal{B}} \text{ (counit)}$$

$$\mu_{a,b}\colon Ta \otimes Tb \to T(a \otimes b), \qquad \eta\colon \mathbb{1} \to T\mathbb{1}$$

The $\mu - \Delta$ compatibility doesn't require a braiding :

$$\begin{array}{ccccc} Ta \otimes Tb & \xrightarrow{\Delta_a \otimes \Delta_b} & T^2a \otimes T^2b & \xrightarrow{\Delta_{Ta,Tb}} & T(Ta \otimes Tb) \\ \downarrow{\scriptstyle \mu_{a,b}} & & & & \downarrow{\scriptstyle T\mu_{a,b}} \\ T(a \otimes b) & & \xrightarrow[\Delta_{a\otimes b}]{} & & T^2(a \otimes b) \end{array}$$

Define the (left and right) *fusion morphisms*

- $H^l_{a,b} = \mu_{a,Tb}(Ta \otimes \Delta_b)\colon Ta \otimes Tb \to T(a \otimes Tb)$,
- $H^r_{a,b} = \mu_{Ta,b}(\Delta_a \otimes Tb)\colon Ta \otimes Tb \to T(Ta \otimes b)$.

T is a *Hopf comonad* if H^l and H^r are isomorphisms.

Hopf comonads and Hopf monoidal adjunctions

T comonad on $\mathcal{B} \rightsquigarrow$ an adjunction $\mathcal{B}_T \underset{R_T}{\overset{U_T}{\rightleftarrows}} \mathcal{B}$

where $\mathcal{B}_T = \{T\text{-comodules } (b, \delta : b \to Tb)\}$, $U_T(b, \delta) = b$ and $R_T(b) = (Tb, \Delta_b)$.

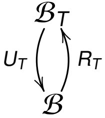
Hopf comonads and Hopf monoidal adjunctions

T comonad on $\mathcal{B} \rightsquigarrow$ an adjunction $$\mathcal{B}_T \underset{R_T}{\overset{U_T}{\rightleftarrows}} \mathcal{B}$$

where $\mathcal{B}_T = \{T\text{-comodules } (b, \delta : b \to Tb)\}$, $U_T(b, \delta) = b$ and $R_T(b) = (Tb, \Delta_b)$.

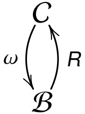
T is a monoidal (resp Hopf) comonad iff its adjunction (U_T, R_T) is a monoidal adjunction (resp a Hopf monoidal adjunction).

Hopf comonads and Hopf monoidal adjunctions

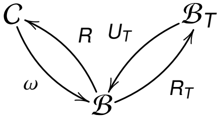
T comonad on $\mathcal{B}$ $\rightsquigarrow$ an adjunction 

where $\mathcal{B}_T = \{T\text{-comodules } (b, \delta : b \to Tb)\}$, $U_T(b, \delta) = b$ and $R_T(b) = (Tb, \Delta_b)$.

T is a monoidal (resp Hopf) comonad iff its adjunction (U_T, R_T) is a monoidal adjunction (resp a Hopf monoidal adjunction).

Conversely let 

be monoidal (Hopf) adjunction ; its comonad $T = \omega R$ is a monoidal (Hopf) comonad.

$\rightsquigarrow$ 

Hopf comonads and Hopf monoidal adjunctions

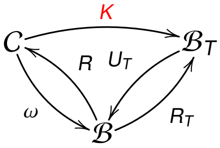
T comonad on $\mathcal{B} \rightsquigarrow$ an adjunction $\mathcal{B}_T \underset{R_T}{\overset{U_T}{\rightleftarrows}} \mathcal{B}$

where $\mathcal{B}_T = \{T\text{-comodules } (b, \delta : b \to Tb)\}$, $U_T(b, \delta) = b$ and $R_T(b) = (Tb, \Delta_b)$.

> T is a monoidal (resp Hopf) comonad iff its adjunction (U_T, R_T) is a monoidal adjunction (resp a Hopf monoidal adjunction).

Conversely let $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$ be monoidal (Hopf) adjunction ; its comonad

$T = \omega R$ is a monoidal (Hopf) comonad.

$\rightsquigarrow$ 

Hopf comonads and Hopf monoidal adjunctions

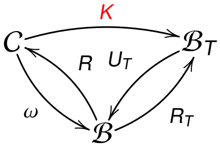
T comonad on $\mathcal{B} \rightsquigarrow$ an adjunction $\mathcal{B}_T \underset{R_T}{\overset{U_T}{\rightleftarrows}} \mathcal{B}$

where $\mathcal{B}_T = \{T\text{-comodules } (b, \delta : b \to Tb)\}$, $U_T(b,\delta) = b$ and $R_T(b) = (Tb, \Delta_b)$.

T is a monoidal (resp Hopf) comonad iff its adjunction (U_T, R_T) is a monoidal adjunction (resp a Hopf monoidal adjunction).

Conversely let $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$ be monoidal (Hopf) adjunction ; its comonad

$T = \omega R$ is a monoidal (Hopf) comonad.

$\rightsquigarrow$ 

the *comparison functor* $K : c \mapsto (\omega c, \omega \eta_c)$ is strong monoidal
The adjunction (ω, R) is *comonadic* if K is an equivalence.

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

- Tannaka dictionary

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals
- Semisimplicity, Maschke criterion

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals
- Semisimplicity, Maschke criterion
- The Drinfeld double of a Hopf (co)monad

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals
- Semisimplicity, Maschke criterion
- The Drinfeld double of a Hopf (co)monad
- Cross-products

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals
- Semisimplicity, Maschke criterion
- The Drinfeld double of a Hopf (co)monad
- Cross-products
- Bosonization for Hopf (co)monads

Outlook of General Theory of Hopf (co)monads

The notion of a Hopf comonad is not self-dual, unlike that of a Hopf algebra. The dual notion is that of Hopf monads. Many classical results of the theory of Hopf algebras extend to Hopf (co)monads [BV 2007, BVL 2011]

- Tannaka dictionary
- Hopf modules and Sweedler decomposition theorem
- Existence of universal integrals
- Semisimplicity, Maschke criterion
- The Drinfeld double of a Hopf (co)monad
- Cross-products
- Bosonization for Hopf (co)monads
- Applications to construction and comparison of quantum invariants (non-braided setting)

Hopf comonads from Hopf algebras

Hopf comonads generalize Hopf algebras in braided categories.

Hopf comonads from Hopf algebras

Hopf comonads generalize Hopf algebras in braided categories.

$\mathcal{B}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{B})$ (which is braided)

$\rightsquigarrow$ a Hopf comonad $T = H \otimes_{\sigma} ?$ on $\mathcal{B}$, defined by $X \mapsto H \otimes X$.

Hopf comonads from Hopf algebras

Hopf comonads generalize Hopf algebras in braided categories.

$\mathcal{B}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{B})$ (which is braided) $\rightsquigarrow$ a Hopf comonad $T = H \otimes_\sigma ?$ on $\mathcal{B}$, defined by $X \mapsto H \otimes X$. The monoidal structure of T is

$$\mu_{a,b} = (m \otimes a \otimes b)(H \otimes \sigma_a \otimes b)$$
$$\eta = \text{unit of } H$$

Hopf comonads from Hopf algebras

Hopf comonads generalize Hopf algebras in braided categories.

$\mathcal{B}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{B})$ (which is braided) $\rightsquigarrow$ a Hopf comonad $T = H \otimes_\sigma ?$ on $\mathcal{B}$, defined by $X \mapsto H \otimes X$. The monoidal structure of T is

$$\mu_{a,b} = (m \otimes a \otimes b)(H \otimes \sigma_a \otimes b)$$
$$\eta = \text{unit of } H$$

Moreover T comes with a Hopf comonad morphism

$$e = (\eta \otimes ?) : \mathrm{id}_{\mathcal{B}} \to T$$

Hopf comonads from Hopf algebras

Hopf comonads generalize Hopf algebras in braided categories.

$\mathcal{B}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{B})$ (which is braided) $\rightsquigarrow$ a Hopf comonad $T = H \otimes_\sigma ?$ on $\mathcal{B}$, defined by $X \mapsto H \otimes X$. The monoidal structure of T is

$$\mu_{a,b} = (m \otimes a \otimes b)(H \otimes \sigma_a \otimes b)$$
$$\eta = \text{unit of } H$$

Moreover T comes with a Hopf comonad morphism

$$e = (\eta \otimes ?) : \mathrm{id}_{\mathcal{B}} \to T$$

Theorem (B Virelizier Lack)

Central Hopf agebras = (co)augmented Hopf (co)monads

Hopf comonads from Hopf algebras

Hopf comonads generalize Hopf algebras in braided categories.

$\mathcal{B}$ monoidal category, (H, σ) a Hopf algebra in $\mathcal{Z}(\mathcal{B})$ (which is braided) $\rightsquigarrow$ a Hopf comonad $T = H \otimes_\sigma ?$ on $\mathcal{B}$, defined by $X \mapsto H \otimes X$. The monoidal structure of T is

$$\mu_{a,b} = (m \otimes a \otimes b)(H \otimes \sigma_a \otimes b)$$
$$\eta = \text{unit of } H$$

Moreover T comes with a Hopf comonad morphism

$$e = (\eta \otimes ?) : \mathrm{id}_{\mathcal{B}} \to T$$

Theorem (B Virelizier Lack)

Central Hopf agebras = (co)augmented Hopf (co)monads

Consequence for Tannaka duality : the neutral case means that ω has a monoidal section $\rightsquigarrow$ the reconstructed Hopf structure is a Hopf algebra in $\mathcal{Z}(\mathcal{B})$.

Hopf comonads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Hopf comonads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear monoidal comonads on ${}_R Mod_R$ which commute to colimits are bialgebroids in the sense of Takeuchi [Szlachányi, Hai]

Hopf comonads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear monoidal comonads on ${}_R Mod_R$ which commute to colimits are bialgebroids in the sense of Takeuchi [Szlachányi, Hai]
- linear Hopf comonads on ${}_R Mod_R$ which commute to colimits are a Hopf algebroids in the sense of Schauenburg.

Hopf comonads as 'quantum groupoids'

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear monoidal comonads on ${}_R Mod_R$ which commute to colimits are bialgebroids in the sense of Takeuchi [Szlachányi, Hai]
- linear Hopf comonads on ${}_R Mod_R$ which commute to colimits are a Hopf algebroids in the sense of Schauenburg.

Hopf algebroids are non-commutative avatars of groupoids.

Hopf comonads as ‘quantum groupoids’

Let R be a unitary ring $\rightsquigarrow$ a monoidal category $({}_R Mod_R, \otimes_R, {}_R R_R)$.

Facts

- linear monoidal comonads on ${}_R Mod_R$ which commute to colimits are bialgebroids in the sense of Takeuchi [Szlachányi, Hai]
- linear Hopf comonads on ${}_R Mod_R$ which commute to colimits are a Hopf algebroids in the sense of Schauenburg.

Hopf algebroids are non-commutative avatars of groupoids. Complicated axioms $\rightsquigarrow$ a Hopf adjunction $\rightsquigarrow$ a Hopf comonad (much easier to manipulate).

Reconstruction witout an adjoint

Let $\omega : \mathcal{C} \to \mathscr{B}$ be a strong monoidal functor. How can we reconstruct a Hopf comonad from ω?

Reconstruction witout an adjoint

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. How can we reconstruct a Hopf comonad from ω ?

Assume that $\mathcal{B}$ has pertinent colimits (small filtered colimits if ω is right exact, small colimits otherwise) and $\otimes_{\mathcal{B}}$ preserves them.

Reconstruction witout an adjoint

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. How can we reconstruct a Hopf comonad from ω ?

Assume that $\mathcal{B}$ has pertinent colimits (small filtered colimits if ω is right exact, small colimits otherwise) and $\otimes_{\mathcal{B}}$ preserves them. Then the following Kan extension exists :

$$T_\omega = \begin{cases} \mathcal{B} & \to & \mathcal{B} \\ b & \mapsto & \lim_{(c,\phi)\in \mathcal{C}/b} \omega c \end{cases}$$

Reconstruction witout an adjoint

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. How can we reconstruct a Hopf comonad from ω ?

Assume that $\mathcal{B}$ has pertinent colimits (small filtered colimits if ω is right exact, small colimits otherwise) and $\otimes_{\mathcal{B}}$ preserves them. Then the following Kan extension exists :

$$T_\omega = \begin{cases} \mathcal{B} & \to & \mathcal{B} \\ b & \mapsto & \lim_{(c,\phi)\in\mathcal{C}/b} \omega c \end{cases}$$

T_ω is a comonad coacting universally on ω

Reconstruction witout an adjoint

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. How can we reconstruct a Hopf comonad from ω ?

Assume that $\mathcal{B}$ has pertinent colimits (small filtered colimits if ω is right exact, small colimits otherwise) and $\otimes_{\mathcal{B}}$ preserves them. Then the following Kan extension exists :

$$T_\omega = \begin{cases} \mathcal{B} & \to & \mathcal{B} \\ b & \mapsto & \lim_{(c,\phi)\in \mathcal{C}/b} \omega c \end{cases}$$

T_ω is a comonad coacting universally on ω, in fact a **monoidal comonad.**

Reconstruction witout an adjoint

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. How can we reconstruct a Hopf comonad from ω ?

Assume that $\mathcal{B}$ has pertinent colimits (small filtered colimits if ω is right exact, small colimits otherwise) and $\otimes_{\mathcal{B}}$ preserves them. Then the following Kan extension exists :

$$T_\omega = \begin{cases} \mathcal{B} & \to & \mathcal{B} \\ b & \mapsto & \lim_{(c,\phi)\in\mathcal{C}/b} \omega c \end{cases}$$

T_ω is a comonad coacting universally on ω, in fact a **monoidal comonad.** If for all $c \in \mathcal{C}$ ωc is 'small' in $\mathcal{B}$ then T preserves pertinent colimits.

Reconstruction witout an adjoint

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. How can we reconstruct a Hopf comonad from ω ?

Assume that $\mathcal{B}$ has pertinent colimits (small filtered colimits if ω is right exact, small colimits otherwise) and $\otimes_{\mathcal{B}}$ preserves them. Then the following Kan extension exists :

$$T_\omega = \begin{cases} \mathcal{B} & \to & \mathcal{B} \\ b & \mapsto & \lim_{(c,\phi)\in \mathcal{C}/b} \omega c \end{cases}$$

T_ω is a comonad coacting universally on ω, in fact a **monoidal comonad.**
If for all $c \in \mathcal{C}$ ωc is 'small' in $\mathcal{B}$ then T preserves pertinent colimits.
T_ω is a Hopf comonad if, in addition $\mathcal{C}$ has (enough) duals or $\mathcal{C}$ and ω are coclosed.

Reconstruction witout an adjoint

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor. How can we reconstruct a Hopf comonad from ω ?

Assume that $\mathcal{B}$ has pertinent colimits (small filtered colimits if ω is right exact, small colimits otherwise) and $\otimes_{\mathcal{B}}$ preserves them. Then the following Kan extension exists :

$$T_\omega = \begin{cases} \mathcal{B} & \to & \mathcal{B} \\ b & \mapsto & \lim_{(c,\phi)\in\mathcal{C}/b} \omega c \end{cases}$$

T_ω is a comonad coacting universally on ω, in fact a **monoidal comonad.**
If for all $c \in \mathcal{C}$ ωc is 'small' in $\mathcal{B}$ then T preserves pertinent colimits.
T_ω is a Hopf comonad if, in addition $\mathcal{C}$ has (enough) duals or $\mathcal{C}$ and ω are coclosed.

The Hopf comonad T_ω will play the rôle of $\mathrm{Aut}(\omega)$.

Comonadicity criterion

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor, with $\mathcal{B}$ having small filtered colimits which are exact and preserved by $\otimes_{\mathcal{B}}$. Assume that

1. $\mathcal{C}$ has finite limits and colimits and ω is exact ;
2. $\mathcal{C}$ has mono-epi factorizations and has strict monos ;
3. $\mathcal{C}$ is coartinian ;
4. ω is conservative ;

Comonadicity criterion

Let $\omega : \mathcal{C} \to \mathcal{B}$ be a strong monoidal functor, with $\mathcal{B}$ having small filtered colimits which are exact and preserved by $\otimes_{\mathcal{B}}$. Assume that

1. $\mathcal{C}$ has finite limits and colimits and ω is exact ;
2. $\mathcal{C}$ has mono-epi factorizations and has strict monos ;
3. $\mathcal{C}$ is coartinian ;
4. ω is conservative ;

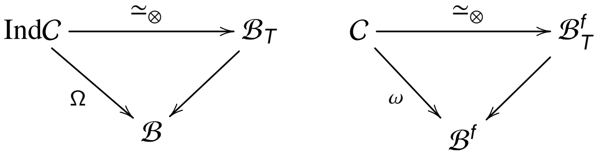
Then

a) ω extends uniquely to a strong monoidal functor $\Omega : \mathrm{Ind}\mathcal{C} \to \mathcal{B}$ which preserves filtered colimits and has a right adjoint R ;

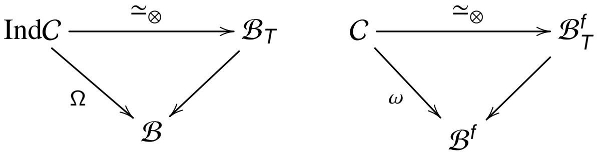
b) Ω is comonadic with comonad $T = T_\omega = \Omega R$, so that $\mathrm{Ind}\mathcal{C} \cong_\otimes \mathcal{B}_T$;

c) Moreover, if $\mathcal{B}^f \subset \mathcal{B}$ is a full monoidal subcategory containing $\omega(\mathcal{C})$ and whose objects have finite type in $\mathcal{B}$ (e. g. $\mathcal{B} = \mathrm{Ind}\mathcal{B}^f$), we have $\mathcal{C} \cong_\otimes \mathcal{B}^f_T$.

Hopf


Ind$\mathcal{C}$
$\simeq_\otimes$
$\mathcal{B}_T$
Ω
$\mathcal{B}$
$\mathcal{C}$
$\simeq_\otimes$
$\mathcal{B}^f_T$
ω
$\mathcal{B}^f$

Hopf



If in addition

1. $\mathcal{C}$ has duals or $\mathcal{C}$ and ω are coclosed,
2. for any mono *i* of $\mathcal{C}$, $\omega(i)$ is a tensor-universal mono of $\mathcal{B}$,

then *T* is a Hopf comonad.

Note that, if ω has a monoidal section, then the Hopf comonad *T* is co-augmented, so there exists a Hopf algebra (H, σ) in $\mathcal{Z}(\mathcal{B})$ such that $T = H \otimes_\sigma ?$ and $\mathcal{C}$ is the category of *H*-comodules in $\mathcal{B}$.

In its direct form the main result yields tannaka duality in the neutral case and in the general case (even non-commutative), with $\mathcal{B} = \mathrm{Vect}_{\Bbbk}$, $\mathcal{B} = \mathrm{Mod}B$, $\mathcal{B} =_B \mathrm{Bimod}_B$.

In its direct form the main result yields tannaka duality in the neutral case and in the general case (even non-commutative), with $\mathcal{B} = \mathrm{Vect}_{\Bbbk}$, $\mathcal{B} = \mathrm{Mod}B$, $\mathcal{B} =_B \mathrm{Bimod}_B$.
In its opposite form, it yields Nori's construction when $\mathcal{B} = \mathrm{Pro}$ mod $_{\Bbbk}$.

In its direct form the main result yields tannaka duality in the neutral case and in the general case (even non-commutative), with $\mathcal{B} = \mathrm{Vect}_{\mathbb{k}}$, $\mathcal{B} = \mathrm{Mod}B$, $\mathcal{B} =_B \mathrm{Bimod}_B$.
In its opposite form, it yields Nori's construction when $\mathcal{B} = \mathrm{Pro}$ mod $_{\mathbb{k}}$. It yields Galois-Grothendieck duality when $\mathcal{B} = \mathrm{Proset}$ (≅ category of compact totally disconnected topological spaces),

In its direct form the main result yields tannaka duality in the neutral case and in the general case (even non-commutative), with $\mathcal{B} = \mathrm{Vect}_{\Bbbk}$, $\mathcal{B} = \mathrm{Mod}B$, $\mathcal{B} =_B \mathrm{Bimod}_B$.
In its opposite form, it yields Nori's construction when $\mathcal{B} = \mathrm{Pro}$ mod $_{\Bbbk}$. It yields Galois-Grothendieck duality when $\mathcal{B} = \mathrm{Proset}$ (≅ category of compact totally disconnected topological spaces), with a proviso.

In its direct form the main result yields tannaka duality in the neutral case and in the general case (even non-commutative), with $\mathcal{B} = \mathrm{Vect}_{\Bbbk}$, $\mathcal{B} = \mathrm{Mod}B$, $\mathcal{B} =_B \mathrm{Bimod}_B$.
In its opposite form, it yields Nori's construction when $\mathcal{B} = \mathrm{Pro}$ mod $_{\Bbbk}$. It yields Galois-Grothendieck duality when $\mathcal{B} = \mathrm{Proset}$ (≅ category of compact totally disconnected topological spaces), with a proviso.
The first part of the theorem reconstructs a comonoidal monad T on Proset, which is in fact a bialgebra G because set is generated by the unit object and T preserves finite sums and filtering limits. Since we are in a cartesian category, bialgebras are just monoids.

In its direct form the main result yields tannaka duality in the neutral case and in the general case (even non-commutative), with $\mathcal{B} = \mathrm{Vect}_{\Bbbk}$, $\mathcal{B} = \mathrm{Mod}B$, $\mathcal{B} =_B \mathrm{Bimod}_B$.
In its opposite form, it yields Nori's construction when $\mathcal{B} = \mathrm{Pro}$ mod $_{\Bbbk}$. It yields Galois-Grothendieck duality when $\mathcal{B} = \mathrm{Proset}$ ($\cong$ category of compact totally disconnected topological spaces), with a proviso.
The first part of the theorem reconstructs a comonoidal monad T on Proset, which is in fact a bialgebra G because set is generated by the unit object and T preserves finite sums and filtering limits. Since we are in a cartesian category, bialgebras are just monoids.
One verifies that the last axiom of Grothendieck implies that C and ω are closed, so the monoid G is a group in Proset, that is a profinite group.

In its direct form the main result yields tannaka duality in the neutral case and in the general case (even non-commutative), with $\mathcal{B} = \mathrm{Vect}_{\mathbb{k}}$, $\mathcal{B} = \mathrm{Mod}B$, $\mathcal{B} =_B \mathrm{Bimod}_B$.
In its opposite form, it yields Nori's construction when $\mathcal{B} = \mathrm{Pro}$ mod $_{\mathbb{k}}$. It yields Galois-Grothendieck duality when $\mathcal{B} = \mathrm{Proset}$ (≅ category of compact totally disconnected topological spaces), with a proviso.
The first part of the theorem reconstructs a comonoidal monad T on Proset, which is in fact a bialgebra G because set is generated by the unit object and T preserves finite sums and filtering limits. Since we are in a cartesian category, bialgebras are just monoids.
One verifies that the last axiom of Grothendieck implies that C and ω are closed, so the monoid G is a group in Proset, that is a profinite group.
One can refine the criterion for categories with enough duals.

Hopf adjunctions revisited

Consider a **Hopf monoidal adjunction** $\omega : \mathcal{C} \rightleftarrows \mathcal{B} : R$

Theorem (BVL)

Hopf adjunctions revisited

Consider a **Hopf monoidal adjunction** $\omega : \mathcal{C} \rightleftarrows \mathcal{B} : R$

Theorem (BVL)

$A = R(\mathbb{1})$ is an algebra in $\mathcal{C}$, and comes with a canonical half-braiding σ which makes it a **commutative algebra in** $\mathcal{Z}(\mathcal{C})$ called the *induced central algebra.*

Hopf adjunctions revisited

Consider a **Hopf monoidal adjunction** $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$

Theorem (BVL)

$A = R(\mathbb{1})$ is an algebra in $\mathcal{C}$, and comes with a canonical half-braiding σ which makes it a **commutative algebra in $\mathcal{Z}(\mathcal{C})$** called the *induced central algebra.*

The monoidal monad $R\omega$ is isomorphic to $A \otimes_\sigma ?$ (that is $A \otimes ?$ endowed with a monoidal structure defined by σ).

Hopf adjunctions revisited

Consider a **Hopf monoidal adjunction** $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$

Theorem (BVL)

$A = R(\mathbb{1})$ is an algebra in $\mathcal{C}$, and comes with a canonical half-braiding σ which makes it a **commutative algebra in $\mathcal{Z}(\mathcal{C})$** called the *induced central algebra.*

The monoidal monad $R\omega$ is isomorphic to $A \otimes_\sigma ?$ (that is $A \otimes ?$ endowed with a monoidal structure defined by σ).

If the adjunction is monadic, $\mathcal{B} \cong_\otimes \mathcal{C}_{A,\sigma}$, where $\mathcal{C}_{A,\sigma}$ is the monoidal category of A-modules in $\mathcal{C}$, with tensor product $\otimes_A$ and unit object A.

Hopf adjunctions revisited

Consider a **Hopf monoidal adjunction** $\mathcal{C} \underset{R}{\overset{\omega}{\rightleftarrows}} \mathcal{B}$

Theorem (BVL)

$A = R(\mathbb{1})$ is an algebra in $\mathcal{C}$, and comes with a canonical half-braiding σ which makes it a **commutative algebra in $\mathcal{Z}(\mathcal{C})$** called the *induced central algebra.*

The monoidal monad $R\omega$ is isomorphic to $A \otimes_\sigma ?$ (that is $A \otimes ?$ endowed with a monoidal structure defined by σ).

If the adjunction is monadic, $\mathcal{B} \cong_\otimes \mathcal{C}_{A,\sigma}$, where $\mathcal{C}_{A,\sigma}$ is the monoidal category of A-modules in $\mathcal{C}$, with tensor product $\otimes_A$ and unit object A. Via this equivalence, ω is the functor 'free module' $c \mapsto A \otimes c$.

Hopf adjunctions revisited

Consider a **Hopf monoidal adjunction** $\omega : \mathcal{C} \rightleftarrows \mathcal{B} : R$

Theorem (BVL)

$A = R(\mathbb{1})$ is an algebra in $\mathcal{C}$, and comes with a canonical half-braiding σ which makes it a **commutative algebra in $\mathcal{Z}(\mathcal{C})$** called the *induced central algebra.*

The monoidal monad $R\omega$ is isomorphic to $A \otimes_\sigma ?$ (that is $A \otimes ?$ endowed with a monoidal structure defined by σ).

If the adjunction is monadic, $\mathcal{B} \cong_\otimes \mathcal{C}_{A,\sigma}$, where $\mathcal{C}_{A,\sigma}$ is the monoidal category of A-modules in $\mathcal{C}$, with tensor product $\otimes_A$ and unit object A. Via this equivalence, ω is the functor 'free module' $c \mapsto A \otimes c$.

This result is the generalization to Hopf adjunctions of Sweedler's theorem on the structure of Hopf modules.

Hopf adjunctions revisited

Consider a **Hopf monoidal adjunction** $\mathcal{C} \overset{\omega}{\underset{R}{\rightleftarrows}} \mathcal{B}$

Theorem (BVL)

$A = R(\mathbb{1})$ is an algebra in $\mathcal{C}$, and comes with a canonical half-braiding σ which makes it a **commutative algebra in $\mathcal{Z}(\mathcal{C})$** called the *induced central algebra.*

The monoidal monad $R\omega$ is isomorphic to $A\otimes_\sigma ?$ (that is $A\otimes ?$ endowed with a monoidal structure defined by σ).

If the adjunction is monadic, $\mathcal{B} \cong_\otimes \mathcal{C}_{A,\sigma}$, where $\mathcal{C}_{A,\sigma}$ is the monoidal category of A-modules in $\mathcal{C}$, with tensor product $\otimes_A$ and unit object A. Via this equivalence, ω is the functor 'free module' $c \mapsto A \otimes c$.

This result is the generalization to Hopf adjunctions of Sweedler's theorem on the structure of Hopf modules.

The central commutative algebra generalizes Deligne's trivializing algebra.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between tensor categories. Then *IndF* : $\mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{B}$ has a right adjoint R.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between tensor categories. Then *IndF* : $\mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{B}$ has a right adjoint R.

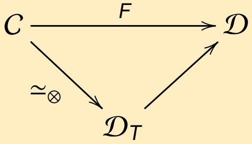
Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. There exists a $\Bbbk$- linear left exact comonad on $\mathrm{Ind}\mathcal{D}$ such that we have a commutative diagram :

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between tensor categories. Then $IndF : \mathrm{Ind}\mathcal{C} \to \mathrm{Ind}\mathcal{B}$ has a right adjoint R.

Theorem

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor. There exists a $\mathbb{k}$- linear left exact comonad on $\mathrm{Ind}\mathcal{D}$ such that we have a commutative diagram :



Theorem

If R is faithful exact (F dominant), with induced central algebra (A, σ), we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F_A} & \mathcal{C}_{A,\sigma} \\ & {\scriptstyle F}\searrow & \nearrow{\scriptstyle \simeq_{\otimes}} \\ & \mathcal{D} & \end{array}$$

where $\mathcal{C}_A$ is the category of A-modules 'of finite type' and F_A is the tensor functor $X \mapsto A \otimes X$.

If $\mathcal{D} = \mathrm{vect}_{\Bbbk}$ and $\mathcal{C}$, F are symmetric, then A is Deligne's trivializing algebra.

Existence of fibre functors

Theorem [BLV]

Let $\mathcal{C}$ be a finite tensor category. Then there exists a finite dimensional $\mathbb{k}$ algebra B and a fibre functor $\omega : \mathcal{C} \to_B \mathrm{Bimod}_B$. Hence $\mathcal{C} \cong_\otimes \mathrm{rep}A$ for a certain Hopf algebroid with base B.

24. More on Hopf monads

BV1. *Hopf Diagrams and Quantum Invariants,* AGT **5** (2005) 1677-1710.

Where Hopf diagram are introduced as a means for computing the Reshetikhin-Turaev invariant in terms of the coend of a ribbon category and its structural morphisms.

BV2. *Hopf Monads,* Advances in Math. **215** (2007), 679-733.

Where the notion of Hopf monad is introduced, and several fundamental results of the theory of finite dimensional Hopf algebras are extended thereto.

BV3. *Categorical Centers and Reshetikhin-Turaev Invariants,* Acta Mathematica Vietnamica **33** 3, 255-279

Where the coend of the center of a fusion spherical category over a ring is described, the modularity of the center, proven, and the corresponding Reshetikhin-Turaev invariant, constructed.

BV4. *Quantum Double of Hopf monads and Categorical Centers,* arXiv :0812.2443, to appear in Transactions of the American Mathematical Society (2010)

Where the general theory of centralizers and doubles of Hopf monads is expounded.

BLV. *Hopf Monads on Monoidal Categories,* arXiv :1003.1920.

Where Hopf monads are defined anew in the monoidal world

BN. *Exact sequences of tensor categories*, arXiv :1006.0569.

See also : `http://www.math.univ-montp2.fr/~bruguieres/recherche.html`