On a tannakian theorem due to Nori

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Résumé

M. V. Nori a construit en 1997 une catégorie de motifs en caractéristique nulle. Sa construction s'appuie sur un nouveau théorème de reconstruction tannakienne valide sur un anneau noethérien K. Étant donné un carquois Q et une représentation T de Q sur K, Nori construit une catégorie abélienne K-linéaire $\mathcal{C}(T)$, qui satisfait une certaine propriété universelle (Th. 1).
Dans le présent article, nous donnons une démonstration de ce résultat, à ce jour non publié, pour un anneau K cohérent. En fait, nous établissons un résultat un peu plus fort sur $\mathcal{C}(T)$ (Th. 2) dont le théorème de Nori est un corollaire. Enfin, nous donnons une version monoïdale de la construction de Nori (Th. 3, Cor. 3).

Abstract

M. V. Nori constructed in 1997 a category of motives in characteristic 0. This construction relies on a new theorem of tannakian reconstruction over a noetherian ring K. Given a quiver Q and a representation T of Q over K, Nori constructs an abelian K-linear category $\mathcal{C}(T)$, which satisfies a certain universal property (Th. 1).
In this paper, we give a proof of this as yet unpublished result for K coherent ring. In fact, we prove a somewhat stronger result on $\mathcal{C}(T)$ (Th. 2) of which Nori's theorem is a corollary. Lastly, we give a monoidal version of Nori's construction (Th. 3, Cor. 3).

Introduction

In 1997, M. V. Nori proposed in a yet unpublished text a very promising candidate for a category of motives in characteristic 0. In a nutshell, Nori's approach goes as follows. He introduces a certain quiver (or diagram) Q_a and a representation T_a of this quiver with values in the category of $\mathbb{Z}$-modules of finite type. The quiver is defined in terms of algebraic varieties over $\mathbb{C}$ (or, more generally, a subfield of $\mathbb{C}$), and the representation is defined by singular (co-)homology. This part of the construction is surprisingly straightforward. Now comes a remarkable result : given a noetherian commutative ring K and a representation T of a quiver Q with values in the category $K\text{-}\mathsf{mod}$ of K-modules of finite type, one may construct an abelian K-linear category $\mathcal{C}(T)$ over $K\text{-}\mathsf{mod}$ such that T lifts to $\mathcal{C}(T)$, and this construction is universal. As a corollary, if Q is a (small) abelian category over K, and T an abelian K-linear faithful exact functor, then $Q \to \mathcal{C}(T)$ is an equivalence. This is a striking result, in that, until then, this type of tannakian reconstruction was known only over a field ([S], [D], [B]).
Applied to the quiver Q_a and the representation T_a, Nori's theorem yields an abelian category $\mathcal{C}(T_a)$ which is his candidate for a category of effective (co)-homological motives.

In Nori's text, the universal property is stated, but the proof is not written down. The aim of the present paper is to provide a written proof, together with minor improvements :
1) we work over a coherent commutative ring;
2) we deduce Nori's theorem from a somewhat stronger statement;
3) we give a monoidal version of Nori's theorem, which requires a rather strong condition on K (essentially, the global dimension of K is at most 2).

I wish to express my deepest gratitude to Madhav Nori for allowing me (and indeed, warmly encouraging me) to publish this text which owes so much to his work.

Plan

In the first section, we recall the definitions of quivers and representations of quivers, give Nori's construction of the category $\mathcal{C}(T)$, and state Nori's theorem (Theorem 1) together with another theorem (Theorem 2) of which theorem 1 will be a consequence.
In section 2, we rehearse a number of constructions related with categories of modules, namely tensor products and external Hom's.

In section 3, we introduce a notion of external Hom of quiver representations, and explore its properties.
In section 4, we prove theorem 2 as a direct application of the notion of external Hom of quiver representations, and deduce theorem 1.
In section 5, we recall the notion of pro-objects and re-interpret the category $\mathcal{C}(T)$ as the category of left modules on the 'pro-' version the algebra of endomorphisms of T. We then give a monoidal version of Nori's reconstruction (Theorem 3 and corollary 3).

Conventions and notations

Let A be a ring. We denote $A\text{-}\mathsf{Mod}$ the category of left A-modules, and $\overline{A}$ the ring A seen as a left A-module.
We denote $A\text{-}\mathsf{mod} \subset A\text{-}\mathsf{Mod}$ the full subcategory of finitely presented A-modules. The ring A is *coherent (on the left)* if the category $A\text{-}\mathsf{mod}$ is abelian and the inclusion functor $A\text{-}\mathsf{mod} \hookrightarrow A\text{-}\mathsf{Mod}$ is exact.
From now on, K is a commutative coherent ring.

1 Quivers and Nori's theorem

1.1 Quivers

Quivers (or *diagrams* in Nori's terminology) are presheaves on the category $\{0 \rightrightarrows 1\}$. As such they form a category, and even a topos.
More concretely, a quiver D consists in the following data : a set D_0 (the *vertices*, or *objects* of the quiver), a set D_1 (the *edges*, or *arrows*), and two maps $s, t : D_1 \longrightarrow D_0$ respectively called *source* and *target*.
EXAMPLE. If $\mathcal{C}$ is a category, $\mathsf{Fl}(\mathcal{C}) \rightrightarrows \mathsf{Ob}(\mathcal{C})$ is a quiver. Any functor is a morphism of quivers. We will denote in the same way a category and the underlying quiver.
Let D be a quiver and $\mathcal{C}$ a category. A *representation of D with values in $\mathcal{C}$* is a morphism of quivers $T : D \to \mathcal{C}$. Representations of a quiver D with values in a category $\mathcal{C}$ form a category. Given two such representations T, T', a morphism $\phi : T \to T'$ is a family $(\phi_i)_{i \in D_0}$, $\phi_i \in \mathsf{Hom}_{\mathcal{C}}(T_i, T'_i)$, such that the following squares are commutative:

$$\begin{array}{ccc} T_j & \xrightarrow{T_a} & T_k \\ {\scriptstyle \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_k} \\ T'_j & \xrightarrow[T'_a]{} & T'_k \end{array}$$

for any $(j \xrightarrow{a} k) \in D_1$ $(j = s(a),\ k = t(a))$.
This is a generalization of the notion of natural morphism.

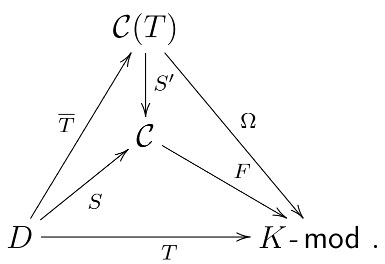
1.2 Nori's theorem

Let K be a coherent commutative ring. Let D be a quiver, and $T : D \to K\text{-}\mathsf{mod}$ a representation of D.
If D is finite, $E = \mathsf{End}\,T$ is a K-algebra in $K\text{-}\mathsf{mod}$. For $i \in D_0$, T_i has a natural structure of left E-module, denoted $\overline{T}_i$, and for $(j \xrightarrow{a} k) \in D_1$, $T_a : T_j \to T_k$ is an E-linear map. Thus T lifts to a representation $\overline{T} : D \to \mathsf{End}\,T\text{-}\mathsf{mod}$.
For D arbitrary, let $\mathcal{F}$ be the set of finite subsheaves of D, ordered by inclusion. For $D' \in \mathcal{F}$, let $E_{D'} = \mathsf{End}(T_{|D'})$. If $D'' \in \mathcal{F}$, $D'' \subset D$, we have a canonical morphism $E_{D'} \to E_{D''}$, hence a functor $E_{D''}\text{-}\mathsf{mod} \to E_{D'}\text{-}\mathsf{mod}$ which is K-linear, faithful, exact. We let

$$\mathcal{C}(T) = \varinjlim_{D' \in \mathcal{F}} E_{D'}\text{-}\mathsf{mod}\ .$$

This 2-limit being filtered, $\mathcal{C}(T)$ is an abelian K-linear category, and the forgetful functor $\Omega : \mathcal{C}(T) \to K\text{-}\mathsf{mod}$ is K-linear, faithful, exact. [1] Moreover T lifts to a representation $\overline{T} : D \to \mathcal{C}(T)$.
Remark. In $\mathcal{C}(T)$, Hom's are finitely presented K-modules if D is finite or K noetherian.
The representation $\overline{T}$, together with the forgetful functor Ω, satisfies a universal property.

Theorem 1 *(Nori) Let D be a quiver, and $T : D \to K\text{-}\mathsf{mod}$ a representation of D. On the other hand, let $\mathcal{C}$ be an abelian K-linear category, $F : \mathcal{C} \to K\text{-}\mathsf{mod}$ a K-linear faithful exact functor, and $S : D \to \mathcal{C}$ a representation of D in $\mathcal{C}$ such that $FS = T$. Then there exists a functor $S' : \mathcal{C}(T) \to \mathcal{C}$, unique (up to unique isomorphism) such that the following diagram is commutative (up to isomorphism) :*



[1]See section 5 for a more conceptual approach

Moreover the functor S' is K-linear faithful exact.

Remark. Let us clarify the statement of the theorem. The claim is that there exist a functor S', an isomorphism $\alpha : S'\overline{T} \xrightarrow{\sim} S$, and an isomorphism $\beta : FS' \xrightarrow{\sim} \Omega$, with the compatibility condition: $F(\alpha) = \beta\overline{T}$. Uniqueness means that if (S'', α', β') is another solution to the problem, there exists a unique isomorphism $\gamma : S'' \xrightarrow{\sim} S'$ such that $\alpha' = (\gamma\overline{T})\alpha$ and $\beta' = \beta F(\gamma)$.

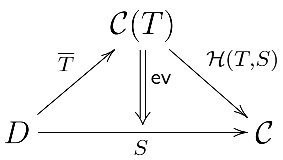
Corollary 1 *If D is a (small) abelian K-linear category and $T : D \to K\text{-}\mathsf{mod}$ is a K-linear, faithful exact functor, then the functor $\overline{T} : D \to \mathcal{C}(T)$ is an equivalence.*

Remark. This corollary generalizes over a coherent ring the fact that, over a field K, the following sets of data are equivalent :
(1) a coalgebra L;
(2) an abelian K-linear category $\mathcal{C}$ together with a K-linear faithful exact functor $F : \mathcal{C} \to \mathsf{vect}\, K$.
This is the fundamental fact which allows one to construct the tannakian dictionary (in the neutral case).

Proof. Observe that the data $(1_D : D \to D, T)$ satisfies the same universal property as the data $(\overline{T}, \Omega)$. By the uniqueness assertion, the functor $\overline{T} : D \to \mathcal{C}(T)$ is an equivalence. □

We will deduce theorem 1 from the following theorem.

Theorem 2 *Let D be a finite quiver, $\mathcal{C}$ an abelian K-linear category, T (resp. S) a representation of D with values in $K\text{-}\mathsf{mod}$ (resp. $\mathcal{C}$).*
There exists a K-linear right exact functor $\mathcal{H}(T,S) : \mathcal{C}(T) \to \mathcal{C}$, equipped with a morphism $\mathsf{ev} : \mathcal{H}(T,S) \circ \overline{T} \to S$, and universal (final) for this property.



Remark. In the statement, universal means that, given any K-linear right exact functor $H : \mathcal{C}(T) \to \mathcal{C}$, the canonical map

$$\mathsf{Hom}(H, \mathcal{H}(T,S)) \to \mathsf{Hom}(H \circ \overline{T}, S)$$

defined by $f \mapsto \mathsf{ev} \circ (f\overline{T})$, is a bijection.

2 Categories of modules, tensor products and external Hom

The results presented here are more or less classical, see [P] for instance.
In this section, $\mathcal{C}$ is an abelian K-linear category. Recall that if A is a K-algebra, a left (resp. right) A-module in $\mathcal{C}$ is an object M of $\mathcal{C}$ equipped with a K-algebra morphism $A \to \mathsf{End}_{\mathcal{C}}(M)$ (resp. $A^o \to \mathsf{End}_{\mathcal{C}}(M)$). We will denote A-$\mathcal{C}$ (resp. $\mathcal{C}$-A) the category of left (resp. right) A-modules in $\mathcal{C}$.

Proposition 1 *Let A be a K-algebra in K-mod, $\mathcal{C}$ an abelian K-linear category, and M an object of $\mathcal{C}$-A.*
1)There exists a K-linear right exact functor $F : A\text{-}\mathsf{mod} \to \mathcal{C}$ such that $F(\overline{A}) \simeq M$ in $\mathcal{C}$-A.
2) Given such a functor F, there exits functorial isomorphism (the 'adjonction' isomorphism) [2]

$$\mathsf{Hom}_{\mathcal{C}}(F(V), X) \simeq \mathsf{Hom}_A(V, \mathsf{Hom}_{\mathcal{C}}(M, X))\,.$$

Notation. The functor F, unique up to unique isomorphism thanks to assertion 2), will be denoted $M \otimes_A ?$.

Corollary 2 *The category $\mathcal{C}$-A is equivalent to the category of K-linear right exact functors from A-mod to $\mathcal{C}$.*

Proof. If M is a right A-module, there is a canonical map $\mathsf{Hom}_A(\overline{A}^m, \overline{A}^n) \to \mathsf{Hom}_{\mathcal{C}}(M^m, M^n)$, and the aim is to extend this to a functor $A\text{-}\mathsf{mod} \to \mathcal{C}$.
Let $\mathcal{A} = A\text{-}\mathsf{mod}$. Define a new category $\mathcal{A}'$ as follows :

- the objects of $\mathcal{A}'$ are the exact sequences $\overline{A}^m \xrightarrow{R} \overline{A}^n \xrightarrow{s} V \to 0$ in $\mathcal{A}$;
- for $V_\bullet = (\overline{A}^m \xrightarrow{R} \overline{A}^n \to V \to 0)$, $V'_\bullet = (\overline{A}^{m'} \xrightarrow{R'} \overline{A}^{n'} \to V' \to 0)$ objects of $\mathcal{A}'$, $\mathsf{Hom}_{\mathcal{A}'}(V_\bullet, V'_\bullet) = \mathsf{Hom}_{\mathcal{A}}(V, V')$.

The forgetful functor $U : \mathcal{A}' \to \mathcal{A}$, $(\overline{A}^m \to \overline{A}^n \to V \to 0) \mapsto V$ is an equivalence, a quasi-inverse thereof is given by arbitrary choice of a finite presentation for each object in A-mod.
For a start, we construct a functor $F' : \mathcal{A}' \to \mathcal{C}$ as follows. If $V_\bullet = (\overline{A}^m \xrightarrow{R} \overline{A}^n \to V \to 0)$ is an object of $\mathcal{A}'$, then $F'(V_\bullet) = \mathsf{coker}(M^m \xrightarrow{R} M^n)$.

[2] If in $\mathcal{C}$, Hom's are finitely presented K-modules, $\mathsf{Hom}_{\mathcal{C}}(?, X)$ is right adjoint to F.

If ϕ is a morphism from $V_\bullet$ to another object $V'_\bullet = (\overline{A}^{m'} \xrightarrow{R'} \overline{A}^{n'} \to V' \to 0)$ in $\mathcal{A}'$, that is, an element of $\mathsf{Hom}_A(V, V')$, pick morphisms ϕ_1, ϕ_2 so that the following diagram commutes in $A\text{-}\mathsf{mod}$:

$$\begin{array}{ccccccc}
\overline{A}^m & \xrightarrow{R} & \overline{A}^n & \longrightarrow & V & \longrightarrow & 0 \\
{\scriptstyle \phi_2}\downarrow & & {\scriptstyle \phi_1}\downarrow & & {\scriptstyle \phi}\downarrow & & \\
\overline{A}^{m'} & \xrightarrow{R'} & \overline{A}^{n'} & \longrightarrow & V' & \xrightarrow{R} & 0\,,
\end{array}$$

and define $F'(\phi)$ as being the morphism in $\mathcal{C}$ which makes the following diagram commutative :

$$\begin{array}{ccccccc}
M^m & \xrightarrow{R} & M^n & \longrightarrow & F'(V_\bullet) & \longrightarrow & 0 \\
{\scriptstyle \phi_2}\downarrow & & {\scriptstyle \phi_1}\downarrow & & {\scriptstyle F'(\phi)}\downarrow & & \\
M^{m'} & \xrightarrow{R'} & M^{n'} & \longrightarrow & F(V'_\bullet) & \xrightarrow{R} & 0\,.
\end{array}$$

This construction, being clearly independent of choices, defines a functor F'.
Let F be as in the statement of the proposition. For $V_\bullet = (\overline{A}^m \xrightarrow{R} \overline{A}^n \xrightarrow{s} V \to 0)$ object of $\mathcal{A}'$, we have via F a short exact sequence $M^m \xrightarrow{R} M^n \to F(V) \to 0$ in $\mathcal{C}$, hence a canonical isomorphism $F'(V_\bullet) \simeq F(V)$. Thus $FU \simeq F'$, and $F \simeq F'Q$.
Left exactness of $F = F'Q$ will immediately follow from assertion 2), which we now prove.
For $V_\bullet = (\overline{A}^m \to \overline{A}^n \to V \to 0)$ object of $\mathcal{A}'$ and X object of $\mathcal{C}$, the lines in the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathsf{Hom}(F(V),X) & \longrightarrow & \mathsf{Hom}(M^n,X) & \longrightarrow & \mathsf{Hom}(M^m,X) \\
& & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathsf{Hom}_A(V,\mathsf{Hom}_{\mathcal{C}}(M,X)) & \longrightarrow & \mathsf{Hom}_A(\overline{A}^n,\mathsf{Hom}_{\mathcal{C}}(M,X)) & \longrightarrow & \mathsf{Hom}_A(\overline{A}^n,\mathsf{Hom}_{\mathcal{C}}(M,X))
\end{array}$$

are exact, and both vertical arrows are isomorphisms; hence an isomorphism

$$\mathsf{Hom}(F(V), X) \xrightarrow{\sim} \mathsf{Hom}_A(V, \mathsf{Hom}_{\mathcal{C}}(M, X))\,,$$

which is functorial in X and in $V_\bullet$, so in V, too. □

Definition. Let $\mathcal{C}$ be an abelian K-linear category, and X an object of $\mathcal{C}$. Define the functor $\underline{\mathsf{Hom}}_{\mathcal{C}}(?, X) : (K\text{-}\mathsf{mod})^o \to \mathcal{C}$ to be the opposite functor to $X \otimes_K ? : K\text{-}\mathsf{mod} \to \mathcal{C}^o$. Thus, for X, Y objects of $\mathcal{C}$ and V object of $K\text{-}\mathsf{mod}$, one has a functorial isomorphism

$$\mathsf{Hom}_{\mathcal{C}}(X, \underline{\mathsf{Hom}}_{\mathcal{C}}(V, Y)) \simeq \mathsf{Hom}_K(V, \mathsf{Hom}_{\mathcal{C}}(X, Y))\,.$$

This defines a K-linear left exact functor

$$\underline{\mathsf{Hom}}_{\mathcal{C}}(?,?) : (K\text{-}\mathsf{mod})^o \times \mathcal{C} \to \mathcal{C} \,.$$

called *external* Hom.

REMARKS.

1) The formation of $M \otimes_A ?$ commutes with K-linear right exact functors; in particular, the construction of $\underline{\mathsf{Hom}}\,(V,?)$ commutes with K-linear left exact functors.

2) Let A be a K-algebra in $K\text{-}\mathsf{mod}$, and $\mathcal{C} = A\text{-}\mathsf{mod}$. Given a finitely presented K-module V, and a finitely presented A-module M, $\underline{\mathsf{Hom}}_{\mathcal{C}}(V,X)$ is nothing but $\mathsf{Hom}(V,M)$, seen as a left A-module.

3 External Hom for quiver representations

Let $\mathcal{C}$ be an abelian K-linear category, D a finite quiver, S a representation of D with values in $\mathcal{C}$, and T a representation $D \to K\text{-}\mathsf{mod}$.

To this data we associate a morphism $\Delta_{T,S}$ of $\mathcal{C}$:

$$\Delta_{T,S} = \partial^0 - \partial^1 : \prod_{i \in D_0} \underline{\mathsf{Hom}}_{\mathcal{C}}(T_i, S_i) \longrightarrow \prod_{(j \xrightarrow{a} k) \in D_1} \underline{\mathsf{Hom}}_{\mathcal{C}}(T_j, S_k) \,.$$

Interpreting the $\prod$'s as $\bigoplus$'s, ∂^0, ∂^1 are defined blockwise as follows. For $j \xrightarrow{a} k$, consider

$$S_{a*} : \underline{\mathsf{Hom}}_{\mathcal{C}}(T_j, S_j) \to \underline{\mathsf{Hom}}_{\mathcal{C}}(T_j, S_k),$$
$$T_a{}^* : \underline{\mathsf{Hom}}_{\mathcal{C}}(T_k, S_k) \to \underline{\mathsf{Hom}}_{\mathcal{C}}(T_j, S_k) \,;$$

then $\partial^0_{i,a} = \delta_{i,j} S_{a*}$ and $\partial^1_{i,a} = \delta_{i,k} T_a{}^*$. Notice that these are morphisms of right $\mathsf{End}\, T$-modules in $\mathcal{C}$.

NOTATION. Set $\underline{\mathsf{Hom}}_{\mathcal{C}}(T,S) = \mathsf{ker}(\Delta_{T,S})$, so that we have the following exact sequence in $\mathcal{C}\text{-}\mathsf{End}\,T$:

$$(\mathcal{E}_{T,S}) \qquad 0 \to \underline{\mathsf{Hom}}_{\mathcal{C}}(T,S) \to \prod_{i \in D_0} \underline{\mathsf{Hom}}_{\mathcal{C}}(T_i, S_i) \to \prod_{(j \xrightarrow{a} k) \in D_1} \underline{\mathsf{Hom}}_{\mathcal{C}}(T_j, S_k) \,.$$

The object $\underline{\mathsf{Hom}}_{\mathcal{C}}(T,S)$ deserves the title of external Hom; indeed we have the following lemma.

Lemma 1 *For any object Σ of $\mathcal{C}\text{-}\mathsf{End}\,T$, there is a canonical isomorphism*

$$\mathsf{Hom}_{\mathcal{C}\text{-}\mathsf{End}\,T}(\Sigma, \underline{\mathsf{Hom}}_{\mathcal{C}}(T,S)) \xrightarrow{\sim} \mathsf{Hom}(\Sigma \otimes_{\mathsf{End}\,T} \overline{T}, S) \,.$$

Proof. Let E be a K-algebra in $K\text{-}\mathsf{mod}$, M an object of $E\text{-}\mathsf{mod}$, Σ an object of $\mathcal{C}$-E, and X an object of $\mathcal{C}$.
By adjunction, we have $\mathsf{Hom}_K(M, \mathsf{Hom}(\Sigma, X)) \simeq \mathsf{Hom}_\mathcal{C}(\Sigma, \underline{\mathsf{Hom}}_\mathcal{C}(M, X)$ hence (by functoriality) $\mathsf{Hom}_E(M, \mathsf{Hom}(\Sigma, X)) \simeq \mathsf{Hom}_{\mathcal{C}\text{-}E}(\Sigma, \underline{\mathsf{Hom}}_\mathcal{C}(M, X))$. By adjunction, $\mathsf{Hom}_\mathcal{C}(\Sigma \otimes_E M, X) \simeq \mathsf{Hom}_E(M, \mathsf{Hom}(\Sigma, X))$ hence a canonical isomorphism :

$$\mathsf{Hom}_\mathcal{C}(\Sigma \otimes_E M, X) \simeq \mathsf{Hom}_{\mathcal{C}\text{-}E}(\Sigma, \underline{\mathsf{Hom}}_\mathcal{C}(M, X)) .$$

Applied to $E = \mathsf{End}\, T$, this yields a commutative diagram :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathsf{Hom}(\Sigma \otimes_E \overline{T}, S) & \longrightarrow & \prod_i \mathsf{Hom}(\Sigma \otimes_E \overline{T}_i, S_i) & \longrightarrow & \prod_a \mathsf{Hom}(\Sigma \otimes_E \overline{T}_j, S_k) \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathsf{Hom}_{\mathcal{C}\text{-}E}(\Sigma, \underline{\mathsf{Hom}}_\mathcal{C}(T, S)) & \longrightarrow & \prod_i \mathsf{Hom}_{\mathcal{C}\text{-}E}(\Sigma, \underline{\mathsf{Hom}}_\mathcal{C}(T_i, S_i)) & \longrightarrow & \prod_a \mathsf{Hom}_{\mathcal{C}\text{-}E}(\Sigma, \underline{\mathsf{Hom}}_\mathcal{C}(T_j, S_k))
\end{array}$$

where lines are exact and vertical arrows are the canonical isomorphisms, hence the isomorphism we were looking for. □
In particular, for $\Sigma = \underline{\mathsf{Hom}}\,(T, S)$, the identity of $\underline{\mathsf{Hom}}_\mathcal{C}(T, S)$ corresponds with a canonical representation morphism

$$\mathsf{ev}_{T,S} = \mathsf{ev} : \underline{\mathsf{Hom}}_\mathcal{C}(T, S) \otimes_{\mathsf{End}\, T} \overline{T} \to S .$$

Lemma 2 *The formation of* $\underline{\mathsf{Hom}}\,(T, S)$ *enjoys the following properties.*
1) (Functoriality) if $\mathcal{C}'$ *is an abelian* K*-linear category and* $F : \mathcal{C} \to \mathcal{C}'$ *a* K*-linear exact left functor, then* $\underline{\mathsf{Hom}}_{\mathcal{C}'}(T, FS) = F(\underline{\mathsf{Hom}}_\mathcal{C}(T, S))$ *and* $\mathsf{ev}_{T,FS} = F(\mathsf{ev}_{T,S})$.
2) (Trivial case) for $\mathcal{C} = K\text{-}\mathsf{mod}$ *and* $S = T$, $\underline{\mathsf{Hom}}\,(T, T) = \mathsf{End}\, T$ *and* ev *is the identity.*

Proof. The functor F commutes with the formation $\underline{\mathsf{Hom}}_\mathcal{C}$, and preserves left short exact sequences, hence $F(\mathcal{E}_{T,S}) = (\mathcal{E}_{T,FS})$, and in particular $\underline{\mathsf{Hom}}_{\mathcal{C}'}(T, FS) = F(\underline{\mathsf{Hom}}_\mathcal{C}(T, S))$. Now one checks that F commutes with the formation of the bijection of lemma 1, and so, of $\mathsf{ev}_{T,S}$. The trivial case is straightforward. □

4 Proof of theorems 1 and 2

Let us prove theorem 2. Setting $E = \mathsf{End}\, T$, we have $\mathcal{C}(T) = E\text{-}\mathsf{mod}$. Let $\mathcal{H}(T, S) = \underline{\mathsf{Hom}}_\mathcal{C}(T, S) \otimes_E ? : \mathcal{C}(T) \longrightarrow \mathcal{C}$, and let ev be the evaluation morphism $\underline{\mathsf{Hom}}_\mathcal{C}(T, S) \otimes_E \overline{T} \to S$. Any right exact K-linear functor S' :

$\mathcal{C}(T) \to \mathcal{C}$ is of the form $\Sigma \otimes_E ?$ for some right E-module Σ in $\mathcal{C}$. We have $\mathsf{Hom}(S' \circ \overline{T}, S) \simeq \mathsf{Hom}(\Sigma \otimes_E \overline{T}, S) \simeq \mathsf{Hom}_E(\Sigma, \underline{\mathsf{Hom}}_{\mathcal{C}}(T, S))$ (by remark 3) $\simeq \mathsf{Hom}(S', \mathcal{H}(T, S))$ (corollary to proposition 1). Hence the theorem.

Now let us prove theorem 1 (Nori's theorem).
Notice that if the functor S' exists, it is K-linear faithful exact. Indeed, the categories $\mathcal{C}(T)$, $\mathcal{C}$ and $K\text{-}\mathsf{mod}$ are abelian, K-linear, and both F and $F \circ S' \simeq \Omega$ are K-linear faithful exact.
Notice also that, since $\mathcal{C}(T)$ is the filtered direct limit of $\mathsf{End}(T_{|D'})$'s for $D' \subset D$ finite, we may assume D finite, and so do we.
Theorem 2 provides a functor $\overline{S} = \mathcal{H}(T, S)$, and we will show that $S' = \overline{S}$ answers our problem. In order to do so, we will check that $\alpha := \mathsf{ev}_{T,S} : \overline{S}\,\overline{T} \to S$ is an isomorphism, and construct an isomorphism $\beta : F\overline{S} \xrightarrow{\sim} \Omega$ such that $\beta\overline{T} = F\alpha$.
Let $E = \mathsf{End}\,T$. Recall that, in the short exact sequence in $K\text{-}\mathsf{mod}$:

$$(\mathcal{E}_{T,T}) \qquad 0 \to E \to \prod_{i \in D_0} \mathsf{Hom}(T_i, T_i) \longrightarrow \prod_{(j \xrightarrow{a} k) \in D_1} \mathsf{Hom}(T_j, T_k)\,,$$

each object is naturally an (E, E)-bimodule, and the maps are (E, E)-linear, so $(\mathcal{E}_{T,T})$ lifts to a short exact sequence of right E-modules in $E\text{-}\mathsf{mod}$:

$$(\overline{\mathcal{E}}) \qquad 0 \to \overline{E} \to \prod_{i \in D_0} \underline{\mathsf{Hom}}\,(T_i, \overline{T}_i) \longrightarrow \prod_{(j \xrightarrow{a} k) \in D_1} \underline{\mathsf{Hom}}\,(T_j, \overline{T}_k)$$

which is none other than $(\mathcal{E}_{T,\overline{T}})$.
Now let $\alpha = \mathsf{ev}_{T,S} : \overline{S}\,\overline{T} \to S$. Since F is exact, $F(\alpha) = \mathsf{ev}_{T,FS} = \mathsf{ev}_{T,T} = 1_T$ (lemma 2); F being also faithful, and therefore conservative [3], we see that α is an isomorphism.
On the other hand, $F\overline{S}$ is right exact, so by proposition 1, it is of the form $\Sigma \otimes_E ?$, where $\Sigma = F\overline{S}(\overline{E}) = F(\underline{\mathsf{Hom}}_{\mathcal{C}}(T, S)) = \mathsf{Hom}(T, FS) = E$ (by exactness of F, $FS = T$, and lemma 2) $= \Omega(\overline{E})$. Hence a canonical isomorphism $\beta : F\overline{S} \xrightarrow{\sim} \Omega$ by proposition 1 again. One checks $F(\alpha) = \beta\overline{T}$, hence existence.
Now for uniqueness. Assume S' is a functor as required; it comes equipped with compatible morphisms $\alpha' : S'\overline{T} \xrightarrow{\sim} S$, $\beta' : FS' \xrightarrow{\sim} \Omega$. By the preliminary remark, S' is K-linear faithful exact. In particular, $S' \simeq \Sigma \otimes_E ?$, with $\Sigma = S'(\overline{E})$ (by right exactness of F). Now $S'(\overline{E}) = S'(\underline{\mathsf{Hom}}_{\mathcal{C}}(T, \overline{T})) =$

[3] A functor F is conservative if any morphism whose image by F is an isomorphism, is an isomorphism.

$\underline{\mathsf{Hom}}_{\mathcal{C}}(T, S'\overline{T})$ (by left exactness of S') $\simeq \underline{\mathsf{Hom}}_{\mathcal{C}}(T, S)$ via α'. Hence an isomorphism $\gamma : S' \xrightarrow{\sim} \overline{S}$. By construction, $\alpha' = (\gamma\overline{T})\alpha$, and one checks : $\beta' = F(\gamma)\beta$. □

We will also need the following result.

Proposition 2 *Let $T : D \to K-\mathsf{mod}$ be a representation of a quiver D. Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}(T)$ stable under direct sum, kernels and cokernels, and containing $\overline{T}_i$ for each $i \in D_0$. Then $\mathcal{A} = \mathcal{C}(T)$.*

Proof. This can be deduced from Th. 1 by abstract nonsense. In a more pedestrian way, one checks that $\mathcal{A}$ contains $\underline{\mathsf{Hom}}\,(N, \overline{T}_i)$ for any N in K-mod, then $\mathcal{A}$ contains $\overline{\mathsf{End}(T_{|D_0})}$ for any finite subquiver D_0 of D, and finally, all of $\mathcal{C}(T)$. □

5 Monoidal version

Throughout this section, we assume a certain familiarity with monoidal categories. See for instance [M], [B].

If K is a field, tannakian theory states that given a monoidal category $\mathcal{C}$ and a monoidal functor $F : \mathcal{C} \to \mathsf{vect}\, K$, the associated coalgebra $L(F)$ actually is a bialgebra. If $\mathcal{C}$ has duals, it is a Hopf bialgebra. Here, we will adapt these results to Nori's setting. Since we are dealing with quivers, instead of categories, we will define monoidal quivers and monoidal representations. Since K is no longer a field, we will have to make rather strong assertions : we will consider representations taking values in projective modules (**P1**), together with a mild hypothesis to compensate for the absence of identities in a quiver (**P2**), and we will make a homological assumption ($\star$) on K, which essentially says that the global dimension of K is at most 2.

Instead of the coalgebra of coendomorphisms, we are dealing with the algebra of endomorphisms, which in this setting is a Pro-object ; so we will now rehearse a few general facts on Pro-objects.

5.1 Pro-objects.

Let $\mathcal{C}$ be a category. There is a universal way of constructing a completion of $\mathcal{C}$ with respect to (small) filtered inverse limits, and this is the category of Pro-objects of $\mathcal{C}$, denoted $\mathsf{Pro}\,\mathcal{C}$. Here follows a sketch of this construction.

A Pro-*object of* $\mathcal{C}$ is a small filtered inverse system $(X_i)_{i\in I}$ in $\mathcal{C}$, which we will denote "$\varprojlim_{i\in I} X_i$" for convenience. The Pro-objects of $\mathcal{C}$ form a catgory

$\mathsf{Pro}\,\mathcal{C}$, with morphisms defined as follows :

$$\mathsf{Hom}_{\mathsf{Pro}\,\mathcal{C}}(\text{“}\varprojlim_{i\in I} X_i\text{”}, \text{“}\varprojlim_{j\in J} Y_j\text{”}) = \varprojlim_j \varinjlim_i \mathsf{Hom}_{\mathcal{C}}(X_i, Y_j)\,.$$

The canonical functor $\mathcal{C} \to \mathsf{Pro}\,\mathcal{C}$, $X \mapsto$ “$\varprojlim X$” being fully faithful, we will identify $\mathcal{C}$ to a full subcategory of $\mathsf{Pro}\,\mathcal{C}$. Small filtered inverse limits exist in $\mathsf{Pro}\,\mathcal{C}$, and $\mathsf{Pro}\,\mathcal{C}$ enjoys the following universal property.
Let $\mathcal{E}$ be a category where small filtered inverse limits exist. Any functor $F : \mathcal{C} \to \mathcal{E}$ extends uniquely (up to unique isomorphsm) to a functor $\tilde{F} : \mathsf{Pro}\,\mathcal{C} \to \mathcal{E}$ which commutes to small filtered inverse limits; $\tilde{F}$ is defined by “$\varprojlim_i X_i$” $\mapsto \varprojlim_i F(X_i)$.
The category $\mathsf{Pro}\,K\text{-}\mathsf{mod}$ is abelian, K-linear, monoidal (indeed, if $\mathcal{C}$ is abelian or monoidal, so is $\mathsf{Pro}\,\mathcal{C}$). If A is an algebra in $\mathsf{Pro}\,K\text{-}\mathsf{mod}$, we denote $A\text{-}\mathsf{mod}$ the category of A-modules whose underlying Pro-object belongs to $K\text{-}\mathsf{mod}$.
If D is quiver and $T : D \to K\text{-}\mathsf{mod}$ a représentation of T, denote

$$\mathcal{E}nd(T) = \text{“}\varprojlim_{D\in F} \mathsf{End}(T\mid_{D_0})\text{”}\,.$$

Then $\mathcal{E}nd(T)$ is an algebra in $\mathsf{Pro}\,K\text{-}\mathsf{mod}$, and $\mathcal{C}(T) = \mathcal{E}nd(T)\text{-}\mathsf{mod}$.
Remark. Dually, one may construct a completion of a category $\mathcal{C}$ with respect to direct filtered limits : this is the category $\mathsf{Ind}\,\mathcal{C}$ of Ind-objects in $\mathcal{C}$. In particular, for a coherent ring K, $\mathsf{Ind}\,K\text{-}\mathsf{mod}$ is just the category $K\text{-}\mathsf{Mod}$ of all K-modules. Unfortunately there is no such simple description for $\mathsf{Pro}\,K\text{-}\mathsf{mod}$: it seems that this is the price to pay for Nori's theorem.

5.2 The monoidal setting

A *(strict) monoidal category* is a data $(\mathcal{M}, \otimes, I)$, where
— $\mathcal{C}$ is a category;
— $\otimes$ is a associative functor $\mathcal{M} \times \mathcal{M} \to \mathcal{M}$ (*tensor product*);
— I is a *unit object*, i. e. $? \otimes I = 1_{\mathcal{M}} = I \otimes ?$.
In other words, it is just a monoid in the category of categories. (Thanks to MacLane's coherence theorem we may assume all monoidal categories to be strict without loss of generality, see [M].)
By analogy, we define a monoidal quiver.
Definition. A *monoidal quiver* is a quiver $D = (D_1 \rightrightarrows D_0)$ with monoid structures on D_0 and D_1 such that $s, t : D_1 \to D_0$ are morphisms of monoids. In other words, it is a quiver in monoids, that is, a data (D, μ, η), where D is a quiver, $\mu : D \times D \to D$ is associative, and $\eta : 1 \to D$ is a unit for μ.

Let $(\mathcal{M}, \otimes, I)$ be a monoidal category. Define a new category $\mathcal{T}/\mathcal{M}$ as follows. Objects of $\mathcal{T}/\mathcal{M}$ are representations of quivers with values in $\mathcal{M}$. Morphisms from $T : D \to \mathcal{M}$ to $T' : D' \to \mathcal{M}$ are couples (f, α), where f is a quiver morphism $D \to D'$, and α an isomorphism of representations : $\alpha : T \xrightarrow{\sim} T' \circ f$, with obvious composition.
Define on $\mathcal{T}/\mathcal{M}$ a tensor product $\boxtimes$: given two objects $T : D \to \mathcal{M}$, $T' : D \to \mathcal{M}$, $T \boxtimes T'$ is the representation $D \times D' \to \mathcal{M}$, $(i, j) \mapsto T_i \otimes T'_j$. Thus, $\mathcal{T}/M$ becomes a monoidal category, whose unit object is the representation $\mathbb{I} : 1 \to \mathcal{M}$ which sends the only vertex of the initial quiver 1 to I and the only edge to 1_I.
DEFINITION. Let D be a monoidal quiver, and $(\mathcal{M}, \otimes, I)$ a monoidal category. A *monoidal representation of D with values in $\mathcal{M}$* is a representation $T : D \to \mathcal{M}$ equipped with morphisms $(\mu, \Phi) : T \boxtimes T \to T$ and $(\eta, \Phi_0) : \mathbb{I} \to T$ in the category $\mathcal{T}/\mathcal{M}$, such that :

- $\Phi_{ij,k}(\Phi_{i,j} \otimes 1_{T_k}) = \Phi_{i,jk}(1_{T_i} \otimes \Phi_{j,k})$;
- $\Phi_{i,e} = 1_{T_i} \otimes \Phi_0$ and $\Phi_{e,i} = \Phi_0 \otimes \Phi_{e,i}$ (where e denotes the unit of D_0).

The representation is usually referred to as (T, Φ, Φ_0), or just T.

Lemma 3 *Monoidal representations of monoidal quivers with values in $\mathcal{M}$ are exactly monoids in $\mathcal{T}/\mathcal{M}$.*

PROOF. A monoid in $\mathcal{T}/\mathcal{M}$ is an data (T, μ, η), where T is an object of $\mathcal{T}/\mathcal{M}$, and μ, η are morphisms, $\mu : T \boxtimes T \to T$, $\eta : \mathbb{I} \to T$, with the usual axioms of associativity : $\mu(\mu \otimes 1_T) = \mu(1_T \otimes \mu)$ and unity : $\mu(\eta \otimes 1_T) = 1_T = \mu(1_T \otimes \eta)$. Worked out explicitly, this boils down to a monoidal representation of a monoidal quiver. $\square$
DEFINITION. Let (T, Φ, Φ_0), (T', Φ', Φ'_0) be two monoidal representations of a monoidal quiver D in a monoidal category $\mathcal{C}$. A morphism of representations $\alpha : T \to T'$ is *monoidal* if :
— $\Phi'_{i,j}(\alpha_i \otimes \alpha_j) = \alpha_{ij}\Phi_{i,j}$ for $i, j \in D_0$;
— $\Phi'_0 = \alpha_e \Phi_0$.

5.3 The theorem in the monoidal setting

Let $\mathcal{T} = \mathcal{T}/K\text{-}\mathsf{mod}$.
We have a canonical functor $T \mapsto \mathcal{E}nd(T)$ from $\mathcal{T}$ to the category of algebras in $\mathsf{Pro}\, K\text{-}\mathsf{mod}$.
Given any two objects T, T' in $\mathcal{T}_K$, we have a canonical morphism $\Phi_{T,T'} : \mathcal{E}nd(T) \otimes \mathcal{E}nd(T') \to \mathcal{E}nd(T \otimes T')$. This is not an isomorphism in general.

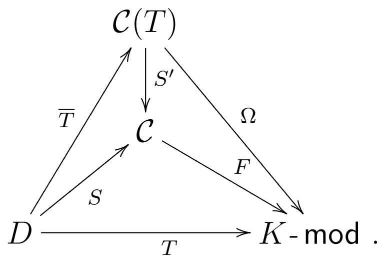
Consider the following example. Let V a finitely presented K-module, $f \in \mathsf{End}(V)$, and denote $T_{V,f}$ the representation of the initial quiver which sends the unique vertex to V and the unique edge to f. Then $\mathcal{E}nd(T_{V,f}) = \mathsf{End}(T_{V,f})$ is the centralizer $Z(f) = \{g \in End(V) \mid gf = fg\}$. We have $\mathsf{End}(T_{V,f}) \otimes \mathsf{End}(T_{K,0}) = Z(f)$, whereas $\mathsf{End}(T_{V,f} \otimes T_{K,0}) = \mathsf{End}(T_{V,0}) = \mathsf{End}(V)$; if f is not central, the canonical morphism is not an isomorphism.
However we shall see that, under certain assumptions, the canonical morphism is an isomorphism.
Definition. We say that the ring K satisfies condition $(\star)$ if for any morphism of K-modules $f : P \to P'$, with P and P' projective of finite type, $\mathsf{ker}(f)$ is projective.

Notice that, since K is coherent, $\mathsf{ker}(f)$ is finitely presented, and so, 'projective' just means 'flat' here. Condition $(\star)$ means that the homological dimension of the category $\mathsf{mod}\, K$ is at most 2. Recall that, if K is noetherian, the homological dimension of $\mathsf{mod}\, K$ is the global dimension of K. In particular $(\star)$ holds for $K = \mathbb{Z}$.
Now, consider the following conditions on a representation $T : D \to K\text{-}\mathsf{mod}$:
P1) $\forall i \in D_0$, T_i is a projective K-module of finite type;
P2) $\forall i \in D_0$, $\exists a \in D_1$, $i \xrightarrow{a} i$, such that $T_a = 1_{T_i}$.
Remark. Condition **P2** is automatic if D is a category and T a functor.

Definition. By *monoidal right exact abelian category over K*, we mean an abelian K-linear category $\mathcal{C}$ equipped with a monoidal structure $(\otimes, I)$ where $\otimes$ is K-linear right exact in each variable, and $\mathsf{End}(I) = K$.

Theorem 3 *Let $T : D \to K\text{-}\mathsf{mod}$ be a monoidal representation of a monoidal quiver D. Assume that T satisfies* **P1** *and* **P2**, *and $(\star)$ holds for K. Then :*
1) the algebra $\mathcal{E}nd(T)$ has a natural structure of bialgebra in $\mathsf{Pro}\, K\text{-}\mathsf{mod}$, so that $\mathcal{C}(T)$ is a monoidal right exact abelian category over K, $\overline{T}$ is a monoidal representation, and the forgetful functor $\Omega : \mathcal{C}(T) \to K\text{-}\mathsf{mod}$ is monoidal strict;
2) the above construction satisfies the following universal property.
Let $\mathcal{C}$ be a monoidal right exact abelian category over K, $F : \mathcal{C} \to K\text{-}\mathsf{mod}$ a K-linear faithful exact monoidal functor, and $S : D \to \mathcal{C}$ a monoidal representation of D in $\mathcal{C}$ such that $FS = T$ as monoidal representations. Then there exists a monoidal functor $S' : \mathcal{C}(T) \to \mathcal{C}$, unique (up to unique monoidal isomorphism) such that the following diagram is commutative (up

to monoidal isomorphism) :



The functor S' is (monoidal) K-linear faithful exact.

Remark. The statement of the universal property is to be understood as follows : there exist a monoidal functor S', a monoidal isomorphism $\alpha : S'\overline{T} \xrightarrow{\sim} S$, and a monoidal isomorphism $\beta : FS' \xrightarrow{\sim} \Omega$, with the compatibility condition: $F(\alpha) = \beta\overline{T}$. Uniqueness means that if (S'', α', β') is another solution to the problem, there exists a unique monoidal isomorphism $\gamma : S'' \xrightarrow{\sim} S'$ such that $\alpha' = (\gamma\overline{T})\alpha$ and $\beta' = \beta F(\gamma)$.

Proof.

Let $\mathcal{T}_0 \subset \mathcal{T}$ be the full subcategory whose objects are representations $T : D \to K\text{-}\mathsf{mod}$ satisfying **P1** and **P2**. Notice that $\mathcal{T}_0$ is a monoidal subcategory of $\mathcal{T}$.

Lemma 4 *The correspondence $T \mapsto \mathcal{E}nd(T)$ is a monoidal functor from ${\mathcal{T}_0}^o$ to the category of algebras in $\mathsf{Pro}\,K\text{-}\mathsf{mod}$.*

Proof.

Let $T : D \to \mathsf{mod}\,K$ and $T' : D \to \mathsf{mod}\,K$ be objects of $\mathcal{T}$, and let $T'' = T \otimes T'$. The whole point is to show that, if T and T' satisfy **P1** and **P2**, the natural morphism $\mathcal{E}nd(T) \otimes \mathcal{E}nd(T') \to \mathcal{E}nd(T'')$ is an isomorphism; in order to prove this, we may and do assume D and D' finite.

Recall that $\mathsf{End}(T)$ is given by the short exact sequence

$$(\mathcal{E}_{T,T}) \qquad 0 \to \mathsf{End}(T) \longrightarrow L(T) \xrightarrow{\Delta_T} M(T)\,,$$

where $L(T) = \prod_{i \in D_0} \mathsf{Hom}(T_i, T_i)$, $M(T) = \prod_{(j \xrightarrow{a} k) \in D_1} \mathsf{Hom}(T_j, T_k)$, $\Delta_T = \Delta_{T,T}$.

By **P1**, the K-modules $L(T)$ and $M(T)$ are projective of finite type, and so is $\mathsf{End}(T)$ by $(\star)$. We view $\mathsf{End}(T)$ as a submodule of $L(T)$. The same holds for T' and T''. All modules involved being flat, we have

$$\mathsf{End}(T) \otimes \mathsf{End}(T') = (\mathsf{End}(T) \otimes L') \cup (L \otimes \mathsf{End}(T'))\,.$$

Let $\phi = (\phi_{i,i'}) \in L(T'') \simeq L(T) \otimes L(T')$. Then :

- $\phi \in \mathsf{End}(T'')$ if an only if the following condition holds :

$$\forall (a,a') \in D_1 \times D_1', \quad (T_a \otimes T_{a'})\, \phi_{s(a),s(a')} = \phi_{t(a),t(a')}(T_a \otimes T_{a'})\,;$$

- $\phi \in \mathsf{End}(T) \otimes \mathsf{End}(T')$ if and only if the following two conditions hold :

$$\forall a \in D_1, \forall i' \in D_0', \quad (T_a \otimes 1_{T_{i'}})\, \phi_{s(a),i'} = \phi_{t(a),i'}(T_a \otimes 1_{T_{i'}})\,,$$
$$\forall a' \in D_1', \forall i \in D_0, \quad (1_{T_i} \otimes T_{a'})\, \phi_{i,s(a')} = \phi_{i,t(a')}(1_{T_i} \otimes T_{a'})\,.$$

Now clearly, $\mathsf{End}(T) \otimes \mathsf{End}(T') \subset \mathsf{End}(T'')$, and equality holds because T and T' satisfy **P2**. $\square$

Now we prove assertion 1) of the theorem. The assumptions made on T mean that it is a monoid in $\mathcal{T}_0$, hence a co-monoid in $\mathcal{T}_0{}^o$; therefore its image by the monoidal functor $\mathcal{E}nd(?)$ is a co-monoid in the category of algebras, that is, a bialgebra. The coproduct Δ and counit ε are obtained by applying $\mathcal{E}nd(?)$ respectively to the multiplication $T \boxtimes T \to T$ and the unit $\eta : \mathbb{I} \to T$. Therefore $\mathcal{C}(T) = \mathcal{E}nd(\mathcal{C}) - \mathsf{mod}$ is a monoidal category, the tensor product being defined as usual for a category of left modules on a bialgebra. The functor Ω and the representation $\overline{T}$ are clearly monoidal.

As for assertion 2) (universal property) : Th. 1 yields S' as a 'naked' functor S', which is K-linear faithful exact. The whole point is to check that S' can be endowed with an appropriate monoidal structure.
For simplicity, we assume that F is monoidal strict. The difficult point is to construct for X, Y in $\mathcal{C}(T)$ a functorial isomorphism $\Phi_{X,Y} : S'(X) \otimes S'(Y) \xrightarrow{\sim} S'(X \otimes Y)$. Since Ω is monoidal and $FS' \xrightarrow{\sim} \Omega$, we dispose of a functorial isomorphism $\Psi_{X,Y} : F(S'(X) \otimes S'(Y)) \xrightarrow{\sim} F(S'(X \otimes Y))$.
Consider the property :
$L(X,Y)$: there exists $\Phi_{X,Y} : S'(X) \otimes S'(Y) \xrightarrow{\sim} S'(X \otimes Y)$ such that $F(\Phi_{X,Y}) = \Psi_{X,Y}$.
Notice that if $L(X,Y)$ holds, $\Phi_{X,Y}$ is uniquely determined because F is faithful.
Now $L(X,Y)$ holds for $X = \overline{T}_i$, $Y = \overline{T}_j$. Indeed we have $S'\overline{T} \xrightarrow{\sim} S$, and S, $\overline{T}$ are monoidal, hence our isomorphism

$$S'(\overline{T}_i) \otimes S'(\overline{T}_i) \xrightarrow{\sim} S_i \otimes S_j \xrightarrow{\sim} S_{i,j} \xrightarrow{\sim} S'(\overline{T}_{ij}) \xrightarrow{\sim} S'(\overline{T}_i \otimes \overline{T}_j)\,.$$

In order to prove $L(X,Y)$ in general, we need the following lemmas.

Lemma 5 *In $\mathcal{C}(T)$, any object is a quotient of an object which is projective as a K-module.*

Proof. For any X in $\mathcal{C}(T)$, there exists a finite subquiver $D_0 \subset D$ such that X is an E_0-module for $E_0 = \mathsf{End}(T_{|D_0})$. Now, thanks to conditions $(\star)$ and **P1**, E_0 is a projective K-module, and X is a quotient of $\overline{E_0}^n$ for some integer n. □

Lemma 6 *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories, $G, G' : \mathcal{A} \to \mathcal{B}$ be additive functors, and $F : \mathcal{B} \to \mathcal{C}$ be a faithful exact functor. Let α be a morphism of functors : $FG \to FG'$. We say that $X \in \mathsf{Ob}\,\mathcal{A}$ has the* lifting *propery if there exists a (necessarily unique) morphism $\beta_X : G(X) \to G'(X)$ such that $F(\beta_X) = \alpha_X$. We denote $\mathcal{A}_0 \subset \mathcal{A}$ the full subcategory of objects having the lifting property. Then :*
1) β is a morphism of functors $G_{|\mathcal{A}_0} \to G'_{|\mathcal{A}_0}$;
2) $\mathcal{A}_0$ is stable under direct sums and direct summands;
3) if G and G' are right- (resp left-) exact, $\mathcal{A}_0$ is stable under cokernels (resp. kernels).

Proof. 1) Holds because F is faithful. 2) Let X, Y be in $\mathcal{A}_0$. Then $F(\beta_X \oplus \beta_Y) = \alpha_X \oplus \alpha_Y = \alpha_{X \oplus Y}$ so $X \oplus Y$ is in $\mathcal{A}_0$. The assertion on summands is equally trivial. 3) Assume for instance that G is right exact. Let $f : X \to Y$ be a morphism in $\mathcal{A}_0$, and let $Q = \mathsf{coker}(f)$. We have the following commutative diagram, where lines are exact :

$$\begin{array}{ccccccc} G(X) & \longrightarrow & G(Y) & \longrightarrow & G(Q) & \longrightarrow & 0\ , \\ \downarrow{\scriptstyle \beta_X} & & \downarrow{\scriptstyle \beta_Y} & & & & \\ G'(X) & \longrightarrow & G'(Y) & \longrightarrow & G'(Q) & \longrightarrow & 0 \end{array}$$

hence a morphism $\beta_Q : G(Q) \to G'(Q)$. Applying the exact functor F, we see that $F(\beta_Q) = \alpha_Q$, so $\mathsf{coker}\, f$ is in $\mathcal{A}_0$. □

Let Y be an object of $\mathcal{C}(T)$. We will apply lemma 6 three times to $G = S'(?) \otimes S'(Y)$, $G'_Y = S'(? \otimes Y)$ and $\alpha = \Psi_{?,Y}$. Notice that G, G' are right exact, and they are actually exact if Y is projective as a K-module.

Consider first the case $Y = \overline{T}_j$. Here G and G' are exact; $\overline{T}_i$'s have the lifting property, and so by proposition 2 and the lemma, we deduce $L(X, \overline{T}_j)$ for any X. We also have $L(\overline{T}_j, X)$ by symmetry.

Now assume Y is projective as a K-module. Here again G and G' are exact, and $\overline{T}_i$'s have the lifting property, so by the same argument we have $L(X, Y)$ provided that X or Y is projective as a K-module.

Now consider arbitrary Y, so that G and G' are just right exact. If X is projective as a K-module, X has the lifting property. Now by lemma 5, any X in $\mathcal{C}(T)$ fits in a short exact sequence $P' \to P \to X \to 0$, with P, P'

projective as K-modules. By lemma 6, X has the lifting property, hence $L(X,Y)$ for X,Y arbitrary.
The rest is then a fastidious but straightforward verification which we shall omit. □

For $\mathcal{E}nd(T)$ to be a Hopf bialgebra, that is, a bialgebra with bijective antipode, we need duals. Recall that a duality in a monoidal category $\mathcal{C}$ is a data (X,Y,e,h), where X, Y are objects, $e: X\otimes Y \to I$, $h: I \to Y\otimes X$ are morphisms satisfying :

$$(e\otimes 1_X)(1_X \otimes h) = 1_X \qquad \text{and} \qquad (1_Y \otimes e)(h\otimes 1_Y) = 1_Y\ .$$

If (X,Y,e,h) is a duality, we say that X is a left dual of Y, and Y is a right dual of X. We say that X has duals if X has both a left- and a right dual.

Definition. Let $\mathcal{C}$ be a monoidal right exact abelian category over K. We say that $\mathcal{C}$ *has enough duals* if any object of $\mathcal{C}$ is a quotient of an object having duals.

Proposition 3 *Let $\mathcal{C}$ be a monoidal right exact abelian category over K and let $\omega : \mathcal{C} \to K\text{-}\mathsf{mod}$ be a K-linear faithful monoidal functor.*
1) Let X be an object of $\mathcal{C}$. If $\omega(X)$ is projective, both $?\otimes X$ and $X\otimes ?$ are exact functors. If X has a left dual in $\mathcal{C}$, $\omega(X)$ is projective.
2) Consider a morphism $f : X \to Y$, and let $Z = \mathsf{coker}(f)$. Assume that X and Y have left duals ${}^\vee X$, ${}^\vee Y$, and that $\omega(Z)$ is projective. Then Z has a left dual.
3) If $\mathcal{C}$ has enough duals, any object X such that $\omega(X)$ is projective has both a left- and a right dual.

Proof. If for some object T in $\mathcal{C}$, $\omega(T)$ is a projective K-module we say for short that T is *projective over* K.
1) As a monoidal functor, ω preserves duals, and an object of $K\text{-}\mathsf{mod}$ has a dual if and only if it is projective, hence the second assertion. The first results from the fact that ω is faithful exact and projective modules are flat.
2) We have $\omega({}^\vee X) \simeq \omega(X)^*$, $\omega({}^\vee Y) \simeq \omega(Y)^*$, and $\omega({}^\vee f) = \omega(f)^*$ via these isomorphisms. Since Z is projective over K, $\omega(Y) \twoheadrightarrow \omega(Z)$ has a section. Therefore $\omega(Z') \hookrightarrow \omega({}^\vee Y)$ is a direct summand. In particular Z' is projective over K. Moreover, $\omega(Z')$ is canonically isomorphic to $\omega(Z)^*$. All that remains to see is that the evaluation and coevaluation morphisms $\omega(Z)\otimes\omega(Z') \to K$ and $K \to \omega(Z')\otimes\omega(Z)$ lift to an evaluation and a coevaluation $Z\otimes Z' \to I$, $I \to Z'\otimes Z$. This is an easy diagram-chase, using assertion 1) and the fact that all objects involved are projective over K.

3) Let Z be projective over Z. If $\mathcal{C}$ has enough duals, there exists a short exact sequence $X \to Y \to Z \to 0$, where X, Y have duals. By (2), Z has duals. □

Corollary 3 *Assume that K satisfies $(\star)$. Let $\mathcal{C}$ be a monoidal right exact abelian category over K having enough duals, and let $\omega : \mathcal{C} \to K\text{-}\mathsf{mod}$ be a K-linear faithful monoidal functor. Then $\mathcal{E}nd(\omega)$ is a Hopf algebra in* $\mathsf{Pro}\text{-}\mathsf{mod}$*, and $\overline{\omega} : \mathcal{C} \to \mathcal{C}(\omega)$ is a K-linear monoidal equivalence.*

Proof.
Let $\mathcal{C}_0$ be the full subcategory of $\mathcal{C}$ of objects having duals, and $\omega_0 = \omega_{|\mathcal{C}_0}$. By assumption, any object of $\mathcal{C}$ is a cokernel of a morphism of $\mathcal{C}_0$. The canonical bialgebra morphism

$$\mathcal{E}nd(\omega) \to \mathcal{E}nd(\omega_0)$$

is therefore an isomorphsm.
Now $\mathcal{C}_0$ is a monoidal category, and ω_0 is a monoidal functor satisfying **P1** and **P2** as a quiver representation, so $\mathcal{E}nd(\omega_0)$ is a bialgebra by Th. 3. Moreover duals exist in $\mathcal{C}_0$, so by standard tannakian theory, $\mathcal{E}nd(\omega_{|\mathcal{C}_0})$ admits a bijective antipode : it is a Hopf bialgebra.
Consider the commutative square :

$$\begin{array}{ccc} \mathcal{C}_0 & \xrightarrow{\overline{\omega}_0} & \mathcal{C}(\omega_0) \\ {\scriptstyle \text{incl.}}\downarrow & & \downarrow{\scriptstyle \text{can.}} \\ \mathcal{C} & \xrightarrow[\overline{\omega}]{} & \mathcal{C}(\omega) \end{array}$$

Since $\mathcal{E}nd(\omega) \xrightarrow{\sim} \mathcal{E}nd(\omega_0)$, the canonical functor $\mathcal{C}(\omega_0) \to \mathcal{C}(\omega)$ is an equivalence. By Th. 3, $\mathcal{C}(\omega_0)$ is a monoidal right exact abelian category over K, so $\mathcal{C}(\omega)$ inherits such a structure. By corollary 1, $\overline{\omega}$ is a K-linear equivalence. We only have to check that $\overline{\omega}$ is monoidal, in other words we need an isomorphism $\Phi_{X,Y} : \overline{\omega}(X) \otimes \overline{\omega}(Y) \simeq \overline{\omega}(X \otimes Y)$. By construction we have such an isomorphism for X, Y in $\mathcal{C}_0$; for arbitrary X, Y we have $\omega(X) \otimes \omega(Y) \simeq \omega(X \otimes Y)$. Using lemma 6 we can lift this to define $\Phi_{X,Y}$.
□

Bibliography

[A-K] Y. André, B. Kahn, *Nilpotence, radicaux et structures monoïdales* Rendiconti del Seminario Matematico dell'Universitá di Padova **108**, pp. 107–291 (2002).

[A-K] Y. André, B. Kahn, *Construction inconditionnelle de groupes de Galois motiviques* C. R. Acad. Sci. Paris **234**, pp. 989-994 (2002).
[B] A. Bruguières, *Théorie tannakienne non commutative*, Comm. in Algebra, **22**(**14**), pp. 5817—5860 (1994).

[D] P. Deligne, *Catégories tannakiennes*, in *The Grothendieck Festschrift*, II, Progress in Mathematics **87,** Birkhäuser, pp. 111–195 (1990).

[M] S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Math. **5,** Springer Verlag (1971).

[P] N. Popescu, Abelian categories with applications to rings and modules, Academic Press (1973).

[S] N. Saavedra Rivano, Catégories tannakiennes, Springer L. N. M. **265** (1972).