DOUBLE BRAIDINGS, TWISTS AND TANGLE INVARIANTS

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ABSTRACT. A tortile (or ribbon) category defines invariants of ribbon (framed) links and tangles. We observe that these invariants, when restricted to links, string links, and more general tangles which we call *turbans*, do not actually depend on the braiding of the tortile category. Besides duality, the only pertinent data for such tangles are the double braiding and twist. We introduce the general notions of twine, which is meant to play the rôle of the double braiding (in the absence of a braiding), and the corresponding notion of twist. We show that the category of (ribbon) pure braids is the free category with a twine (a twist). We show that a category with duals and a self-dual twist defines invariants of stringlinks. We introduce the notion of *turban* category, so that the category of turban tangles is the free turban category. Lastly we give a few examples and a tannaka dictionary for twines and twists.

'Just the place for a Snark!', the Bellman cried,
As he landed his crew with care;
Supporting each man at the top of the tide
By a finger entwined in his hair.

Lewis Carroll, *The Hunting of the Snark*

CONTENTS

INTRODUCTION

It is now well understood that certain categorical notions are very closely related to low dimensional topology. For instance, braids form a braided monoidal category, and the category of braids is the free braided category. The category Tang of oriented ribbon tangles is a tortile (or ribbon) category [JS93], and indeed, it has been proved by Shum [Shu94] (also [Tur94]) that Tang is the free tortile category. This theorem is a powerful tool for constructing invariants of ribbon links in S^3, since ribbon links up to isotopy are the endomorphisms of the unit object in Tang. Via Kirby calculus, Shum's theorem underlies the construction of the Reshetikhin-Turaev invariants of closed 3-manifolds. Kirby calculus can also be used to describe cobordisms of 3-manfolds in terms of certain tangles, and this allowed Turaev to construct a TQFT associated with a modular category [Tur94].

The present work explores certain consequences of the following observation. Let $\mathcal{C}$ be a tortile category. Recall that $\mathcal{C}$ is a braided category with duals, and a (self-dual) twist θ. Denoting $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ the braiding, define the *double braiding* by $D_{X,Y} = c_{Y,X}c_{X,Y}$. Notice that θ satisfies certain axioms where c appears only in the form of its double D, and conversely, θ determines D. It turns out that many significant notions apparently related to c actually depend only on D or θ. The S-matrix, and the subcategory of transparent objects [Bru00], which play an important role in the construction of invariants of 3-manifolds, are defined purely in terms of the double braiding D. More surprisingly, the invariants of ribbon links defined by $\mathcal{C}$ via Shum's Theorem do not depend on the actual braiding, but only on D (see Proposition 1, and section 5); and this result extends to a much larger class of ribbon tangles, namely those whose linking matrix is diagonal $\mathsf{mod} 2$. Since these tangles play an important role here, we give them a name: we call them *turban* tangles.

All this suggests that the double braiding and the twist deserve to be studied for their own sake, and that the universal property of the category of tangles, that is, Shum's Theorem, should have an analogue for the category of turban tangles.

The first step is to axiomatize the notion of double braiding. We observe that a double braiding satisfies certain formal properties (TW0)-(TW2). An operator D satisfying these properties will be called a *twine*. An *entwined* category is a monoidal category with a twine. The category of pure braids is the free entwined category (section 2, Theorem 1). We also introduce a general notion of twist, in such a way that the category of ribbon pure braids is the free category with twist (section 3, Theorem 2).

In section 4, we bring duality into the picture, and find out that a category with duals and (self-dual) twist defines invariants of ribbon string links (theorem 3).

The heart of the matter is to extend these constructions to the largest possible subcategory of the category of tangles. The natural candidate is the subcategory generated by the twist and duality: this is precisely the category of turban tangles (proposition 1). In section 5, we define a *turban category* to be a category with a twist and good duals (sovereign structure), satisfying certain additional conditions. We show that the category of turban tangles is the free turban category (theorem 4).

Section 6 gives a few examples of twines, twists and turban categories, as well as the tannaka dictionary for twines and twists.

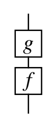
The definition of a turban category proposed in this paper is certainly not definitive, but I believe that this notion could lead to new topological invariants, including 3-manifold invariants and related TQFT's. The land of twines and twists is 'full of crags and chasms', and exploring it sometimes feels like snark hunting. For instance, the fact that if c, c' are braidings, then $c'_{Y,X}c_{X,Y}$ is a twine, came to me as a real surprise! While so far I have few examples of twines or twists, there are many indications that the class of entwined category is much larger that the class of braided category.

After completion of this work, I was informed that in a recent and unpublished work, S. Lack had given a categorical characterization of the category of pure braids ([Lac04]) which is essentially the same as mine (section 2). I also learned about M. Staic's recent work ([Sta04]), where the notions of *pure braided structure* and *pure ribbon structure* on a Hopf algebra are introduced; those are the *twinor* and *twistor* notions my section 6. M. Staic shows that such algebraic data yield invariants of pure braids and long knots respectively. Moreover, in a final remark, M. Staic suggests definitions for a *pure braided structure* and a *pure ribbon structure* on a monoidal category. The former is more complicated than, but equivalent to, my notion of *twine*; the latter is essentially the same as my notion of *twist*. Our contributions seem to be complementary, and, hopefully, one need not conclude that 'the Snark *was* a Boojum, you see!'.

I wish to thank Alexis Virelizier for many enlightening discussions.

1. Conventions and notations

1.1. **Monoidal categories.** Unless otherwise specified, all categories will be small and all monoidal categories will be strict. We will use Penrose graphical calculus, with the ascending convention: diagrams are to be read from bottom to top, *e. g.* given $X \xrightarrow{f} Y \xrightarrow{g} Z$, we represent gf as



If $\mathcal{C}$ is a monoidal category, with tensor product $\otimes$ and unit object I, we denote $\otimes^n$ the n-uple tensor product

$$\mathcal{C}^n \longrightarrow \mathcal{C}\,,$$
$$(X_1, \ldots X_n) \mapsto X_1 \otimes \cdots \otimes X_n\,.$$

In particular $\otimes^0 = I$, $\otimes^1 = 1_{\mathcal{C}}$ and $\otimes^2 = \otimes$.

Let $\mathcal{C}$ be a monoidal category. A *duality of* $\mathcal{C}$ is a data (X, Y, e, h), where X, Y are objects, and $e : X \otimes Y \to I$, $h : I \to Y \otimes X$ morphisms of $\mathcal{C}$, satisfying:

$$(e \otimes 1_X)(1_X \otimes h) = 1_X \text{ and } (1_Y \otimes e)(h \otimes 1_Y) = 1_Y\ .$$

If (X, Y, e, h) is a duality, we say that (Y, e, h) is a right dual of X, and (X, e, h) is a left dual of X. If a right or left dual of an object exists, it is unique up to unique isomorphism.

By *monoidal category with right duals (resp. left duals, resp. duals)*, we mean a monoidal category $\mathcal{C}$ where each object X admits a right dual (resp. a left dual, resp. both a right and a left dual).

If $\mathcal{C}$ has right duals, we may pick a right dual $(X^\vee, e_X, h_X)$ for each object X (the actual choice is inocuous, in that a right dual is unique up to unique isomorphism). This defines a monoidal functor

$$?^\vee : \mathcal{C}^o \to \mathcal{C}$$

where $\mathcal{C}^o$ denotes the category with opposite composition and tensor product.

Similarly a choice of left duals $({}^\vee X, \varepsilon_X, \eta_X)$ for all $X \in \mathsf{Ob}\mathcal{C}$ defines a monoidal functor ${}^\vee ? : \mathcal{C}^o \to \mathcal{C}$.

A *(strict) sovereign structure on* $\mathcal{C}$ is the choice, for each object X, of a right dual (X^*, e_X, h_X) and a left dual $(X^*, \varepsilon_X, \eta_X)$, with same underlying object X^*, in such a way that ${}^\vee ? = ?^\vee$ as monoidal functors. Essentially, left duals and right duals coincide. By *sovereign category,* we mean a monoidal category with a sovereign structure. This is an appropriate categorical setting for a good notion of trace; however one must distinguish a left- and a right trace tr_l and tr_r. If X is an object of $\mathcal{C}$ and $f \in \mathsf{End}(X)$,

$$\mathsf{tr}_l(f) = \varepsilon_X(1_{X^*} \otimes f)h_X, \quad \mathsf{tr}_r(f) = e_X(f \otimes 1_{X^*})\eta_X \quad \text{in } \mathsf{End}(I)\,.$$

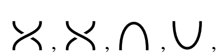
Definition. Let $\mathcal{C}$ be a braided category, with braiding c; the *double braiding* is the functorial isomorphism

$$D_{X,Y} = c_{Y,X}c_{X,Y} : X \otimes Y \xrightarrow{\sim} X \otimes Y\,.$$

A *tortile category* is a monoidal braided category with duals, equipped with a *twist,* that is, a functorial isomorphism $\theta_X : X \xrightarrow{\sim} X$ $(X \in \mathsf{Ob}\mathcal{C})$ such that $\theta_X \otimes \theta_Y = \theta_{X\otimes Y} D_{X,Y}$ and $\theta_I = 1_I$. Moreover the twist is assumed to be *self-dual, i. e.* $\theta_{X^\vee} = \theta_X^\vee$.

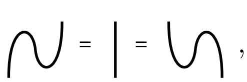
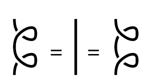
If $\mathcal{C}$ is a tortile category, and if one makes the (inocuous) choice of right duals $(X^\vee = X^*, e_X, h_X)$, there is a canonical choice of left duals $(X^*, \varepsilon_X, \eta_X)$ which defines a sovereign structure. The self-duality of the twist implies that the left- and right trace coincide (a property often referred to as *sphericity*).

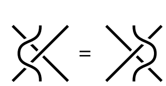
1.2. **Tangles.** We will often represent tangles by tangle diagrams, which we view as drawings made up of the following pictograms:



called positive crossing, negative crossing, local max an local min respectively (linked up by smooth arcs without horizontal tangents).

Two tangle diagrams represent the same isotopy class of ribbon tangles (also called framed tangles) if and only if one may be obtained from the other by deformation and a finite number of ribbon Reidemeister moves:

(R0)  , (R1)  ,

(R2)  , (R3)  .

Note that isotopy of non-ribbon tangles is obtained by adding the Reidemeister move  = to this list.

We will denote $\{D\}$ (and sometimes just D) the isotopy class of ribbon tangles represented by a tangle diagram D.

Let D be a tangle diagram, C a component of D. We denote $\Delta_C D$ the tangle diagram obtained from D replacing C by 2 parallel copies of D.

A tangle may be oriented, and/or coloured by elements of a set.

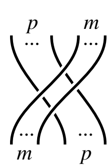
We denote Tang the category of isotopy classes of oriented ribbon tangles. This is a tortile category, whose objects are words on the letters $[+]$ and $[-]$. We denote $\mathsf{Tang}[\Lambda]$ the category of isotopy classes of oriented ribbon tangles coloured by elements of the set Λ, which is another tortile category. In $\mathsf{Tang}[\Lambda]$, we denote $[+]_\lambda$ (resp. $[-]_\lambda$) the object $[+]$ (resp. $[-]$) coloured by the element $\lambda \in \Lambda$. In the unoriented case, the point will be denoted $[\bullet]$.

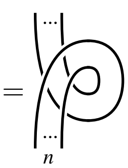
Shum's theorem may be formulated as follows : if $\mathcal{C}$ is a tortile category and $\Lambda = \mathsf{Ob}\mathcal{C}$, there exists a unique strict monoidal functor

$$F_{\mathcal{C}} : \mathsf{Tang}[\Lambda] \to \mathcal{C}$$

preserving the twist and dualities, and sending $[+]_X$ ($X \in \mathsf{Ob}\mathcal{C}$) to the object X itself. We will refer to $F_{\mathcal{C}}$ as Shum's functor; one may view it as a ribbon tangle invariant.

1.3. **Ribbon Tangles and Turbans Tangles.** Recall that the braiding c and the twist θ of the category of ribbon tangles are defined by

$c_{m,p} =$ 

, $\theta_n =$ 

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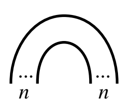
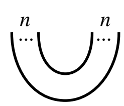
evaluation and coevaluation morphisms e_n and h_n being given by

$e_n =$ 

, $h_n =$ 

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the ribbon structure on $\mathsf{Tang}[\Lambda]$ is defined by the same tangles, with appropriate orientation and Λ-colouring.

Definition. A ribbon tangle T is *turban* (resp. *even*) if its linking matrix is diagonal $\mathsf{mod}2$ (resp. zero $\mathsf{mod}2$).

For instance, ribbon links, ribbon pure braids, ribbon string links are turban.

Turban tangles (resp. even tangles) form a monoidal subcategory of Tang which we denote Turb (resp. eTang). The following proposition, with its corollary, is the main motivation for the rest of this work.

Proposition 1. *The category* Turb *(resp.* eTang*) is the smallest monoidal subcategory of* Tang *having the same objects as* Tang*, and containing all evaluations, coevaluations and twists (resp. double braidings).*

Corollary 1. *Any invariant of ribbon links or turban tangles arising from a tortile category $\mathcal{C}$ is independent of the braiding: it depends only on the twist and the duality.*

We will prove proposition 1 and its corollary in section 5.

2. Twines and pure links

Definition. Let $\mathcal{C}$ be a monoidal category. A *twine of* $\mathcal{C}$ is an automorphism D of $\otimes$, that is, a functorial isomorphism

$$D_{X,Y} : X \otimes Y \xrightarrow{\sim} X \otimes Y \qquad (X, Y \in \mathcal{C})$$

satisfying the following axioms:

(DB0) $$D_{I,I} = 1_I\,;$$

(DB1) $$(D_{X,Y} \otimes 1_Z) D_{X\otimes Y,Z} = (1_X \otimes D_{Y,Z}) D_{X,Y\otimes Z}\,;$$

(DB2) $$\begin{aligned}(D_{X\otimes Y,Z} \otimes 1_T)(1_X \otimes D^{-1}_{Y,Z} \otimes 1_T)(1_X \otimes D_{Y,Z\otimes T})\\ = \quad (1_X \otimes D_{Y,Z\otimes T})(1_X \otimes D^{-1}_{Y,Z} \otimes 1_T)(D_{X\otimes Y,Z} \otimes 1_T)\,.\end{aligned}$$

An *entwined category* is a monoidal category equipped with a twine.

If $\mathcal{C}$, $\mathcal{C}'$ are two entwined categories, with twines D, D', a strict entwined functor $F : \mathcal{C} \to \mathcal{C}'$ is a strict monoidal functor $\mathcal{C} \to \mathcal{C}'$ such that for all $X, Y \in \mathsf{Ob}\mathcal{C}$,

$$F(D_{X,Y}) = D'_{FX,FY}\,.$$

Example. Let $\mathcal{C}$ be a monoidal category, and c a braiding of $\mathcal{C}$. Then $D_{X,Y} = c_{Y,X} c_{X,Y}$ (the double of c) is a twine of $\mathcal{C}$. In particular, let B be the category of braids, with its canonical braiding c. Recall that the canonical braiding c is characterised by the fact that $c_{1,1}$ is the standard generator of B_2. Let D be the double of c. Let PB be the category of pure braids. Then for any integers m, p, $D_{m,p}$ is a morphism of P and this defines a twine of PB. We will therefore consider PB as an entwined category.

Remark. This example admits of the following surprising generalization, which was pointed out to me by A. Virelizier: if c, c' are two braidings in $\mathcal{C}$, then $D_{X,Y} = c'_{Y,X} c_{X,Y}$ is a twine.

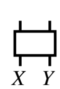
Here are a few comments on the axioms.

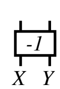
The first two axioms (DB0) and (DB1) imply the following:

(a) $D_{X,I} = 1_X = D_{I,X}$.

(b) $(D^{-1}_{X,Y} \otimes 1_Z) D_{X,Y\otimes Z} = D_{X\otimes Y,Z}(1_X \otimes D^{-1}_{Y,Z})$ and $(1_X \otimes D^{-1}_{Y,Z}) D_{X\otimes Y,Z} = D_{X,Y\otimes Z}(D^{-1}_{X,Y} \otimes 1_Z)$.

It will be very convenient to depict $D_{X,Y}$, $D^{-1}_{X,Y}$ as follows:

$D_{X,Y} =$ 

, $D^{-1}_{X,Y} =$ 

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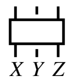
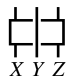
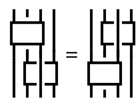
Similarly, let

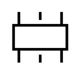
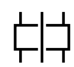
$$D^f_{X,Y,Z} =$$ 

$$= (D^{-1}_{X,Y} \otimes 1_Z) D_{X,Y\otimes Z} = D_{X\otimes Y,Z}(1_X \otimes D^{-1}_{Y,Z})\,,$$

$$D^b_{X,Y,Z} =$$ 

$$= (1_X \otimes D^{-1}_{Y,Z}) D_{X\otimes Y,Z} = D_{X,Y\otimes Z}(D^{-1}_{X,Y} \otimes 1_Z)\,.$$

Now (DB2) can be re-interpreted in a nice way. Indeed, composing each side of (DB2) on the right by $(1_X \otimes D^{-1}_{Y,Z} \otimes 1_T)$ and using (b), we obtain the sliding property:  .

Notice that the notion of twine is invariant under left-right symmetry (tensor product reversal) and under top-bottom symmetry (composition reversal). In both cases front and back (i. e.  and ) are exchanged. In particular central symmetry preserves front and back.

The following theorem justifies, in a sense, the axioms for a twine.

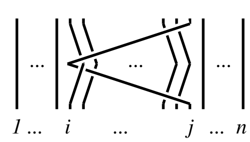
Theorem 1. *The category of pure braids is the universal entwined category. More precisely, let $\mathcal{C}$ be an entwined category, $\Lambda = \mathsf{Ob}\mathcal{C}$, and denote $\mathsf{PB}(\Lambda)$ the category of Λ-coloured pure braids. There exists a unique strict entwined functor $\mathsf{PB}(\Lambda) \to \mathcal{C}$ sending $[\bullet]_X$ $(X \in \mathsf{Ob}\mathcal{C})$ to the object X itself.*

Proof of Theorem 1. The proof relies on a presentation of the group of pure braids P_n by generators and relations, due to Markov [Mar45]. (See also [Ver03]). Let $\sigma_i \in B_n$ $(1 \le i < n)$ be the standard generator:

$$\sigma_i =$$ 

.

For $1 \le i < j \le n$, let $s_{i,j} = \sigma_{j-1} \dots \sigma_i \sigma_i \dots \sigma_{j-1}$; pictorially:

$$s_{i,j} =$$ 

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Then the $s_{i,j}$'s generate P_n, subject to the Burau relations:

(Bu1) $s_{i,j}s_{k,l} = s_{k,l}s_{i,j}$ for $i < j < k < l$ or $i < k < j < l$;
(Bu2) $s_{i,j}s_{i,k}s_{j,k} = s_{i,k}s_{j,k}s_{i,j} = s_{j,k}s_{i,j}s_{i,k}$ for $i < j < k$;
(Bu3) $s_{i,k}s_{j,k}s_{j,l}s^{-1}_{j,k} = s_{j,k}s_{j,l}s^{-1}_{j,k}s_{i,k}$ for $i < j < k < l$.

In the entwined category PB of pure braids, $s_{i,j} = 1_{i-1} \otimes D^f_{1,j-i-1,1} \otimes 1_{n-j}$.

Proposition 2. *Let $\mathcal{C}$ be an entwined category. There exists a unique group morphism*

$$\begin{aligned} \mathsf{PB}_n &\to \mathsf{Aut}(\otimes^n) \\ P &\mapsto [P] \end{aligned}$$

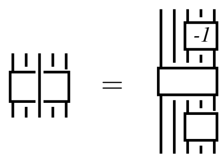
such that for all $X_1, \ldots, X_n \in \mathsf{Ob}\mathcal{C}$ and $1 \leq i < j \leq n$,

$$[\sigma_{i,j}]_{X_1,\ldots,X_n} = 1_{X_1 \otimes \cdots \otimes X_{i-1}} \otimes D^f_{X_i, X_{i+1} \otimes \cdots \otimes X_{j-1}, X_j} \otimes 1_{X_{j+1} \otimes \cdots \otimes X_n} \ .$$

Proof. Since the $s_{i,j}$'s generate P_n, we only have to check compatibility with the Burau relations.

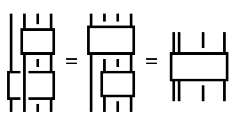
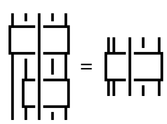
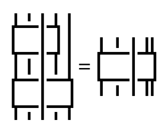
Now the first case of (Bu1) is functoriality of the tensor product, and the second case of (Bu2) is functoriality of $D^f_{X,Y,Z}$ with respect to Y.

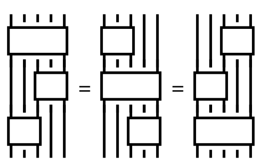
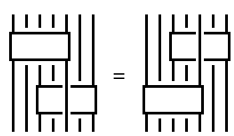
In order to check the other relations, we will have to perform certain computations in an entwined category. Let us adopt the notation:

 .

It is understood that each strand is coloured by an object of $\mathcal{C}$, so this is an identity of morphisms of $\mathcal{C}$.

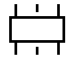
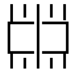
Lemma 1. *The following identities hold in an entwined category:*

a)  ; *b)*  ; *c)*  ;

d)  ; *e)*  .

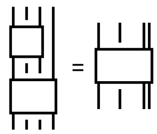
N. B.: strings which are drawn very close represent one entry coloured by the tensor product of the colours of the strings.

Proof. The computations would be very awkward in algebraic form; they are much easier to conduct using Penrose graphical calculus. Here is a sketch of the proof.

Assertions a) and b): the first identity of a) holds by definition; the second results from the definition of  by straightforward computation, and implies b) by definition of  .

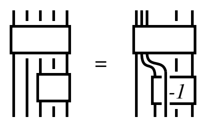
Consider assertion e), and denote X, A, Y, B, Z, C, T the objects of $\mathcal{C}$ used to colour the seven strands, listed from left to right. Then the case $A = B = C = I$ is just the sliding property, which is a consequence of the twine axioms. Now using a), we deduce e) in the case $B = C = I$.

c) Using the definitions and elementary manipulations, assertion c) can be easily reduced to assertion e) in the case $B = C = I$, which we just proved,

and the identity  , analogous to the second identity of a).

Assertion e): the case $C = I$ can now be deduced from the case $B = C = I$ using b). Hence the general case, using a) and c).

Let us prove assertion d). By reason of symmetry, it is enough to check the first identity. Now one computes easily



and one concludes using e) and functoriality of the twine. Thus ends the proof of the lemma. □

Relations (Bu2) and (Bu3) are direct consequences of assertions d) and e) of the lemma, hence the proposition. □

Now the lemma clearly defines a monoidal functor $\mathsf{PB}[\Lambda] \to \mathcal{C}$ which sends $[X]$ to X ($X \in \mathsf{Ob}\mathcal{C}$), the pure braid on n strands coloured by $X_1, \dots X_n$ to $[P]_{X_1,\dots,X_n}$, and preserves the twine. Uniqueness results form the fact that the $s_{i,j}$'s generate PB.

3. Twists and ribbon pure braids

Definition. Let $\mathcal{C}$ be a monoidal category. A *twist of* $\mathcal{C}$ is an automorphism θ of $1_{\mathcal{C}}$, that is, a functorial isomorphism

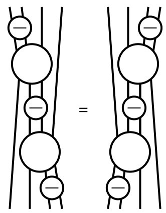
$$\theta_X : X \xrightarrow{\sim} X \qquad (X \in \mathcal{C})$$

satisfying the following axioms:

(TW0) $$\theta_I = 1_I\,;$$

(TW1) $$(\theta^{-1}_{X\otimes Y} \otimes 1_{Z\otimes T})(\theta_{X\otimes Y\otimes Z} \otimes 1_T)(1_X \otimes \theta^{-1}_{Y\otimes Z} \otimes 1_T)$$
$$(1_X \otimes \theta_{Y\otimes Z\otimes T})(1_{X\otimes Y} \otimes \theta^{-1}_{Z\otimes T}) = (1_{X\otimes Y} \otimes \theta^{-1}_{Z\otimes T})(1_X \otimes \theta_{Y\otimes Z\otimes T})$$
$$(1_X \otimes \theta^{-1}_{Y\otimes Z} \otimes 1_T)(\theta_{X\otimes Y\otimes Z} \otimes 1_T)(\theta^{-1}_{X\otimes Y} \otimes 1_{Z\otimes T})\,.$$

Graphically, axiom (TW1) may be represented as



, with $\bigcirc = \theta$ and $\ominus = \theta^{-1}$.

A *twisted category* is a monoidal category equipped with a twist.

If $\mathcal{C}$, $\mathcal{C}'$ are two twisted categories, with twists θ, θ', a strict twisted functor $F : \mathcal{C} \to \mathcal{C}'$ is a strict monoidal functor satisfying for all $X \in \mathsf{Ob}\mathcal{C}$:

$$F(\theta_X) = \theta_{FX} .$$

Proposition 3. *Let $\mathcal{C}$ be a monoidal category and θ an automorphism of $1_\mathcal{C}$. Define an automorphism D of $\otimes$ by*

$$D_{X,Y} = (\theta_X^{-1} \otimes \theta_Y^{-1})\theta_{X\otimes Y} .$$

Then θ is a twist if and only if D is a twine.

Proof. By its very form, D satisfies (DB1), and one checks easily that (DB0) and (DB2) are equivalent respectively to (TW0) and (TW1). □

As a result, a twisted category is canonically entwined, and a strict twisted functor is entwined.

Example. Let $\mathcal{C}$ be a braided category, and let θ be a balanced structure, that is an automorphism of $1_\mathcal{C}$ satisfying

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) R_{Y,X} R_{X,Y} .$$

Then θ is a twist.

In particular, the category of ribbon braids is twisted, and so is the category of ribbon pure braids. Moreover we have a canonical group isomorphism

$$(u, t_1, \ldots, t_n) : \mathsf{RPB}_n \xrightarrow{\sim} \mathsf{PB}_n \times \mathbb{Z}^n ,$$

where u denotes the forgetful morphism $\mathsf{RPB}_n \to \mathsf{PB}_n$, and t_i the self-linking number of the i-th component.

Remark. Let $\mathcal{C}$ be an entwined category, with twine D. Just like in the braided case (*c. f.* [Str94]), there is a canonical way of adjoining a twist to $\mathcal{C}$. Indeed, define a category $\tilde{\mathcal{C}}$ as follows. The objects of $\tilde{\mathcal{C}}$ are data (X, t), with $X \in \mathsf{Ob}\mathcal{C}$ and $t \in \mathsf{Aut}(X)$. Morphisms from (X, t) to (X', t') are morphisms $f : X \to X'$ in $\mathcal{C}$ such that $t'f = ft$. Define a tensor product on $\tilde{\mathcal{C}}$, on objects, by

$$(X, t) \otimes (X', t') = (X \otimes X', (t \otimes t')D_{X,X'}) ,$$

and on morphisms, by the tensor product of $\mathcal{C}$. One checks easily that this makes $\tilde{\mathcal{C}}$ a strict monoidal category (using axioms TW0 and TW1), and that setting $\theta_{(X,t)} = t$ defines a twist θ on $\tilde{\mathcal{C}}$ (using TW2). The forgetful functor $\tilde{\mathcal{C}} \to \mathcal{C}$ is entwined, and this construction is universal.

Theorem 2. *Let $\mathcal{C}$ be a twisted category, $\Lambda = \mathsf{Ob}\mathcal{C}$, and $\mathsf{RPB}(\Lambda)$ be the category of Λ-coloured ribbon pure braids. There exists a unique strict twisted functor*

$$[?] : \mathsf{RPB}[\Lambda] \to \mathcal{C}$$

sending $[\bullet]_X$ ($X \in \mathsf{Ob}\mathcal{C}$) to the object X itself.

Proof. Clearly the image of the coloured ribbon pure braid $P[X_1, \dots, X_n]$ can be no other than $(\theta_{X_1}^{t_1(P)} \otimes \dots \theta_{X_n}^{t_n(P)})[u(P)]_{X_1,\dots,X_n}$, and this defines indeed a strict twisted functor. □

Notation. Let $\mathcal{C}$ be a twisted category; for $P \in \mathsf{RPB}_n$, we let $[P] = (\theta^{t_1(P)} \otimes \dots \theta^{t_n(P)})[u(P)] \in \mathsf{End}(\otimes^n)$.

4. Twists, duality, and invariants of string links

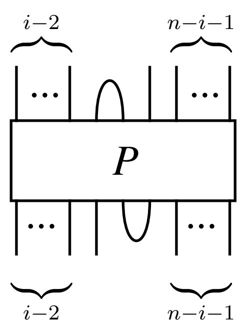
Definition. Let n be a non-negative integer. A *(ribbon) string link* on n strands is an oriented (ribbon) tangle from $n[+]$ to $n[+]$, without closed components, and such that the i-th input is connected to the i-th output.

We denote RStL the monoidal subcategory of Tang whose morphisms are isotopy classes of ribbon string links.

The category of ribbon pure braids RPB is naturally embedded as a monoidal subcategory of RStL.

Definition. Let P be a ribbon string link on n strands, and $1 < i < n$. We define the *i-th right contraction* of P to be the ribbon string link on $n-2$ strands $c_i(P)$ defined by

$c_i(P) =$



.

We will now mimick this construction in a categorical setting, using the notion of duality.

Proposition 4. *Let $\mathcal{C}$ be a monoidal category, and let θ be a twist of $\mathcal{C}$. Let X be an object of $\mathcal{C}$ and (Y, e, h) be a right dual of X.*

The following assertions are equivalent:

(i) θ_X and θ_Y are dual morphisms;

(ii) $\theta_X^2 = (eD_{X,Y}^{-1} \otimes 1_X)(1_X \otimes h)$;

(ii') $\theta_Y^2 = (1_Y \otimes e\, D_{Y,X}^{-1})(h \otimes 1_Y)$;

Moreover if they hold for one right dual of X, they hold for all.

Definition. Let $\mathcal{C}$ be a monoidal category with right duals. Let θ be a twist of $\mathcal{C}$. We say that θ is *self-dual* if for any object X of $\mathcal{C}$ the equivalent assertions of the previous proposition hold.

Let $\mathcal{C}$ be a twisted with right duals and a self-dual twist. Assume that right duals are chosen.

Notation. Let $\mathcal{C}$ be a monoidal category with right duals. Let X, Y, Z be objects of $\mathcal{C}$. For $f \in \mathsf{End}(X \otimes Y \otimes Y^\vee \otimes Y \otimes Z)$, let

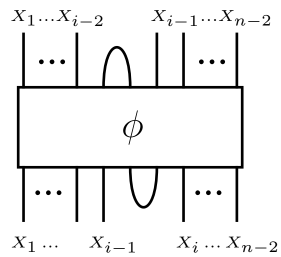
$$c_{X,Y,Z}(f) = (1_X \otimes e \otimes 1_Y \otimes 1_Z) f (1_X \otimes 1_Y \otimes h \otimes 1_Z)\,.$$

Now let $\phi \in \mathsf{End}(\otimes^n)$ and $1 < i < n$. Define $c_i\phi \in \mathsf{End}(\otimes^{n-2})$ by

$$(c_i\ \phi)_{X_1,\dots,X_{n-2}} = c_{X_1\otimes\cdots\otimes X_{i-2}, X_{i-1}, X_i\otimes\dots X_{n-2}} \phi_{X_1,\dots,X_{i-1},X_{i-1}^\vee,X_{i-1},\dots,X_n}\,.$$

Pictorially,

$$(c_i\phi)_{X_1,\dots,X_{n-2}} =$$



.

Notice that $c_i\phi$ is in fact independent of the choice of a right dual for X_{i-1}.

Theorem 3. *Let $\mathcal{C}$ be a monoidal category with right duals and a self-dual twist. There exists a unique way of associating to each isotopy class of ribbon string link $P \in \mathsf{RStL}_n$ a functorial endomorphim $\overrightarrow{P} \in \mathsf{End}(\otimes^n)$ in such a way that:*

(i) *$\overrightarrow{P} = [P]$ for any ribbon pure braid P;*
(ii) *$c_i\overrightarrow{P} = \overrightarrow{c_iP}$ for any $P \in \mathsf{RStL}_n$ and $1 < i < n$ such that the i-th component of P is trivial.*

Corollary 2. *Let $\mathcal{C}$ be a monoidal category with right duals and a self-dual twist. Let $\Lambda = \mathsf{Ob}\mathcal{C}$ and denote $\mathsf{RStL}[\Lambda] \subset \mathsf{Tang}[\Lambda]$ the category of Λ-coloured ribbon string links. There exists a canonical strict twisted functor*

$$\overrightarrow{?} : \mathsf{RStL}[\Lambda] \longrightarrow \mathcal{C}$$

which sends a coloured ribbon string link $P[X_1,\dots,X_n]$ to $\overrightarrow{P}_{X_1,\dots,X_n}$.

If $\mathcal{C}$ is a twisted category with left duals, one may (by left-right symmetry) associate with any ribbon string link $P \in \mathsf{RStL}_n$ an element $\overleftarrow{P} \in \mathsf{End}(\otimes^n)$. If both right- and left duals exist, it is not at all clear whether $\overleftarrow{P} = \overrightarrow{P}$. This suggests the following definition.

Definition. Let $\mathcal{C}$ be a monoidal category with left and right duals, and θ a twist of $\mathcal{C}$. We say that θ is *ambidextrous* if it is self-dual, and we have

$$\forall P \in \mathsf{RStL},\ \overleftarrow{P} = \overrightarrow{P}\,.$$

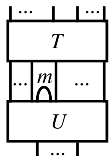
If such is the case, we set $[P] = \overleftarrow{P} = \overrightarrow{P}$.

When the twist is ambidextrous, we have $[c_iP] = c_i[P]$ for any $P \in \mathsf{RStL}_n$ and any $1 < i < n$.

Remark. Theorem 3, while it provides a means of constructing invariants of ribbon string links, has a serious drawback : it is not a universal property, because the category of ribbon string links has no duals. The aim of section 5 will be to mend this matter.

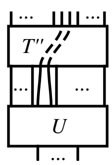
Proof of Theorem 3. If P is a ribbon pure braid, $\overrightarrow{P} = [P]$ is already well-defined. The point is now to see that a string link can be obtained from a pure braid by a sequence of 'nice' contractions. This will at least show that $\overrightarrow{?}$ is unique, and suggest a construction for it. We then must check the coherence of this construction, *i. e.* its independence from the choices made.

The main trick we use consists in 'pulling a max to the top line'. Let D be a tangle diagram with a local max m, with p outputs. We may write

$D =$  ,

where T, U are tangle diagrams.

Let i be an integer, $1 \leq i \leq n+1$. Let j be the number of strands to the left of m on the same horizontal line, plus 1. Let T' be a tangle diagram obtained from T by inserting a new component C going from a point between the $(j-1)$-th and j-th inputs of T to a point between the $(i-1)$-th and i-th outputs of T. We assume also that C has no local extrema. Note that we have $T = T' - C$. Let $T'' = \Delta_C T'$ be the tangle diagram obtained from T' by doubling C. Set

$D' =$  .

We say that D' *is obtained from* D *by pulling* m *to the top in the* i*-th position (along the path* C*).*

One defines similarly the action of *pulling a local min to the bottom.*

Now let D be a n-string link diagram, oriented from bottom to top. We say that D is *right-handed* if all local extrema point to the right.

Assume D is right-handed. Pulling all local max to the top and all local min to the bottom, one may obtain a pure braid diagram. Here is an algorithm. Denote m_i the number of local max (which is equal to the number of local min) on the i-th component of D. Let $m = m(D) = m_1 + \cdots + m_n$ be the number of local max of D. If $m(D) = 0$, we are already done. Otherwise, chose i minimal so that $m_i > 0$. Denote c the i-th component, and let m be the first max, and m' the first min you meet on c, going from bottom to top. Pull m to the top, in the i-th position (just to the left of c), and m' to the bottom, in the $i+1$-th position (just to the right of c). Let D' be the diagram so constructed. Then D' is a string link diagram, with $m(D') = m(D) - 1$. Moreover, $\{D\} = c_{i+1}\{D'\}$, and the $(i+1)$-th component of D' is unknotted. Repeated m times, this transformation yields a pure braid diagram P with $n + 2m$ threads, and we have

$$\{D\} = c_{j_m} \dots c_{j_1} P \,,$$

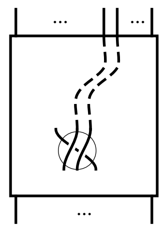
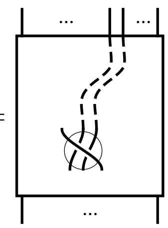
where $1 \leq j_1 \leq \cdots \leq j_m \leq n$, and j_k takes m_i times the value $i+1$.

We therefore set

$$\overrightarrow{D} = c_{j_m} \dots c_{j_1}[P]\,,$$

and we now proceed to show that this is independent of the choice made in the construction of P, that is, the actual paths along which the local extrema are being pulled.

Lemma 2. *Let P, P' be two pure braid diagrams:*

$P =$  , $P' =$  ,

which differ only inside a circle. Inside the circle, the i-th and $i+1$ strands pass respectively to the front and the back of the k-th strand; above the circle, the i-th and $(i+1)$-th strands run parallel. Let $\mathcal{C}$ be a entwined category, $X_1, \dots, X_n$ objects of $\mathcal{C}$, and let $e : X_i \otimes X_{i+1} \to I$ be any morphism. Let $E = 1_{X_1 \otimes \cdots \otimes X_{i-1}} \otimes e \otimes 1_{X_{i+2} \otimes \cdots \otimes X_n}$. Then $E[P]_{X_1,\dots,X_n} = E[P']_{X_1,\dots,X_n}$.

Proof. We will use the following fact, which is an immediate consequence of Proposition 1. If $A \in \mathsf{PB}_n$ and $1 \leq i \leq n$, construct $\Delta_i P \in \mathsf{PB}_{n+1}$ by doubling the i-th strand of P. Given $n+1$ objects $X_1, \dots, X_i, X_i', X_{i+1}, \dots X_n$ in $\mathcal{C}$, we have:

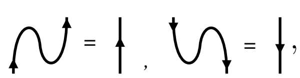
$$[\Delta_i P]_{X_1,\dots,X_i,X_i',X_{i+1},\dots,X_n} = [P]_{X_1,\dots,X_i \otimes X_i',X_{i+1},\dots,X_n}\,.$$

Now let us prove the lemma, and assume for instance $k < i$. One may represent P and P' as

$$P = \Delta_i A \Delta_i s_{k,i} B,\ P' = \Delta_i A B\,,$$

with $A \in \mathsf{PB}_{n-1}$, $B \in \mathsf{PB}_n$, and $s_{k,i}$ is the Burau generator. Using the above-mentioned fact, we may assume A and B trivial. The lemma then results from elementary properties of the twine. The case $k > i+1$ can be treated in a similar way. □

From the lemma, we see not only that $\overrightarrow{D}_{X_1,\dots,X_n}$ is independent of the choices made, but also that it is invariant under Reidemeister moves of type 2 and 3. In addition, it is invariant under 'right-handed moves of type 0', namely

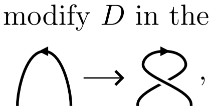
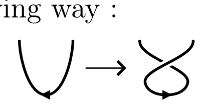


In the first case, it is an easy consequence of the identity

$$c_{i+1}[\Delta_i^3 P] = [P]\,,$$

where P is a pure braid diagram and $\Delta_i^3 P$ is is obtained by tripling the i-th strand of P. The second case is deduced from the first, using type 2 moves.

Now let D be a arbitrary n-string link diagram. For each local extremum pointing to the left, modify D in the following way :

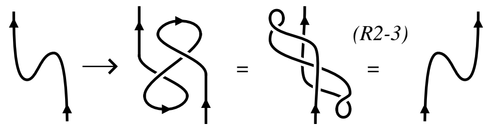
(1)  , (2)  .

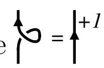
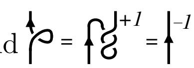
This operation yields a right-handed diagram D^r.

For $1 \leq i \leq n$, let t_i be the algebraic number of modifications made on the i-th component, with (1) counting as -1 and (2) as $+1$.

Set $\overrightarrow{D} = (\theta^{t_1} \otimes \cdots \otimes \theta^{t_n})D^r$.

Clearly this is invariant under Reidemeister moves of type 2 and 3. As for invariance under type 0 moves, the case when the extrema point to the left reduces to the right-handed case (already proved) via:



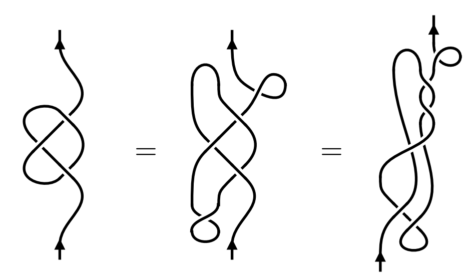
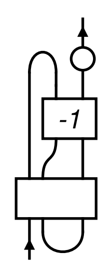
Moreover, we have  (where the $+1$ denotes the twist), and  by self-duality of the twist, hence invariance under moves of type 1.

Let us summarize: given $P \in \mathsf{RStL}_n$, we have constructed $\overrightarrow{P} \in \mathsf{End}(\otimes^n)$. Now notice that when one forms the i-th contraction $c_i D$ of a string link diagram D, the orientation of its i-th component is reversed; in particular, if D is right-handed, $c_i D$ is not, unless there are no local extrema on the i-th component, that is, the i-th component is unknotted. In that case, we do have $c_i\overrightarrow{P} = \overrightarrow{c_i P}$. Indeed, we may represent P by a diagram whose i-th component has no local extrema, hence $P = c_{j_m} \dots c_{j_1} Q$, with Q ribbon pure braid, $j_1 \leq \cdots \leq j_m$, and $j_k \neq i+1$ for all k. We conclude by the following straightforward lemma.

Lemma 3. *The contraction operators c_i satisfy the following relations :*
(a) for $i \leq j-2$, $c_i c_j = c_{j-2} c_i$;
(b) for $i \geq j$, $c_i c_j = c_j c_{i+2}$.

Assume $j_{k-1} \leq i \leq j_k - 2$. By the lemma, $c_i P = c_{j_m-2} \dots c_{j_k-2} c_i c_{j_{k-1}} \dots c_1 Q$, so $\overrightarrow{c_i P} = c_{j_m-2} \dots c_{j_k-2} c_i c_{j_{k-1}} \dots c_1[Q] = c_i c_{j_m} \dots c_{j_1}[Q] = c_i \overrightarrow{P}$, hence the theorem.

As an illustration, let us compute $\overrightarrow{P}$ in the case of the trefoil :

$P =$  , hence : $\overrightarrow{P} =$ 

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5. Turban categories

By virtue of Shum's theorem, the category of ribbon oriented tangles is the universal tortile category. On the other hand, we have just seen that any category with right duals and a self-dual twist defines invariants of ribbon string links. Recall proposition 1:

Proposition 1. *The category* Turb *(resp.* eTang*) is the smallest monoidal subcategory of* Tang *having the same objects as* Tang*, and containing all evaluations, coevaluations and twists (resp. double braidings).*

This strongly suggests that Shum's theorem has an analogue for turbans. In other words, one should be able to define a notion of 'turban category', in such a way that Turb is the universal turban category. Before we proceed to do so, let us prove proposition 1.

Proof of Proposition 1 and its corollary. We denote $\mathcal{E}$ the monoidal subcategory of Tang generated by the evaluation morphisms. A tangle in $\mathcal{E}$ may be represented by a diagram with $2n+k$ input and k output, without crossings and local min. Here is a typical example:

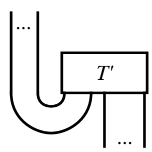
.

Dually, we denote $\mathcal{E}^*$ the monoidal subcategory of Tang generated by the coevaluation morphisms.

Lemma 4. *Any turban tangle T may be factorized as $T = EPH$, where P is a ribbon pure braid and $E \in \mathcal{E}$, $H \in \mathcal{E}^*$. Moreover, if T is even, we may assume that each component of P has trivial self-linking number.*

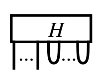
Proof. Let T be an (oriented) turban tangle, that is, an oriented ribbon tangle whose linking matrix has only even entries outside the diagonal.

We may write T as

$T =$ 

,

where T' is a turban with $2n$ input and no output. Assume T' has k closed component. Pulling one local min per closed component down to the bottom line on the right-hand side, we may represent T' as

$T' =$ 

,

where H is a turban with $2N = 2n + 2k$ input, no output and no closed components.

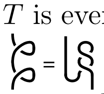
Now the turban condition on H excludes a configuration of four legs $i < j < k < l$ with (i, j) and (k, l) connected in H.

By pulling one local max per component to the top line, we may therefore write H as

$$H = EQ\,,$$

where $Q \in \mathsf{RStL}_{2N}$, and E is an element of $\mathcal{E}$ with $2N$ input and 0 output.

So we may write $P = E'QH'$, with $Q \in \mathsf{RStL}$, $E' \in \mathcal{E}$, $H' \in \mathcal{E}^*$. Now any ribbon string link may be obtained from a ribbon pure braid by a finite number of contractions, so we may write $Q = E''PH''$, with $P \in \mathsf{RPB}$, $E'' \in \mathcal{E}$ and $H'' \in \mathcal{E}^*$. Setting $E = E'E''$, $H = H''H'$, we have $T = EPH$.

If T is even, Q may be assumed even. Now by self-duality of the twist we have 

so we may factor Q as $E'PH'$, with $E' \in \mathcal{E}$, $H' \in \mathcal{E}^*$ and P a ribbon pure braid with trivial self-linking numbers. □

Clearly, the lemma implies proposition 1. Indeed, we know that the category of ribbon pure braids is generated (as a monoidal category) by the twist, and the subcategory of ribbon pure braids with trivial self-linking numbers is generated by the double braiding.

Remark. Proposition 1 has the following straightforward consequence. If $\mathcal{C}$ is a twisted category with chosen right and left duals, and $\mathsf{Ob}\mathcal{C} = \Lambda$, there exists at most one strict twisted, dual-preserving functor $F_{\mathcal{C}} : \mathsf{Turb}[\Lambda] \to \mathcal{C}$ sending $[+]_X$ ($X \in \mathsf{Ob}\mathcal{C}$) to the object X itself.

Now let us prove the corollary:

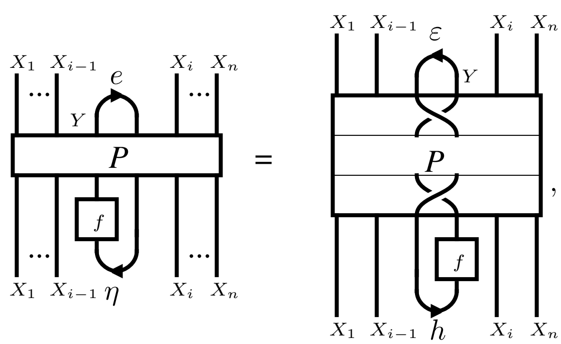
Corollary 1. *Any invariant of ribbon links or turban tangle arising from a ribbon category* $\mathcal{C}$ *is independent of the braiding: it depends only on the twist and the duality.*

Let $\mathcal{C}$ be a ribbon category, and let $F_{\mathcal{C}} : \mathsf{Tang}[\Lambda] \to \mathcal{C}$ be Shum's functor. We may view $F_{\mathcal{C}}$ as an invariant of Λ-colored tangles, and by 'tangle invariants arising from $\mathcal{C}$', we mean tangle invariants which may be expressed in terms of $F_{\mathcal{C}}$. Now the restriction of $F_{\mathcal{C}}$ to Turb is a strict twisted, dual-preserving functor sending $[+]_X$ to X for any $X \in \mathsf{Ob}\mathcal{C}$. There is at most one such functor. Now let $\mathcal{C}'$ be another tortile category with same underlying monoidal category, choices of duals and twist (that is, we only allow for a change of braiding, and the double braiding must be the same). Then the restriction of $F_{\mathcal{C}'}$ to Turb satisfies the same conditions as $F_{\mathcal{C}}$, hence they coincide; and this proves the assertion on turban invariants, as well as link invariants since any ribbon link is turban.

The notion of turban category. Naturally, one could define a turban category to be a twisted category with chosen left and right duals, and such that the functor $F_{\mathcal{C}}$ exists. We would have the universal property for free! However, such a definition would not be of great practical use: we need a more concrete criterion. Also, it seems reasonable to assume that the choices of left and right duals define a sovereign structure on $\mathcal{C}$ (indeed, such is the case in Turb).

Let $\mathcal{C}$ be a sovereign category with ambidextrous twist. For any ribbon stringlink P, denote $[P] = \overrightarrow{P} = \overleftarrow{P}$. Pictorially, we will represent $[P]$ as $\boxed{\mathrm{P}}$.

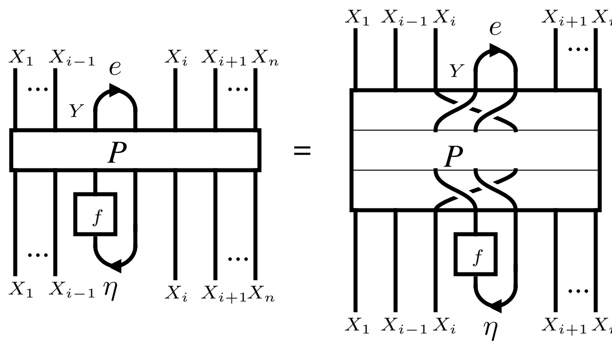
We say that the *strong sphericity condition* is satisfied if for any $P \in \mathsf{RStL}_{n+2}$, $1 \le i \le n+1$, $X_1, \ldots, X_n, Y \in \mathsf{Ob}\mathcal{C}$, and $f \in \mathsf{End}(Y)$, we have

(Sph) 

,

where e, h, ε, η denote the evaluation and coevaluation morphisms for Y.

We say that the *strong interchange condition* is satisfied if for any $P \in \mathsf{RStL}_{n+2}$, $1 \leq i \leq n$, $X_1, \ldots, X_n, Y \in \mathsf{Ob}\mathcal{C}$, and $f \in \mathsf{End}(Y)$, we have :

(Int) 

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We say that the *weak sphericity condition* (resp. the *weak interchange condition*) holds when we have (Sph) (resp. (Int)) whenever $f = 1_Y$.

Definition. A turban category is a twisted sovereign category with ambidextrous twist satisfying the strong sphericity and the strong interchange conditions.

Examples.
1) For any set Λ, $\mathsf{Turb}[\Lambda]$ is a turban category.
2) Any tortile category is a turban category.
3) If $\mathcal{C}$ is a turban category, and $\mathcal{D} \subset \mathcal{C}$ is a twisted sovereign subcategory of $\mathcal{C}$, then $\mathcal{D}$ is a turban category.

We can now state the analogue of Shum's theorem.

Theorem 4. *The category of oriented turban tangles is the universal turban category. In other words, if $\mathcal{C}$ is a turban category and $\Lambda = \mathsf{Ob}\mathcal{C}$, there exists a unique turban functor*

$$F_{\mathcal{C}} : \mathsf{Turb}[\Lambda] \to \mathcal{C}$$

sending $[+]_X$ ($X \in \mathsf{Ob}\mathcal{C}$) to the object X itself.

Proof.
We must construct a twisted, dual-preserving functor

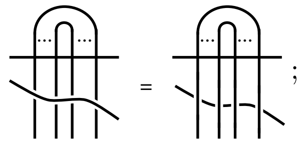
$$F : \mathsf{Turb}[\Lambda] \to \mathcal{C}$$

sending $[+]_X$ ($X \in \mathsf{Ob}\mathcal{C}$) to the object X itself. The proof of proposition 1 gives us a construction for F, and we have to check that it is unambiguous. We will now outline the proof.

1) The assumption that the twist is ambidextrous tells us that the functor F is well-defined on ribbon string links. It is also well-defined on $\mathcal{E}$ and $\mathcal{E}^*$.

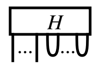
2) We check that F is well-defined on a turban tangle T with $2n$ input, no output, and no closed component. Such a T may be factorized as $T = EP$, with $P \in \mathsf{RStL}_{2n}$ and $E \in \mathcal{E}$, so we should set $F(T) = F(E)F(P)$. We have to check that this is independent of the actual factorization. The proof of this fact is similar to that of Theorem 3: starting from a suitable diagram representing T, a factorization is obtained by pulling certain local max to the top line in the right order. Just as in the proof of theorem 3, each of

these local max may have to be modified so as to point in the appropriate direction. We need an analogue of lemma 2, graphically:



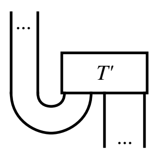
this is easy to check using theorem 3. This tells us that $F(E)F(P)$ is independent of the choices of pathes, and it is then easy to check that it depends only on the isotopy class of the tangle.

3) We now define F on turban tangles T with $2n$ input, no output, and closed components $L_1, \dots L_k$. Such a tangle may be factorized as

$T =$  ,

where H is a turban with $2N = 2n+2k$ input, no output and no closed components. Such a factorization is obtained by pulling a local min of each of the L_i to the bottom line, and to the right. This defines $F(T)$ with possibly two types of ambiguities: we use a numbering of closed components, and for each closed component we must decide whether the local minimum points to the right or to the left. However, the (weak) exchange and sphericity conditions say precisely that the value for $F(T)$ is independent of the numbering of components, and the direction of each min.

4) We may write an arbitrary turban tangle T as

$T =$  ,

where T' is a turban with $2n$ input and no output, and this defines F on T in an unambiguous way.

This defines a monoidal functor which has the required properties. □

Remark. Theorem 4 remains true if we replace the strong sphericity and strong interchange condition by their weak counterparts. However the strong version will probably prove more useful.

6. Construction of twisted categories

6.1. **Toy example: the group-like case.** Let G be a group and $\mathcal{C} = G-\mathsf{vect}$ the category of G-graded vector spaces over a field k. Denote k_g the simple object consisting of one copy of k in degree g. The dual of k_g is $k_{g^{-1}}$, and the canonical evaluation and coevaluation morphisms define a sovereign structure on $\mathcal{C}$. Each simple object has left and right dimension equal to 1.

An automorphism of $\otimes$ is characterized by its values on simple objects, that is, a map $\delta : G \otimes G \to k^*$. It is a twine if and only if $\delta(e, e) = 1$ and $\delta(g, h)\delta(gh, k) = \delta(h, k)\delta(g, hk)$ (in other words, δ is a 2-cocycle).

Notice that if G is not commutative, $\mathcal{C}$ is not braided; and if G is commutative, a braiding on $\mathcal{C}$ corresponds to a bicharacter $c : G \times G \to k^*$. The double braiding, $c(h, g)c(g, h)$, is a symmetric bicharacter. Twines are far more numerous than double braidings.

An automorphism of $1_{\mathcal{C}}$ is given by a map $\theta : G \to k^*$. It is a twist if and only if $\theta(e) = 1$. Self-duality is equivalent to $\theta(g^{-1}) = \theta(g)$. Now any self-dual twist actually defines a turban structure on $\mathcal{C}$.

The invariants of ribbon links and turban tangles associated with such turban categories contain no more information than the linking matrix.

6.2. **Toy example: the infinitesimal case, first order.** Let $\mathcal{C}$ be a k-monoidal category, where k is a field, and define $\mathcal{C}[\varepsilon] = \mathcal{C} \otimes_k k[\varepsilon]$ by extending the scalars of $\mathcal{C}$ to the ring of dual numbers $k[\varepsilon] = k[X]/(X^2)$.

Let $d_{X,Y} : X \otimes Y \to X \otimes Y$ be a functorial morphism, $X, Y \in \mathcal{C}$. Set $D_{X,Y} = 1_{X \otimes Y} + \varepsilon\, d_{X,Y}$. Then D is a twine on $\mathcal{C}[\varepsilon]$ if and only if d satisfies the following conditions :
(a) $d_{I,I} = 0$;
(b) $d_{X,Y} \otimes 1_Z + d_{X \otimes Y, Z} = 1_X \otimes d_{Y,Z} + d_{X, Y \otimes Z}$.
We say that d is an *infinitesimal twine* if it satisfies (a) and (b).

When is such a twine D a double braiding? Assume that c is a braiding on $\mathcal{C}[\varepsilon]$ such that $D_{X,Y} = c_{Y,X}\, c_{Y,X}$. We may write

$$c_{X,Y} = S_{X,Y}(1_{X,Y} + \varepsilon\, \tau_{X,Y}),$$

where S is a symmetry of $\mathcal{C}$, and $\tau_{X,Y}$ is an infinitesimal braiding of the symmetric category $(\mathcal{C}, S)$. Let us recall the axioms of an infinitesimal braiding:

$$\tau_{X, Y \otimes Z} = \tau_{X,Y} \otimes 1_Z + (1_X \otimes S_{Z,Y})^{-1}(\tau_{X,Z} \otimes 1_Y)(1_X \otimes S_{Z,Y}),$$
$$\tau_{X \otimes Y, Z} = 1_X \otimes \tau_{Y,Z} + (S_{X,Y} \otimes 1_Z)^{-1}(1_Y \otimes \tau_{X,Z})(S_{X,Y} \otimes 1_Z).$$

We have $d = \tau + \tau^o$, where $\tau^o_{X,Y} = S^{-1}_{X,Y} \tau_{Y,X} S_{X,Y}$. As a result d is an infinitesimal braiding satisfying $d^o = d$. We may therefore conclude that, if D is a double braiding, d is an infinitesimal braiding such that $d^o = d$ for some symmetry S on $\mathcal{C}$. The converse is true in characteristic $\neq 2$, with $c = S + \frac{1}{2}\, \varepsilon S\, d$.

Now let us consider twists. Let $t_X : X \to X$ be a functorial morphism, $X \in \mathsf{Ob}\mathcal{C}$. Set $\theta_X = 1_X + \varepsilon\, t_X$. Then θ is a twist if and only if t satisfies the condition $t_I = 0$. If $\mathcal{C}$ has duals, θ is self-dual if and only if $t_{X^\vee} = -t_X$.

6.3. **The infinitesimal case, higher order.** Let k be a field of characteristic $\neq 2$, and let $\mathcal{C}$ be a k-monoidal category with duals. Define $\mathcal{C}[[h]] = \mathcal{C} \otimes_k k[[h]]$ by extending the scalars of $\mathcal{C}$ to the ring of formal series $k[[h]]$. Let D be a twine of $\mathcal{C}[[h]]$, of the form

$$D_{X,Y} = 1_{X \otimes Y} + \sum_{n>0} h^n\, d^{(n)}_{X,Y}$$

with $d^{(n)}_{X,Y} \in \mathsf{End}_{\mathcal{C}}(X \otimes Y)$.

The relations between the $d^{(n)}$'s expressing the axioms of a twine are rather complicated; however we have the following result, which generalizes a theorem of Deligne ([Yet92]).

Theorem 5. *With the above notations, assume D is self-dual, i. e. $D^\vee_{X,Y} = D_{Y^\vee, X^\vee}$ for all $X, Y \in \mathsf{Ob}\mathcal{C}$.*

Then the entwined category $(\mathcal{C}[[h]], D)$ *admits a unique self-dual twist of the form*

$$\theta_X = 1_X + \sum_{n>0} h^n \, t_X^{(n)}$$

with $t_X^{(n)} \in \mathsf{End}_{\mathcal{C}}(X)$.

Proof. Let $W_X = e_X \, D_{X,X^\vee}$ and $U_X = (W_X \otimes 1_X)(1_X \otimes h_X)$. A twist θ is self-dual if and only if $\theta^{-2} = U$, so, if θ is a formal series with constant term 1, it has to be the inverse of the formal square root of U : $\theta = \sqrt{U}^{-1}$. We must now check that this defines a twist for D, *i. e.* that we have :

$$\theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) D_{X,Y} \, .$$

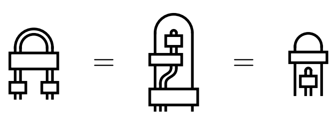
Since these formal series have constant term 1, it is enough to check the square of that identity, which amounts to :

$$U_X \otimes U_Y = U_{X \otimes Y} D^2_{X,Y} \, .$$

Now using duality and self-duality of D, this is equivalent to :

$$W_{X \otimes Y}(D_{X,Y} \otimes D_{Y^\vee, X^\vee}) = W_X(1_X \otimes W_Y \otimes 1_{X^\vee}) \, .$$

Now let us denote T_n the central pure braid on n strands defined inductively by $T_1 = 1$ and $T_{p+q} = D_{p,q}(T_p \otimes T_q)$. We have $D_{2,2}(D_{1,1} \otimes D_{1,1}) = T_4 = (1 \otimes D_{1,1} \otimes 1)(D_{1,2} \otimes 1)D_{3,1}$, so that, pictorially, we have :


(by functoriality of D and **DB0**) ,

hence the theorem. □

Infinitesimal twists are expected to define turban invariants of finite type.

6.4. **Tannaka theory for twined and twisted categories.** Let k be a field, and H a bialgebra over k, with coproduct Δ, counit ε, product μ and unit η. Denote $\mathsf{comod}H$ the monoidal category of finite dimensional right H-comodules.

Definition. A *cotwinor* of H is a linear form $d : H \otimes H \to K$ satisfying the following axioms :

(codt-1) d is invertible (for the convolution product on $H \otimes H$), and $d * \mu = \mu * d$ in $\mathsf{Hom}(H^{\otimes 2}, H)$;

(codt0) $d(\eta \otimes \eta) = 1$;

(codt1) $(d \otimes \varepsilon) * d(\mu \otimes 1_H) = (\varepsilon \otimes d) * d(1_H \otimes \mu)$;

(codt2) $d_{(12)3} d_{23}^{-1} d_{2(34)} = d_{2(34)} d_{23}^{-1} d_{(12)3}$ in $\mathsf{Hom}(H^{\otimes 4}, k)$.

Definition. A *cotwistor* of H is a linear form $\theta : H \to k$ satisfying the following axioms:

(cotw-1) θ is invertible (for the convolution product on H), and $(\theta \otimes 1_H)\Delta = (1_H \otimes \theta)\Delta$;

(cotw0) $\theta\eta = 1$;

(cotw1) $(\theta^{-1}\mu)_{12}(\theta\mu^3)_{123}(\theta^{-1}\mu)_{23}(\theta\mu^3)_{234}(\theta^{-1}\mu)_{34} =$.

Theorem 6. *The set of twines (resp twists) of* $\mathsf{comod}H$ *is in 1—1 correspondence with the set of cotwinors (resp. cotwistors) on* H*. Moreover, when* H *admits an antipode* S *(that is, when* $\mathsf{comod}H$ *has right duals) self-dual twists of* $\mathsf{comod}H$ *correspond exactly with cotwistors* θ *such that* $\theta S = \theta$*.*

Proof. This is straightforward tannakian translation. Given $d : H \otimes H \to k$, and two H-comodules V, V', with coactions ∂, ∂', define

$$D_{V,V'} = (1_{V\otimes V'} \otimes d)(1_X \otimes \sigma_{H,X'} \otimes 1_H)(\partial \otimes \partial')\,.$$

Axiom (codt-1) means that $D_{V,V'}$ is an isomorphism of comodules, and (codt0)-(codt2) translate axioms (DT0)-(DT2) of twines. Similarly, given $\theta : H \to k$, and a H-comodule V, define

$$\theta_V = (1_H \otimes \theta)\,\partial\,.$$

Axiom (cotw-1) means that θ_V is an isomorphism of comodules, and (cotw0), (cotw1) translate axioms (TW0), (TW1) of twists. $\square$

Should the reader prefer modules to comodules, here are the dual notions.

Definition. A *twinor* of H is an element $d \in H \otimes H$ satisfying the following axioms :

(dt-1) d is invertible, and $\forall x \in H, d\Delta(x) = \Delta(x)d$;

(dt0) $(\varepsilon \otimes \varepsilon)d = 1$;

(dt1) $(d \otimes \eta)(\Delta \otimes 1_H)d = (\eta \otimes d)(1_H \otimes \Delta)d$;

(dt2) $d_{(12)3}d_{23}^{-1}d_{2(34)} = d_{2(34)}d_{23}^{-1}d_{(12)3}$ in $H^{\otimes 4}$.

Definition. A *twistor* of H is an element $\theta \in H$ satisfying the following axioms:

(tw-1) θ is invertible and central;

(tw0) $\varepsilon\theta = 1$;

(tw1) $(\Delta\theta^{-1})_{12}(\Delta^3\theta)_{123}(\Delta\theta^{-1})_{23}(\Delta^3\theta)_{234}(\Delta\theta^{-1})_{34} = (\Delta\theta^{-1})_{34}(\Delta^3\theta)_{234}(\Delta\theta^{-1})_{23}(\Delta^3\theta)_{123}(\Delta\theta^{-1})_{12}$.

If d is a twinor (resp. twistor) of H, the monoidal category H-mod of finite dimensional left H-modules is entwined (resp. twisted).

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