A Hybrid High-Order method for locally degenerate advection-diffusion-reaction

Daniele A. Di Pietro

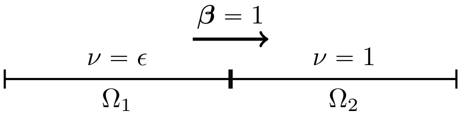
joint work with J. Droniou and A. Ern

Université de Montpellier
Institut Montpelliérain Alexander Grothendieck

X-DMS 2015

Locally degenerate advection-diffusion-reaction I

- We consider locally degenerate advection-diffusion-reaction
- Let us start with the following 1d problem:



- As $\epsilon \to 0^+$, a boundary layer develops at $x = 1/2$
- When $\epsilon = 0$, it turns into a jump discontinuity

Locally degenerate advection-diffusion-reaction II

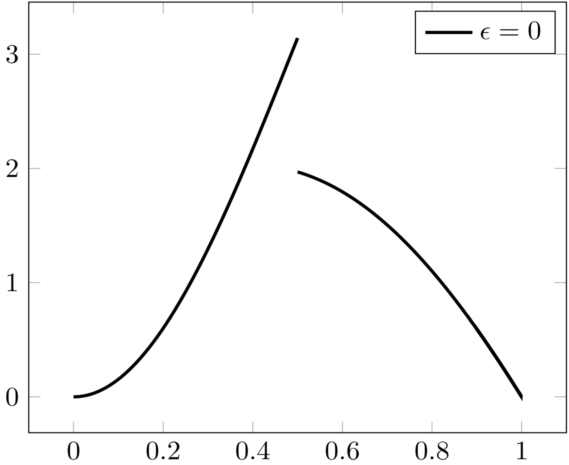


Figure: Solutions for different values of ϵ

Locally degenerate advection-diffusion-reaction III

- Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$. The diffusion coefficient $\nu : \Omega \to \mathbb{R}$ is s.t.

$$\nu \text{ is piecewise constant and } \nu \geqslant \underline{\nu} \geqslant 0 \text{ a.e. in } \Omega$$

- The velocity field $\boldsymbol{\beta} : \Omega \to \mathbb{R}^d$ is s.t.

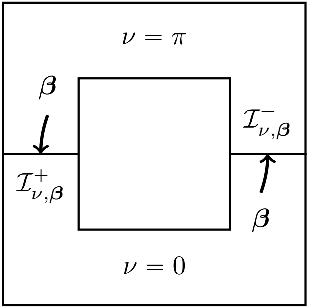
$$\boldsymbol{\beta} \in \mathrm{Lip}(\Omega)^d, \qquad \boldsymbol{\nabla}\cdot\boldsymbol{\beta} \equiv 0$$

- For the reaction coefficient $\mu : \Omega \to \mathbb{R}$, we assume

$$\mu \in L^\infty(\Omega) \text{ and } \mu \geqslant \mu_0 > 0 \text{ a.e. in } \Omega$$

- **Generalizations possible for both ν and β!**

Locally degenerate advection-diffusion-reaction IV



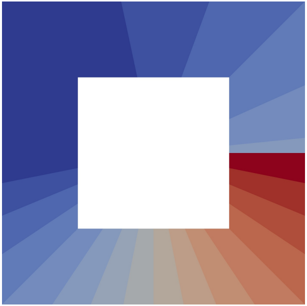


Figure: Two-dimensional example from [Di Pietro et al., 2008]

Locally degenerate advection-diffusion-reaction V

- Let $f \in L^2(\Omega)$. We seek $u : \Omega \to \mathbb{R}$ s.t.

$$\boldsymbol{\nabla}\cdot(-\nu\boldsymbol{\nabla}u + \boldsymbol{\beta}u) + \mu u = f \text{ in } \Omega\backslash(\mathcal{I}^+_{\nu,\boldsymbol{\beta}} \cup \mathcal{I}^-_{\nu,\boldsymbol{\beta}})$$

- Boundary conditions are enforced setting

$$u = g \text{ on } \Gamma_{\nu,\boldsymbol{\beta}} := \{\boldsymbol{x} \in \partial\Omega \mid \nu > 0 \text{ or } \boldsymbol{\beta}\cdot\boldsymbol{n} < 0\}$$

- Transmission conditions on $\mathcal{I}^\pm_{\nu,\boldsymbol{\beta}}$ are required to close the problem

$$[-\nu\boldsymbol{\nabla}u + \boldsymbol{\beta}u]\cdot\boldsymbol{n}_{\Omega_i} = 0 \text{ on } \mathcal{I}^\pm_{\nu,\boldsymbol{\beta}}, \qquad [u] = 0 \text{ on } \mathcal{I}^+_{\nu,\boldsymbol{\beta}}$$

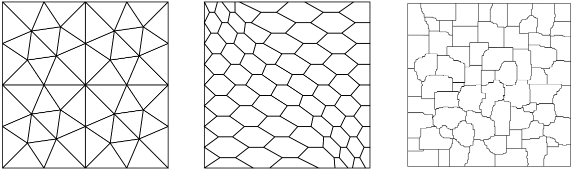
- **The solution $u \in U$ can jump across $\mathcal{I}^-_{\nu,\boldsymbol{\beta}}$!**

A few references on ADR

- Several works on the diffusion-dominated case, including, e.g.,
  - Hybridizable DG (standard meshes) [Cockburn et al., 2009]
  - Mimetic Finite Differences [Beirão da Veiga, Droniou, Manzini, 2010]
  - Weak Galerkin [Wang and Ye, 2013]
  - Virtual Elements [Beirão da Veiga, Brezzi, Marini, Russo, 2014]
  - (Non)conforming Virtual Elements [Cangiani, Manzini, Sutton, 2015]
  - ...
- Fewer tackle the advection-dominated and locally degenerate cases
  - 1d domain decomposition [Gastaldi and Quarteroni, 1989]
  - DG (only numerics) [Houston, Schwab, Süli, 2002]
  - DG (weak formultation + full analysis) [DP, Ern, Guermond, 2008]

DP, Droniou, Ern, *SINUM*, **2015**, DOI: 10.1137/140993971

Mesh regularity



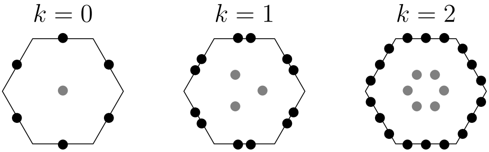
Definition (Admissible mesh sequence)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polytopal meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the usual sense of Ciarlet;
- contact-regular, i.e., every simplex $S \subset T$ is s.t. $h_S \approx h_T$;

Additionally, we assume every $\mathcal{T}_h$ compliant with ν, so that $\nu \in \mathbb{P}^0(\mathcal{T}_h)$.

Hybrid degrees of freedom



- For all $k \geqslant 0$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\bigtimes_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right)$$

- The global space has single-valued interface DOFs

$$\underline{U}_h^k := \left(\bigtimes_{T \in \mathcal{T}_h} \mathbb{P}^k(T) \right) \times \left(\bigtimes_{F \in \mathcal{F}_h} \mathbb{P}^k(F) \right)$$

- **Grey DOFs can be condensed ("discontinuous skeletal")!**

Key ideas and main features

- Diffusion terms of order $(k+1)$, cf. [DP, Ern, Lemaire, 2014]
- Element-face upwind stabilization of advection
- Automatic enforcement of the conditions on $\Gamma_{\nu,\boldsymbol{\beta}}$ and $\mathcal{I}^{\pm}_{\nu,\boldsymbol{\beta}}$

- Arbitrary order $k \geqslant 0$ in any dimension $d \geqslant 1$
- Method valid for the full range of local Peclet numbers
- Analysis capturing the variation in the convergence rate
- Reduced cost through static condensation

- **No need to duplicate interface unknowns on $\mathcal{I}^{-}_{\nu,\boldsymbol{\beta}}$ (!)**

Diffusion I

- Let $T \in \mathcal{T}_h$. The local potential reconstruction operator

$$p_T^{k+1} : \underline{U}_T^k \to \mathbb{P}^{k+1}(T)$$

is s.t. for all $\underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$ and all $w \in \mathbb{P}^{k+1}(T)$,

$$\boxed{(\boldsymbol{\nabla} p_T^{k+1} \underline{v}_T, \boldsymbol{\nabla} w)_T := -(v_T, \triangle w)_T + \sum_{F \in \mathcal{F}_T} (v_F, \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{TF})_F}$$

- Let $\underline{I}_T^k : H^1(T) \ni v \mapsto (\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$
- $(p_T^{k+1} \circ \underline{I}_T^k)$ has optimal approximation properties in $\mathbb{P}^{k+1}(T)$

Diffusion II

- Let $T \in \mathcal{T}_h$. We define the local bilinear form $a_{\nu,T}$ on $\underline{U}_T^k \times \underline{U}_T^k$:

$$\boxed{a_{\nu,T}(\underline{u}_T, \underline{v}_T) := (\nu_T \boldsymbol{\nabla} p_T^{k+1}\underline{u}_T, \boldsymbol{\nabla} p_T^{k+1}\underline{v}_T)_T + \sum_{F \in \mathcal{F}_T} \frac{\nu_T}{h_F}(r_{TF}^k \underline{u}_T, r_{TF}^k \underline{v}_T)_F}$$

- We stabilize by least-square penalty of the high-order face residual

$$r_{TF}^k(\underline{v}_T) := \pi_F^k(v_F - p_T^{k+1}\underline{v}_T) - \pi_T^k(v_T - p_T^{k+1}\underline{v}_T)$$

- $a_{\nu,T}$ is polynomially consistent up to degree $(k+1)$

Diffusion III

- The last step is to assembly and weakly enforce BCs
- The global bilinear form $a_{\nu,h}$ on $\underline{U}_h^k \times \underline{U}_h^k$ is defined as

$$\boxed{a_{\nu,h}(\underline{w}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_{\nu,T}(\underline{w}_T, \underline{v}_T) + s_{\partial,\nu,h}(\underline{w}_h, \underline{v}_h)}$$

where, for a user-defined penalty parameter $\varsigma > 0$,

$$s_{\partial,\nu,h}(\underline{w}_h, \underline{v}_h) := \sum_{F \in \mathcal{F}_h^{\mathrm{b}}} \left\{ -(\nu_F \boldsymbol{\nabla} p_T^{k+1} \underline{w}_T \cdot \boldsymbol{n}_{TF}, v_F)_F + \frac{\varsigma \nu_F}{h_F} (w_F, v_F)_F \right\}$$

- Symmetric and skew-symmetric variants can be devised (cf. DG)

Diffusion IV

Lemma (Coercivity of $a_{\nu,h}$)

Assuming that $\varsigma > C_{\rm tr}^2 N_\partial/4$ it holds, for all $\underline{v}_h \in \underline{U}_h^k$,

$$a_{\nu,h}(\underline{v}_h, \underline{v}_h) =: \|\underline{v}_h\|_{\nu,h}^2 \simeq \sum_{T\in\mathcal{T}_h} \nu_T \|\underline{v}_T\|_{1,T}^2 + \sum_{F\in\mathcal{F}_h^{\rm b}} \frac{\nu_F}{h_F}\|v_F\|_F^2,$$

where, for all $T \in \mathcal{T}_h$, we have defined the $H^1(T)$-like seminorm on $\underline{U}_T^k$:

$$\|\underline{v}_T\|_{1,T}^2 := \|\boldsymbol{\nabla} v_T\|_T^2 + \sum_{F\in\mathcal{F}_T} \frac{1}{h_F}\|v_F - v_T\|_F^2.$$

Advection-reaction I

- The discrete advective derivative operator

$$G^k_{\boldsymbol{\beta},T} : \underline{U}^k_T \to \mathbb{P}^k(T)$$

is s.t., for all $\underline{v}_T \in \underline{U}^k_T$ and all $w \in \mathbb{P}^k(T)$,

$$(G^k_{\boldsymbol{\beta},T}\underline{v}_T, w)_T = -(v_T, \boldsymbol{\beta}\cdot\boldsymbol{\nabla} w)_T + \sum_{F\in\mathcal{F}_T} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})v_F, w)_F$$

- We have the following global IBP formula: For all $\underline{w}_h, \underline{v}_h \in \underline{U}^k_h$,

$$\sum_{T\in\mathcal{T}_h} \Big((G^k_{\boldsymbol{\beta},T}\underline{w}_T, v_T)_T + (w_T, G^k_{\boldsymbol{\beta},T}\underline{v}_T)_T\Big) = \sum_{F\in\mathcal{F}^{\rm b}_h} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_F)w_F, v_F)_F$$
$$- \sum_{T\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_h} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})(w_F - w_T), v_F - v_T)_F$$

- To control the term in red, we use element-face upwinding

Advection-reaction II

- For all $T \in \mathcal{T}_h$, we define the bilinear form $a_{\boldsymbol{\beta},\mu,T}$ on $\underline{U}_T^k \times \underline{U}_T^k$ s.t.

$$\boxed{a_{\boldsymbol{\beta},\mu,T}(\underline{w}_T, \underline{v}_T) := -(w_T, G_{\boldsymbol{\beta},T}^k \underline{v}_T)_T + \mu(w_T, v_T)_T + s_{\boldsymbol{\beta},T}^-(\underline{w}_T, \underline{v}_T)}$$

with local element-face upwind stabilization given by

$$s_{\boldsymbol{\beta},T}^-(\underline{w}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})^-(w_F - w_T), v_F - v_T)_F$$

- Assembling and including the weak enforcement of BCs, we have

$$\boxed{a_{\boldsymbol{\beta},\mu,h}(\underline{w}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_{\boldsymbol{\beta},\mu,T}(\underline{w}_h, \underline{v}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{b}}} ((\boldsymbol{\beta}\cdot\boldsymbol{n})^+ w_F, v_F)_F}$$

Advection-reaction III

Lemma (Coercivity of $a_{\boldsymbol{\beta},\mu,h}$)

Let $\eta := \min_{T\in\mathcal{T}_h}(1, \tau_{\text{ref},T}\mu)$, $\tau_{\text{ref},T} := \{\max(\|\mu\|_{L^\infty(T)}, L_{\boldsymbol{\beta},T})\}^{-1}$. *Then,*

$$\boxed{\forall \underline{v}_h \in \underline{U}_h^k, \qquad \eta\|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 \leqslant a_{\boldsymbol{\beta},\mu,h}(\underline{v}_h, \underline{v}_h),}$$

with global advection-reaction norm

$$\|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{\boldsymbol{\beta},\mu,T}^2 + \frac{1}{2}\sum_{F\in\mathcal{F}_h^{\text{b}}} \||\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF}|^{1/2} v_F\|_F^2,$$

and $\|\underline{v}_T\|_{\boldsymbol{\beta},\mu,T}^2 := \frac{1}{2}\sum_{F\in\mathcal{F}_T} \||\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF}|^{1/2}(v_F - v_T)\|_F^2 + \tau_{\text{ref},T}^{-1}\|v_T\|_T^2$.

Discrete problem I

- Define the following RHS linear form l_h on $\underline{U}_h^k$:

$$l_h(\underline{v}_h) := \sum_{T\in\mathcal{T}_h} (f, v_T)_T + \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \left(((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})^- g, v_F)_F + \frac{\nu_F\varsigma}{h_F}(g, v_F)_F \right)$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_h^k$ s.t., $\forall \underline{v}_h \in \underline{U}_h^k$,

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) := a_{\nu,h}(\underline{u}_h, \underline{v}_h) + a_{\boldsymbol{\beta},\mu,h}(\underline{u}_h, \underline{v}_h) = l_h(\underline{v}_h)}$$

Discrete problem II

Lemma (inf-sup stability of a_h)

There is $\gamma_\varrho > 0$ independent of h, ν, $\boldsymbol{\beta}$ and μ s.t.

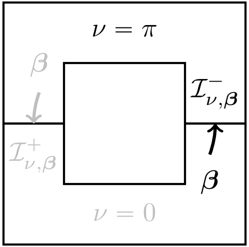
$$\forall \underline{w}_h \in \underline{U}_h^k, \qquad \|\underline{w}_h\|_{\sharp,h} \leqslant \gamma_\varrho \zeta^{-1} \sup_{\underline{v}_h \in \underline{U}_h^k \setminus \{0\}} \frac{a_h(\underline{w}_h, \underline{v}_h)}{\|\underline{v}_h\|_{\sharp,h}},$$

with $\zeta := \tau_{\mathrm{ref},T}\mu$ and augmented global stability norm

$$\|\underline{v}_h\|_{\sharp,h}^2 := \|\underline{v}_h\|_{\nu,h}^2 + \|\underline{v}_h\|_{\boldsymbol{\beta},\mu,h}^2 + \sum_{T \in \mathcal{T}_h} h_T \beta_{\mathrm{ref},T}^{-1} \|G_{\boldsymbol{\beta},T}^k \underline{v}_h\|_T^2$$

The $\|\cdot\|_{\sharp,h}$-norm adds control for the discrete advective derivative!

A tailored reduction map



- We need a reduction map $\underline{I}_h^k : U \to \underline{U}_h^k$. For $T \in \mathcal{T}_h$, simply set

$$(\underline{I}_h^k v)_T := \pi_T^k v$$

- For faces $F \in \mathcal{F}_h$, taking $\gamma_F v$ from the diffusive side if $F \subset \mathcal{I}^-_{\nu,\boldsymbol{\beta}}$,

$$(\underline{I}_h^k v)_F := \pi_F^k(\gamma_F v)$$

- **Hence, interface DOFs on $\mathcal{I}^-_{\nu,\boldsymbol{\beta}}$ represent the diffusive trace!**

Convergence I

Theorem (Error estimate)

Assume that, for all $T \in \mathcal{T}_h$, $u \in H^{k+2}(T)$ and

$$h_T L_{\boldsymbol{\beta},T} \leqslant \beta_{\mathrm{ref},T} \quad \text{and} \quad h_T \mu \leqslant \beta_{\mathrm{ref},T},$$

Then, there is $C > 0$ independent of h, ν, $\boldsymbol{\beta}$, and μ s.t.

$$\boxed{\|\underline{u}_h - \underline{I}_h^k u\|_{\sharp,h}^2 \leqslant C \sum_{T \in \mathcal{T}_h} \left\{ B_T^{\mathrm{d}}(u,k) h_T^{2(k+1)} + B_T^{\mathrm{a}}(u,k) \min(1, \mathrm{Pe}_T) h_T^{2(k+\frac{1}{2})} \right\},}$$

with Pe_T denoting the local Péclet number.

Convergence II

- This estimate holds across the entire range of Pe_T
- For diffusion-dominated elements with $\mathrm{Pe}_T \leqslant h_T$, the contribution is

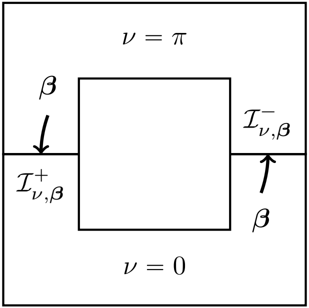
$$\mathcal{O}(h_T^{k+1})$$

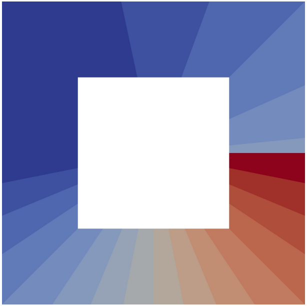
- For advection-dominated elements with $\mathrm{Pe}_T \geqslant 1$, the contribution is

$$\mathcal{O}(h_T^{k+1/2})$$

- In between, we have intermediate orders of convergence

Numerical example I





$$u(\theta, r) = \begin{cases} (\theta - \pi)^2 & \text{if } 0 < \theta < \pi \\ 3\pi(\theta - \pi) & \text{if } \pi < \theta < 2\pi \end{cases}$$

Numerical example II

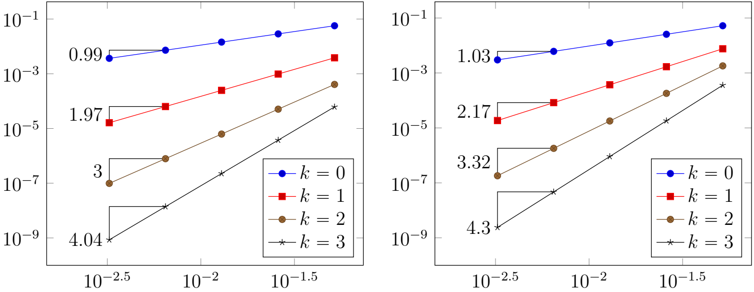


Figure: Energy (left) and L^2-norm (right) of the error vs. h

References I

Beirão da Veiga, L., Brezzi, F., Marini, L., and Russo, A. (2014).
Virtual Element Methods for general second order elliptic problems on polygonal meshes.
Submitted. Preprint arXiv:1412.2646.

Beirão da Veiga, L., Droniou, J., and Manzini, M. (2010).
A unified approach for handling convection terms in finite volumes and mimetic discretization methods for elliptic problems.
IMA J. Numer. Anal., 31(4):1357–1401.

Cangiani, A., Manzini, G., and Sutton, O. J. (2015).
Conforming and nonconforming virtual element methods for elliptic problems.
ArXiV preprint arXiv:1507.03543.

Cockburn, B., Dong, B., Guzmán, J., Restelli, M., and Sacco, R. (2009).
A hybridizable discontinuous Galerkin method for steady-state convection-diffusion-reaction problems.
SIAM J. Sci. Comput., 31(5):3827–3846.

Di Pietro, D. A., Droniou, J., and Ern, A. (2015).
A discontinuous-skeletal method for advection-diffusion-reaction on general meshes.
SIAM J. Numer. Anal.
Published online. DOI: 10.1137/140993971.

Di Pietro, D. A. and Ern, A. (2015).
A hybrid high-order locking-free method for linear elasticity on general meshes.
Comput. Methods Appl. Mech. Engrg., 283:1–21.

Di Pietro, D. A., Ern, A., and Guermond, J.-L. (2008).
Discontinuous Galerkin methods for anisotropic semi-definite diffusion with advection.
SIAM J. Numer. Anal., 46(2):805–831.

Di Pietro, D. A., Ern, A., and Lemaire, S. (2014).
An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators.
Comput. Methods Appl. Math., 14(4):461–472.

References II

Gastaldi, F. and Quarteroni, A. (1989).
On the coupling of hyperbolic and parabolic systems: Analytical and numerical approach.
Appl. Numer. Math., 6:3–31.

Houston, P., Schwab, C., and Süli, E. (2002).
Discontinuous hp-finite element methods for advection-diffusion-reaction problems.
SIAM J. Numer. Anal., 39(6):2133–2163.

Wang, J. and Ye, X. (2013).
A weak Galerkin element method for second-order elliptic problems.
J. Comput. Appl. Math., 241:103–115.