Hybrid High-Order methods

Introduction and fully discrete analysis framework

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Outline

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Setting

- Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open connected polytopal domain
- We focus on the Poisson problem: Given $f : \Omega \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- Let $f \in L^2(\Omega)$. A possible weak formulation reads: Find $u \in H^1_0(\Omega)$ s.t.

$$a(u,v) \coloneqq \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H^1_0(\Omega)$$

- The well-posedness of this problem hinges on the Poincaré inequality

$$\|v\|_{L^2(\Omega)} \leq C_\Omega \|\nabla v\|_{L^2(\Omega)^d}$$

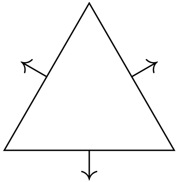
Local polynomial spaces and L^2-orthogonal projector

- Denote by $\mathbb{P}_d^\ell$ the space of d-variate polynomials of total degree $\le \ell$
- Let $Y \subset \mathbb{R}^d$ and denote by $\mathcal{P}^\ell(Y)$ the restriction of $\mathbb{P}_d^\ell$ to Y
- Given $v \in L^2(Y)$, its L^2-orthogonal projection on $\mathcal{P}^\ell(Y)$ is s.t.

$$\int_Y (v - \pi_{\mathcal{P}^\ell(Y)} v) w = 0 \quad \forall w \in \mathcal{P}^\ell(Y)$$

- $\pi_{\mathcal{P}^\ell(Y)} v$ optimally approximates v in all Sobolev seminorms under mild assumptions on Y; see [DP and Droniou, 2020, Chapter 1] for details

Raviart–Thomas–Nédélec and Crouzeix–Raviart elements I

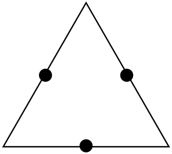


- Denote by T a d-simplex and by $\mathcal{F}_T$ the set collecting its faces
- Let $\mathcal{RTN}^1(T) := \mathcal{P}^0(T)^d + x\mathcal{P}^0(T)$
- Define the degrees of freedom $\sigma := (\sigma_F)_{F \in \mathcal{F}_T}$ s.t.

$$\sigma_F : H^1(T)^d \ni \tau \mapsto \frac{1}{|F|} \int_F \tau \cdot n_{TF} \in \mathbb{R}$$

- Notice that $\tau \cdot n_{TF} \in \mathcal{P}^0(F)$ for all $\tau \in \mathcal{RTN}^l(T)$ and all $F \in \mathcal{F}_T$
- $(T, \mathcal{RTN}^1(T), \sigma)$ is a FE [Raviart and Thomas, 1977, Nédélec, 1980]

Raviart–Thomas–Nédélec and Crouzeix–Raviart elements II



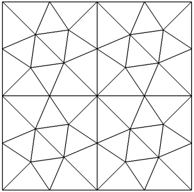
- Let now $\mathcal{P}^1(T)$ be the space of affine functions on T
- Define the degrees of freedom $\sigma := (\sigma_F)_{F\in\mathcal{F}_T}$ s.t.

$$\sigma_F : H^1(T) \ni v \mapsto \frac{1}{|F|}\int_F v \in \mathbb{R}$$

- $(T, \mathcal{P}^1(T), \sigma)$ is a FE [Crouzeix and Raviart, 1973]

Outline

A non-conforming finite element scheme I



- Let $\mathcal{T}_h$ be a conforming simplicial mesh of Ω
- Let $\mathcal{CR}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$ be the Crouzeix–Raviart space on $\mathcal{T}_h$ and set

$$\mathcal{CR}_0(\mathcal{T}_h) \coloneqq \left\{ v_h \in \mathcal{CR}(\mathcal{T}_h) \,:\, \pi_{\mathcal{P}^0(F)} v = 0 \text{ for all } F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- Notice that $\mathcal{CR}_0(\mathcal{T}_h) \not\subset H_0^1(\Omega)$!
- With ∇_h broken gradient, the scheme reads: Find $u_h \in \mathcal{CR}_0(\mathcal{T}_h)$ s.t.

$$a_h(u_h, v_h) \coloneqq \int_\Omega \nabla_h u_h \cdot \nabla_h v_h = \int_\Omega f v_h \quad \forall v_h \in \mathcal{CR}_0(\mathcal{T}_h)$$

Stability analysis I

Lemma (Discrete Poincaré inequality in the Crouzeix–Raviart space)

For all $v_h \in \mathcal{CR}_0(\mathcal{T}_h)$*,*

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$$

where $a \lesssim b$ *means* $a \le Cb$ *with* C *independent of* h*.*

- The Poincaré–Wirtinger inequality gives, setting $v_T \coloneqq (v_h)_{|T}$,

$$\|v_T\|_{L^2(T)} \lesssim \|\pi_{\mathcal{P}^0(T)} v_T\|_{L^2(T)} + h_T \|\nabla v_T\|_{L^2(T)^d}$$

- Hence, letting $\overline{v}_h \coloneqq \pi_{\mathcal{P}^0(\mathcal{T}_h)} v_h$, it suffices to prove that

$$\|\overline{v}_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$$

- Since $\mathrm{div} : \mathcal{RTN}^1(\mathcal{T}_h) \to \mathcal{P}^0(\mathcal{T}_h)$ is surjective: $\exists \tau_h \in \mathcal{RTN}^1(\mathcal{T}_h)$ s.t.

$$\mathrm{div}\, \tau_h = \overline{v}_h \text{ and } \|\tau_h\|_{H(\mathrm{div};\Omega)} \lesssim \|\overline{v}_h\|_{L^2(\Omega)}$$

Stability analysis II

- We write, letting $v_T := (v_h)_{|T}$ for all $T \in \mathcal{T}_h$,

$$\|\overline{v}_h\|_{L^2(\Omega)}^2 \overset{\text{⚠}}{=} \sum_{T\in\mathcal{T}_h} \int_T v_T \operatorname{div} \tau_h \overset{\text{IBP}}{=} -\int_\Omega \nabla_h v_h \cdot \tau_h + \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F v_T (\tau_h \cdot n_{TF})$$

- Replacing $v_T \leftarrow \pi_{\mathcal{P}^0(F)} v_T =: v_F$ in the boundary term and rearranging,

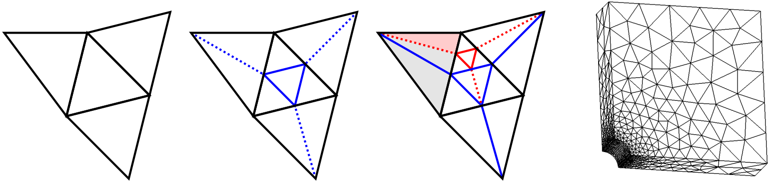
$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F v_T(\tau_h \cdot n_{TF}) &= \sum_{F\in\mathcal{F}_h} \sum_{T\in\mathcal{T}_F} \int_F v_T(\tau_h \cdot n_{TF}) \\ &= \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \sum_{T\in\mathcal{T}_F} \int_F v_F(\tau_h \cdot n_{TF}) + \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \int_F v_F(\tau_h \cdot n_F) = 0,\end{aligned}$$

by single-valuedness of v_F for $F \in \mathcal{F}_h^{\mathrm{i}}$ and $v_F = 0$ for $F \in \mathcal{F}_h^{\mathrm{b}}$

- Using Cauchy–Schwarz inequalities, we get

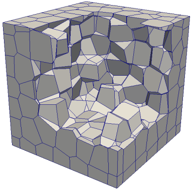
$$\|\overline{v}_h\|_{L^2(\Omega)}^2 \le \|\nabla_h v_h\|_{L^2(\Omega)^d} \|\tau_h\|_{L^2(\Omega)^d} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d} \|\overline{v}_h\|_{L^2(\Omega)}$$

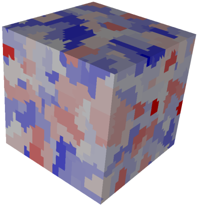
Limitations of the finite element approach

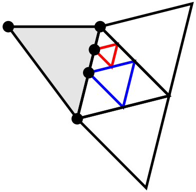


- Approach limited to conforming meshes with standard elements
  - ⟹ local refinement requires to trade mesh size for mesh quality
  - ⟹ complex geometries may require a large number of elements
  - ⟹ the element shape cannot be adapted to the solution
- Treating more general meshes in the FE spirit would significantly increase the space dimension [Droniou et al., 2021]
- The extension to high-order is not straightforward

Fully discrete polytopal approach







- Key idea: replace both spaces and operators by discrete counterparts
- Support of polyhedral meshes and high-order
- Several strategies to reduce the number of unknowns on general shapes
- Elegant analysis framework available

A few key references

- Introduction of Hybrid High-Order (HHO) methods [DP et al., 2014]
- Fully discrete analysis framework [DP and Droniou, 2018]
- A monograph on HHO methods [DP and Droniou, 2020]
- Introduction of Discrete de Rham (DDR) methods [DP et al., 2020]
- DDR for the de Rham complex of differential forms [Bonaldi et al., 2023]

Outline

1 Preliminaries

2 A non-conforming finite element scheme on standard meshes

3 An Hybrid High-Order scheme on polytopal meshes

4 Fully discrete analysis framework

A paradigm shift

- Let $v_T \in \mathcal{P}^1(T)$ and set $v_F := \pi_{\mathcal{P}^0(F)}(v_T)_{|F}$ for all $F \in \mathcal{F}_T$
- We have, for all $\tau \in \mathcal{P}^0(T)^d$,

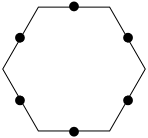
$$\int_T \nabla v_T \cdot \tau = -\int_T v_T \cancel{\operatorname{div} \tau} + \sum_{F \in \mathcal{F}_T} \int_F v_T (\tau \cdot n_{TF}) = \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF})$$

- Moreover, with $\overline{x}_Y$ center of mass of $Y \in \{T\} \cup \mathcal{F}_T$, noticing that $\overline{x}_T = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \overline{x}_F$ and using linearity,

$$\pi_{\mathcal{P}^0(T)} v = v_T(\overline{x}_T) = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} v_T(\overline{x}_F) = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F$$

- These formulas remain valid when T is a general polytope!

Generalization to polytopes

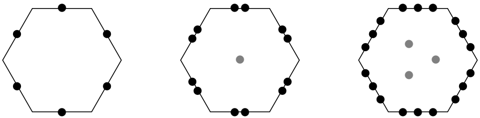


- Let $\underline{V}_T^0 := \left\{ \underline{v}_T := (v_F)_{F \in \mathcal{F}_T} \ : \ v_F \in \mathcal{P}^0(F) \text{ for all } F \in \mathcal{F}_T \right\}$
- We can define a potential reconstruction $r_T^1 : \underline{V}_T^0 \to \mathcal{P}^1(T)$ enforcing

$$\int_T \nabla r_T^1 \underline{v}_T \cdot \tau = \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF}) \quad \forall \tau \in \mathcal{P}^0(T)^d$$

$$\pi_{\mathcal{P}^0(T)}(r_T^1 \underline{v}_T) = \frac{1}{\mathrm{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F$$

Extension to arbitrary-order I



- Let $k \geq 0$ and define the Hybrid High-Order (HHO) space

$$\underline{V}_T^k := \left\{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) \; : \; v_T \in \mathcal{P}^{k-1}(T) \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_T \right\}$$

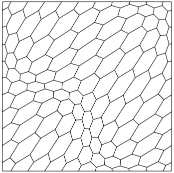
- We define $r_T^{k+1} : \underline{V}_T^k \to \mathcal{P}^k(T)^d$ s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\int_T \nabla r_T^{k+1} \underline{v}_T \cdot \nabla w = -\int_T v_T \, \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T),$$

$$\pi_{\mathcal{P}^0(T)}(r_T^1 \underline{v}_T) = \pi_{\mathcal{P}^0(T)} v_T,$$

where $v_T := \frac{1}{\mathrm{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F$ if $k = 0$

Global HHO space and H^1-like seminorm



- Given a polytopal mesh $\mathcal{T}_h$ of Ω, define the global HHO space

$$\underline{V}_h^k \coloneqq \Big\{\underline{v}_h = ((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_T}) \in \underline{V}_h^k \;: \\ v_T \in \mathcal{P}^{k-1}(T) \text{ for all } T\in\mathcal{T}_h \text{ and } v_F\in\mathcal{P}^k(F) \text{ for all } F\in\mathcal{F}_h\Big\}$$

- We define on $\underline{V}_h^k$ the norm

$$\|\underline{v}_h\|_{1,h}^2 \coloneqq \sum_{T\in\mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2$$

where $\|\underline{v}_T\|_{1,T}^2 \coloneqq \|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F\in\mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2$ for all $T\in\mathcal{T}_h$

Discrete Poincaré inequality in HHO spaces I

Lemma (Poincaré inequality in HHO spaces)

Denote by $\underline{V}_{h,0}^k$ the subspace of $\underline{V}_h^k$ with vanishing boundary values. For any $\underline{v}_h \in \underline{V}_{h,0}^k$, letting $v_h \in \mathcal{P}^{\max(k-1,0)}(\mathcal{T}_h)$ be s.t. $(v_h)_{|T} := v_T$ for all $T \in \mathcal{T}_h$, it holds

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}_h\|_{1,h},$$

hence $\|\cdot\|_{1,h}$ is a norm on $\underline{V}_{h,0}^k$.

- Since $\operatorname{div} : H^1(\Omega)^d \to L^2(\Omega)$ is surjective, there is $\tau \in H^1(\Omega)^d$ s.t.

$$\operatorname{div} \tau = v_h \text{ and } \|\tau\|_{H^1(\Omega)^d} \lesssim \|v_h\|_{L^2(\Omega)}$$

- Notice that here we cannot seek τ in $\mathcal{RTN}^{k+1}(\mathcal{T}_h)$ since $\mathcal{T}_h$ is not conforming simplicial!

Discrete Poincaré inequality in HHO spaces II

- We go on writing

$$
\begin{aligned}
\|v_h\|_{L^2(\Omega)}^2 &= \int_\Omega v_h \operatorname{div} \tau = \sum_{T\in\mathcal{T}_h} \int_T v_T \operatorname{div} \tau \\
&\overset{\text{IBP}}{=} \sum_{T\in\mathcal{T}_h} \left(-\int_T \nabla v_T \cdot \tau + \sum_{F\in\mathcal{F}_T} \int_F (v_T - {\color{red} v_F})(\tau\cdot n_{TF}) \right) \\
&\overset{\text{C–S}}{\le} \left[\sum_{T\in\mathcal{T}_h} \left(\|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F\in\mathcal{F}_T} \|v_T - v_F\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}} \\
&\quad \times \left[\sum_{T\in\mathcal{T}_h} \left(\|\tau\|_{L^2(T)^2}^2 + h_T \|\tau\|_{L^2(\partial T)}^2 \right) \right]^{\frac{1}{2}}
\end{aligned}
$$

- Recalling the definition of $\|\cdot\|_{1,h}$ and using trace inequalities along with $h_T \le h_\Omega \lesssim 1$, we get

$$
\|v_h\|_{L^2(\Omega)}^2 \lesssim \|\underline{v}_h\|_{1,h} \|\tau\|_{H^1(T)^d} \lesssim \|\underline{v}_h\|_{1,h} \|v_h\|_{L^2(\Omega)}
$$

An HHO scheme

We consider the following scheme: Find $\underline{u}_h \in \underline{V}_{h,0}^k$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \int_\Omega f v_h \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

where, for all $T \in \mathcal{T}_h$,

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T \nabla r_T^{k+1} \underline{u}_T \cdot \nabla r_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

and the symmetric semi-definite bilinear form s_T satisfies

$$\|\underline{v}_T\|_{1,T} \lesssim a_T(\underline{v}_T, \underline{v}_T)^{\frac{1}{2}} \lesssim \|\underline{v}_T\|_{1,T} \quad \forall \underline{v}_T \in \underline{V}_T^k \tag{ST1}$$

Stability analysis

Lemma (Well-posedness of the HHO discrete problem)

The HHO problem admits a unique solution that satisfies

$$\|\underline{u}_h\|_{1,h} \lesssim \|f\|_{L^2(\Omega)}.$$

- Squaring and summing (ST1) over $T \in \mathcal{T}_h$, we have

$$\|\underline{v}_h\|_{1,h}^2 \lesssim a_h(\underline{v}_h, \underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k,$$

  which expresses the coercivity of a_h
- Using the Cauchy–Schwarz and discrete Poincaré inequalities,

$$\int_\Omega f v_h \le \|f\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} \|\underline{v}_h\|_{1,h}$$

- Letting $\underline{v}_h = \underline{u}_h$ above, the conclusion follows

Error analysis I

- Let $\underline{I}_h^k : H^1(\Omega) \to \underline{V}_h^k$ be s.t.

$$\underline{I}_h^k v := \left((\pi_{\mathcal{P}^{k-1}(T)} v)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{P}^k(F)} v)_{F \in \mathcal{F}_h} \right) \quad \forall v \in H^1(\Omega)$$

- We aim at estimating the error

$$\underline{e}_h := \underline{u}_h - \underline{I}_h^k u \in \underline{V}_{h,0}^k$$

- It holds, for all $\underline{v}_h \in \underline{V}_{h,0}^k$,

$$a_h(\underline{e}_h, \underline{v}_h) = a_h(\underline{u}_h, \underline{v}_h) - a_h(\underline{I}_h^k u, \underline{v}_h) = \int_\Omega f v_h - a_h(\underline{I}_h^k u, \underline{v}_h) =: \mathcal{E}_h(u; \underline{v}_h)$$

- A straightforward modification of the stability proof gives

$$\|\underline{e}_h\|_{1,h} \le \sup_{\underline{v}_h \in \underline{V}_{h,0}^k \setminus \{\underline{0}\}} \frac{\mathcal{E}_h(u; \underline{v}_h)}{\|\underline{v}_h\|_{1,h}}$$

Error analysis II

We reformulate the components of the consistency error $\mathcal{E}_h(\underline{v}_h)$:

$$\int_\Omega f v_h = -\sum_{T\in\mathcal{T}_h}\int_T \Delta u\, v_h = \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla u\cdot\nabla v_T + \sum_{F\in\mathcal{F}_T}\int_F (\nabla u\cdot n_{TF})(v_F - v_T)\right]$$

$$\begin{aligned}
a_h(\underline{I}_h^k u, \underline{v}_h) &= \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla r_T^{k+1}(\underline{I}_T^k u)\cdot\nabla r_T^{k+1}\underline{v}_T + s_T(\underline{I}_T^k u, \underline{v}_T)\right] \\
&= \sum_{T\in\mathcal{T}_h}\left[\int_T \nabla r_T^{k+1}(\underline{I}_T^k u)\cdot\nabla v_T + \sum_{F\in\mathcal{F}_T}\int_F (\nabla r_T^{k+1}(\underline{I}_T^k u)\cdot n_{TF})(v_F - v_T)\right] \\
&\quad + \sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T)
\end{aligned}$$

Error analysis III

Gathering the above results, we get

$$
\begin{aligned}
\mathcal{E}_h(\underline{v}_h) &= \sum_{T\in\mathcal{T}_h} \int_T \left[\nabla u - \nabla r_T^{k+1}(\underline{I}_T^k u)\right] \cdot \nabla v_T \\
&+ \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \int_F \left[\nabla u - (\nabla r_T^{k+1}(\underline{I}_T^k u) \cdot n_{TF})\right] (v_F - v_T) \\
&- \sum_{T\in\mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T) =: \mathfrak{T}_1 + \mathfrak{T}_2 + \mathfrak{T}_3
\end{aligned}
$$

Approximation properties of the potential reconstruction I

- By definition, for all $T \in \mathcal{T}_h$, all $v \in H^1(T)$, and all $w \in \mathcal{P}^{k+1}(T)$,

$$\int_T \nabla r_T^{k+1}(\underline{I}_T^k v) \cdot \nabla w = -\int_T \pi_{\mathcal{P}^{k-1}(T)} v \, \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \pi_{\mathcal{P}^k(F)} v (\nabla w \cdot n_{TF})$$

- Noticing that $\Delta w \in \mathcal{P}^{k-1}(T)$ and $\nabla w \cdot n_{TF} \in \mathcal{P}^k(F)$, we can remove the projectors and integrate by parts to obtain

$$\int_T \nabla r_T^{k+1}(\underline{I}_T^k v) \cdot \nabla w = \int_T \nabla v \cdot \nabla w \quad \forall w \in \mathcal{P}^{k+1}(T)$$

- This shows that $\nabla r_T^{k+1} \circ \underline{I}_T^k = \pi_{\nabla \mathcal{P}^{k+1}(T)} \circ \nabla$
- It can be proved that $\pi_{\nabla \mathcal{P}^{k+1}(T)} \circ \nabla$ optimally approximates gradients

Approximation properties of the potential reconstruction II

- Noticing that $\nabla v_T \in \nabla\mathcal{P}^k(T) \subset \nabla\mathcal{P}^{k+1}(T)$,

$$\int_T \left[\nabla u - \nabla r_T^{k+1}(\underline{I}_T^k u)\right] \cdot \nabla v_T = \int_T \left[\nabla u - \pi_{\nabla\mathcal{P}^{k+1}(T)}(\nabla u)\right] \cdot \nabla v_T = 0$$

  we infer

$$\mathfrak{T}_1 = 0$$

- Using Cauchy–Schwarz inequalities and the definition of $\|\cdot\|_{1,h}$,

$$\mathfrak{T}_2 \leq \left(\sum_{T\in\mathcal{T}_h} h_T \|\nabla u - \pi_{\nabla\mathcal{P}^{k+1}(T)}(\nabla u)\|_{\partial T}^2\right)^{\frac{1}{2}} \|\underline{v}_h\|_{1,h}$$

- If, additionally, $u \in H^{k+2}(\mathcal{T}_h)$,

$$\mathfrak{T}_2 \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

Polynomial consistency of the stabilization I

- To have $\mathfrak{T}_3$ scale as $\mathfrak{T}_2$, we further assume polynomial consistency:

$$s_T(\underline{I}_T^k w, \underline{v}_T) = 0 \quad \forall (w, \underline{v}_T) \in \mathcal{P}^{k+1}(T) \times \underline{V}_T^k \tag{ST2}$$

- For all $w \in H^{k+2}(T)$, setting $|\cdot|_{s,T} \coloneqq s_T(\cdot,\cdot)^{\frac{1}{2}}$, we have

$$\begin{aligned} |\underline{I}_T^k w|_{s,T} &\overset{\text{(ST2)}}{=} \min_{v \in \mathcal{P}^{k+1}(T)} |\underline{I}_T^k (w - v)|_{s,T} \\ &\overset{\text{(ST1)}}{\lesssim} \min_{v \in \mathcal{P}^{k+1}(T)} \|\underline{I}_T^k (w - v)\|_{1,T} \lesssim h_T^{k+1} |w|_{H^{k+2}(T)} \end{aligned}$$

hence, by Cauchy–Schwarz inequalities and again (ST1),

$$\mathfrak{T}_3 \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

Polynomial consistency of the stabilization II

Theorem (Error estimate for the HHO scheme)

Denote by $u \in H^1_0(\Omega)$ *the solution to the Poisson problem and by* $\underline{u}_h \in \underline{V}^k_h$ *its HHO approximation. Then, under* (ST1)–(ST2)*, and further assuming* $u \in H^{k+2}(\mathcal{T}_h)$*, it holds*

$$\|\underline{u}_h - \underline{I}^k_h u\|_{1,h} \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)}.$$

An example of stabilization bilinear form

Example

Let, for all $T \in \mathcal{T}_h$ and all $\underline{v}_T \in \underline{V}_T^k$,

$$(\delta_T^k \underline{v}_T, (\delta_{TF}^k \underline{v}_T)_{F \in \mathcal{F}_T}) := \underline{v}_T - \underline{I}_T^k (r_T^{k+1} \underline{v}_T).$$

The stabilization bilinear form

$$s_T(\underline{w}_T, \underline{v}_T) := h_T^{-2} \int_T \delta_T^k \underline{w}_T \, \delta_T^k \underline{v}_T + h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F \delta_{TF}^k \underline{w}_T \, \delta_{TF}^k \underline{v}_T$$

satisfies properties (ST1)–(ST2).

Numerical example

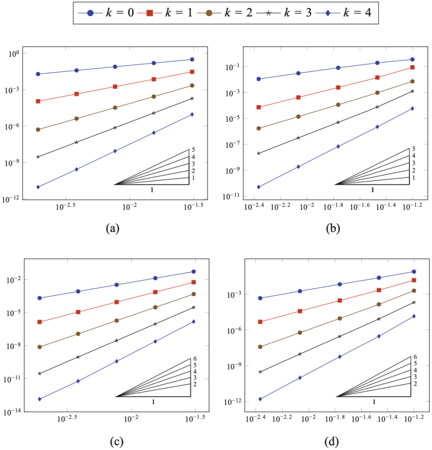


Figure: $\|\underline{e}_h\|_{1,h}$ and $\|e_h\|_{L^2(\Omega)}$ as a function of h for uniformly refined triangular (top) and hexagonal (bottom) mesh families

Outline

An abstract problem and its discretization

$$\boxed{\text{Find } u \in U \text{ s.t. } a(u,v) = l(v) \text{ for all } v \in U} \qquad (\Pi)$$

- U Hilbert space
- $a : U \times U \to \mathbb{R}$ continuous bilinear form
- $l : U \to \mathbb{R}$ continuous linear form

$$\boxed{\text{Find } u_h \in U_h \text{ s.t. } a_h(u_h,v_h) = l_h(v_h) \text{ for all } v_h \in U_h} \qquad (\Pi_h)$$

- (finite-dimensional) vector space U_h with norm $\|\cdot\|_{U,h}$
- $a_h : U_h \times U_h \to \mathbb{R}$ bilinear form
- $l_h : U_h \to \mathbb{R}$ linear form

Approximation error

- We want to compare u and u_h without assuming that $U_h \subset U$
- The difference $u - u_h$ does not make sense in general
- Instead, we can consider

$$u - R_h u_h \text{ with } R_h : U_h \to U \text{ reconstruction}$$

or

$$u_h - I_h u \text{ with } I_h : U \to U_h \text{ interpolator}$$

- We will explore the second choice, which leads to a simpler framework

Stability and a priori estimate I

Definition (Inf-sup stability)

The bilinear form a_h is inf-sup stable if there is $\gamma > 0$ s.t.

$$\gamma \|w_h\|_{U,h} \leq \sup_{v_h \in U_h \setminus \{0\}} \frac{a_h(w_h, v_h)}{\|v_h\|_{U,h}} \quad \forall w_h \in U_h. \tag{inf-sup}$$

Remark (Uniform inf-sup stability)

One typically requires, in practice, γ independent of h.

Stability and a priori estimate II

Lemma (A priori estimate)

If a_h is inf-sup stable, $b_h : U_h \to \mathbb{R}$ is a linear form, and $w_h \in U_h$ is s.t.

$$a_h(w_h, v_h) = b_h(v_h) \quad \forall v_h \in U_h,$$

then

$$\|w_h\|_{U,h} \le \gamma^{-1} \|b_h\|_{U,h,*}.$$

Proof.

It suffices to write

$$\gamma \|w_h\|_{U,h} \overset{\text{(inf-sup)}}{\le} \sup_{v_h \in U_h \setminus \{0\}} \frac{a_h(w_h, v_h)}{\|v_h\|_{U,h}} = \sup_{v_h \in U_h \setminus \{0\}} \frac{b_h(v_h)}{\|v_h\|_{U,h}} =: \|b_h\|_{U,h,*}. \quad \square$$

Error equation

- We want to characterize $e_h := u_h - I_h u$ as the solution to a problem
- Using linearity, we have, for all $v_h \in U_h$,

$$a_h(e_h, v_h) = a_h(u_h, v_h) - a_h(I_h u, v_h) \overset{(\Pi_h)}{=} l_h(v_h) - a_h(I_h u, v_h)$$

- Thus, setting

$$\mathcal{E}_h(u; v_h) := l_h(v_h) - a_h(I_h u, v_h)$$

  e_h is the solution to the following error equation:

$$a_h(e_h, v_h) = \mathcal{E}_h(u; v_h) \quad \forall v_h \in U_h$$

Third Strang Lemma I

Lemma (Basic a priori error estimate)

Assume that a_h *is inf-sup stable and let* $u_h \in U_h$ *solve* (Π_h)*. Then,*

$$\|e_h\|_{U_h} \leq \gamma^{-1} \|\mathcal{E}_h(u; \cdot)\|_{U,h,*}.$$

Proof.

Apply the stability result with $w_h \leftarrow e_h$ and $b_h \leftarrow \mathcal{E}_h(u; \cdot)$. □

Definition (Consistency)

We say that consistency holds if

$$\|\mathcal{E}_h(u; \cdot)\|_{U,h,*} \to 0 \text{ as } h \to 0.$$

Quasi-optimality of the error estimate

- Assume that both the following quantities are independent of h:

$$\gamma \text{ and } \alpha := \sup_{w_h,\, v_h \in U_h \setminus \{0\}} \frac{a_h(w_h, v_h)}{\|w_h\|_{U,h} \|v_h\|_{U,h}}$$

- Then, we have

$$\alpha^{-1} \|\mathcal{E}_h(u; \cdot)\|_{U,h,*} \le \|e_h\|_{U_h} \le \gamma^{-1} \|\mathcal{E}_h(u; \cdot)\|_{U,h,*}$$

- This expresses the quasi-optimality of the error estimate

Key properties and their role

- Uniform inf-sup stability $\implies$ basic error estimate
- Consistency $\implies$ convergence
- Uniform continuity $\implies$ quasi-optimality of the estimate

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Thank you for your attention!

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