Hybrid High-Order methods on general meshes

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Minimalistic bibliography on high-order polyhedral methods

- Discontinuous Galerkin (DG)
  - Basic analysis tools [Di Pietro and Ern, 2012]
  - Adaptive coarsening [Bassi et al., 2012]
  - Locally degenerate ADR [Di Pietro et al., 2008]
- Hybridizable Discontinuous Galerkin (HDG)
  - Pure diffusion [Cockburn et al., 2009]
  - Diffusion-dominated ADR [Chen and Cockburn, 2014]
- Virtual elements (VEM)
  - Pure diffusion [Beirão da Veiga et al., 2013]
  - Diffusion-dominated ADR [Beirão da Veiga et al., 2014]
- Hybrid High-Order (HHO)
  - Pure diffusion [Di Pietro et al., 2014b]
  - Locally degenerate ADR [Di Pietro et al., 2014a]
  - HHO as HDG on steroids [Cockburn et al., 2015]

Features of HHO

- Capability of handling general polyhedral meshes
- Construction valid for arbitrary space dimensions
- Arbitrary approximation order (including $k = 0$)
- Reproduction of desirable continuum properties
  - Integration by parts formulas
  - Kernels of operators
  - Symmetries
- Reduced computational cost after hybridization

$$N_{\text{dof}}^{\text{hho}} \approx \frac{1}{2}k^2 \,\text{card}(\mathcal{F}_h) \qquad N_{\text{dof}}^{\text{dg}} \approx \frac{1}{6}k^3 \,\text{card}(\mathcal{T}_h)$$

Outline

Outline

Mesh regularity I

Definition (Mesh regularity)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polyhedral meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the sense of Ciarlet;
- contact-regular: every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences:

- Trace and inverse inequalities
- Optimal approximation for broken polynomial spaces

Mesh regularity II

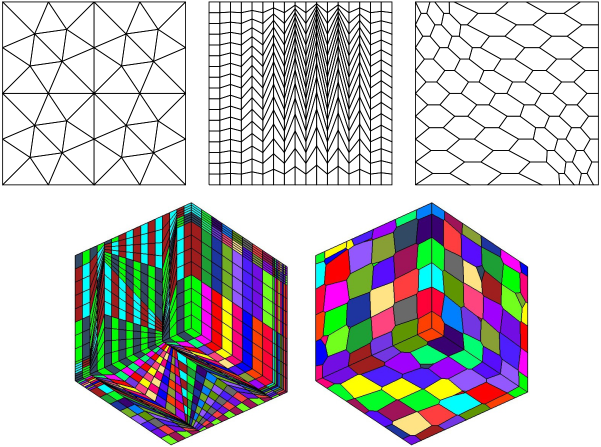


Figure : Admissible meshes in 2d and 3d: [Herbin and Hubert, 2008, FVCA5] and [Di Pietro and Lemaire, 2015] (above) and [Eymard et al., 2011, FVCA6] (below)

Model problem

- Let Ω denote a bounded, connected polyhedral domain
- For $f \in L^2(\Omega)$, we consider the Poisson problem

$$\begin{aligned} -\triangle u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- In weak form: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) := (\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

HHO (The power and the grip)

- DOFs: polynomials of degree $k \geqslant 0$ at elements and faces
- Differential operators reconstructions taylored to the problem:

$$\boxed{a_{|T}(u, v) \approx (\nabla p_T^k \underline{\mathsf{u}}_T, \nabla p_T^k \underline{\mathsf{v}}_T) + \text{stab.}}$$

  with
  - high-order reconstruction p_T^k from local Neumann solves
  - stabilization via face-based penalty
- Construction yielding superconvergence on general meshes

DOFs

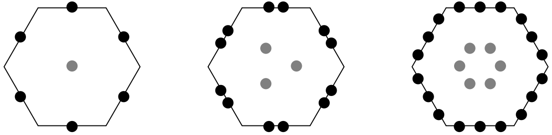


Figure : $\underline{\mathsf{U}}_T^k$ for $k \in \{1, 2\}$

- For $k \geqslant 0$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{\mathsf{U}}_T^k := \mathbb{P}_d^k(T) \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}$$

- The global space has single-valued interface DOFs

$$\underline{\mathsf{U}}_h^k := \left\{ \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}_d^k(T) \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}_{d-1}^k(F) \right\}$$

Local potential reconstruction (The power) I

- Let $T \in \mathcal{T}_h$. The local potential reconstruction operator

$$p_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}_d^{k+1}(T)$$

is s.t. $\forall \underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^k$, $(p_T^k \underline{\mathsf{v}}_T, 1)_T = (\mathsf{v}_T, 1)_T$ and $\forall w \in \mathbb{P}_d^{k+1}(T)$,

$$\boxed{(\boldsymbol{\nabla} p_T^k \underline{\mathsf{v}}_T, \boldsymbol{\nabla} w)_T := -(\mathsf{v}_T, \triangle w)_T + \sum_{F \in \mathcal{F}_T} (\mathsf{v}_F, \boldsymbol{\nabla} w \boldsymbol{n}_{TF})_F}$$

- SPD linear system of size

$$N_{k,d} := \binom{k+1+d}{k+1}$$

Local potential reconstruction (The power) II

k	$d=1$	$d=2$	$d=3$
0	2	3	4
1	3	6	10
2	4	10	20
3	5	15	35

Table : Size $N_{k,d}$ of the local matrix to invert to compute $p_T^k \underline{\mathsf{v}}_T$

Local potential reconstruction (The power) III

Lemma (Approximation properties for $p_T^k \underline{\mathsf{I}}_T^k$)

Define the local interpolator $\underline{\mathsf{I}}_T^k : H^1(T) \to \underline{\mathsf{U}}_T^k$ *s.t.*

$$\underline{\mathsf{I}}_T^k : v \mapsto \left(\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T}\right).$$

Then, for all $T \in \mathcal{T}_h$ *and all* $v \in H^{k+2}(T)$*,*

$$\|v - p_T^k \underline{\mathsf{I}}_T^k v\|_T + h_T \|\boldsymbol{\nabla}(v - p_T^k \underline{\mathsf{I}}_T^k v)\|_T \lesssim h_T^{k+2} \|v\|_{k+2,T}.$$

Local potential reconstruction (The power) IV

- Since $\triangle w \in \mathbb{P}_d^{k-1}(T)$ and $\nabla w_{|F}\cdot\boldsymbol{n}_{TF} \in \mathbb{P}_{d-1}^k(F)$,

$$
\begin{aligned}
(\nabla p_T^k \underline{\mathsf{I}}_T^k v, \nabla w)_T &= -(\pi_T^k v, \triangle w)_T + \sum_{F\in\mathcal{F}_T} (\pi_F^k v, \nabla w\cdot\boldsymbol{n}_{TF})_F \\
&= -(v, \triangle w)_T + \sum_{F\in\mathcal{F}_T} (v, \nabla w\cdot\boldsymbol{n}_{TF})_F \\
&= (\nabla v, \nabla w)_T
\end{aligned}
$$

- This shows that $p_T^k \underline{\mathsf{I}}_T^k$ is the elliptic projector on $\mathbb{P}_d^{k+1}(T)$:

$$
(\nabla p_T^k \underline{\mathsf{I}}_T^k v - \nabla v, \nabla w)_T = 0 \qquad \forall w \in \mathbb{P}_d^{k+1}(T)
$$

- The approximation properties follow

Stabilization (The grip) I

- We would be tempted to approximate

$$a_{|T}(u,v) \approx (\nabla p_T^k \underline{\mathsf{u}}_T, \nabla p_T^k \underline{\mathsf{v}}_T)_T$$

- However, this choice is not stable
- To remedy, we add a local stabilization term

$$a_{|T}(u,v) \approx a_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) := (\nabla p_T^k \underline{\mathsf{u}}_T, \nabla p_T^k \underline{\mathsf{v}}_T)_T + s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T)$$

- Coercivity and boundedness are expressed w.r.t. to

$$\|\underline{\mathsf{v}}_T\|_{1,T}^2 := \|\nabla \mathsf{v}_T\|_T^2 + \sum_{F \in \mathcal{F}_T} \frac{1}{h_F} \|\mathsf{v}_F - \mathsf{v}_T\|_F^2$$

Stabilization (The grip) II

- Define, for $T \in \mathcal{T}_h$, the stabilization bilinear form s_T as

$$s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\pi_F^k(P_T^k \underline{\mathsf{u}}_T - \mathsf{u}_F), \pi_F^k(P_T^k \underline{\mathsf{v}}_T - \mathsf{v}_F))_F,$$

with P_T^k high-order correction of cell DOFs based on p_T^k

$$P_T^k \underline{\mathsf{v}}_T := \mathsf{v}_T + (p_T^k \underline{\mathsf{v}}_T - \pi_T^k p_T^k \underline{\mathsf{v}}_T)$$

- With this choice, a_T satisfies for all $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}_T^k$,

$$\|\underline{\mathsf{v}}_h\|_{1,T}^2 \lesssim a_T(\underline{\mathsf{v}}_T, \underline{\mathsf{v}}_T) \lesssim \|\underline{\mathsf{v}}_T\|_{1,T}^2$$

Stabilization (The grip) III

- Key point: s_T preserves the approximation properties of $\boldsymbol{\nabla} p_T^k$
- For all $u \in H^{k+2}(T)$, letting $\underline{\widehat{\mathsf{u}}}_T := \underline{\mathsf{I}}_T^k u = \left(\pi_T^k u, (\pi_F^k u)_{F\in\mathcal{F}_T}\right)$,

$$\begin{aligned}\|\pi_F^k(P_T^k\underline{\widehat{\mathsf{u}}}_T - \widehat{\mathsf{u}}_F)\|_F &= \|\pi_F^k\left(\pi_T^k u + p_T^k\underline{\widehat{\mathsf{u}}}_T - \pi_T^k p_T^k\underline{\widehat{\mathsf{u}}}_T - \pi_F^k u\right)\|_F \\ &\leqslant \|\pi_F^k\left(p_T^k\underline{\widehat{\mathsf{u}}}_T - u\right)\|_F + \|\pi_T^k\left(u - p_T^k\underline{\widehat{\mathsf{u}}}_T\right)\|_F \\ &\lesssim h_T^{-1/2}\|p_T^k\underline{\widehat{\mathsf{u}}}_T - u\|_T\end{aligned}$$

- Recalling the approximation properties of p_T^k, this yields

$$\boxed{\left\{\|\boldsymbol{\nabla} p_T^k\underline{\widehat{\mathsf{u}}}_T - \boldsymbol{\nabla} u\|_T^2 + s_T(\underline{\widehat{\mathsf{u}}}_T, \underline{\widehat{\mathsf{u}}}_T)\right\}^{1/2} \lesssim h_T^{k+1}\|u\|_{k+2,T}}$$

Discrete problem

- We enforce boundary conditions strongly considering the space

$$\underline{\mathsf{U}}_{h,0}^{k} := \left\{ \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k \mid \mathsf{v}_F \equiv 0 \quad \forall F \in \mathcal{F}_h^{\mathrm{b}} \right\}$$

- The discrete problem reads: Find $\underline{\mathsf{u}}_h \in \underline{\mathsf{U}}_{h,0}^{k}$ s.t.

$$\boxed{a_h(\underline{\mathsf{u}}_h, \underline{\mathsf{v}}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) = \sum_{T \in \mathcal{T}_h} (f, \mathsf{v}_T)_T \qquad \forall \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_{h,0}^{k}}$$

- Well-posedness follows from the coercivity of a_h

Convergence I

Theorem (Energy-norm error estimate)

Assume $u \in H^{k+2}(\mathcal{T}_h)$ *and let*

$$\widehat{\underline{\mathsf{u}}}_h := \left((\pi_T^k u)_{T\in\mathcal{T}_h}, (\pi_F^k u)_{F\in\mathcal{F}_h}\right) \in \underline{\mathsf{U}}_{h,0}^k.$$

Then, we have the following energy error estimate:

$$\boxed{\max\left(\|\underline{\mathsf{u}}_h - \widehat{\mathsf{u}}_h\|_{1,h}, \|\underline{\mathsf{u}}_h - \widehat{\mathsf{u}}_h\|_{a,h}\right) \lesssim h^{k+1}\|u\|_{H^{k+2}(\Omega)},}$$

with

$$\|\underline{\mathsf{v}}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{\mathsf{v}}_T\|_{1,T}^2.$$

Convergence II

Theorem (L^2-norm error estimate)

Further assuming elliptic regularity and $f \in H^1(\Omega)$ if $k = 0$,

$$\boxed{\max\left(\|\check{u}_h - u\|, \|\widehat{u}_h - u_h\|\right) \lesssim h^{k+2}\mathcal{N}_k,}$$

with $\mathcal{N}_0 := \|f\|_{H^1(\Omega)}$, $\mathcal{N}_k := \|u\|_{H^{k+2}(\mathcal{T}_h)}$ if $k \geqslant 1$, and

$$\forall T \in \mathcal{T}_h, \qquad \check{u}_{h|T} := p_T^k \underline{\mathsf{u}}_T, \quad \widehat{u}_{h|T} := p_T^k \underline{\mathsf{I}}_T^k u, \quad u_{h|T} := \mathsf{u}_T.$$

Numerical example

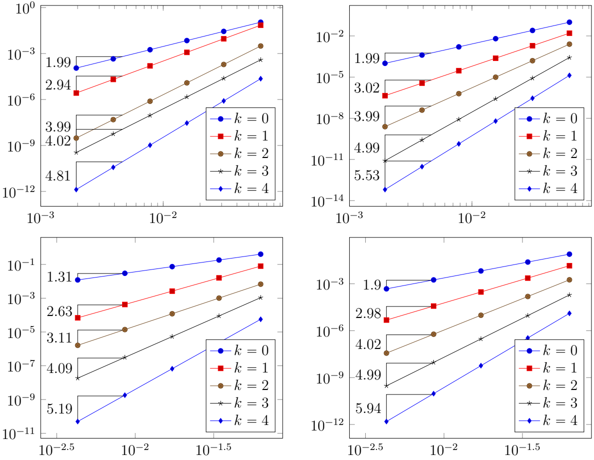


Figure : Energy (left) and L^2-norm (right) of the error vs. h for uniformly refined triangular (top) and hexagonal (bottom) mesh families

Outline

Variable diffusion I

- Let $\boldsymbol{\kappa} : \Omega \to \mathbb{R}^{d\times d}$ be a SPD tensor-valued field
- We consider the variable diffusion problem

$$\begin{aligned} -\boldsymbol{\nabla}\cdot(\boldsymbol{\kappa}\boldsymbol{\nabla} u) &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- In weak form: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u,v) := (\boldsymbol{\kappa}\boldsymbol{\nabla} u, \boldsymbol{\nabla} v) = (f,v) \qquad \forall v \in H_0^1(\Omega)$$

- We confer built-in homogeneization features to p_T^k

$$\boxed{(\boldsymbol{\kappa}\boldsymbol{\nabla} p_T^k \underline{\mathsf{v}}_T, \boldsymbol{\nabla} w)_T = (\boldsymbol{\kappa}\boldsymbol{\nabla}\mathsf{v}_T, \boldsymbol{\nabla} w)_T + \sum_{F\in\mathcal{F}_T} (\mathsf{v}_F - \mathsf{v}_T, \boldsymbol{\nabla} w\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF})_F}$$

Variable diffusion II

Lemma (Approximation properties of $p_T^k \underline{\mathrm{I}}_T^k$)

There is C independent of h_T and $\boldsymbol{\kappa}$ s.t., for all $v \in H^{k+2}(T)$, it holds with $\alpha = \frac{1}{2}$ if $\boldsymbol{\kappa}$ is piecewise constant and $\alpha = 1$ otherwise:

$$\boxed{\|v - p_T^k \underline{\mathrm{I}}_T^k v\|_T + h_T \|\boldsymbol{\nabla}(v - p_T^k \underline{\mathrm{I}}_T^k v)\|_T \leqslant C \rho_T^\alpha h_T^{k+2} \|v\|_{k+2,T},}$$

with heterogeneity/anisotropy ratio

$$\rho_T := \frac{\kappa_T^\sharp}{\kappa_T^\flat} \geqslant 1.$$

Discrete problem and convergence I

- We define the local bilinear form $a_{\boldsymbol{\kappa},T}$ on $\underline{\mathsf{U}}_T^k \times \underline{\mathsf{U}}_T^k$ as

$$a_{\boldsymbol{\kappa},T}(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) := (\boldsymbol{\kappa}\boldsymbol{\nabla} p_T^k \underline{\mathsf{u}}_T, \boldsymbol{\nabla} p_T^k \underline{\mathsf{v}}_T)_T + s_{\boldsymbol{\kappa},T}(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T)$$

where, letting $\kappa_F := \|\boldsymbol{n}_{TF}\cdot\boldsymbol{\kappa}\cdot\boldsymbol{n}_{TF}\|_{L^\infty(F)}$,

$$s_{\boldsymbol{\kappa},T}(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) := \sum_{F\in\mathcal{F}_T} \frac{\kappa_F}{h_F} (\pi_F^k(P_T^k \underline{\mathsf{u}}_T - \mathsf{u}_F), \pi_F^k(P_T^k \underline{\mathsf{v}}_T - \mathsf{v}_F))_F$$

- The discrete problem reads: Find $\underline{\mathsf{u}}_h \in \underline{\mathsf{U}}_{h,0}^k$ s.t.

$$\boxed{a_{\boldsymbol{\kappa},h}(\underline{\mathsf{u}}_h, \underline{\mathsf{v}}_h) := \sum_{T\in\mathcal{T}_h} a_{\boldsymbol{\kappa},T}(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) = \sum_{T\in\mathcal{T}_h} (f, \mathsf{v}_T)_T \quad \forall \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_{h,0}^k}$$

Discrete problem and convergence II

Theorem (Energy-error estimate)

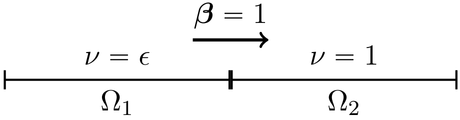
Assume that $u \in H^{k+2}(\mathcal{T}_h)$. Then, with $\widehat{\underline{\mathsf{u}}}_h$ and α as above,

$$\|\widehat{\underline{\mathsf{u}}}_h - \underline{\mathsf{u}}_h\|_{\boldsymbol{\kappa},h} \lesssim \left\{ \sum_{T\in\mathcal{T}_h} \kappa_T^\sharp \rho_T^{1+2\alpha} h_T^{2(k+1)} \|u\|_{k+2,T}^2 \right\}^{1/2}.$$

Outline

Degenerate diffusion-advection-reaction I

- Let us start by the following 1d problem:



- As $\epsilon \to 0^+$, a boundary layer develops at $x = 1/2$
- When $\epsilon = 0$, it turns into a jump discontinuity
- This was already observed in [Gastaldi and Quarteroni, 1989]

Degenerate diffusion-advection-reaction II

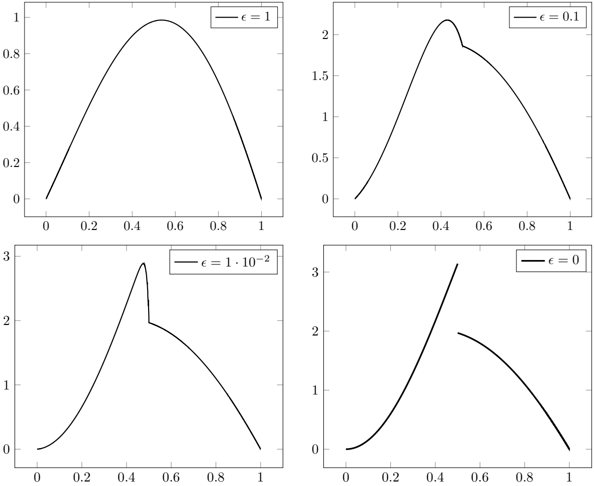


Figure : Solutions for different values of ϵ

Degenerate diffusion-advection-reaction III

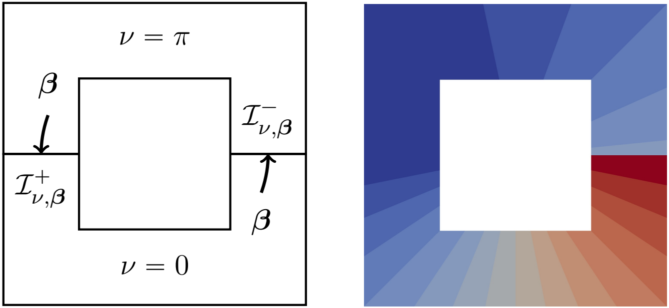


Figure : Example of degenerate diffusion-advection-reaction problem in 2d from [Di Pietro et al., 2008]. The diffusive/non-diffusive interface is $\mathcal{I}_{\nu,\boldsymbol{\beta}} := \mathcal{I}^-_{\nu,\boldsymbol{\beta}} \cup \mathcal{I}^+_{\nu,\boldsymbol{\beta}}$.

Degenerate diffusion-advection-reaction IV

- Define the diffusive/inflow portion of $\partial\Omega$

$$\Gamma_{\nu,\boldsymbol{\beta}} := \{\boldsymbol{x} \in \partial\Omega \mid \nu > 0 \text{ or } \boldsymbol{\beta}\cdot\boldsymbol{n} < 0\}$$

- Consider the possibly degenerate problem

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{\Phi}(u) + \mu u &= f && \text{in } \Omega\backslash\mathcal{I}_{\nu,\boldsymbol{\beta}}, \\ \boldsymbol{\Phi}(u) &= -\nu\boldsymbol{\nabla}u + \boldsymbol{\beta}u && \text{in } \Omega, \\ u &= g && \text{on } \Gamma_{\nu,\boldsymbol{\beta}}, \end{aligned}$$

supplemented with the interface conditions on $\mathcal{I}_{\nu,\boldsymbol{\beta}}$

$$[\![\boldsymbol{\Phi}(u)]\!]\cdot\boldsymbol{n}_I = 0 \text{ on } \mathcal{I}_{\nu,\boldsymbol{\beta}} \quad \text{and} \quad [\![u]\!] = 0 \text{ on } \mathcal{I}^{+}_{\nu,\boldsymbol{\beta}}$$

Key ideas

- Discrete advective derivative satisfying a discrete IBP formula
- Weakly enforced boundary conditions
  - Extension of Nietsche's ideas to HHO
  - Automatic detection of $\Gamma_{\nu,\boldsymbol{\beta}}$
- Upwind stabilization using cell- and face-unknowns
  - Independent control for the advective part
  - Consistency also on $\mathcal{I}^{-}_{\nu,\boldsymbol{\beta}}$, where u jumps

Features

- Polyhedral meshes and arbitrary approximation order $k \geqslant 0$
- Method valid for the full range of Peclet numbers
- Analysis capturing the variation in the order of convergence in the diffusion-dominated and advection-dominated regimes
- **No need to duplicate interface unknowns on $\mathcal{I}^-_{\nu,\boldsymbol{\beta}}$ (!)**

Advective derivative I

- The discrete advective derivative $G^k_{\boldsymbol{\beta},T} : \underline{\mathsf{U}}^k_T \to \mathbb{P}^k_d(T)$ is s.t.

$$\boxed{(G^k_{\boldsymbol{\beta},T}\underline{\mathsf{v}}_T, w)_T = -(\mathsf{v}_T, \boldsymbol{\beta}\cdot\boldsymbol{\nabla} w)_T + \sum_{F\in\mathcal{F}_T} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})\mathsf{v}_F, w)_F}$$

  for all $\underline{\mathsf{v}}_T \in \underline{\mathsf{U}}^k_T$ and all $w \in \mathbb{P}^k_d(T)$
- For advective stability, we need a discrete IBP mimicking

$$(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} w, v)_\Omega + (w, \boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)_\Omega = ((\boldsymbol{\beta}\cdot\boldsymbol{n})w, v)_{\partial\Omega}$$

Advective derivative II

Lemma (Discrete IBP)

For all $\underline{\mathsf{w}}_h, \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$ *it holds*

$$\sum_{T\in\mathcal{T}_h} \left\{ (G_{\boldsymbol{\beta},T}^k \underline{\mathsf{w}}_T, \mathsf{v}_T)_T + (\mathsf{w}_T, G_{\boldsymbol{\beta},T}^k \underline{\mathsf{v}}_T)_T \right\} = \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_F)\mathsf{w}_F, \mathsf{v}_F)_F - \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_h} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})(\mathsf{w}_F - \mathsf{w}_T), \mathsf{v}_F - \mathsf{v}_T)_F.$$

Diffusion I

- We modify the diffusion bilinear form to weakly enforce BCs
- The new bilinear form $a_{\nu,h}$ reads (after setting $\boldsymbol{\kappa} = \nu \boldsymbol{I}_d$),

$$a_{\nu,h}(\underline{\mathsf{w}}_h, \underline{\mathsf{v}}_h) := \sum_{T \in \mathcal{T}_h} a_{\nu,T}(\underline{\mathsf{w}}_T, \underline{\mathsf{v}}_T) + s_{\partial,\nu,h}(\underline{\mathsf{w}}_h, \underline{\mathsf{v}}_h)$$

with, for a user-defined parameter ς,

$$s_{\partial,\nu,h}(\underline{\mathsf{w}}_h, \underline{\mathsf{v}}_h) := \sum_{F \in \mathcal{F}_h^{\mathrm{b}}} \left\{ -(\nu_F \boldsymbol{\nabla} p_{T(F)}^k \underline{\mathsf{w}}_T \cdot \boldsymbol{n}_{TF}, \mathsf{v}_F)_F + \frac{\varsigma \nu_F}{h_F} (\mathsf{w}_F, \mathsf{v}_F)_F \right\}$$

Diffusion II

Lemma (inf-sup stability of $a_{\nu,h}$)

Assuming that

$$\varsigma > \frac{C_{\rm tr}^2 N_\partial}{4}$$

it holds for all $\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$

$$a_{\nu,h}(\underline{\mathsf{v}}_h, \underline{\mathsf{v}}_h) =: \|\underline{\mathsf{v}}_h\|_{\nu,h}^2 \simeq \sum_{T\in\mathcal{T}_h} \nu_T \|\underline{\mathsf{v}}_T\|_{1,T}^2 + \sum_{F\in\mathcal{F}_h^{\rm b}} \frac{\nu_F}{h_F} \|\mathsf{v}_F\|_F^2.$$

Advection-reaction I

- For all $T \in \mathcal{T}_h$, we let

$$a_{\boldsymbol{\beta},\mu,T}(\underline{\mathsf{w}}_T, \underline{\mathsf{v}}_T) := -(\mathsf{w}_T, G^k_{\boldsymbol{\beta},T}\underline{\mathsf{v}}_T)_T + \mu(\mathsf{w}_T, \mathsf{v}_T)_T + s^-_{\boldsymbol{\beta},T}(\underline{\mathsf{w}}_T, \underline{\mathsf{v}}_T)$$

  with local upwind stabilization bilinear form s.t.

$$s^-_{\boldsymbol{\beta},T}(\underline{\mathsf{w}}_T, \underline{\mathsf{v}}_T) := \sum_{F \in \mathcal{F}_T} ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})^- (\mathsf{w}_F - \mathsf{w}_T), \mathsf{v}_F - \mathsf{v}_T)_F,$$

- Including weak enforcement of BCs, we let

$$\boxed{a_{\boldsymbol{\beta},\mu,h}(\underline{\mathsf{w}}_h, \underline{\mathsf{v}}_h) := \sum_{T \in \mathcal{T}_h} a_{\boldsymbol{\beta},\mu,T}(\underline{\mathsf{w}}_h, \underline{\mathsf{v}}_h) + \sum_{F \in \mathcal{F}_h^{\mathrm{b}}} ((\boldsymbol{\beta}\cdot\boldsymbol{n})^+ \mathsf{w}_F, \mathsf{v}_F)_F}$$

Advection-reaction II

Lemma (Stability of $a_{\boldsymbol{\beta},\mu,h}$)

Let $\eta := \min_{T\in\mathcal{T}_h}(1,\tau_{\mathrm{ref},T}\mu)$ *with* $\tau_{\mathrm{ref},T} := \{\max(\|\mu\|_{L^\infty(T)}, L_{\boldsymbol{\beta},T})\}^{-1}$. *Then,*

$$\forall \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k, \qquad \eta\|\underline{\mathsf{v}}_h\|_{\boldsymbol{\beta},\mu,h}^2 \leqslant a_{\boldsymbol{\beta},\mu,h}(\underline{\mathsf{v}}_h,\underline{\mathsf{v}}_h),$$

with global advection-reaction norm

$$\|\underline{\mathsf{v}}_h\|_{\boldsymbol{\beta},\mu,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{\mathsf{v}}_T\|_{\boldsymbol{\beta},\mu,T}^2 + \frac{1}{2}\sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \||\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF}|^{1/2}\mathsf{v}_F\|_F^2,$$

and, for all $T\in\mathcal{T}_h$,

$$\|\underline{\mathsf{v}}_T\|_{\boldsymbol{\beta},\mu,T}^2 := \frac{1}{2}\sum_{F\in\mathcal{F}_T} \||\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF}|^{1/2}(\mathsf{v}_F-\mathsf{v}_T)\|_F^2 + \tau_{\mathrm{ref},T}^{-1}\|\mathsf{v}_T\|_T^2.$$

Discrete problem I

- Let, accounting for boundary conditions,

$$l_h(\underline{\mathsf{v}}_h) := \sum_{T\in\mathcal{T}_h} (f, \mathsf{v}_T)_T + \sum_{F\in\mathcal{F}_h^{\mathrm{b}}} \left\{ ((\boldsymbol{\beta}\cdot\boldsymbol{n}_{TF})^- g, \mathsf{v}_F)_F + \frac{\nu_F \varsigma}{h_F}(g, \mathsf{v}_F)_F \right\}$$

- The discrete problem reads: Find $\underline{\mathsf{u}}_h \in \underline{\mathsf{U}}_h^k$ s.t., $\forall \underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k$,

$$\boxed{a_h(\underline{\mathsf{u}}_h, \underline{\mathsf{v}}_h) := a_{\nu,h}(\underline{\mathsf{u}}_h, \underline{\mathsf{v}}_h) + a_{\boldsymbol{\beta},\mu,h}(\underline{\mathsf{u}}_h, \underline{\mathsf{v}}_h) = l_h(\underline{\mathsf{v}}_h)}$$

Discrete problem II

Lemma (Stability of a_h)

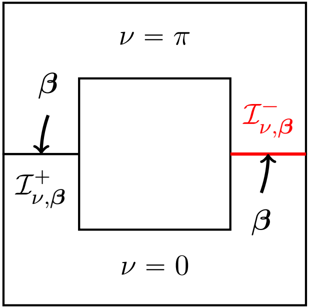
There is $\gamma_{\varrho,\varsigma} > 0$ independent of h, ν, $\boldsymbol{\beta}$ and μ s.t., for all $\underline{\mathsf{w}}_h \in \underline{\mathsf{U}}_h^k$,

$$\|\underline{\mathsf{w}}_h\|_{\sharp,h} \leqslant \gamma_{\varrho,\varsigma}\zeta^{-1} \sup_{\underline{\mathsf{v}}_h \in \underline{\mathsf{U}}_h^k \setminus \{\mathsf{0}\}} \frac{a_h(\underline{\mathsf{w}}_h, \underline{\mathsf{v}}_h)}{\|\underline{\mathsf{v}}_h\|_{\sharp,h}},$$

with $\zeta := \tau_{\mathrm{ref},T}\mu$ and stability norm

$$\|\underline{\mathsf{v}}_h\|_{\sharp,h}^2 := \|\underline{\mathsf{v}}_h\|_{\nu,h}^2 + \|\underline{\mathsf{v}}_h\|_{\boldsymbol{\beta},\mu,h}^2 + \sum_{T\in\mathcal{T}_h} h_T \beta_{\mathrm{ref},T}^{-1} \|G_{\boldsymbol{\beta},T}^k \underline{\mathsf{v}}_h\|_T^2$$

A modified interpolator



- Let $F \in \mathcal{F}_h^{\mathrm{i}}$ be such that $F \subset \mathcal{I}^-_{\nu,\boldsymbol{\beta}}$
- The trace of u is two-valued on F
- We interpolate the face unknown **from the diffusive side**

Convergence I

Theorem (Error estimate)

Assume that, for all $T \in \mathcal{T}_h$, $u \in H^{k+2}(T)$ *and*

$$h_T L_{\boldsymbol{\beta},T} \leqslant \beta_{\mathrm{ref},T} \quad \textit{and} \quad h_T \mu \leqslant \beta_{\mathrm{ref},T},$$

Then, there is $C > 0$ *independent of* h, ν, $\boldsymbol{\beta}$, *and* μ *s.t.*

$$\|\widehat{\underline{\mathsf{u}}}_h - \underline{\mathsf{u}}_h\|_{\sharp,h} \leqslant C \left\{ \sum_{T \in \mathcal{T}_h} \left[(\nu_T \|u\|_{k+2,T}^2 + \tau_{\mathrm{ref},T}^{-1} \|u\|_{k+1,T}^2) h_T^{2(k+1)} + \beta_{\mathrm{ref},T} \min(1, \mathrm{Pe}_T) h_T^{2(k+1/2)} \|u\|_{k+1,T}^2 \right] \right\}^{1/2},$$

where $\mathrm{Pe}_T = \max_{F \in \mathcal{F}_T} \|\mathrm{Pe}_{TF}\|_{L^\infty(F)}$.

Convergence II

- This estimate holds across the entire range for Pe_T
- In the diffusion-dominated regime ($\mathrm{Pe}_T \leqslant h_T$), we have

$$\|\widehat{\underline{\mathsf{u}}}_h - \underline{\mathsf{u}}_h\|_{\sharp,h} = \mathcal{O}(h^{k+1})$$

- In the advection-dominated regime ($\mathrm{Pe}_T \geqslant 1$), we have

$$\|\widehat{\underline{\mathsf{u}}}_h - \underline{\mathsf{u}}_h\|_{\sharp,h} = \mathcal{O}(h^{k+1/2})$$

- In between, we have intermediate orders of convergence

Numerical example I

- Let $\Omega = (-1,1)^2 \backslash [-0.5, 0.5]^2$ and set

$$\nu(\theta, r) = \begin{cases} \pi & \text{if } 0 < \theta < \pi, \\ 0 & \text{if } \pi < \theta < 2\pi, \end{cases} \quad \boldsymbol{\beta}(\theta, r) = \frac{\boldsymbol{e}_\theta}{r}, \quad \mu = 1 \cdot 10^{-6}$$

- We consider the exact solution

$$u(\theta, r) = \begin{cases} (\theta - \pi)^2 & \text{if } 0 < \theta < \pi \\ 3\pi(\theta - \pi) & \text{if } \pi < \theta < 2\pi \end{cases}$$

Numerical example II

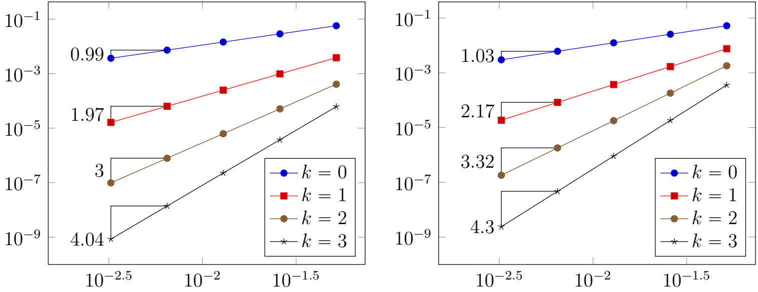


Figure : Energy (left) and L^2-norm (right) of the error vs. h

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