Hybrid and mixed high-order methods

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References used in this presentation

- For hybrid high-order (HHO)
  - [Di Pietro, Ern, and Lemaire, 2014]
  - complements: [Di Pietro and Ern, 2015c, Di Pietro and Ern, 2015a]
- For mixed high-order (MHO)
  - [Di Pietro and Ern, 2013]
  - complements: [Aghili, Boyaval, and Di Pietro, 2014]
- Links with other methods
  - VEM [Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, 2013]
  - HOM/NCVEM [Ayuso de Dios, Lipnikov, and Manzini, 2014]
  - $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming VEM [Beirão da Veiga, Brezzi, Marini, Russo, 2014]
  - HDG [Cockburn et al., 2009, Lehrenfeld, 2010]
  - MHM [Harder, Paredes, Valentin, 13]

Mesh

Definition (Mesh regularity)

We consider a sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of polyhedral meshes s.t., for all $h \in \mathcal{H}$, $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{T}_h$ and $(\mathfrak{T}_h)_{h\in\mathcal{H}}$ is

- shape-regular in the sense of Ciarlet;
- contact-regular: every simplex $S \subset T$ is s.t. $h_S \approx h_T$.

Main consequences

- Trace and inverse inequalities
- Optimal approximation for broken polynomial spaces

Setting

- Let $\Omega \subset \mathbb{R}^d$, $d \geqslant 1$, be a bounded connected polyhedral domain
- For a given $f \in L^2(\Omega)$, we consider the model problem

$$\begin{aligned} -\triangle u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

- The weak formulation reads: Find $u \in V := H_0^1(\Omega)$ s.t.

$$a(u, v) = (f, v) \qquad \forall v \in V$$

where $a(u, v) := (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v)$

Outline

Outline

DOFs and reduction map I

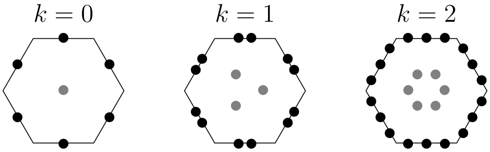

Figure : Degrees of freedom (DOFs)

- For $k \geqslant 0$ and all $T \in \mathcal{T}_h$, we define the local space of DOFs

$$\underline{U}_T^k := \mathbb{P}_d^k(T) \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}$$

- The global space is obtained enforcing single-valuedness at interfaces

$$\underline{U}_h^k := \left\{ \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}_d^k(T) \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}_{d-1}^k(F) \right\}$$

DOFs and reduction map II

- For a collection of DOFs $\underline{v}_h \in \underline{U}_h^k$ we use the underlined notation

$$\underline{v}_h = \left((v_T)_{T\in\mathcal{T}_h}, (v_F)_{F\in\mathcal{F}_h}\right)$$

- For all $T \in \mathcal{T}_h$, $\underline{v}_T \in \underline{U}_T^k$ denotes its restriction to $\underline{U}_T^k$ s.t.

$$\underline{v}_T = \left(v_T, (v_F)_{F\in\mathcal{F}_T}\right)$$

- Finally, we define the local reduction map $\underline{I}_T^k : H^1(T) \to \underline{U}_T^k$ s.t.

$$\underline{I}_T^k : v \mapsto \left(\pi_T^k v, (\pi_F^k v)_{F\in\mathcal{F}_T}\right)$$

Local potential reconstruction I

- Let $T \in \mathcal{T}_h$. The local potential reconstruction operator

$$p_T^k : \underline{U}_T^k \to \mathbb{P}_d^{k+1}(T)$$

is s.t. for all $\underline{v}_T \in \underline{U}_T^k$ and $w \in \mathbb{P}_d^{k+1}(T)$, $\int_T p_T^k \underline{v}_T = \int_T v_T$ and

$$\begin{aligned}(\nabla p_T^k \underline{v}_T, \nabla w)_T &:= (\nabla v_T, \nabla w)_T + \sum_{F \in \mathcal{F}_T} (v_F - v_T, \nabla w \cdot \boldsymbol{n}_{TF})_F \\ &= -(v_T, \triangle w)_T + \sum_{F \in \mathcal{F}_T} (v_F, \nabla w \cdot \boldsymbol{n}_{TF})_F\end{aligned}$$

- SPD linear system of size $\binom{k+1+d}{k+1}$

Local potential reconstruction II

Lemma (Approximation properties for $p_T^k \underline{I}_T^k$)

For all $v \in H^{k+2}(T)$, it holds

$$\|v - p_T^k \underline{I}_T^k v\|_T + h_T \|\nabla(v - p_T^k \underline{I}_T^k v)\|_T \lesssim h_T^{k+2} \|v\|_{H^{k+2}(T)}.$$

- Let $v \in H^{k+2}(T)$. Since $\triangle w \in \mathbb{P}_d^{k-1}(T)$ and $\nabla w_{|F} \cdot \boldsymbol{n}_{TF} \in \mathbb{P}_{d-1}^k(F)$,

$$\begin{aligned}(\nabla p_T^k \underline{I}_T^k v, \nabla w)_T &= -(\pi_T^k v, \triangle w)_T + \sum_{F \in \mathcal{F}_T} (\pi_F^k v, \nabla w \cdot \boldsymbol{n}_{TF})_F \\ &= -(v, \triangle w)_T + \sum_{F \in \mathcal{F}_T} (v, \nabla w \cdot \boldsymbol{n}_{TF})_F = (\nabla v, \nabla w)_T\end{aligned}$$

- Hence, $p_T^k \underline{I}_T^k$ is the elliptic projector on $\mathbb{P}_d^{k+1}(T)$ since

$$(\nabla v - \nabla p_T^k \underline{I}_T^k v, \nabla w)_T = 0 \qquad \forall w \in \mathbb{P}_d^{k+1}(T)$$

Stabilization I

- Define, for $T \in \mathcal{T}_h$, the stabilization bilinear form s_T^{hho} as

$$s_T^{\mathrm{hho}}(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (\pi_F^k (P_T^k \underline{u}_T - u_F), \pi_F^k (P_T^k \underline{v}_T - v_F))_F,$$

with local potential reconstruction $P_T^k : \underline{U}_T^k \to \mathbb{P}_d^{k+1}(T)$ s.t.

$$P_T^k \underline{v}_T := v_T + (p_T^k \underline{v}_T - \pi_T^k p_T^k \underline{v}_T)$$

where v_T is corrected using the highest-order part of $p_T^k \underline{v}_T$

Stabilization II

Lemma (Stability and continuity)

For all $T \in \mathcal{T}_h$, define the $H^1(T)$-like local seminorm

$$\|\underline{v}_T\|_{1,T}^2 := \|\boldsymbol{\nabla} v_T\|_T^2 + \sum_{F \in \mathcal{F}_T} h_F^{-1} \|v_F - v_T\|_F^2.$$

Then, for all $\underline{v}_T \in \underline{U}_T^k$, it holds that

$$\|\underline{v}_T\|_{1,T}^2 \lesssim \underbrace{\|\boldsymbol{\nabla} p_T^k \underline{v}_T\|_T^2 + s_T^{\rm hho}(\underline{v}_T, \underline{v}_T)}_{:= \|\underline{v}_T\|_{a,T}^2} \lesssim \|\underline{v}_T\|_{1,T}^2.$$

Stabilization III

Lemma (Consistency of s_T^{hho})

For all $T \in \mathcal{T}_h$ and all $v \in H^{k+2}(T)$, it holds

$$\left\{ \|\boldsymbol{\nabla} p_T^k \underline{I}_T^k v - \boldsymbol{\nabla} v\|_T^2 + s_T^{\mathrm{hho}}(\underline{I}_T^k v, \underline{I}_T^k v) \right\}^{1/2} \lesssim h_T^{k+1} \|v\|_{H^{k+2}(T)}.$$

- We have

$$\begin{aligned}
\|\pi_F^k(P_T^k \underline{I}_T^k v - \pi_F^k v)\|_F &= \|\pi_F^k\big(\pi_T^k v - \pi_T^k p_T^k \underline{I}_T^k v + p_T^k \underline{I}_T^k v - \pi_F^k v\big)\|_F \\
&= \|\pi_F^k\big(p_T^k \underline{I}_T^k v - v\big) + \pi_T^k\big(v - p_T^k \underline{I}_T^k v\big)\|_F \\
&\leqslant \|\pi_F^k\big(p_T^k \underline{I}_T^k v - v\big)\|_F + \|\pi_T^k\big(v - p_T^k \underline{I}_T^k v\big)\|_F \\
&\lesssim h_T^{-1/2} \|p_T^k \underline{I}_T^k v - v\|_T
\end{aligned}$$

- The result follows from the approximation properties of p_T^k

Stabilization IV

- One could wonder why we did not use, in place of s_T^{hho},

$$j_T(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{-1} (u_T - u_F, v_T - v_F)_F$$

- This choice yields in fact similar stability and boundedness results
- However, j_T has suboptimal approximation properties w.r.t. to p_T^k

$$j_T(\underline{I}_T^k v, \underline{I}_T^k v)^{1/2} \lesssim h_T^k \|v\|_{H^{k+1}(T)},$$

Discrete problem I

- We define the local bilinear form a_T on $\underline{U}_T^k \times \underline{U}_T^k$ as

$$a_T(\underline{u}_T, \underline{v}_T) := (\boldsymbol{\nabla} p_T^k \underline{u}_T, \boldsymbol{\nabla} p_T^k \underline{v}_T)_T + s_T^{\text{hho}}(\underline{u}_T, \underline{v}_T)$$

- The discrete problem reads: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ s.t., $\forall \underline{v}_h \in \underline{U}_{h,0}^k$,

$$\boxed{a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \sum_{T \in \mathcal{T}_h} (f, v_T)_T,}$$

with $\underline{U}_{h,0}^k := \left\{ \underline{v}_h \in \underline{U}_h^k \mid v_F \equiv 0 \; \forall F \in \mathcal{F}_h^{\text{b}} \right\}$ incorporating bcs

Discrete problem II

Lemma (Well-posedness)

The map $\underline{U}_h^k \ni \underline{v}_h \mapsto \|\underline{v}_h\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\underline{v}_h\|_{1,T}^2$ *is a norm on* $\underline{U}_{h,0}^k$ *and*

$$\|\underline{v}_h\|_{1,h}^2 \lesssim a_h(\underline{v}_h,\underline{v}_h) \lesssim \|\underline{v}_h\|_{1,h}^2 \qquad \forall \underline{v}_h \in \underline{U}_{h,0}^k.$$

As a consequence, the discrete problem is well-posed.

Error estimates I

Theorem (Energy-norm error estimate)

Assume $u \in H_0^1(\Omega) \cap H^{k+2}(\Omega)$. *With* $\underline{\widehat{u}}_h \in \underline{U}_{h,0}^k$ *s.t.* $\underline{\widehat{u}}_T = \underline{I}_T^k u \ \forall T \in \mathcal{T}_h$,

$$\|\underline{u}_h - \underline{\widehat{u}}_h\|_{a,h} \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)}.$$

Corollary (Energy estimate for $p_T^k \underline{u}_T$)

With $\breve{u}_h \in \mathbb{P}_d^{k+1}(\mathcal{T}_h)$ *s.t.* $\breve{u}_{h|T} = p_T^k \underline{u}_T$ *for all* $T \in \mathcal{T}_h$, *it holds*

$$\|\boldsymbol{\nabla}(u - \breve{u}_h)\| \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)}.$$

Error estimates II

Theorem (Supercloseness of the potential)

Further assuming elliptic regularity and $f \in H^1(\Omega)$ if $k = 0$, we have

$$\left\{ \sum_{T \in \mathcal{T}_h} \|u_T - \widehat{u}_T\|_T^2 \right\}^{1/2} \lesssim h^{k+2} \mathcal{N}_k,$$

with $\mathcal{N}_k := \|u\|_{H^{k+2}(\Omega)}$ if $k \geqslant 1$ while $\mathcal{N}_k := \|f\|_{H^1(\Omega)}$ if $k = 0$.

Corollary (L^2-error estimate for $p_T^k \underline{u}_T$)

With $\breve{u}_h \in \mathbb{P}_d^{k+1}(\mathcal{T}_h)$ s.t. $\breve{u}_{h|T} = p_T^k \underline{u}_T$ for all $T \in \mathcal{T}_h$, it holds

$$\|u - \breve{u}_h\| \lesssim h^{k+2} \mathcal{N}_k.$$

Local conservation and numerical fluxes I

- Let $T \in \mathcal{T}_h$, $v_T \in \mathbb{P}_d^k(T)$. Partial integration yields

$$(f, v_T)_T = -(\triangle u, v_T)_T = (\boldsymbol{\nabla} u, \boldsymbol{\nabla} v_T)_T - \sum_{F \in \mathcal{F}_T} (\boldsymbol{\nabla} u {\cdot} \boldsymbol{n}_{TF}, v_T)_F$$

- Hence, denoting by $\Phi_{TF}(u) := -\boldsymbol{\nabla} u {\cdot} \boldsymbol{n}_{TF}$ the conservative flux,

$$(\boldsymbol{\nabla} u, \boldsymbol{\nabla} v_T)_T + \sum_{F \in \mathcal{F}_T} (\Phi_{TF}(u), v_T)_F = (f, v_T)_T$$

- Our goal is to identify a discrete counterpart of this relation for HHO
- The difficulty comes from the coupling among faces in s_T^{hho}

Local conservation and numerical fluxes II

- Define the auxiliary bilinear form $\widetilde{a}_T : \underline{U}_T^k \times \underline{U}_T^k \to \mathbb{R}$ s.t.

$$\widetilde{a}_T(\underline{u}_T, \underline{v}_T) := (\boldsymbol{\nabla} p_T^k \underline{u}_T, \boldsymbol{\nabla} p_T^k \underline{v}_T)_T + j_T(\underline{u}_T, \underline{v}_T)$$

where $j_T(\underline{u}_T, \underline{v}_T) := \sum_{F\in\mathcal{F}_T} h_F^{-1}(w_T - w_F, v_T - v_F)_F$

- For all $T \in \mathcal{T}_h$, we define the isomorphism $\underline{c}_T^k : \underline{U}_T^k \to \underline{U}_T^k$ s.t.,

$$\boxed{\widetilde{a}_T(\underline{c}_T^k \underline{u}_T, \underline{v}_T) = a_T(\underline{u}_T, \underline{v}_T) + j_T(\underline{u}_T, \underline{v}_T) \qquad \forall \underline{v}_T \in \underline{U}_T^k}$$

- We also introduce the discrete gradient $\boldsymbol{\mathcal{G}}_T^k : \underline{U}_T^k \to \boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T)$ s.t.

$$\boldsymbol{\mathcal{G}}_T^k := \boldsymbol{\nabla}(p_T^k \circ \underline{c}_T^k)$$

Local conservation and numerical fluxes III

Lemma (Discrete local conservation and convergence for $\underline{c}_T^k \underline{u}_T$)

The solution $\underline{u}_h \in \underline{U}_{h,0}^k$ *satisfies, for all* $\underline{v}_h \in \underline{U}_{h,0}^k$ *and all* $T \in \mathcal{T}_h$,

$$(\boldsymbol{\mathcal{G}}_T^k \underline{u}_T, \boldsymbol{\nabla} v_T)_T + \sum_{F \in \mathcal{F}_T} (\Phi_{TF}^{\rm hho}(\underline{u}_T), v_T - v_F)_F = (f, v_T)_T,$$

with conservative numerical flux $\Phi_{TF}^{\rm hho} : \underline{U}_T^k \to \mathbb{P}_{d-1}^k(F)$ *s.t.*

$$\boxed{\Phi_{TF}^{\rm hho}(\underline{u}_T) = -\boldsymbol{\mathcal{G}}_T^k \underline{u}_T \cdot \boldsymbol{n}_{TF} + h_F^{-1} \Big[\big(w_F - (\underline{c}_T^k \underline{u}_T)_F\big) - \big(w_T - (\underline{c}_T^k \underline{u}_T)_T\big) \Big].}$$

Additionally, $\boldsymbol{\mathcal{G}}_T^k \underline{u}_T$ *is optimally convergent.*

Variable diffusion I

- With $\boldsymbol{\nu}$ diffusion tensor, assume $(\mathcal{T}_h)_{h\in\mathcal{H}}$ compatible with $\boldsymbol{\nu}$
- Let us briefly consider the Darcy problem

$$-\boldsymbol{\nabla}\cdot(\boldsymbol{\nu}\boldsymbol{\nabla}u) = f \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega$$

- In weak form: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u,v) := (\boldsymbol{\nu}\boldsymbol{\nabla}u, \boldsymbol{\nabla}v) = (f,v) \qquad \forall v \in H_0^1(\Omega)$$

- We include built-in homogeneization in p_T^k

$$(\boldsymbol{\nu}\boldsymbol{\nabla}p_T^k\underline{v}_T, \boldsymbol{\nabla}w)_T = (\boldsymbol{\nu}\boldsymbol{\nabla}v_T, \boldsymbol{\nabla}w)_T + \sum_{F\in\mathcal{F}_T}(v_F - v_T, \boldsymbol{\nabla}w\cdot\boldsymbol{\nu}\cdot\boldsymbol{n}_{TF})_F$$

Variable diffusion II

Lemma (Approximation properties of $p_T^k \underline{I}_T^k$)

There is C independent of h_T and $\boldsymbol{\nu}$ s.t., for all $v \in H^{k+2}(T)$, it holds with $\alpha = \frac{1}{2}$ if $\boldsymbol{\nu}$ is piecewise constant and $\alpha = 1$ otherwise,

$$\|v - p_T^k \underline{I}_T^k v\|_T + h_T \|\boldsymbol{\nabla}(v - p_T^k \underline{I}_T^k v)\|_T \lesssim \rho_T^{\alpha} h_T^{k+2} \|v\|_{k+2,T},$$

with heterogeneity/anisotropy ratio $\rho_T := \nu_T^\sharp / \nu_T^\flat \geqslant 1$.

Variable diffusion III

We modify the local bilinear form as follows:

$$a_{\boldsymbol{\nu},T}(\underline{u}_T,\underline{v}_T) := (\boldsymbol{\nu}\boldsymbol{\nabla} p_T^k\underline{u}_T, \boldsymbol{\nabla} p_T^k\underline{v}_T)_T + s_{\boldsymbol{\nu},T}(\underline{u}_T,\underline{v}_T)$$

where, letting $\nu_{TF} := \|\boldsymbol{n}_{TF}\cdot\boldsymbol{\nu}_T\cdot\boldsymbol{n}_{TF}\|_{L^\infty(F)}$,

$$s_{\boldsymbol{\nu},T}(\underline{u}_T,\underline{v}_T) := \sum_{F\in\mathcal{F}_T}\frac{\nu_{TF}}{h_F}(\pi_F^k(P_T^k\underline{u}_T - u_F), \pi_F^k(P_T^k\underline{v}_T - v_F))_F$$

Theorem (Energy-error estimate)

Assume that $u \in H^{k+2}(\mathcal{T}_h)$. Then, with $\widehat{\underline{u}}_h$ and α as above,

$$\|\widehat{\underline{u}}_h - \underline{u}_h\|_{\boldsymbol{\nu},h} \lesssim \left\{\sum_{T\in\mathcal{T}_h}\nu_T^\sharp\rho_T^{1+2\alpha}h_T^{2(k+1)}\|u\|_{k+2,T}^2\right\}^{1/2}.$$

In summary

- Dimension-independent construction on general polyhedral meshes
- Arbitrary order (including $k = 0$)
- Naturally handles variable coefficients
- Possibility to incorporate advection robustly for $0 \leqslant \mathrm{Pe}_T \leqslant +\infty$
  - [Di Pietro et al., 2014b]
- Extension to a locking-free method for linear elasticity
  - [Di Pietro and Ern, 2015b]
- Conservative fluxes obtained via element-wise post-processing

Outline

Mixed high-order methods

- With $\boldsymbol{\nu}$ diffusion tensor, assume $(\mathcal{T}_h)_{h\in\mathcal{H}}$ compatible with $\boldsymbol{\nu}$
- Define the spaces

$$\boldsymbol{\Sigma} := \boldsymbol{H}(\mathrm{div};\Omega) \qquad U := L^2(\Omega)$$

- The mixed Darcy problem reads: Find $(\boldsymbol{\sigma}, u) \in \boldsymbol{\Sigma} \times U$ s.t.

$$\begin{aligned} (\boldsymbol{\nu}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) + (u, \boldsymbol{\nabla}\cdot\boldsymbol{\tau}) &= 0 \qquad && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma} \\ (\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}, v) &= -(f, v) && \forall v \in U \end{aligned}$$

- We separate the constitutive law from the equilibrium equation

DOFs and reduction map I

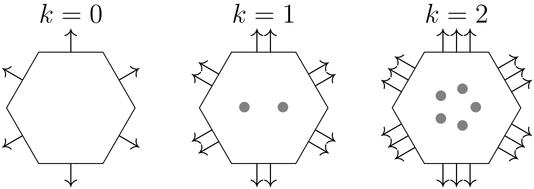

Figure : Flux DOFs for MHO

- Let, for $k \geqslant 0$, $\boldsymbol{\Gamma}_T^k := \boldsymbol{\nu}_T \boldsymbol{\nabla} \mathbb{P}_d^k$ and define the local spaces of DOFs

$$\underline{\boldsymbol{\Sigma}}_T^k := \boldsymbol{\Gamma}_T^k \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}, \qquad U_T^k := \mathbb{P}_d^k(T)$$

- The global space has single-valued flux unknowns at interfaces

$$\underline{\boldsymbol{\Sigma}}_h^k := \left\{ \bigtimes_{T \in \mathcal{T}_h} \boldsymbol{\Gamma}_T^k \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}_h} \mathbb{P}_{d-1}^k(F) \right\}, \qquad U_h^k := \bigtimes_{T \in \mathcal{T}_h} \mathbb{P}_d^k(T)$$

DOFs and reduction map II

- Let an element $T \in \mathcal{T}_h$ be fixed. For $s > 2$, we set

$$\boldsymbol{\Sigma}^+(T) := \{\boldsymbol{\tau} \in L^s(T)^d \mid \boldsymbol{\nabla}\cdot\boldsymbol{\tau} \in L^2(T)\}$$

- The reduction map $\underline{I}_{\boldsymbol{\Sigma},T}^k : \boldsymbol{\Sigma}^+(T) \to \underline{\boldsymbol{\Sigma}}_T^k$ is s.t., for all $\boldsymbol{\tau} \in \boldsymbol{\Sigma}^+(T)$,

$$\boxed{(\underline{I}_{\boldsymbol{\Sigma},T}^k\boldsymbol{\tau})_T = \boldsymbol{\nu}_T\boldsymbol{\nabla}v, \qquad (\underline{I}_{\boldsymbol{\Sigma},T}^k\boldsymbol{\tau})_F = \pi_F^k(\boldsymbol{\tau}\cdot\boldsymbol{n}_F) \quad \forall F \in \mathcal{F}_T,}$$

where $v \in \mathbb{P}_d^k(T)$ solves the following Neumann problem:

$$(\boldsymbol{\nu}_T\boldsymbol{\nabla}v, \boldsymbol{\nabla}w)_T = (\boldsymbol{\tau}, \boldsymbol{\nabla}w)_T \qquad \forall w \in \mathbb{P}_d^k(T)$$

Local flux reconstruction I

- The discrete divergence $D_T^k : \underline{\boldsymbol{\Sigma}}_T^k \to U_T^k$ is s.t., $\forall(\underline{\boldsymbol{\tau}}_T, v) \in \underline{\boldsymbol{\Sigma}}_T^k \times U_T^k$,

$$\boxed{(D_T^k \underline{\boldsymbol{\tau}}_T, v)_T = -(\boldsymbol{\nabla} v, \boldsymbol{\tau}_T)_T + \sum_{F \in \mathcal{F}_T} (v, \tau_F \epsilon_{TF})_F,}$$

  where $\epsilon_{TF} := \boldsymbol{n}_{TF} \cdot \boldsymbol{n}_F$
- By construction, we have the commuting property

$$\begin{array}{ccc} \boldsymbol{\Sigma}^+(T) & \xrightarrow{\boldsymbol{\nabla}\cdot} & L^2(T) \\ {\scriptstyle \underline{I}_{\boldsymbol{\Sigma},T}^k}\big\downarrow & & \big\downarrow{\scriptstyle \pi_T^k} \\ \underline{\boldsymbol{\Sigma}}_T^k & \xrightarrow{D_T^k} & U_T^k \end{array}$$

Local flux reconstruction II

- The consistent flux reconstruction $\mathfrak{C}_T^{k+1} : \underline{\mathbf{\Sigma}}_T^k \to \mathbf{\Gamma}_T^{k+1}$ is s.t.

$$\forall \underline{\boldsymbol{\tau}}_T \in \underline{\mathbf{\Sigma}}_T^k, \qquad \mathfrak{C}_T^{k+1} \underline{\boldsymbol{\tau}}_T = \boldsymbol{\nu}_T \boldsymbol{\nabla} v$$

where $v \in \mathbb{P}_d^{k+1}(T)$ solves: For all $w \in \mathbb{P}_d^{k+1}(T)$,

$$\boxed{(\mathfrak{C}_T^{k+1} \underline{\boldsymbol{\tau}}_T, \boldsymbol{\nabla} w)_T = (\boldsymbol{\nu}_T \boldsymbol{\nabla} v, \boldsymbol{\nabla} w)_T = -(w, D_T^k \underline{\boldsymbol{\tau}}_T)_T + \sum_{F \in \mathcal{F}_T} (w, \tau_F \epsilon_{TF})_F}$$

- SPD linear system of size $\binom{k+1+d}{k+1}$
- $\mathfrak{C}_T^{k+1}$ is polynomial-preserving up to degree $k+1$:

$$\boxed{\forall \boldsymbol{\tau} \in \mathbf{\Gamma}_T^{k+1}, \qquad \mathfrak{C}_T^{k+1}(\underline{I}_{\mathbf{\Sigma},T}^k \boldsymbol{\tau}) = \boldsymbol{\tau}}$$

Discrete $(\boldsymbol{\nu}^{-1}\cdot,\cdot)_T$-product and stabilization

- We define the following $(\boldsymbol{\nu}^{-1}\cdot,\cdot)_T$-product on $\underline{\boldsymbol{\Sigma}}_T^k$

$$\boxed{H_T(\underline{\boldsymbol{\sigma}}_T,\underline{\boldsymbol{\tau}}_T) := (\boldsymbol{\nu}_T^{-1}\mathfrak{C}_T^{k+1}\underline{\boldsymbol{\sigma}}_T, \mathfrak{C}_T^{k+1}\underline{\boldsymbol{\tau}}_T)_T + S_T(\underline{\boldsymbol{\sigma}}_T,\underline{\boldsymbol{\tau}}_T),}$$

where positive definiteness is ensured by the stabilizing bilinear form

$$S_T(\underline{\boldsymbol{\sigma}}_T,\underline{\boldsymbol{\tau}}_T) := \sum_{F\in\mathcal{F}_T}\frac{h_F}{\nu_{TF}}((\mathfrak{C}_T^{k+1}\underline{\boldsymbol{\sigma}}_T)\cdot\boldsymbol{n}_F-\sigma_F, (\mathfrak{C}_T^{k+1}\underline{\boldsymbol{\tau}}_T)\cdot\boldsymbol{n}_F-\tau_F)_F$$

- S_T enjoys polynomial consistency up to degree $(k+1)$:

$$\boxed{\forall\boldsymbol{\sigma}\in\boldsymbol{\Gamma}_T^{k+1},\qquad S_T(\underline{I}_{\boldsymbol{\Sigma},T}^k(\boldsymbol{\sigma}),\underline{\boldsymbol{\tau}}_T)=0\qquad\forall\underline{\boldsymbol{\tau}}_T\in\underline{\boldsymbol{\Sigma}}_T^k}$$

Discrete problem

- Find $(\underline{\boldsymbol{\sigma}}_h, u_h) \in \underline{\boldsymbol{\Sigma}}_h^k \times U_h^k$ s.t., $\forall (\underline{\boldsymbol{\tau}}_h, v_h) \in \underline{\boldsymbol{\Sigma}}_h^k \times U_h^k$, $\forall T \in \mathcal{T}_h$,

$$
\begin{aligned}
H_T(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_T) + (u_T, D_T^k \underline{\boldsymbol{\tau}}_T)_T &= 0 \\
(D_T^k \underline{\boldsymbol{\sigma}}_T, v_T)_T &= -(f, v_T)_T
\end{aligned}
$$

- Well-posedness follows from coercivity + LBB
- In what follows, we establish a link with HHO

Mixed hybrid formulation I

- For simplicity, let us take $\boldsymbol{\nu} \equiv \boldsymbol{I}_d$ in what follows
- We enforce single-valuedness at interfaces by Lagrange multipliers in

$$\Lambda_h^k := \bigtimes_{F \in \mathcal{F}_h} \Lambda_F^k,$$

with

$$\Lambda_F^k := \begin{cases} \mathbb{P}_{d-1}^k(F) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}} \\ \{0\} & \text{if } F \in \mathcal{F}_h^{\mathrm{b}} \end{cases}$$

- Recalling the definition of $\underline{U}_{h,0}^k$ from HHO, we can rewrite

$$\boxed{\underline{U}_{h,0}^k = U_h^k \times \Lambda_h^k}$$

Mixed hybrid formulation II

- Define the bilinear form $B_T \in \underline{\boldsymbol{\Sigma}}_T^k \times \underline{U}_T^k$ s.t.,

$$\boxed{B_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T) := (v_T, D_T^k \underline{\boldsymbol{\tau}}_T)_T - \sum_{F \in \mathcal{F}_T} (v_F, \tau_{TF})_F}$$

- Introducing the fully discontinuous flux space

$$\underline{\check{\boldsymbol{\Sigma}}}_h^k := \bigtimes_{T \in \mathcal{T}_h} \underline{\boldsymbol{\Sigma}}_T^k$$

we look for $(\underline{\boldsymbol{\sigma}}_h, \underline{u}_h) \in \underline{\check{\boldsymbol{\Sigma}}}_h^k \times \underline{U}_h^k$ s.t., $\forall (\underline{\boldsymbol{\tau}}_h, \underline{v}_h) \in \underline{\check{\boldsymbol{\Sigma}}}_h^k \times \underline{U}_h^k$, $\forall T \in \mathcal{T}_h$,

$$\boxed{\begin{aligned} H_T(\underline{\boldsymbol{\sigma}}_T, \underline{\boldsymbol{\tau}}_h) + B_T(\underline{\boldsymbol{\tau}}_T, \underline{u}_T) &= 0 \\ B_T(\underline{\boldsymbol{\sigma}}_T, \underline{v}_T) &= -(f, v_T)_T \end{aligned}}$$

Potential lifting I

- We define a potential lifting $\varsigma_T^k : \underline{U}_T^k \to \underline{\mathbf{\Sigma}}_T^k$ s.t., for all $\underline{\boldsymbol{\tau}}_T \in \underline{\mathbf{\Sigma}}_T^k$,

$$\boxed{H_T(\varsigma_T^k \underline{v}_T, \underline{\boldsymbol{\tau}}_T) = -B_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T)}$$

- Using the definition of D_T^k and partial integration, we have

$$\begin{aligned} B_T(\underline{\boldsymbol{\tau}}_T, \underline{v}_T) &:= (v_T, D_T^k \underline{\boldsymbol{\tau}}_T) - \sum_{F \in \mathcal{F}_T} (v_F, \tau_{TF})_F \\ &= -(\boldsymbol{\nabla} v_T, \boldsymbol{\tau}_T)_T + \sum_{F \in \mathcal{F}_T} (v_T - v_F, \tau_{TF})_F \end{aligned}$$

Potential lifting II

- Using the previous relation, and letting $\boldsymbol{G}_T^k : \underline{U}_T^k \to \mathbb{P}_d^{k+1}(T)$ be s.t.

$$\boldsymbol{G}_T^k := \mathfrak{C}_T^{k+1} \circ \varsigma_T^k,$$

we have from polynomial consistency: $\forall \underline{v}_T \in \underline{U}_T^k$, $\forall w \in \mathbb{P}_d^{k+1}(T)$,

$$\boxed{(\boldsymbol{G}_T^k \underline{v}_T, \boldsymbol{\nabla} w)_T = (\boldsymbol{\nabla} v_T, \boldsymbol{\nabla} w)_T + \sum_{F \in \mathcal{F}_T} (v_F - v_T, \boldsymbol{\nabla} w \cdot \boldsymbol{n}_{TF})_F}$$

- Comparing with HHO, it is readily inferred that

$$\boxed{\boldsymbol{G}_T^k \underline{v}_T = \boldsymbol{\nabla} p_T^k \underline{v}_T}$$

Primal formulation and link with HHO

- Let us go back to the mixed-hybrid formulation
- By definition of ς_T^k, the constitutive law rewrites

$$\boxed{\underline{\boldsymbol{\sigma}}_T = \varsigma_T^k \underline{u}_T \qquad \forall T \in \mathcal{T}_h}$$

- The equilibrium equation thus becomes: $\forall \underline{v}_h \in \underline{U}_{h,0}^k$, $\forall T \in \mathcal{T}_h$,

$$-B_T(\varsigma_T^k \underline{u}_T, \underline{v}_T) = (f, v_T)_T,$$

or, by definition of $\varsigma_T^k \underline{v}_T$, letting $s_T^{\mathrm{mho}}(\underline{u}_T, \underline{v}_T) := S_T(\varsigma_T^k \underline{u}_T, \varsigma_T^k \underline{v}_T)$,

$$\boxed{H_T(\varsigma_T^k \underline{u}_T, \varsigma_T^k \underline{v}_T) = \underbrace{(\boldsymbol{\nabla} p_T^k \underline{u}_T, \boldsymbol{\nabla} p_T^k \underline{v}_T) + s_T^{\mathrm{mho}}(\underline{u}_T, \underline{v}_T)}_{:= a_T^{\mathrm{mho}}(\underline{u}_T, \underline{v}_T)} = (f, v_T)_T}$$

- Thus, after hybridization, $\mathrm{MHO} \simeq \mathrm{HHO}$

Outline

Variants of HHO

- Let $T \in \mathcal{T}_h$, $k-1 \leqslant l \leqslant k+1$, and consider the local space

$$\underline{U}_T^{k,l} := \mathbb{P}_d^l(T) \times \left\{ \bigtimes_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}$$

- In all the cases, convergence rates as for the original HHO method
- We have that
  - $l = k-1$ yields HOM (up to variants in stabilization)
  - $l = k$ yields the original HHO method
  - $l = k+1$ yields a new HDG-like method

Bridging HOM and HHO I

- A (minor) extension of [Ayuso de Dios et al., 2014] bridges HHO and HOM when

$$\boldsymbol{\nu} = \boldsymbol{I}_d$$

- For a given $T \in \mathcal{T}_h$, define the local space

$$\boxed{V_T^{k,l} := \left\{ \varphi \in H^1(T) \mid \boldsymbol{\nabla}\varphi_{|F}\cdot\boldsymbol{n}_F \in \mathbb{P}^k_{d-1}(F)\ \forall F \in \mathcal{F}_T \text{ and } \triangle\varphi \in \mathbb{P}^l_d(T) \right\}}$$

- We next study the relation between $V_T^{k,l}$ and $\underline{U}_T^{k,l}$

Bridging HOM and HHO II

- Let $\Phi_T : \underline{U}_T^{k,l} \to V_T^{k,l}$ be s.t. $\Phi_T(\underline{v}_T)$ solves the Neumann problem

$$\boxed{\triangle \Phi_T(\underline{v}_T) = v_T - \frac{1}{|T|_d}\left(\int_T v_T - \sum_{F \in \mathcal{F}_T} \int_F v_F \right)}$$

with

$$\boldsymbol{\nabla}\Phi_T(\underline{v}_T)_{|F} \cdot \boldsymbol{n}_{TF} = v_F \quad \forall F \in \mathcal{F}_T, \qquad \int_T \Phi_T(\underline{v}_T) = \int_T v_T$$

- Clearly, both Φ_T and $\underline{I}_T^{k,l} : V_T^{k,l} \to \underline{U}_T^{k,l}$ are injective
- Therefore, $\underline{I}_T^{k,l} : V_T^{k,l} \to \underline{U}_T^{k,l}$ is an isomorphism and we can identify

$$\boxed{V_T^{k,l} \sim \underline{U}_T^{k,l}}$$

Bridging HOM and HHO III

- Define the computable projection $\Pi_T^{k+1} : V_T^{k,l} \to \mathbb{P}_d^{k+1}(T)$ s.t.

$$\boxed{\Pi_T^{k+1}\varphi := p_T^k \underline{I}_T^{k,l}\varphi}$$

- For all $F \in \mathcal{F}_T$, one can readily verify, with $\delta_{TF}^k\varphi := \pi_F^k\varphi - \pi_T^k\varphi$,

$$\pi_F^k(P_T^{k,l}\underline{I}_T^{k,l}\varphi - (\underline{I}_T^{k,l}\varphi)_F) = \delta_{TF}^k(\Pi_T^{k+1}\varphi - \varphi) \quad \forall\varphi \in V_T^{k,l}$$

- Therefore, the HHO stabilization is a least-square penalty of

$$\delta_{TF}^k(\Pi_T^{k+1}\varphi - \varphi)$$

Bridging HOM and HHO IV

- Define on $V_T^{k,l} \times V_T^{k,l}$ the local bilinear form

$$\widetilde{a}_T(\psi,\varphi) := (\boldsymbol{\nabla}\Pi_T^{k+1}\psi, \boldsymbol{\nabla}\Pi_T^{k+1}\varphi)_T + \widetilde{s}_T(\psi,\varphi)$$

  with stabilization bilinear form

$$\widetilde{s}_T(\psi,\varphi) := \sum_{F\in\mathcal{F}_T} h_F^{-1}(\delta_{TF}^k(\Pi_T^{k+1}\psi-\psi), \delta_{TF}^k(\Pi_T^{k+1}\varphi-\varphi))_F$$

- Then, using $V_T^{k,l} \sim \underline{U}_T^{k,l}$, with $\underline{w}_T = \underline{I}_T^{k,l}\psi$ and $\underline{v}_T = \underline{I}_T^{k,l}\varphi$,

$$\boxed{\widetilde{a}_T(\psi,\varphi) = a_T(\underline{w}_T,\underline{v}_T), \qquad \widetilde{s}_T(\psi,\varphi) = s_T(\underline{w}_T,\underline{v}_T)}$$

The case $l = k + 1$

- Let $k = l + 1$ in $\underline{U}_T^{k,l}$ and leave p_T^k unaltered. Then, P_T^k becomes

$$P_T^{k,l}\underline{v}_T = v_T + p_T^k\underline{v}_T - \pi_T^{k+1}p_T^k\underline{v}_T = v_T$$

- Hence, the stabilization bilinear form s_T simply rewrites

$$s_T^{\mathrm{hdg}}(\underline{u}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} \frac{1}{h_F}(\pi_F^k(u_T - u_F), \pi_F^k(v_T - v_F))_F$$

- This yields a (new) HDG-like method based on the local spaces

$$\boldsymbol{V}(T) = \boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T), \qquad W(T) = \mathbb{P}_d^{k+1}(T)$$

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