An extension of the Crouzeix–Raviart and Raviart–Thomas spaces to general meshes

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Outline

Quasi-compressible materials and numerical locking I

- Let $\Omega \subset \mathbb{R}^d$ denote a bounded polygonal or polyhedral domain
- We consider the linear elasticity equations

$$\begin{aligned} -\boldsymbol{\nabla}\cdot\underline{\underline{\sigma}}(\boldsymbol{u}) &= \boldsymbol{f} \qquad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \qquad \text{on } \partial\Omega, \end{aligned}$$

where, for $\mu,\ \lambda \in \mathbb{R}$, $\underline{\underline{\sigma}}(\boldsymbol{u})$ is the Cauchy stress tensor,

$$\underline{\underline{\sigma}}(\boldsymbol{u}) := 2\mu\underline{\underline{\epsilon}}(\boldsymbol{u}) + \lambda\boldsymbol{\nabla}\cdot\boldsymbol{u}\underline{\underline{I_d}}, \qquad \underline{\underline{\epsilon}}(\boldsymbol{u}) := \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^t\right)$$

- When $\lambda \to +\infty$, numerical locking can be observed
- To avoid locking: uniform convergence w.r.t. λ

Quasi-compressible materials and numerical locking II

- If Ω convex, in $d = 2$ there holds, cf. [Brenner and Sung, 1992],

$$\mathcal{N}_{\text{el}} := \left(\|\boldsymbol{u}\|^2_{H^2(\Omega)} + |\lambda \boldsymbol{\nabla}\cdot\boldsymbol{u}|^2_{H^1(\Omega)} \right)^{1/2} \leqslant \|\boldsymbol{f}\|_{L^2(\Omega)^d},$$

- Locking-free methods satisfy an error estimate of the form

$$\|\!|\boldsymbol{u} - \boldsymbol{u}_h|\!\|_{\text{el}} \leqslant C\mathcal{N}_{\text{el}} h, \qquad C \neq f(\lambda, h, \boldsymbol{u})$$

- Key point: approximation of non-trivial solenoidal fields
- Idea: extend the Crouzeix–Raviart space to general meshes

Essential bibliography

- Lowest-order methods on general meshes
  - Mimetic Finite Differences [Brezzi et al., 2005b, Brezzi et al., 2005a]
  - Hybrid/Mixed FV [Droniou and Eymard, 2006, Eymard et al., 2010]
  - Cell centered Galerkin [DP, 2010, DP, 2012, DP et al., 2012]
  - . . . and many more
- Linear elasticity and locking
  - Crouzeix–Raviart [Brenner and Sung, 1992]
  - Stabilized CR/dG [Hansbo and Larson, 2002]
  - Mixed MFD [Beirão Da Veiga, 2010]
  - . . . and many more
- And, of course, the classical works
  - [Crouzeix and Raviart, 1973]
  - [Raviart and Thomas, 1977]

Outline

Admissible mesh sequences I

Trace and inverse inequalities

- Every $\mathcal{T}_h$ admits a simplicial submesh $\mathfrak{S}_h$
- $(\mathfrak{S}_h)_{h\in\mathcal{H}}$ is shape-regular in the sense of Ciarlet
- $(\mathfrak{S}_h)_{h\in\mathcal{H}}$ is contact regular: every simplex $S \subset T$ is s.t. $h_S \approx h_T$

Optimal polynomial approximation (for error estimates)

Every element T is star-shaped w.r.t. a ball of diameter $\delta_T \approx h_T$

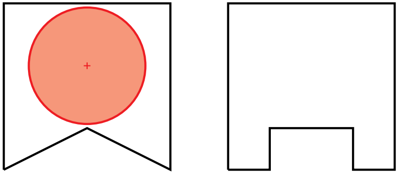


Figure: Admissible (*left*) and non-admissible (*right*) mesh elements

Admissible mesh sequences II

Cell centers

We fix a family of points $(\boldsymbol{x}_T)_{T\in\mathcal{T}_h}$ s.t.

- all $T \in \mathcal{T}_h$ is star-shaped w.r.t. $\boldsymbol{x}_T$
- for all $T \in \mathcal{T}_h$, and all $F \in \mathcal{F}_T$, $d_{T,F} := \text{dist}(\boldsymbol{x}_T, F) \approx h_T$

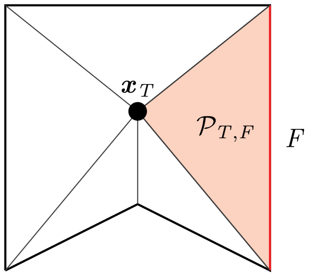


Figure: Cell center and face-based pyramid $\mathcal{P}_{T,F}$

Admissible mesh sequences III

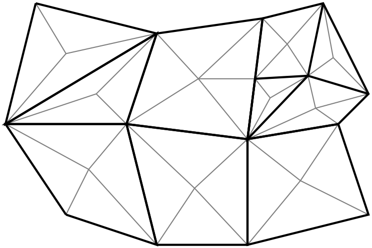


Figure: Pyramidal submesh $\mathcal{P}_h := \{\mathcal{P}_{T,F}\}_{T\in\mathcal{T}_h,\, F\in\mathcal{F}_T}$. $\Sigma_h := \{\text{faces of } \mathcal{P}_h\}$

Lemma (Shape- and contact-regularity of $\mathcal{P}_h$)

Let $\mathcal{T}_h$ admit a set of cell centers. Then, if $\mathcal{T}_h$ is shape- and contact-regular, the same holds for $\mathcal{P}_h$.

Extending the Crouzeix–Raviart space to general meshes I

- Following [Eymard et al., 2010], we consider the space of DOFs

$$\mathbb{V}_h := \mathbb{R}^{\mathcal{T}_h} \times \mathbb{R}^{\mathcal{F}_h}$$

- Define the gradient reconstruction $\mathfrak{G}_h : \mathbb{V}_h \to \mathbb{P}^0_d(\mathcal{P}_h)^d$ s.t.

$$\forall \mathcal{P}_{T,F} \in \mathcal{P}_h, \quad \mathfrak{G}_h(\mathbb{v}_h)_{|\mathcal{P}_{T,F}} = \boldsymbol{G}_T(\mathbb{v}_h) + \boldsymbol{R}_{T,F}(\mathbb{v}_h) := \boldsymbol{G}_{T,F}(\mathbb{v}_h)$$

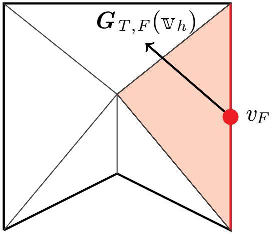
where

$$\boldsymbol{G}_T(\mathbb{v}_h) := \sum_{F \in \mathcal{F}_T} \frac{|F|}{|T|} v_F \boldsymbol{n}_{T,F},$$

$$\boldsymbol{R}_{T,F}(\mathbb{v}_h) := \frac{\eta}{d_{T,F}} \left[v_F - (v_T + \boldsymbol{G}_T(\mathbb{v}_h) \cdot (\overline{\boldsymbol{x}}_F - \boldsymbol{x}_T)) \right] \boldsymbol{n}_{T,F}$$

- Observe that $\boldsymbol{R}_{T,F}(\mathbb{v}_h) \in (\mathbb{P}^0_d(T)^d)^\perp$

Extending the Crouzeix–Raviart space to general meshes II



- In the spirit of ccG methods, define $\mathfrak{R}_h : \mathbb{V}_h \to \mathbb{P}^1_d(\mathcal{T}_h)$ s.t.

$$\forall \mathcal{P}_{T,F} \in \mathcal{P}_h, \qquad \mathfrak{R}_h(\mathbb{v}_h)_{|\mathcal{P}_{T,F}}(\boldsymbol{x}) = v_F + \mathfrak{G}_h(\mathbb{v}_h)_{|\mathcal{P}_{T,F}} \cdot (\boldsymbol{x} - \overline{\boldsymbol{x}}_F)$$

- Following [DP, 2012] we let

$$\boxed{\mathfrak{CR}(\mathcal{T}_h) := \mathfrak{R}_h(\mathbb{V}_h) \subset \mathbb{P}^1_d(\mathcal{P}_h)}$$

Continuity of face-averaged values I

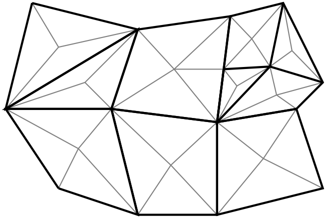


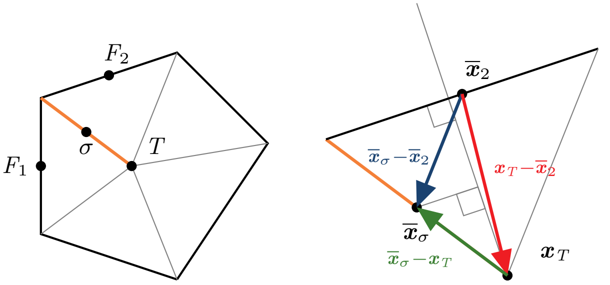
Figure: Primal mesh faces (thick lines) and lateral pyramidal faces (thin lines)

Lemma (Continuity of face-averaged values)

Assume $\eta = d$. *There holds for all* $v_h \in \mathfrak{CR}(\mathcal{T}_h)$ *and all* $\sigma \in \Sigma_h$,

$$\langle [\![v_h]\!] \rangle_\sigma = 0.$$

Continuity of face-averaged values II



- Choice of the starting point: $\langle [\![v_h]\!] \rangle_F = v_F = 0$ for all $F \in \mathcal{F}_h$
- For $\sigma \in \Sigma_h \backslash \mathcal{F}_h$, there holds with $\mathbb{v}_h \in \mathbb{V}_h$ s.t. $\mathfrak{R}_h(\mathbb{v}_h) = v_h$,

$$\begin{aligned}\langle [\![v_h]\!] \rangle_\sigma &= v_{h|\mathcal{P}_{T,F_1}}(\overline{\boldsymbol{x}}_\sigma) - v_{h|\mathcal{P}_{T,F_2}}(\overline{\boldsymbol{x}}_\sigma) \\ &= v_{F_1} - v_{F_2} - \boldsymbol{G}_T(\mathbb{v}_h)\cdot(\overline{\boldsymbol{x}}_{F_1} - \overline{\boldsymbol{x}}_{F_2}) + \alpha_1 - \alpha_2,\end{aligned}$$

with $\alpha_i := \boldsymbol{R}_{T,F_i}(\mathbb{v}_h)\cdot(\overline{\boldsymbol{x}}_\sigma - \overline{\boldsymbol{x}}_i) = -\frac{\eta}{d}\left(v_{F_i} - v_T - \boldsymbol{G}_T(\mathbb{v}_h)\cdot(\overline{\boldsymbol{x}}_i - \boldsymbol{x}_T)\right)$

Continuity of face-averaged values III

- Hence,

$$\langle [\![v_h]\!] \rangle_\sigma = \left(1 - \frac{\eta}{d}\right)(v_{F_1} - v_{F_2} - \boldsymbol{G}_T(\mathbb{v}_h)\cdot(\overline{\boldsymbol{x}}_{F_1} - \overline{\boldsymbol{x}}_{F_2})) = 0$$

  since

$$\alpha_1 - \alpha_2 = -\frac{\eta}{d}(v_{F_1} - v_{F_2} - \boldsymbol{G}_T(\mathbb{v}_h)\cdot(\overline{\boldsymbol{x}}_1 - \overline{\boldsymbol{x}}_2))$$

- Clearly, $\langle [\![v_h]\!] \rangle_\sigma = 0$ if $\eta = d$

Link with the classical Crouzeix–Raviart space

Lemma (The matching simplicial case)

Assume $\mathcal{T}_h$ matching simplicial. Then, for all $\eta > 0$ there holds

$$\mathbb{CR}(\mathcal{T}_h) \subset \mathfrak{CR}(\mathcal{T}_h),$$

where $\mathbb{CR}(\mathcal{T}_h)$ denotes the classical Crouzeix–Raviart space on $\mathcal{T}_h$.

Proof.

- Let $v_h \in \mathbb{CR}(\mathcal{T}_h)$ and set $\mathbb{v}_h := ((v_h(\boldsymbol{x}_T))_{T\in\mathcal{T}_h}, (v_h(\overline{\boldsymbol{x}}_F))_{F\in\mathcal{F}_h})$
- For all $T \in \mathcal{T}_h$, $\boldsymbol{G}_T(\mathbb{v}_h) = \boldsymbol{\nabla}_h v_{h|T}$ and $\boldsymbol{R}_{T,F}(\mathbb{v}_h) = \boldsymbol{0}\ \forall F \in \mathcal{F}_T$
- Hence, $\mathfrak{R}_h(\mathbb{v}_h) = v_h$, which concludes the proof □

Approximation properties I

Lemma (Approximation in $\mathfrak{CR}(\mathcal{T}_h)$)

Let $\eta > 0$. For $v \in H^1(\Omega)$ let $\mathcal{I}_h v \in \mathfrak{CR}(\mathcal{T}_h)$ be s.t.

$$\mathcal{I}_h v = \mathfrak{R}_h(\mathbb{v}_h) \text{ with } \mathbb{v}_h = \left((\pi_h^1 v(\boldsymbol{x}_T))_{T \in \mathcal{T}_h}, (\langle v \rangle_F)_{F \in \mathcal{F}_h} \right).$$

Then there holds

$$\pi_h^0(\boldsymbol{\nabla}_h \mathcal{I}_h v) = \pi_h^0(\boldsymbol{\nabla} v).$$

Moreover, if $v \in H^1(\Omega) \cap H^2(\mathcal{T}_h)$, there holds for all $T \in \mathcal{T}_h$,

$$\|v - \mathcal{I}_h v\|_{L^2(T)} + h_T \|\boldsymbol{\nabla}(v - \mathcal{I}_h v)\|_{L^2(T)^d} \leqslant C h_T^2 \|v\|_{H^2(T)}.$$

Approximation properties II

Proof.

Let $T \in \mathcal{T}_h$. Using Green's formula and since $\boldsymbol{R}_{T,F}(\mathbb{v}_h) \in (\mathbb{P}^0_d(T)^d)^\perp$,

$$\pi^0_h(\boldsymbol{\nabla}_h \mathcal{I}_h v)_{|T} = \boldsymbol{G}_T(\mathbb{v}_h) = \sum_{F \in \mathcal{F}_T} \frac{|F|}{|T|} \langle v \rangle_F \boldsymbol{n}_{T,F}$$

$$= \frac{1}{|T|} \sum_{F \in \mathcal{F}_T} \int_F v \boldsymbol{n}_{T,F} = \frac{1}{|T|} \int_T \boldsymbol{\nabla} v = \langle \boldsymbol{\nabla} v \rangle_T.$$

The second point can be proved as in [DP, 2012]. □

Approximation properties III

Corollary (Divergence approximation)

For $\boldsymbol{v} \in H^1(\Omega)^d$ *with* $\boldsymbol{\nabla}\cdot\boldsymbol{v} \in H^1(\Omega)$ *let* $\boldsymbol{v}_h := \mathcal{I}_h\boldsymbol{v}$ *and*

$$\forall T \in \mathcal{T}_h, \quad D_h(\boldsymbol{v}_h)_{|T} = \pi_h^0(\boldsymbol{\nabla}_h\cdot\boldsymbol{v}_h)_{|T} = \frac{1}{|T|}\sum_{F\in\mathcal{F}_T}|F|\boldsymbol{v}_F\cdot\boldsymbol{n}_{T,F}.$$

For all $T \in \mathcal{T}_h$, *there holds*

$$\|\boldsymbol{\nabla}\cdot\boldsymbol{v} - D_h(\boldsymbol{v}_h)\|_{L^2(T)} + h_T|\boldsymbol{\nabla}\cdot\boldsymbol{v} - D_h(\boldsymbol{v}_h)|_{H^1(T)} \leqslant Ch_T|\boldsymbol{\nabla}\cdot\boldsymbol{v}|_{H^1(T)}.$$

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SCIENCES ET TECHNIQUES

Outline

A locking-free method on general meshes I

$$\boxed{\begin{aligned} -\boldsymbol{\nabla}\cdot\underline{\underline{\sigma}}(\boldsymbol{u}) &= \boldsymbol{f} && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega \end{aligned}}$$

- We seek an approximation of the displacement $\boldsymbol{u}$ in the space

$$\boldsymbol{U}_h := \mathfrak{CR}_0(\mathcal{T}_h)^d$$

- The discrete problem reads

$$\text{Find } \boldsymbol{u}_h \in \boldsymbol{U}_h \text{ s.t. } a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h \text{ for all } \boldsymbol{v}_h \in \boldsymbol{U}_h$$

with Navier–Cauchy discrete bilinear form

$$a_h(\boldsymbol{w}, \boldsymbol{v}) := \int_\Omega \mu \boldsymbol{\nabla}_h \boldsymbol{w} : \boldsymbol{\nabla}_h \boldsymbol{v} + \int_\Omega \mu(\boldsymbol{\nabla}_h\cdot\boldsymbol{w})(\boldsymbol{\nabla}_h\cdot\boldsymbol{v}) + \int_\Omega \lambda D_h(\boldsymbol{w}) D_h(\boldsymbol{v})$$

A locking-free method on general meshes II

Lemma (Coercivity of a_h)

There holds for all $\boldsymbol{v}_h \in \boldsymbol{U}_h$ of h and of λ,

$$a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) =: |\!|\!| \boldsymbol{v}_h |\!|\!|_{\mathrm{el}}^2 \geqslant \mu \|\boldsymbol{\nabla}_h \boldsymbol{v}_h\|_{L^2(\Omega)^{d,d}}^2.$$

Remark (Other boundary conditions)

- *For other bcs, the Navier–Cauchy formulation is no longer valid*
- *Stability then hinges on the discrete Korn's inequality (cf. [Brenner, 2004]), which requires to penalize interface jumps*
- *See [DP and Lemaire, 2012] for further details*

A locking-free method on general meshes III

Lemma (Weak consistency)

Assume $\boldsymbol{u} \in \boldsymbol{U}_* := (H_0^1(\Omega) \cap H^2(\Omega))^d$. *Then*

$$\forall \boldsymbol{v}_h \in \boldsymbol{U}_h, \quad a_h(\boldsymbol{u}, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h + \mathcal{E}_h(\boldsymbol{v}_h),$$

where, for $\underline{\underline{\tau}}(\boldsymbol{u}) := \mu \boldsymbol{\nabla} \boldsymbol{u} + (\mu + \lambda)(\boldsymbol{\nabla} \cdot \boldsymbol{u}) \underline{\underline{I_d}}$,

$$\mathcal{E}_h(\boldsymbol{v}_h) := \sum_{\sigma \in \Sigma_h} \int_\sigma \underline{\underline{\tau}}(\boldsymbol{u}) : [\![\boldsymbol{v}_h]\!] \otimes \boldsymbol{n}_F + \int_\Omega \lambda (D_h(\boldsymbol{u}) - \boldsymbol{\nabla} \cdot \boldsymbol{u})(\boldsymbol{\nabla}_h \cdot \boldsymbol{v}_h).$$

Proof.

Integrate by parts the volumic terms and use $-\boldsymbol{\nabla} \cdot \underline{\underline{\sigma}}(\boldsymbol{u}) = \boldsymbol{f}$ a.e. in Ω. □

A locking-free method on general meshes IV

Theorem (Convergence)

Assume $\boldsymbol{u} \in \boldsymbol{U}_*$. Then the method satisfies the locking-free estimate

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\rm el} \leqslant C\mathcal{N}_{\rm el} h.$$

Proof.

Strang's Second Lemma yields

$$|||\boldsymbol{u} - \boldsymbol{u}_h|||_{\rm el} \lesssim \inf_{\boldsymbol{v}_h \in \boldsymbol{U}_h} |||\boldsymbol{u} - \boldsymbol{v}_h|||_{\rm el} + \sup_{\boldsymbol{v}_h \in \boldsymbol{U}_h \setminus \{\boldsymbol{0}\}} \frac{|\mathcal{E}_h(\boldsymbol{v}_h)|}{|||\boldsymbol{v}_h|||_{\rm el}} := \mathfrak{T}_1 + \mathfrak{T}_2.$$

- $\mathfrak{T}_1$: approximation properties of $\mathcal{I}_h$;
- $\mathfrak{T}_2$: continuity of face-averaged values + regularity estimate. □

Link with FE: The matching simplicial case

Lemma (Link with the method of [Brenner and Sung, 1992])

Assume $\mathcal{T}_h$ matching simplicial and consider the following problems:

$$\textit{Find } \hat{\boldsymbol{u}}_h \in \mathbb{CR}_0(\mathcal{T}_h)^d \textit{ s.t. } a_h(\hat{\boldsymbol{u}}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h \;\forall \boldsymbol{v}_h \in \mathbb{CR}_0(\mathcal{T}_h)^d,$$

and

$$\textit{Find } \boldsymbol{u}_h \in \mathfrak{CR}_0(\mathcal{T}_h)^d \textit{ s.t. } a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) = \int_\Omega \boldsymbol{f}\cdot\mathcal{I}_{\mathbb{CR}}(\boldsymbol{v}_h) \;\forall \boldsymbol{v}_h \in \mathfrak{CR}_0(\mathcal{T}_h)^d.$$

Then, there holds

$$\boldsymbol{u}_h = \hat{\boldsymbol{u}}_h.$$

Link with FV: Numerical fluxes and local conservation I

Lemma (Numerical fluxes)

There holds for all $\boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{U}_h$ *s.t.* $\boldsymbol{v}_h = \mathfrak{R}_h(\mathbb{v}_h)$ *and* $\boldsymbol{w}_h = \mathfrak{R}_h(\mathbb{w}_h)$,

$$a_h(\boldsymbol{w}_h, \boldsymbol{v}_h) = \sum_{i=1}^{d} \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \Phi_{K,F,i}(\mathbb{w}_h)(v_{F,i} - v_{K,i}),$$

where, for vectors $\boldsymbol{y}_{F',F}^{T}$ *only depending on* η *and on geometric quantities,*

$$\Phi_{K,F,i}(\mathbb{w}_h) = \sum_{F' \in \mathcal{F}_T} |\mathcal{P}_{T,F'}| \Bigg\{ \mu \boldsymbol{G}_{T,F'}(\mathbb{w}_{h,i}) \cdot \boldsymbol{y}_{F',F}^{T} + \sum_{j=1}^{d} \Big(\mu \boldsymbol{G}_{T,F'}(\mathbb{w}_{h,j}) + \lambda \boldsymbol{G}_T(\mathbb{w}_{h,j}) \Big) \cdot \boldsymbol{e}_j (\boldsymbol{y}_{F',F}^{T} \cdot \boldsymbol{e}_i) \Bigg\}.$$

Link with FV: Numerical fluxes and local conservation II

- Consider the variation: Find $\mathbb{u}_h \in \mathbb{V}^d_{h,0}$ s.t. for all $\mathbb{v}_h \in \mathbb{V}^d_{h,0}$,

$$\sum_{i=1}^{d} \sum_{T\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_T} \Phi_{T,F,i}(\mathbb{u}_h)(v_{F,i} - v_{T,i}) = \sum_{i=1}^{d} \sum_{T\in\mathcal{T}_h} |T| f_i(\boldsymbol{x}_T) v_{K,i}$$

- A suitable choice for $\mathbb{v}_h$ shows flux continuity

$$\forall F \subset \partial T_1 \cap \partial T_2, \qquad \Phi_{T_1,F,i}(\mathbb{u}_h) = -\Phi_{T_2,F,i}(\mathbb{u}_h),$$

and local conservation on $\mathcal{T}_h$

$$\forall T \in \mathcal{T}_h, \qquad -\sum_{i=1}^{d} \sum_{F\in\mathcal{F}_T} \Phi_{T,F,i}(\mathbb{u}_h) = \sum_{i=1}^{d} |T| f_i(\boldsymbol{x}_T)$$

Numerical example I

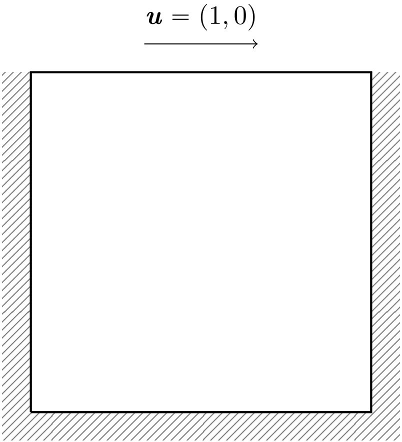

Figure: Closed cavity [Hansbo and Larson, 2003] ($\lambda \approx 1.666 \cdot 10^6$, $\mu \approx 333$)

Numerical example II

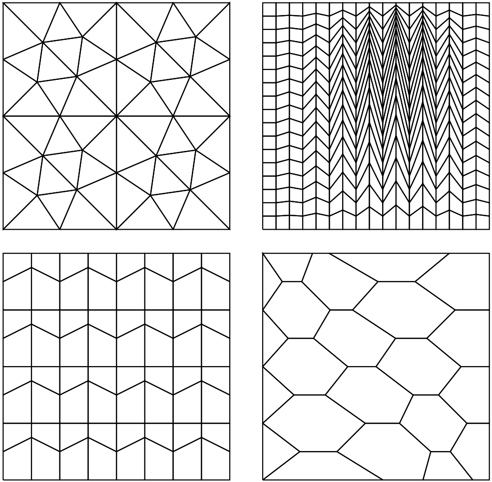

Figure: Meshes for the closed cavity

Numerical example III

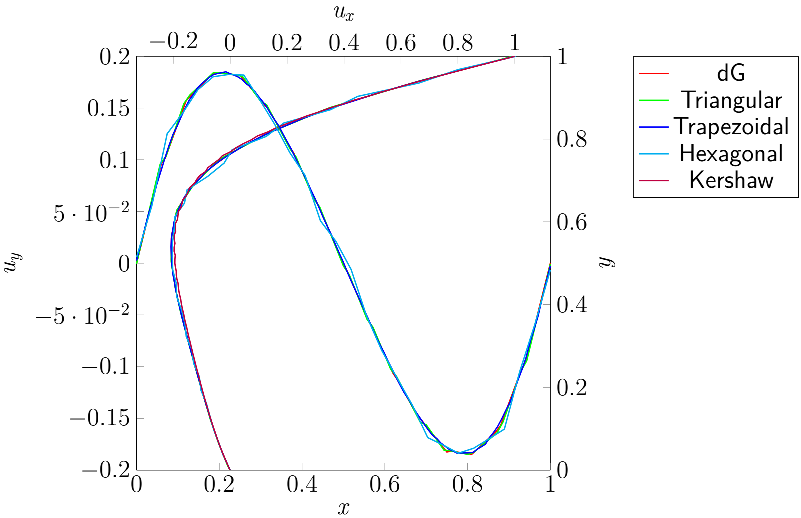

Figure: Closed cavity problem, coarse meshes

Numerical example IV

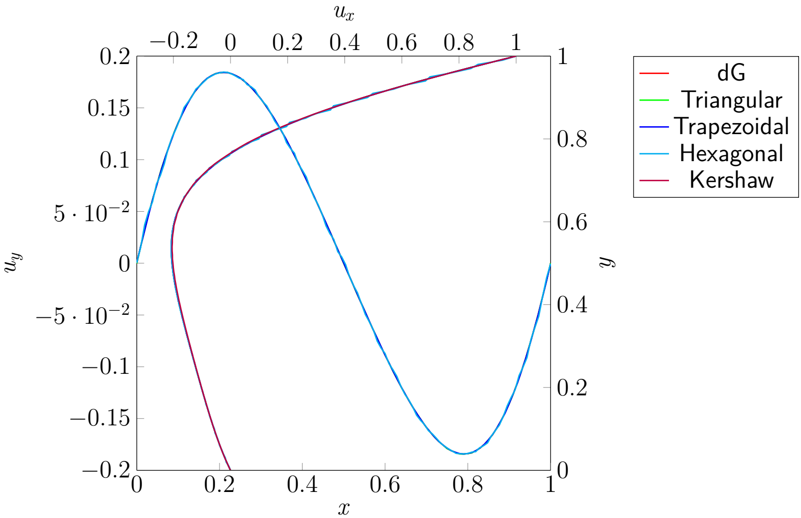

Figure: Closed cavity problem, fine meshes

Outline

Extending the Raviart–Thomas space to general meshes I

- Lowest-order Raviart–Thomas space on a matching simplicial mesh:

$$\mathbb{RT}_d^0(\mathcal{T}_h) := \mathbb{P}_d^0(\mathcal{T}_h)^d + \boldsymbol{x}\mathbb{P}_d^0(\mathcal{T}_h)$$

- The functions in $\mathbb{RT}_d^0(\mathcal{T}_h)$ have a piecewise constant isotropic gradient
- Key feature: $\boldsymbol{H}(\mathrm{div};\Omega)$-conformity,

$$\forall \boldsymbol{v}_h \in \mathbb{RT}_d^0(\mathcal{T}_h), \qquad [\![\boldsymbol{v}_h]\!]\cdot\boldsymbol{n}_F = 0 \qquad \forall F \in \mathcal{F}_h^{\mathrm{i}}$$

- Using the above ideas, we extend this space to general meshes

Extending the Raviart–Thomas space to general meshes II

- Let

$$\mathbb{V}_h := (\mathbb{R}^{\mathcal{T}_h})^d \times \mathbb{R}^{\mathcal{F}_h}$$

- We define an isotropic gradient reconstruction by setting

$$\mathfrak{G}_h(\mathbb{v}_h)_{|\mathcal{P}_{T,F}} := G_T(\mathbb{v}_h) + R_{T,F}(\mathbb{v}_h) \qquad \forall T \in \mathcal{T}_h,\ F \in \mathcal{F}_T,$$

with scalars

$$G_T(\mathbb{v}_h) := \frac{1}{d|T|} \sum_{F \in \mathcal{F}_T} v_F^n \boldsymbol{n}_F \cdot \boldsymbol{n}_{T,F},$$

$$R_{T,F}(\mathbb{v}_h) := \frac{\eta}{d_{T,F}} \left(v_F^n \boldsymbol{n}_F - \boldsymbol{v}_T - G_T(\mathbb{v}_h)(\overline{\boldsymbol{x}}_F - \boldsymbol{x}_T)\right) \cdot \boldsymbol{n}_{T,F}$$

Extending the Raviart–Thomas space to general meshes III

- The piecewise affine reconstruction operator is s.t.

$$\mathfrak{R}_h(\mathtt{v}_h)_{|\mathcal{P}_{T,F}}(\boldsymbol{x}) = \boldsymbol{v}_T + \mathfrak{G}_h(\mathtt{v}_h)_{|\mathcal{P}_{T,F}}(\boldsymbol{x} - \boldsymbol{x}_T) \quad \forall \mathcal{P}_{T,F} \in \mathcal{P}_h$$

- The extended Raviart–Thomas space is obtained letting

$$\boxed{\mathfrak{RT}(\mathcal{T}_h) := \mathfrak{R}_h(\mathbb{V}_h)}$$

Extending the Raviart–Thomas space to general meshes IV

Lemma ($\boldsymbol{H}(\mathrm{div};\Omega)$-conformity)

Assume $\eta = 1$. Then there holds for all $\boldsymbol{v}_h \in \mathfrak{RT}(\mathcal{T}_h)$ and all $\sigma \in \Sigma_h^{\mathrm{i}}$,

$$[\![\boldsymbol{v}_h]\!]\cdot\boldsymbol{n}_F = 0.$$

Remark (The matching simplicial case)

Unlike the Crouzeix–Raviart case, the assumption $\eta = 1$ is required also if $\mathcal{T}_h$ is matching simplicial.

Extending the Raviart–Thomas space to general meshes V

Lemma (Approximation in $\mathfrak{RT}(\mathcal{T}_h)$)

Let $\eta > 0$. *For all* $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega)$ *let*

$$\mathcal{I}_h\boldsymbol{v} = \mathfrak{R}_h(\mathbb{v}_h) \text{ with } \mathbb{v}_h = \left((\pi_h^1\boldsymbol{v}(\boldsymbol{x}_T))_{T\in\mathcal{T}_h}, (\langle\boldsymbol{v}\rangle_F\cdot\boldsymbol{n}_F)_{F\in\mathcal{F}_h}.\right)$$

Then there holds

$$D_h(\boldsymbol{v}_h) = \pi_h^0(\boldsymbol{\nabla}_h\cdot\boldsymbol{v}_h) = \pi_h^0(\boldsymbol{\nabla}\cdot\boldsymbol{v}).$$

Moreover, if $\boldsymbol{v} \in H^1(\Omega)^d$ *et* $\boldsymbol{\nabla}\cdot\boldsymbol{v} \in H^1(\Omega)$, *there holds for all* $T \in \mathcal{T}_h$ *with* $\boldsymbol{v}_h := \mathcal{I}_h\boldsymbol{v}$,

$$\|\boldsymbol{v} - \boldsymbol{v}\|_{L^2(T)^d} + \|\boldsymbol{\nabla}\cdot\boldsymbol{v} - D_h(\boldsymbol{v}_h)\|_{L^2(T)} \lesssim h_T\left(|\boldsymbol{v}|_{H^1(T)^d} + |\boldsymbol{\nabla}\cdot\boldsymbol{v}|_{H^1(T)}\right).$$

References I

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