

Basic principles of polytopal approximations of partial differential equations

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- 1 Preliminaries
- 2 A non-conforming finite element scheme on standard meshes
- 3 An Hybrid High-Order scheme on polytopal meshes

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- Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open connected polytopal domain
- We focus on the Poisson problem: Given $f : \Omega \rightarrow \mathbb{R}$, find $u : \Omega \rightarrow \mathbb{R}$ s.t.

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

- Let $f \in L^2(\Omega)$. A possible weak formulation reads: Find $u \in H_0^1(\Omega)$ s.t.

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v \quad \forall v \in H_0^1(\Omega)$$

- The well-posedness of this problem hinges on the **Poincaré inequality**

$$\|v\|_{L^2(\Omega)} \leq C_{\Omega} \|\nabla v\|_{L^2(\Omega)^d}$$

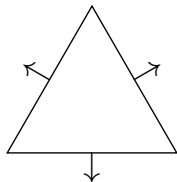
Local polynomial spaces and L^2 -orthogonal projector

- Denote by \mathbb{P}_d^ℓ the space of d -variate polynomials of total degree $\leq \ell$
- Let $Y \subset \mathbb{R}^d$ and denote by $\mathcal{P}^\ell(Y)$ the restriction of \mathbb{P}_d^ℓ to Y
- Given $v \in L^2(Y)$, its L^2 -orthogonal projection on $\mathcal{P}^\ell(Y)$ is s.t.

$$\int_Y (v - \pi_{\mathcal{P}^\ell(Y)} v) w = 0 \quad \forall w \in \mathcal{P}^\ell(Y)$$

- $\pi_{\mathcal{P}^\ell(Y)} v$ optimally approximates v in all Sobolev seminorms under mild assumptions on Y ; see [DP and Droniou, 2020, Chapter 1] for details

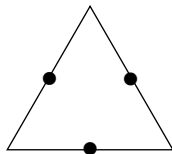
Raviart–Thomas–Nédélec and Crouzeix–Raviart elements I



- Denote by T a d -simplex and by \mathcal{F}_T the set collecting its faces
- Let $\mathcal{RTN}^1(T) := \mathcal{P}^0(T)^d + x\mathcal{P}^0(T)$
- Define the degrees of freedom $\sigma := (\sigma_F)_{F \in \mathcal{F}_T}$ s.t.

$$\sigma_F : H^1(T)^d \ni \tau \mapsto \frac{1}{|F|} \int_F \tau \cdot n_{TF} \in \mathbb{R}$$

- Notice that $\tau \cdot n_{TF} \in \mathcal{P}^0(F)$ for all $\tau \in \mathcal{RTN}^1(T)$ and all $F \in \mathcal{F}_T$
- $(T, \mathcal{RTN}^1(T), \sigma)$ is a FE [Raviart and Thomas, 1977, Nédélec, 1980]



- Let now $\mathcal{P}^1(T)$ be the space of affine functions on T
- Define the degrees of freedom $\sigma := (\sigma_F)_{F \in \mathcal{F}_T}$ s.t.

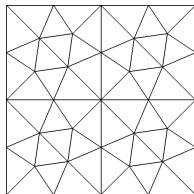
$$\sigma_F : H^1(T) \ni v \mapsto \frac{1}{|F|} \int_F v \in \mathbb{R}$$

- $(T, \mathcal{P}^1(T), \sigma)$ is a FE [Crouzeix and Raviart, 1973]

Outline

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A non-conforming finite element scheme I



- Let \mathcal{T}_h be a conforming simplicial mesh of Ω
- Let $C\mathcal{R}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)$ be the Crouzeix–Raviart space on \mathcal{T}_h and set

$$C\mathcal{R}_0(\mathcal{T}_h) := \{v_h \in C\mathcal{R}(\mathcal{T}_h) : \pi_{\mathcal{P}^0(F)} v = 0 \text{ for all } F \in \mathcal{F}_h^b \}$$

- Notice that $C\mathcal{R}_0(\mathcal{T}_h) \not\subset H_0^1(\Omega)$!
- With ∇_h **broken gradient**, the scheme reads: Find $u_h \in C\mathcal{R}_0(\mathcal{T}_h)$ s.t.

$$a_h(u_h, v_h) := \int_{\Omega} \nabla_h u_h \cdot \nabla_h v_h = \int_{\Omega} f v_h \quad \forall v_h \in C\mathcal{R}_0(\mathcal{T}_h)$$

Stability analysis I

Lemma (Discrete Poincaré inequality in the Crouzeix–Raviart space)

For all $v_h \in C\mathcal{R}_0(\mathcal{T}_h)$,

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$$

where $a \lesssim b$ means $a \leq Cb$ with C independent of h .

- The Poincaré–Wirtinger inequality gives, setting $v_T := (v_h)|_T$,

$$\|v_T\|_{L^2(T)} \lesssim \|\pi_{\mathcal{P}^0(T)} v_T\|_{L^2(T)} + h_T \|\nabla v_T\|_{L^2(T)^d}$$

- Hence, letting $\bar{v}_h := \pi_{\mathcal{P}^0(\mathcal{T}_h)} v_h$, it suffices to prove that

$$\|\bar{v}_h\|_{L^2(\Omega)} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d}$$

- Since $\operatorname{div} : \mathcal{RTN}^1(\mathcal{T}_h) \rightarrow \mathcal{P}^0(\mathcal{T}_h)$ is surjective: $\exists \tau_h \in \mathcal{RTN}^1(\mathcal{T}_h)$ s.t.

$$\operatorname{div} \tau_h = \bar{v}_h \text{ and } \|\tau_h\|_{H(\operatorname{div}; \Omega)} \lesssim \|\bar{v}_h\|_{L^2(\Omega)}$$

Stability analysis II

- We write, letting $v_T := (v_h)|_T$ for all $T \in \mathcal{T}_h$,

$$\|\bar{v}_h\|_{L^2(\Omega)}^2 \stackrel{\diamond}{=} \sum_{T \in \mathcal{T}_h} \int_T v_T \operatorname{div} \tau_h \stackrel{\text{IBP}}{=} - \int_{\Omega} \nabla_h v_h \cdot \tau_h + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F v_T (\tau_h \cdot n_{TF})$$

- Replacing $v_T \leftarrow \pi_{\mathcal{P}^0(F)} v_T =: \mathbf{v}_F$ in the boundary term and rearranging,

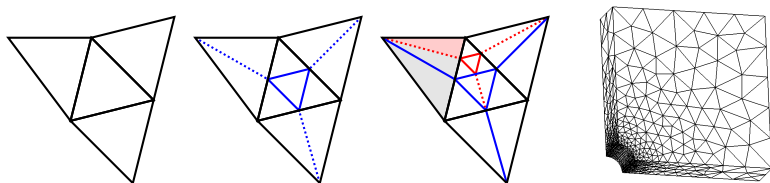
$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F v_T (\tau_h \cdot n_{TF}) &= \sum_{F \in \mathcal{F}_h} \sum_{T \in \mathcal{T}_F} \int_F v_T (\tau_h \cdot n_{TF}) \\ &= \sum_{F \in \mathcal{F}_h^i} \sum_{T \in \mathcal{T}_F} \int_F \mathbf{v}_F (\tau_h \cdot n_{TF}) + \sum_{F \in \mathcal{F}_h^b} \int_F \mathbf{v}_F (\tau_h \cdot n_F) = 0, \end{aligned}$$

by single-valuedness of v_F for $F \in \mathcal{F}_h^i$ and $v_F = 0$ for $F \in \mathcal{F}_h^b$

- Using Cauchy–Schwarz inequalities, we get

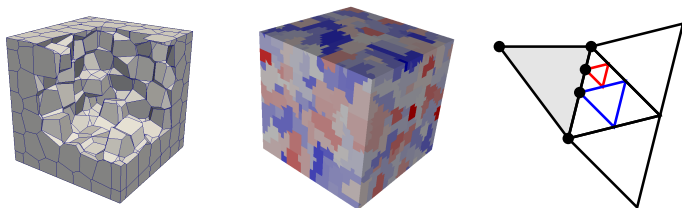
$$\|\bar{v}_h\|_{L^2(\Omega)}^2 \leq \|\nabla_h v_h\|_{L^2(\Omega)^d} \|\tau_h\|_{L^2(\Omega)^d} \lesssim \|\nabla_h v_h\|_{L^2(\Omega)^d} \|\bar{v}_h\|_{L^2(\Omega)}$$

Limitations of the finite element approach



- Approach limited to **conforming meshes** with **standard elements**
 - ⇒ local refinement requires to **trade mesh size for mesh quality**
 - ⇒ complex geometries may require a **large number of elements**
 - ⇒ the element shape cannot be **adapted to the solution**
- Treating more general meshes in the FE spirit would significantly increase the space dimension [Droniou et al., 2021]
- The extension to **high-order** is not straightforward

Fully discrete polytopal approach



- **Key idea:** replace both spaces and operators by discrete counterparts
- Support of **polyhedral meshes** and **high-order**
- Several strategies to **reduce the number of unknowns** on general shapes
- Elegant analysis framework available

A few key references

- Introduction of Hybrid High-Order (HHO) methods [DP et al., 2014]
- Fully discrete analysis framework [DP and Droniou, 2018]
- A monograph on HHO methods [DP and Droniou, 2020]
- Introduction of Discrete de Rham (DDR) methods [DP et al., 2020]
- DDR for the de Rham complex of differential forms [Bonaldi et al., 2023]

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A paradigm shift

- Let $v_T \in \mathcal{P}^1(T)$ and set $\mathbf{v}_F := \pi_{\mathcal{P}^0(F)}(v_T)|_F$ for all $F \in \mathcal{F}_T$
- We have, for all $\tau \in \mathcal{P}^0(T)^d$,

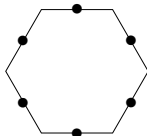
$$\int_T \nabla \mathbf{v}_T \cdot \tau = - \int_T v_T \operatorname{div} \tau + \sum_{F \in \mathcal{F}_T} \int_F v_T(\tau \cdot n_{TF}) = \sum_{F \in \mathcal{F}_T} \int_F \mathbf{v}_F(\tau \cdot n_{TF})$$

- Moreover, with \bar{x}_Y center of mass of $Y \in \{T\} \cup \mathcal{F}_T$, noticing that $\bar{x}_T = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \bar{x}_F$ and using linearity,

$$\pi_{\mathcal{P}^0(T)} \mathbf{v} = v_T(\bar{x}_T) = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} v_T(\bar{x}_F) = \frac{1}{\operatorname{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F \mathbf{v}_F$$

- These formulas remain valid when T is a general polytope!

Generalization to polytopes

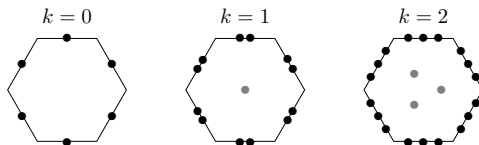


- Let $\underline{V}_T^0 := \{ \underline{v}_T := (v_F)_{F \in \mathcal{F}_T} : v_F \in \mathcal{P}^0(F) \text{ for all } F \in \mathcal{F}_T \}$
- We can define a **potential reconstruction** $r_T^1 : \underline{V}_T^0 \rightarrow \mathcal{P}^1(T)$ enforcing

$$\int_T \nabla r_T^1 \underline{v}_T \cdot \tau = \sum_{F \in \mathcal{F}_T} \int_F v_F (\tau \cdot n_{TF}) \quad \forall \tau \in \mathcal{P}^0(T)^d$$

$$\pi_{\mathcal{P}^0(T)}(r_T^1 \underline{v}_T) = \frac{1}{\text{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F$$

Extension to arbitrary-order I



- Let $k \geq 0$ and define the **Hybrid High-Order (HHO)** space

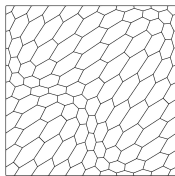
$$\underline{V}_T^k := \{ \underline{v}_T = (v_T, (v_F)_{F \in \mathcal{F}_T}) : \\ v_T \in \mathcal{P}^{k-1}(T) \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_T \}$$

- We define $r_T^{k+1} : \underline{V}_T^k \rightarrow \mathcal{P}^k(T)^d$ s.t., for all $\underline{v}_T \in \underline{V}_T^k$,

$$\int_T \nabla r_T^{k+1} \underline{v}_T \cdot \nabla w = - \int_T v_T \Delta w + \sum_{F \in \mathcal{F}_T} \int_F v_F (\nabla w \cdot n_{TF}) \quad \forall w \in \mathcal{P}^{k+1}(T), \\ \pi_{\mathcal{P}^0(T)}(r_T^1 \underline{v}_T) = \pi_{\mathcal{P}^0(T)} v_T,$$

$$\text{where } v_T := \frac{1}{\text{card}(\mathcal{F}_T)} \sum_{F \in \mathcal{F}_T} \frac{1}{|F|} \int_F v_F \text{ if } k = 0$$

Global HHO space and H^1 -like seminorm



- Given a polytopal mesh \mathcal{T}_h of Ω , define the global HHO space

$$\underline{V}_h^k := \{ \underline{v}_h = ((v_T)_{T \in \mathcal{T}_h}, (v_F)_{F \in \mathcal{F}_T}) \in \underline{V}_h^k : \\ v_T \in \mathcal{P}^{k-1}(T) \text{ for all } T \in \mathcal{T}_h \text{ and } v_F \in \mathcal{P}^k(F) \text{ for all } F \in \mathcal{F}_h \}$$

- We define on \underline{V}_h^k the norm

$$\|\underline{v}_h\|_{1,h}^2 := \sum_{T \in \mathcal{T}_h} \|\underline{v}_T\|_{1,T}^2$$

$$\text{where } \|\underline{v}_T\|_{1,T}^2 := \|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_F - v_T\|_{L^2(F)}^2 \text{ for all } T \in \mathcal{T}_h$$

Discrete Poincaré inequality in HHO spaces I

Lemma (Poincaré inequality in HHO spaces)

Denote by $\underline{V}_{h,0}^k$ the subspace of \underline{V}_h^k with vanishing boundary values. For any $\underline{v}_h \in \underline{V}_{h,0}^k$, letting $v_h \in \mathcal{P}^{\max(k-1,0)}(\mathcal{T}_h)$ be s.t. $(v_h)|_T := v_T$ for all $T \in \mathcal{T}_h$, it holds

$$\|v_h\|_{L^2(\Omega)} \lesssim \|\underline{v}_h\|_{1,h},$$

hence $\|\cdot\|_{1,h}$ is a norm on $\underline{V}_{h,0}^k$.

- Since $\operatorname{div} : H^1(\Omega)^d \rightarrow L^2(\Omega)$ is surjective, there is $\tau \in H^1(\Omega)^d$ s.t.

$$\operatorname{div} \tau = v_h \text{ and } \|\tau\|_{H^1(\Omega)^d} \lesssim \|v_h\|_{L^2(\Omega)}$$

- Notice that here we cannot seek τ in $\mathcal{RTN}^{k+1}(\mathcal{T}_h)$ since \mathcal{T}_h is not conforming simplicial!

Discrete Poincaré inequality in HHO spaces II

- We go on writing

$$\begin{aligned}
 \|v_h\|_{L^2(\Omega)}^2 &= \int_{\Omega} v_h \operatorname{div} \tau = \sum_{T \in \mathcal{T}_h} \int_T v_T \operatorname{div} \tau \\
 &\stackrel{\text{IBP}}{=} \sum_{T \in \mathcal{T}_h} \left(- \int_T \nabla v_T \cdot \tau + \sum_{F \in \mathcal{F}_T} \int_F (v_T - \textcolor{red}{v}_F) (\tau \cdot n_{TF}) \right) \\
 &\stackrel{\text{C-S}}{\leq} \left[\sum_{T \in \mathcal{T}_h} \left(\|\nabla v_T\|_{L^2(T)^d}^2 + h_T^{-1} \sum_{F \in \mathcal{F}_T} \|v_T - v_F\|_{L^2(F)}^2 \right) \right]^{\frac{1}{2}} \\
 &\quad \times \left[\sum_{T \in \mathcal{T}_h} \left(\|\tau\|_{L^2(T)^2}^2 + h_T \|\tau\|_{L^2(\partial T)}^2 \right) \right]^{\frac{1}{2}}
 \end{aligned}$$

- Recalling the definition of $\|\cdot\|_{1,h}$ and using trace inequalities along with $h_T \leq h_{\Omega} \lesssim 1$, we get

$$\|v_h\|_{L^2(\Omega)}^2 \lesssim \|\underline{v}_h\|_{1,h} \|\tau\|_{H^1(T)^d} \lesssim \|\underline{v}_h\|_{1,h} \|v_h\|_{L^2(\Omega)}$$

An HHO scheme

We consider the following scheme: Find $\underline{u}_h \in \underline{V}_{h,0}^k$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} a_T(\underline{u}_T, \underline{v}_T) = \int_{\Omega} f v_h \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k$$

where, for all $T \in \mathcal{T}_h$,

$$a_T(\underline{u}_T, \underline{v}_T) := \int_T \nabla r_T^{k+1} \underline{u}_T \cdot \nabla r_T^{k+1} \underline{v}_T + s_T(\underline{u}_T, \underline{v}_T)$$

and the symmetric semi-definite bilinear form s_T satisfies

$$\|\underline{v}_T\|_{1,T} \lesssim a_T(\underline{v}_T, \underline{v}_T)^{\frac{1}{2}} \lesssim \|\underline{v}_T\|_{1,T} \quad \forall \underline{v}_T \in \underline{V}_T^k \quad (\text{ST1})$$

Lemma (Well-posedness of the HHO discrete problem)

The HHO problem admits a unique solution that satisfies

$$\|\underline{u}_h\|_{1,h} \lesssim \|f\|_{L^2(\Omega)}.$$

- Squaring and summing (ST1) over $T \in \mathcal{T}_h$, we have

$$\|\underline{v}_h\|_{1,h}^2 \lesssim a_h(\underline{v}_h, \underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_{h,0}^k,$$

which expresses the **coercivity of a_h**

- Using the Cauchy–Schwarz and discrete Poincaré inequalities,

$$\int_{\Omega} f v_h \leq \|f\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)} \|\underline{v}_h\|_{1,h}$$

- Letting $\underline{v}_h = \underline{u}_h$ above, the conclusion follows

Error analysis I

- Let $\underline{I}_h^k : H^1(\Omega) \rightarrow \underline{V}_h^k$ be s.t.

$$\underline{I}_h^k v := ((\pi_{\mathcal{P}^{k-1}(T)} v)_{T \in \mathcal{T}_h}, (\pi_{\mathcal{P}^k(F)} v)_{F \in \mathcal{F}_h}) \quad \forall v \in H^1(\Omega)$$

- We aim at estimating the error

$$\underline{e}_h := \underline{u}_h - \underline{I}_h^k u \in \underline{V}_{h,0}^k$$

- It holds, for all $\underline{v}_h \in \underline{V}_{h,0}^k$,

$$a_h(\underline{e}_h, \underline{v}_h) = a_h(\underline{u}_h, \underline{v}_h) - a_h(\underline{I}_h^k u, \underline{v}_h) = \int_{\Omega} f v_h - a_h(\underline{I}_h^k u, \underline{v}_h) =: \mathcal{E}_h(u; \underline{v}_h)$$

- A straightforward modification of the stability proof gives

$$\|\underline{e}_h\|_{1,h} \leq \sup_{\underline{v}_h \in \underline{V}_{h,0}^k \setminus \{\underline{0}\}} \frac{\mathcal{E}_h(u; \underline{v}_h)}{\|\underline{v}_h\|_{1,h}}$$

Error analysis II

We reformulate the components of the **consistency error** $\mathcal{E}_h(\underline{v}_h)$:

$$\int_{\Omega} f v_h = - \sum_{T \in \mathcal{T}_h} \int_T \Delta u v_h = \sum_{T \in \mathcal{T}_h} \left[\int_T \nabla u \cdot \nabla v_T + \sum_{F \in \mathcal{F}_T} \int_F (\nabla u \cdot n_{TF}) (\underline{v}_F - v_T) \right]$$

$$\begin{aligned} a_h(I_h^k u, \underline{v}_h) &= \sum_{T \in \mathcal{T}_h} \left[\int_T \nabla r_T^{k+1}(I_T^k u) \cdot \nabla r_T^{k+1} \underline{v}_T + s_T(I_T^k u, \underline{v}_T) \right] \\ &= \sum_{T \in \mathcal{T}_h} \left[\int_T \nabla r_T^{k+1}(I_T^k u) \cdot \nabla v_T + \sum_{F \in \mathcal{F}_T} \int_F (\nabla r_T^{k+1}(I_T^k u) \cdot n_{TF}) (v_F - v_T) \right] \\ &\quad + \sum_{T \in \mathcal{T}_h} s_T(I_T^k u, \underline{v}_T) \end{aligned}$$

Gathering the above results, we get

$$\begin{aligned}\mathcal{E}_h(\underline{v}_h) &= \sum_{T \in \mathcal{T}_h} \int_T [\nabla u - \nabla r_T^{k+1}(\underline{I}_T^k u)] \cdot \nabla v_T \\ &\quad + \sum_{T \in \mathcal{T}_h} \sum_{F \in \mathcal{F}_T} \int_F [\nabla u - (\nabla r_T^{k+1}(\underline{I}_T^k u) \cdot n_{TF})] (v_F - v_T) \\ &\quad - \sum_{T \in \mathcal{T}_h} s_T(\underline{I}_T^k u, \underline{v}_T) =: \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3\end{aligned}$$

Approximation properties of the potential reconstruction I

- By definition, for all $T \in \mathcal{T}_h$, all $v \in H^1(T)$, and all $w \in \mathcal{P}^{k+1}(T)$,

$$\int_T \nabla r_T^{k+1}(\underline{I}_T^k v) \cdot \nabla w = - \int_T \pi_{\mathcal{P}^{k-1}(T)} v \Delta w + \sum_{F \in \mathcal{F}_T} \int_F \pi_{\mathcal{P}^k(F)} v (\nabla w \cdot n_{TF})$$

- Noticing that $\Delta w \in \mathcal{P}^{k-1}(T)$ and $\nabla w \cdot n_{TF} \in \mathcal{P}^k(F)$, we can remove the projectors and integrate by parts to obtain

$$\int_T \nabla r_T^{k+1}(\underline{I}_T^k v) \cdot \nabla w = \int_T \nabla v \cdot \nabla w \quad \forall w \in \mathcal{P}^{k+1}(T)$$

- This shows that $\nabla r_T^{k+1} \circ \underline{I}_T^k = \pi_{\nabla \mathcal{P}^{k+1}(T)} \circ \nabla$
- It can be proved that $\pi_{\nabla \mathcal{P}^{k+1}(T)} \circ \nabla$ **optimally approximates gradients**

Approximation properties of the potential reconstruction II

- Noticing that $\nabla v_T \in \nabla \mathcal{P}^k(T) \subset \nabla \mathcal{P}^{k+1}(T)$,

$$\int_T [\nabla u - \nabla r_T^{k+1}(\underline{I}_T^k u)] \cdot \nabla v_T = \int_T [\nabla u - \pi_{\nabla \mathcal{P}^{k+1}(T)}(\nabla u)] \cdot \nabla v_T = 0$$

we infer

$$\mathfrak{I}_1 = 0$$

- Using Cauchy–Schwarz inequalities and the definition of $\|\cdot\|_{1,h}$,

$$\mathfrak{I}_2 \leq \left(\sum_{T \in \mathcal{T}_h} h_T \|\nabla u - \pi_{\nabla \mathcal{P}^{k+1}(T)}(\nabla u)\|_{\partial T}^2 \right)^{\frac{1}{2}} \|\underline{v}_h\|_{1,h}$$

- If, additionally, $u \in H^{k+2}(\mathcal{T}_h)$,

$$\mathfrak{I}_2 \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

Polynomial consistency of the stabilization I

- To have \mathfrak{I}_3 scale as \mathfrak{I}_2 , we further assume **polynomial consistency**:

$$s_T(\underline{I}_T^k w, \underline{v}_T) = 0 \quad \forall (w, \underline{v}_T) \in \mathcal{P}^{k+1}(T) \times \underline{V}_T^k \quad (\text{ST2})$$

- For all $w \in H^{k+2}(T)$, setting $|\cdot|_{s,T} := s_T(\cdot, \cdot)^{\frac{1}{2}}$, we have

$$\begin{aligned} |\underline{I}_T^k w|_{s,T} &\stackrel{(\text{ST2})}{=} \min_{v \in \mathcal{P}^{k+1}(T)} |\underline{I}_T^k(w - v)|_{s,T} \\ &\stackrel{(\text{ST1})}{\lesssim} \min_{v \in \mathcal{P}^{k+1}(T)} \|\underline{I}_T^k(w - v)\|_{1,T} \lesssim h_T^{k+1} |w|_{H^{k+2}(T)} \end{aligned}$$

hence, by Cauchy–Schwarz inequalities and again (ST1),

$$\mathfrak{I}_3 \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)} \|\underline{v}_h\|_{1,h}$$

Polynomial consistency of the stabilization II

Theorem (Error estimate for the HHO scheme)

Denote by $u \in H_0^1(\Omega)$ the solution to the Poisson problem and by $\underline{u}_h \in \underline{V}_h^k$ its HHO approximation. Then, under (ST1)–(ST2), and further assuming $u \in H^{k+2}(\mathcal{T}_h)$, it holds

$$\|\underline{u}_h - \underline{I}_h^k u\|_{1,h} \lesssim h^{k+1} |u|_{H^{k+2}(\mathcal{T}_h)}.$$

An example of stabilization bilinear form

Example

Let, for all $T \in \mathcal{T}_h$ and all $\underline{v}_T \in \underline{V}_T^k$,

$$(\delta_T^k \underline{v}_T, (\delta_{TF}^k \underline{v}_T)_{F \in \mathcal{F}_T}) := \underline{v}_T - \underline{I}_T^k(r_T^{k+1} \underline{v}_T).$$

The stabilization bilinear form

$$s_T(\underline{w}_T, \underline{v}_T) := h_T^{-2} \int_T \delta_T^k \underline{w}_T \delta_T^k \underline{v}_T + h_T^{-1} \sum_{F \in \mathcal{F}_T} \int_F \delta_{TF}^k \underline{w}_T \delta_{TF}^k \underline{v}_T$$

satisfies properties (ST1)–(ST2).

Numerical example

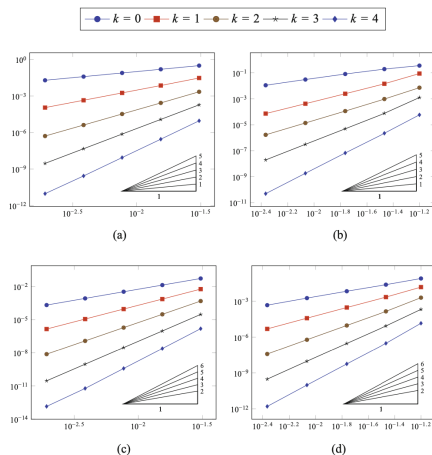


Figure: $\|e_h\|_{1,h}$ and $\|e_h\|_{L^2(\Omega)}$ as a function of h for uniformly refined triangular (top) and hexagonal (bottom) mesh families



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Thank you for your attention!

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