GLOBAL JACQUET-LANGLANDS CORRESPONDENCE FOR DIVISION ALGEBRAS IN CHARACTERISTIC p

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Abstract: We prove a full global Jacquet-Langlands correspondence between $GL(n)$ and division algebras over global fields of non zero characteristic. If D is a central division algebra of dimension n^2 over a global field F of non zero characteristic, we prove that there exists an injective map from the set of automorphic representations of $D^\times$ to the set of automorphic square integrable representations of $GL_n(F)$, compatible at all places with the local Jacquet-Langlands correspondence for unitary representations. We characterize the image of the map. As a consequence we get multiplicity one and strong multiplicity one theorems for $D^\times$.

Contents

1. Introduction

We prove the global Jacquet-Langlands correspondence between GL_n over a global field F of characteristic p and $D^\times$ where D is a central division algebra of dimension n^2 over F. A corollary is the multiplicity one and strong multiplicity one theorem for $D^\times$. We then answer two questions asked by Laumon, Rapoport and Stuhler. The first case of full global Jacquet-Langlands correspondence was proved by Jacquet and Langlands [24], for $n = 2$. This is a monumental work which served as an example for all the other proofs so far. By "full" we mean that there is no condition on the representations to transfer. Notice that "partial" correspondences, say, for automorphic representations which are cuspidal at two places or so, are very useful, but never imply as a corollary the multiplicity one theorem for inner forms.

For $n = 3$ and F of zero characteristic a full global correspondence was proved by Flath in [18]. Then in zero characteristic for D satisfying the additional condition that D is a division algebra at every place where it does not split by Vignéras ([42], never published) and later by Harris and Taylor ([21], Chap. VI). The correspondence for every n and without condition on D is proved in zero characteristic in [9] and [12] (the first paper assumes that D splits at all infinite places, and in the second this condition is dropped). Only some partial cases of the Jacquet-Langlands correspondences were proved in non zero characteristic, mainly for practical purposes (need to construct a representation doing this or that), for instance in [23], Appendix 2, and [29].

As far as we know, our result is the first case of full correspondence in non zero characteristic. The main ingredients lacking in the previous proofs are the local transfer of all unitary representations and a trace formula in non zero characteristic. Laumon and Lafforgue developed the trace formula in [28] and [25]. The formula is not invariant like the one in [5] so it is more difficult to use. This explains why we had to confine ourselves here only to the case when the inner form is a division algebra.

In the second section we recall the local tools we will use. We are very careful to give reference or full arguments for results which are "well known" in zero characteristic, but less well known in non zero characteristic. For instance we work only with functions with support in the regular set (which excludes for example elements whose characteristic polynomial is irreducible but not separable). The submersion theorem of Harish-Chandra allows one to easily transfer these functions in any characteristic.

In the third section we define the automorphic representations we want to transfer (the discrete series). We use the positive characteristic setting (as in [28] and [25]) which is slightly different from the one in zero characteristic but we explain how to switch from one to another. Then we give the precise claim of our main result.

The fourth section is devoted to the proof. The main ingredient is clearly here the trace formula of Lafforgue. Without this non trivial result nothing would be possible. We show that the geometric side and the spectral side of the trace formula for GL_n take simple form when applied to functions coming from $D^\times$.

In the fifth section we give (positive) answer to some questions asked by Laumon, Rapoport and Stuhler in [29].

The correspondence proved here completes also the proofs of Lubotzky, Samuels and Vishne in [32] (see their Remark 1.6).

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2. Local

2.1. **Basic facts.** Let F be a local field and fix an algebraic closure $\bar{F}$ of F. Let D be a central division algebra of dimension d^2 over F. Let O_D be the ring of integers of D.

For r any positive integer, we denote $GL_r(D)$ the group of invertible elements of $M_r(D)$. Let B be the subgroup of upper triangular matrices and let standard parabolic subgroups be the parabolic subgroups containing B. Let $\Delta = \{1, \cdots, r-1\}$, (for $r = 1, \Delta = \emptyset$), to any subset $I \subset \Delta$ one associates an ordered partition $r_I = (r_1, \cdots, r_k)$ of r defined by the condition $\Delta \setminus I = \{r_1, r_1 + r_2, \cdots, r_1 + r_2 + ... + r_{k-1}\}$. This map is a bijection between the set of subsets of Δ and the set of ordered partitions of r. To any $I \subset \Delta$ one associates the subgroup $M_I(D)$ of $GL_r(D)$ which is the group of block diagonal invertible matrices with blocks of size $r_1, r_2, ..., r_k$ (the components of the partition r_I) with coefficients in D, the unipotent sub-group $N_I(D)$ which is the group of corresponding upper block triangular matrices with unit matrices on the diagonal and the associated parabolic subgroup $P_I(D) = M_I(D)N_I(D)$. The groups $M_I(D)$ will be called **standard Levi subgroups of** $GL_r(D)$.

If P is associated to r_I consisting of a k-tuple we denote $|P| := k$. There is a bijection between the set $\mathcal{P}_0^s$ of standard parabolic subgroups of $GL_r(D)$, the set of ordered partition R of r, and the subsets of Δ. If $P = P_I$ is a standard parabolic subgroup of $GL_r(D)$ we will denote $M_P := M_I$ the standard Levi component of P and $N_P := N_I$ its unipotent radical. Two important parabolic subgroups are the one corresponding to $I = \emptyset$ and to $I = \Delta$. We have $r_\emptyset = (1, 1, ..., 1), P_\emptyset$ is the standard minimal parabolic subgroup of $GL_r(D)$ and $M_0 = M_\emptyset$ is the group $diag(D^\times, ..., D^\times)$. We have $r_\Delta = (r), P_\Delta = M_\Delta = GL_r(D)$.

Let K be the maximal compact subgroup $GL_r(O_D)$ of $GL_r(D)$. We endow $GL_r(D)$ with the Haar measure dg such that the volume of K is one, and the center Z of $GL_r(D)$ with the Haar measure such that the volume of $Z \cap K$ is one.

Set now $G := GL_r(D)$, $A := M_r(D)$ and $n := rd$. The theory of central simple algebras allows one to define the characteristic polynomial P_g for elements $g \in A$ in spite D is non commutative. P_g is a monic polynomial of degree n with coefficients in F. It is the main tool for transferring conjugacy classes between $GL_n(F)$ and its inner forms like G.

There are (at least) two ways of defining the characteristic polynomial P_g as we recall hereafter. For details and proofs see, for example, [36] chap. 16 and 17. It is known by class field theory that the division algebra D contains an unramified extension E of F of degree d, with (cyclic) Galois group say $Gal(E/F)$, and that

$A \otimes_F E = M_n(E)$ (Corollary 13.3 and Proposition 17.10 [36]). This gives an embedding of A into $M_n(E)$. If g is an element of A, the characteristic polynomial P_g of the image of g in $M_n(E)$ does not depend on the embedding (by Skolem-Noether theorem). Also, P_g turns out to be stable by all the elements of $Gal(E/F)$, hence $P_g \in F[X]$ and this is the first definition of the characteristic polynomial. An embedding of A in $M_n(E)$ preserves the minimal polynomial, so we have that the minimal polynomial of g divides the characteristic polynomial and the roots of the characteristic polynomial in $\bar{F}$ are also roots of the minimal polynomial.

The other way of defining P_g is the following: left translation with g in $M_r(D)$ is an F linear operator $L(g)$ and it has a characteristic polynomial $P_{L(g)}$. It is a monic polynomial of degree n^2. One can prove that this polynomial is always the power n of a monic polynomial which is, again, P_g.

Let $g \in G$, we say g is **elliptic** if P_g is irreducible and has simple roots in $\bar{F}$. We say g is **regular semisimple** if P_g has simple roots in $\bar{F}$. Let $\tilde{G}$ be the set of regular semisimple elements of G, which we familiarly call the **regular set**. If $g \in \tilde{G}$, then P_g is also the minimal polynomial of g over F. If $g, h \in \tilde{G}$, then h is conjugated to g if and only if $P_g = P_h$, as showed in the following lemma. Let $\mathcal{O}_G$ be the set of conjugacy classes in G, $\tilde{\mathcal{O}}_G$ the set of conjugacy classes of regular semisimple elements and $\tilde{\mathcal{O}}_G^{ell}$ the set of conjugacy classes of elliptic elements.

For $k|n$, let $\mathcal{X}_k \subset F[X]$ be the set of monic polynomials P of degree n with distinct non zero roots in $\bar{F}$ and such that, if $P = \prod_i P_i$ is the decomposition of P in irreducible factors k divides the degree of each P_i.

Lemma 2.1. *The map $g \mapsto P_g$ is a bijection from $\tilde{\mathcal{O}}_G$ to $\mathcal{X}_d$ and from $\tilde{\mathcal{O}}_G^{ell}$ to $\mathcal{X}_n$.*

Proof. We prove only the bijection between $\tilde{\mathcal{O}}_G$ and $\mathcal{X}_d$; the bijection between $\tilde{\mathcal{O}}_G^{ell}$ and $\mathcal{X}_n$ being obvious.

First we show that $g \in \tilde{G}$ implies $P_g \in \mathcal{X}_d$. We do it by induction on r. Let $r = 1$. Then, if g is regular semisimple, then P_g is irreducible. Indeed, we know that P_g is also the minimal polynomial of g, and as D is an integral domain, it has to be irreducible. Now assume $r > 1$. If P_g is irreducible, the result is clear. Assume $P_g = P_1 P_2$ with P_1 and P_2 non constant. As it is pointed out in [27], XVII sect. 1, if D' is the opposite algebra to D and we consider the left-D'-vector space $V := D'^r$ endowed with the canonical basis, then the usual way of associating a matrix to a linear map *in the commutative case* yields here a left-D-linear isomorphism $M_r(D) \simeq End_{D'}V$. If $g \in M_r(D)$ we denote f_g the associated D'-endomorphism. As $P_g(g) = 0$, one has $P_g(f_g) = 0$. Now P_1 and P_2 are mutually prime because P_g has simple roots in $\bar{F}$. Write $UP_1 + VP_2 = 1$ with $U, V \in F[X]$. It is easy to see that, as in the commutative case, $U(f_g)P_1(f_g)$ and $V(f_g)P_2(f_g)$ are associated non zero projectors which both commute to f_g (because all the coefficients of the polynomials involved are in F), and yield a non trivial decomposition of $V = V_1 \oplus V_2$ of V into a direct sum of spaces stable by f_g. Base change implies then that g is conjugated with an element of $M_{r_1}(D) \times M_{r_2}(D) \subset M_r(D)$, $r_1 + r_2 = r$, $r_1 r_2 \neq 0$. We then apply the induction assumption. This proves $P_g \in \mathcal{X}_d$.

We show now that the map $g \mapsto P_g$ is injective. As $g \in \tilde{G}$, the subalgebra $F[g]$ of A generated by g is isomorphic to $F[X]/(P_g)$ by sending g to the class of X. So, if g and g' are such that $P_g = P'_g$, there is an isomorphism $i : F[g] \to F[g']$ sending g to g'. Assume first that P_g is irreducible. Then $F[g]$ is a field and as A is a simple algebra, the result follows by Skolem-Noether theorem which asserts that i is conjugation with an element of A. The general case follows then by induction, as before.

We show the surjectivity. Let first P be an *irreducible* monic non constant polynomial over F of degree divisible by d. Assume P has simple roots in $\bar{F}$. Consider the extension $E := F[X]/(P)$ of F of degree equal to $\deg P$. According to [36], Corollary 13.3, there exists a subfield of $M_{\frac{\deg P}{d}}(D)$ isomorphic to E. So $M_{\frac{\deg P}{d}}(D)$ contains an element g, such that $P_g = P$. Moreover g is (invertible and) regular semisimple by definition. Now pick up any element P of $\mathcal{X}_d$ and decompose it $P = \prod_i P_i$ in irreducible factors. By definition, the degree of each P_i is divisible by d. For each i, let $g_i \in M_{\frac{\deg P_i}{d}}(D)$ such that $P_{g_i} = P_i$. Then let $g \in M_r(D)$ be the element in the Levi subgroup $\prod_i GL_{\frac{\deg P_i}{d}}(D)$ whose blocks are the g_i. Then $P_g = P$. This proves the surjectivity. □

If $g \in A$ (resp. $g \in G$), A_g (resp. G_g) will be the centralizer of g in A (resp. in G). If $g \in \tilde{G}$, then $\bar{X} \mapsto g$ is an embedding of $F[X]/(P_g)$ in A with image A_g. G_g is a maximal torus of G, isomorphic to the group $A_g^\times$ of invertible elements of A_g. The set $\tilde{G}_g$ of regular semisimple elements of G_g is a dense subset of G_g. In the following we will use the lemma:

Lemma 2.2. *Let $g \in \tilde{G}$ and fix a Haar measure on G_g. Let i be a continuous automorphism of G_g such that, for all $h \in \tilde{G}_g$, $i(h)$ has the same characteristic polynomial as h. Then i is measure preserving.*

Proof. If i is conjugation by an element of G this comes from the fact that the Weyl group is finite, the Haar measure is unique up to a scalar in $\mathbb{R}_+^\times$ and a finite subgroup of $\mathbb{R}_+^\times$ is trivial.

In the general case, as g is regular semisimple, $i(g)$ is conjugated to g. So composing i with the appropriate conjugation, which is measure preserving, one may then assume that $i(g) = g$. Now there is an open neighborhood V of g in G_g such that all the elements of V are regular semisimple, and not conjugated to each other ([19] for example). The map $g \mapsto P_g$ is so injective on V. Then $i^{-1}(V)$ has the same property. If $W := V \cap i^{-1}(V)$ then W is an open neighborhood of g, and, as i preserves the characteristic polynomial we have to have $i(h) = h$ for all $h \in W$. So the restriction of i to an open set is identity and i is measure preserving. □

2.2. **Transfer of orbits.** We now change notation in order to fit to the standard literature in this field: we set $A' := M_r(D)$, $G' := GL_r(D)$, like before, and

$A := M_n(F)$, $G := GL_n(F)$. We identify the centers of G and G' by the canonical isomorphism and we call it Z. If d is a positive integer dividing n, we let $\tilde{G}^d$ be the set of elements $g \in \tilde{G}$ such that $P_g \in \mathcal{X}_d$.

We write $g \leftrightarrow g'$ and we say that g **corresponds** to g' if $g \in \tilde{G}^d$, $g' \in \tilde{G}'$ and $P_g = P_{g'}$.

Because $\tilde{\mathcal{O}}_{G'}$ is in bijection with $\mathcal{X}_d$ and $\tilde{\mathcal{O}}_G$ is in bijection with $\mathcal{X}_1$, the inclusion $\mathcal{X}_d \subset \mathcal{X}_1$ induces an injective map from $\tilde{\mathcal{O}}_{G'}$ to $\tilde{\mathcal{O}}_G$ associated to the previous correspondence.

2.3. **Transfer of centralizers.** On tori of type G_g, $g \in \tilde{G}$, of G we fix Haar measures such that if two such tori are conjugated then the measures are compatible with the conjugation. Moreover, if G_g/Z is compact (i.e. g is elliptic), we assume the measure gives volume one to G_g/Z. This is well defined thanks to the lemma 2.2.

We are going to fix Haar measures on tori $G'_{g'}$, $g' \in \tilde{G}'$, of G'. If $g' \in \tilde{G}'$, let $g \in \tilde{G}$ such that $g \leftrightarrow g'$. Then $P_g = P_{g'}$ and we get canonical isomorphisms $A_g \simeq F[X]/(P_g) \simeq A'_{g'}$ which preserve the characteristic polynomial. Then we get an isomorphism $G_g \simeq G'_{g'}$ (both are isomorphic to $(F[X]/(P_g))^\times$) and we use this isomorphism to define a Haar measure on $G'_{g'}$ through transfer from G_g. This is well defined (does not depend of choices) thanks to the lemma 2.2. Moreover, if $G'_{g'}$ and $G'_{h'}$ are conjugated then the measures are compatible with the conjugation and if $G'_{g'}/Z$ is compact (i.e. g' is elliptic), the measure gives volume one to $G'_{g'}/Z$.

2.4. **Transfer of functions.** If C is a non empty subset of G, we denote

- 1_C the characteristic function of C,
- $Ad(G)C$ the set of all elements of G which are conjugated to an element of C,
- $H(C)$ the set of complex functions on G which are locally constant and has compact support included in C.

We denote $Supp(f)$ the support of a function f.

If $f \in H(G)$, then we define the orbital integral of f in a point $g \in \tilde{G}$ by

$$\Phi(f,g) := \int_{G_g \backslash G} f(x^{-1}gx)dx$$

where dx is the quotient measure. The integral is convergent ([28] proposition (4.8.9)). $\Phi(f,\cdot)$ is locally constant on $\tilde{G}$ and stable by conjugation under G. If $f \in H(\tilde{G})$, then we have $Supp(\Phi(f,\cdot)) \subset Ad(G)Supp(f) \subset \tilde{G}$.

According to the Harish-Chandra submersion theorem [19], every $g \in \tilde{G}$ has a neighborhood V in $\tilde{G}$ such that there is an open compact subgroup K_g of G and a neighborhood V_g of g in $G_g \cap \tilde{G}$ such that the map $K_g \times V_g \to V$ defined by $(k,x) \mapsto k^{-1}xk$ is an isomorphism. We will call such a neighborhood a HC-neighborhood. Notice that the orbital integral $\Phi(1_V,\cdot)$ of the characteristic function of V is a scalar multiple of $1_{Ad(G)V}$. A classical application of Harish-Chandra submersion theorem is then the following lemma:

Lemma 2.3. *Let C be an open compact subset of $\tilde{G}$. Let $\Phi : G \to \mathbb{C}$ be a locally constant function stable by conjugation, such that $Supp(\Phi) \subset Ad(G)C$. Then Φ is the orbital integral of a function $f \in H(C)$.*

Proof. Let $C = \cup_{j\in J} V_j$ be a covering of C with open sets V_j such that every set V_j is included in a HC-neighborhood. One may write $C = \coprod_{i=1}^{k} U_i$ where U_i is open compact, Φ is constant on U_i and for every i there exists j such that $U_i \subset V_j$ ([38], Lemma II.1.1.ii). Then the orbital integral of 1_{U_i} is constant and non zero on $Ad(G)U_i$ and so there is a scalar λ_i such that the orbital integral $\Phi(\lambda_i 1_{U_i}, \cdot)$ is equal to Φ on $Ad(G)U_i$. The function $f := \sum_{i=1}^{k} \lambda_i 1_{U_i}$ has the required property. □

We adopt the same notation with G' instead of G. The same results are true for G'. We write $f \leftrightarrow f'$ and say that f **corresponds** to f' if $f \in H(\tilde{G}^d)$, $f' \in H(\tilde{G}')$, and we have

- $\Phi(f, g) = \Phi(f', g'), \forall g \in \tilde{G}, \forall g' \in \tilde{G}, g \leftrightarrow g'$,
- $\Phi(f, g) = 0$ if $g \in \tilde{G} \backslash \tilde{G}^d$.

A consequence of the lemma 2.3 is the following:

Proposition 2.4. *(a) If $f \in H(\tilde{G}^d)$, then there exists $f' \in H(\tilde{G}')$ such that $f \leftrightarrow f'$.*
(b) If $f' \in H(\tilde{G}')$ then there exists $f \in H(\tilde{G}^d)$ such that $f \leftrightarrow f'$.

2.5. **Transfer of unitary representations.** If π is a smooth irreducible representation and $f \in H(G)$, one defines the finite rank operator $\pi(f)$ by the usual formula $\pi(f) := \int_G f(g)\pi(g)dg$. If π and π' are isomorphic, then $\mathrm{tr}\pi(f) = \mathrm{tr}\pi'(f)$. Let $Irr(G)$ be the set of isomorphy classes of smooth irreducible representations of G and $Irr_u(G)$ the subset of unitarizable (classes of) representations. Harish-Chandra ([19]) attached to the smooth irreducible representation π its character χ_π, defined in *any* characteristic, which verifies:

- χ_π is a locally constant function from $\tilde{G}$ to $\mathbb{C}$, which is stable by conjugation
- if $f \in H(\tilde{G})$, then for every representation σ in the isomorphy class of π one has $\mathrm{tr}\sigma(f) = \int_{\tilde{G}} f(g)\chi_\pi(g)dg$

(original result by Harish-Chandra [20]; see also [15] and [33]).

This holds also for G', and we define $Irr(G')$, $Irr_u(G')$ and χ_π for $\pi \in Irr(G')$ in the same way.

Harish-Chandra ([20]) proved, *when the characteristic of F is zero*, that $\mathrm{tr}\sigma(f) = \int_{\tilde{G}} f(g)\chi_\pi(g)dg$ for any $f \in H(G)$ (see also [15]) (resp. $f \in H(G')$). This was also proved to hold for G ([30]) and G' ([10], [31]) *when the characteristic of F is positive.*

We will frequently identify irreducible representations with their class in $Irr(G)$ when using notions which are invariant under isomorphism. Let $Irr_u^d(G)$ be the set of representations of $\pi \in Irr_u(G)$ such that the restriction of χ_π to $\tilde{G}^d$ is not null. We have the following theorem, proved in [11], which is a local Jacquet-Langlands transfer in positive characteristic for all irreducible unitary representations generalizing [16]:

Theorem 2.5. *There is a unique map* $\mathbf{LJ} : Irr_u^d(G) \to Irr_u(G')$ *such that, for every* $\pi \in Irr_u^d(G)$ *there exist* $\varepsilon(\pi) \in \{-1, 1\}$ *such that*

$$\chi_\pi(g) = \varepsilon(\pi)\chi_{\mathbf{LJ}(\pi)}(g')$$

for all $g \leftrightarrow g'$.

In general, the map **LJ** is neither injective nor surjective.

3. Main result

3.1. **Basic facts.** Let F be a global field of characteristic p i.e. a finite extension of the field of fractions $\mathbb{F}_p(X)$. Fix an algebraic closure $\bar{F}$ of F. For each place v of F, let F_v be the completion of F at v, O_v be the ring of integers of F_v, and fix once for all an algebraic closure $\bar{F}_v$ of F_v.

Let D a central division algebra over F of dimension n^2. We set $A = D$, and for each place v of F let $A_v := D \otimes_F F_v$. A_v is a central simple algebra over F_v and by Wedderburn theorem $A_v \simeq M_{n_v}(D_v)$ for some positive integer n_v and some central division algebra D_v of dimension d_v^2 over F_v such that $n_v d_v = n$. We will fix once and for all an isomorphism and identify these two algebras. We will denote O'_v the ring of integers of D_v.

We say that D **splits** at a place v if $d_v = 1$. The set V of places where D does not split is finite and it is known by the class field theory that n is the least common multiple of the d_v over all the places $v \in V$.

Let G be the group $GL_n(F)$, and for each place v of F, let G_v be the group $GL_n(F_v)$. Let then K_v be the maximal compact subgroup $GL_n(O_v)$ of G_v.

Let G' be the group $D^\times$; for every place $v \in V$, set $G'_v = A_v^\times = GL_{r_v}(D_v)$. Set then $K'_v := GL_{r_v}(O'_v)$ a maximal compact subgroup of G'_v. For $v \notin V$, we fix once for all an isomorphism $A_v \simeq M_n(F_v)$ and we identify these algebras. Notice that such an isomorphism is, by Skolem-Noether theorem, unique up to a conjugation by an invertible element of the algebra. Identify consequently G'_v and G_v and set $K'_v := K_v$.

Let $\mathbb{A}$ be the ring of adeles of F and denote $G(\mathbb{A})$ the adelic group of G with respect to the K_v. We consider G as a subgroup of $G(\mathbb{A})$ by the diagonal embedding. Let Z be the center of G; it is identified with $F^\times$, and for each place v, let Z_v be the center of G_v also identified with $F_v^\times$. Let $Z(\mathbb{A})$ be the center of $G(\mathbb{A})$, also identified with the adelic group of Z with respect to open compact subgroups $K_v \cap Z_v$. $Z(\mathbb{A})$ identifies with the group of ideles $\mathbb{A}^\times$ of F. For every place v of F, fix the Haar measure dg_v on G_v such that the volume of K_v is one, and dz_v on Z_v such that the volume of $Z_v \cap K_v$ is one. On $G(\mathbb{A})$ (resp. $Z(\mathbb{A})$) consider then the unique product Haar measure dg (resp. dz).

One defines a group morphism $\deg : \mathbb{A}^\times \to \mathbb{Z}$, as defined page 15 of [25] or Part II of [28] page 3, by

$$deg(a) = \sum_v deg(v)v(a_v)$$

where $a = (a_v)_v$, κ_v is the residual field of F_v, $deg(v)$ denotes the dimension of κ_v over $\mathbb{F}_p$ and the sum is taken over all places v of F.

This morphism is surjective (Lemma 9.1.4 page 3 of [28]). We let $a = (a_v)_v \in \mathbb{A}^\times$ an idele of degree 1. According to the Lemme 1 page 48 of [25], *we may assume that* $a_v = 1$ *outside a finite set of places* T_a *such that* $T_a \cap V = \emptyset$. This is not essential for the proof, but it highly simplifies computations. Let $J := a^{\mathbb{Z}}$ the subgroup of $Z(\mathbb{A}) \simeq \mathbb{A}^\times$ generated by a. It is not a restricted product over all the places, but may be written as a product $J_{T_a} \times \{1\}$, where J_{T_a} is a subgroup of $\times_{v\in T_a} Z_v$ and $\{1\}$ is to be understood as the trivial subgroup of the restricted product $\times'_{v\notin T_a} Z_v$.

We denote $G'(\mathbb{A})$ the adelic group of G' with respect to the K'_v. We consider $G'(F)$ as a subgroup of $G'(\mathbb{A})$ by the diagonal embedding.

There are canonical isomorphisms between the center of G and the center of G', and, for all place v, between the center of G_v and the center of G'_v, so we will identify them. The same is true for the center of $G(\mathbb{A})$ and the center of $G'(\mathbb{A})$ which will be identified. For every place v of F, fix the Haar measure dg'_v on G'_v such that the volume of K'_v is one. On $G'(\mathbb{A})$ consider then the product Haar measure dg'. For the places $v \notin V$, the identification between G_v and G'_v is compatible with these choices.

For the theory of parabolic subgroups of G we adopt the same conventions and notation as in the local case, which are the conventions of [28] for example $\Delta = \{1, \cdots, r-1\}$, and to any subset $I \subset \Delta$ we associate a standard parabolic subgroup $P_I(\mathbb{A})$, with Levi decomposition $P_I(\mathbb{A}) = M_I(\mathbb{A})N_I(\mathbb{A})$ etc.. If $P = P_I$ is a standard parabolic subgroup of $GL_n(\mathbb{A})$, we will sometimes write $M_P := M_I$ for the Levi component of P and $N_P := N_I$ for its unipotent radical, $P = M_P N_P$. Same notation over F: $P_I(F)$ etc.. $M_0 := M_\emptyset$ is the minimal standard Levi subgroup made of diagonal matrices of GL_n and $\mathcal{P}_0$ will denote the finite set of all parabolic subgroups, standard or not, containing M_0. Let $\mathcal{P}_0^s$ be the subset of $\mathcal{P}_0$ made of standard parabolic subgroups. Every $P \in \mathcal{P}_0$ has a Levi component which is a standard Levi subgroup, denoted M_P. Then, if $r_I = (r_1, ..., r_k)$ is the partition associated to M_P, we define a homomorphism $deg_{M_P} : M_P(\mathbb{A}) \to \mathbb{Z}^k$ by:

$$deg_{M_P}(g) = (deg(det(g_1)), ..., deg(det(g_k)))$$

where $g = diag(g_1, ..., g_k)$ is its block decomposition, (note:we use here the normalization of L.Lafforgue [25] p280).

3.2. **Automorphic representations.**

In this subsection we follow [28] and [25]. We will be concerned with the representation of $G(\mathbb{A})$ acting on the space of functions on $G(F)\backslash G(\mathbb{A})/J$ by right translation. We endow $G(F)\backslash G(\mathbb{A})/J$ and $G'(F)\backslash G'(\mathbb{A})/J$ with the quotient measures. According to [25], III.6.Lemme 5, $G'(F)\backslash G'(\mathbb{A})/J$ is compact, $G(F)\backslash G(\mathbb{A})/J$ has finite measure, and they both have the same measure. We denote R_G the representation of $G(\mathbb{A})$ acting on the space $L^2(G(F)\backslash G(\mathbb{A})/J)$ by right translations.

We have a variant, for any parabolic subgroup $P \in \mathcal{P}_0$ of G, of this representation which is the representation of $G(\mathbb{A})$ acting on the space of $L^2(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J)$ by right translation. We denote R_G^P this representation. Note that $R_G^G = R_G$.

We also need the representations of $M_P(\mathbb{A})$ on the space of $L^2(M_P(F)\backslash M_P(\mathbb{A})/J)$ which is important for defining the notion of discrete pair.

Let $P = MN$ be a parabolic subgroup of G, Z_M the center of M and $\chi : Z_M(F)\backslash Z_M(\mathbb{A})/J \to \mathbb{C}$ a smooth character, let $K' \subset K$ an open subgroup of K, we denote $L^2_{K'}(M(F)\backslash M(\mathbb{A})/J, \chi)$ the space of (necessarily locally constant) functions f on $M(\mathbb{A})/J$ with values in $\mathbb{C}$ such that:

- $f(zm) = \chi(z)f(m), \forall z \in Z_M(\mathbb{A})/J, \forall m \in M(\mathbb{A})/J$
- f is invariant on the left by $M(F)$
- f is invariant on the right by $K' \cap M_P(\mathbb{A})$
- f is of finite norm in the sense of Lafforgue [25] page 282.

We denote $L^2_\infty(M(F)\backslash M(\mathbb{A})/J, \chi)$ the inductive limit $\varinjlim L^2_{K'}(M(F)\backslash M(\mathbb{A})/J, \chi)$.

One therefore obtains a representation $R_{M_P,\chi}$ of $M_P(\mathbb{A})$ acting on $L^2_\infty(M(F)\backslash M(\mathbb{A})/J, \chi)$ by right translation.

An irreducible subrepresentation of $R_{M_P,\chi}$ is called a **discrete series** of $M(\mathbb{A})$. The subspace of $R_{M_P,\chi}$ generated by all irreducible subrepresentations is denoted $L^2_\infty(M(F)\backslash M(\mathbb{A})/J, \chi)_{disc}$. The isotypical components of $L^2_\infty(M(F)\backslash M(\mathbb{A})/J, \chi)_{disc}$ are called the **discrete components**.

A **discrete pair** is a couple (P, π) where $P \in \mathcal{P}_0$ and π a discrete component of R_{M_P,χ_π} for some central character χ_π (which is necessarily the central character of π).

Every discrete series π of $G(\mathbb{A})$ is isomorphic with a restricted Hilbertian tensor product of (smooth) irreducible unitary representations π_v of the groups G_v as explained in [17]. Each representation π_v is determined by π up to isomorphism and is called the **local component of π at the place** v. For almost all places v, π_v has a non zero fixed vector under K_v. We say then π_v is **spherical**.

The same definitions and properties hold for $M_P(\mathbb{A})$ and for $G'(\mathbb{A})$.

3.3. **Relation with the classical setting.** This setting is slightly different from the classical one ([34] or [1]) and references therein. This is because the quotient with this subgroup J is very convenient in non zero characteristic. As the cornerstone of our proof is the Theorem 12, VI.2 from [25], we need this definition. Let us explain quickly the link with the classical setting: let us say that a discrete series in the sense of [34] is *cctJ* if it has *central character trivial on* J. Then the discrete series of $G(\mathbb{A})$ as defined here correspond exactly to the cctJ discrete series in the classical setting. In particular, the multiplicity one theorem holds for $G(\mathbb{A})$ and our discrete series (not *a priori* for $G'(\mathbb{A})$ but we will prove it here). The other way round, a discrete series in the classical setting is always a twist with a character of a cctJ (following lemma) so proving the Jacquet-Langlands correspondence in Lafforgue's setting leads also to the desired result in the classical setting.

Lemma 3.1. (a) *Let χ be a character of $F^\times\backslash Z(\mathbb{A})$. Then there exists a character χ' of $G(F)\backslash G(\mathbb{A})$ (or $G(F)\backslash G'(\mathbb{A})$), $\chi' = p^{s \deg \circ \det}$ where s is a complex number, such that $\chi\chi'$ is trivial on J.*

(b) *If π is a discrete series in the sense of* [34] *with central character χ, if χ' is like in (a), then $\chi'^{-1} \otimes \pi$ is cctJ.*

Proof. (a) We search for s such that $\chi'(a)\chi(a) = 1$. Recall $\deg\det(a) = n$. Let z be a n-th root of $\chi(a)$. It is enough to chose $s \in \mathbb{C}$ such that $p^s = z^{-1}$. (b) is obvious. $\square$

3.4. **Claim of the correspondence.** If π is a discrete series of $G(\mathbb{A})$, we say π is D-**compatible** if the local components of π verify : for all $v \in V$, $\pi_v \in Irr_u^{d_v}(G_v)$.

We will prove the following theorems:

Theorem 3.2. Global Jacquet-Langlands correspondence.

There exists a unique map $\mathbf{G}$ *from the set of D-compatible discrete series of $G(\mathbb{A})$ to the set of discrete series of $G'(\mathbb{A})$ such that for all discrete series π of $G(\mathbb{A})$ if $\pi' = \mathbf{G}(\pi)$ then*

- $\mathbf{LJ}_v(\pi_v) = \pi'_v$ *for all places $v \in V$, and*
- $\pi_v = \pi'_v$ *for all places $v \notin V$*

where $\mathbf{LJ}_v$ denote the local Langlands-Jacquet correspondence at place v of theorem 2.5.

The map $\mathbf{G}$ *is bijective.*

Theorem 3.3. Multiplicity one Theorems for $G'(\mathbb{A})$.

(a) *If π' is a discrete series of $G'(\mathbb{A})$, then π' appears with multiplicity one in the discrete spectrum (multiplicity one theorem).*

(b) *If π', π'' are discrete series of $G'(\mathbb{A})$ such that $\pi'_v \simeq \pi''_v$ for almost all place v, then $\pi' = \pi''$ as subrepresentations of $L^2(G'(F)\backslash G'(\mathbb{A})/J)$ (strong multiplicity one theorem).*

The rest of the paper is devoted to the proof of these theorems. We will work with the Laumon-Lafforgue trace formula. Then, the lemma 3.1 (b) allows one to transpose the theorem in the classical setting.

4. The proof

4.1. **Transfer of elliptic global orbits.** Characteristic polynomials are defined in the global case like in the local case. [36] does not treat explicitly the global characteristic p case, but the reader may find it in [43]. If $g \in D$ has characteristic polynomial $P_g \in F[X] \subset F_v[X]$, then P_g is the characteristic polynomial of g as an element of A_v for all v since an embedding $D \hookrightarrow M_k(F)$ uniquely extends to a continuous embedding $A_v \hookrightarrow M_k(F_v)$.

We say an element $g \in G(F)$ (resp. $g \in G'(F)$) is **elliptic** if the characteristic polynomial of g is irreducible over F and has simple roots in $\bar{F}$. Let $\tilde{O}^{ell}_{G(F)}$ (resp. $\tilde{O}^{ell}_{G'(F)}$) be the set of elliptic conjugacy classes in $G(F)$ (resp. $G'(F)$).

Let $\mathbb{X}$ be the set of monic polynomials P of degree n with coefficients in F such that P is irreducible over F and has simple roots in $\bar{F}$. Let $\mathbb{X}_D$ be the subset of polynomials $P \in \mathbb{X}$ such that for all place $v \in V$, if $P = \prod P_i$ is the decomposition in irreducible factors of P over F_v, then for all i the integer d_v divides $\deg P_i$.

Then we have

Lemma 4.1. *(a) The map $g \mapsto P_g$ induces a bijection from $\tilde{O}^{ell}_{G(F)}$ to $\mathbb{X}$.*

(b) The map $g \mapsto P_g$ induces a bijection from $\tilde{O}^{ell}_{G'(F)}$ to $\mathbb{X}_D$.

Proof. (b) The fact that the map $\tilde{O}_{G'(F)} \to \mathbb{X}_D$ is well defined, i.e. takes values in $\mathbb{X}_D$, comes from the local analogous result.

The map is injective (this may be proved as in the local case, using the Skolem-Noether theorem).

The map is surjective: it is consequence of a result of class field theory: Let $P \in \mathbb{X}_D$ and set $L := F[X]/(P)$ which we see as an extension of F. Then $L \otimes F_v$ is a product of fields, isomorphic to $F_v[X]/(P_i)$, where P_i are the prime factors of P over F_v. The condition $P \in \mathbb{X}_D$ implies that the extension L/F verifies the condition (ii) of Proposition 5, [43] XIII sect. 3, page 253. The equivalence stated in this proposition between (ii) and (iii) implies that L is isomorphic to a subfield of D. The element $\bar{X} \in F[X]/(P) = L$ is then sent to an element $g \in D$ whose characteristic polynomial is P, as required. This proves the map is surjective.

(a) is now a particular case of (b). The surjectivity in (a), however, is easier to prove using the companion matrix. □

Let $\tilde{G}(F)^D$ be the set of $g \in G(F)$ such that $P_g \in \mathbb{X}_D$. Let $\tilde{O}^D_{G(F)}$ be the set of conjugacy classes of $\tilde{G}(F)^D$.

4.2. **Transfer of global functions.** Let $H(G(\mathbb{A}))$ (resp. $H(G'(\mathbb{A}))$ be the set of functions $f : G(\mathbb{A}) \to \mathbb{C}$ (resp. $f : G'(\mathbb{A}) \to \mathbb{C}$) such that f is a product $f = \otimes_v f_v$ over all places v of F, where $f_v \in H(G_v)$ (resp. $f_v \in H(G'_v)$) for all v, and, for almost all v, f_v is the characteristic function of K_v (resp. of K'_v). We write then $f = (f_v)_v$.

Let $\tilde{G}(\mathbb{A})^D$ be the set of elements $g \in G(\mathbb{A})$ such that for every place $v \in V$ we have $g_v \in \tilde{G}^{d_v}_v$, which is also the set of elements $g \in G(\mathbb{A})$ such that, for all place $v \in V$, if $P_{g_v} = \prod P_i$ is the decomposition of the characteristic polynomial of g_v in a product of irreducible polynomials over F_v, then d_v divides the degree of every P_i.

Let $H(\tilde{G}(\mathbb{A})^D)$ be the subset of $H(G(\mathbb{A}))$ made of functions $f = (f_v)_v \in H(G(\mathbb{A}))$ such that, for all $v \in V$, $f_v \in H(\tilde{G}^{d_v}_v)$. Let $H(\tilde{G}'(\mathbb{A}))$ be the set of functions $f' = (f'_v)_v \in H(G'(\mathbb{A}))$ such that, for all $v \in V$, $f_v \in H(\tilde{G}'_v)$.

If $f = (f_v)_v \in H(G(\mathbb{A}))$ and $f' \in H(G'(\mathbb{A}))$, we write $f \overset{\mathbb{A}}{\leftrightarrow} f'$ if $f \in H(\tilde{G}(\mathbb{A})^D)$, $f' \in H(\tilde{G}'(\mathbb{A}))$ and, for all $v \in V$, $f_v \leftrightarrow f'_v$, and for all $v \notin V$, $f_v = f'_v$. If

$f \in H(\tilde{G}(\mathbb{A})^D)$, then there exists $f' \in H(\tilde{G}'(\mathbb{A}))$ such that $f \overset{\mathbb{A}}{\leftrightarrow} f'$ and if $f' \in H(\tilde{G}'(\mathbb{A}))$, then there exists $f \in H(\tilde{G}(\mathbb{A})^D)$ such that $f \overset{\mathbb{A}}{\leftrightarrow} f'$. This is a direct consequence of the local transfer of functions.

Proposition 4.2. *If $f \in H(\tilde{G}(\mathbb{A})^D)$, we have*

(a) If $g \in G(\mathbb{A}) \setminus \tilde{G}(\mathbb{A})^D$, then $f(g) = 0$.

(b) If $g \in G(\mathbb{A})$ is conjugated to an element of a standard proper parabolic subgroup $P(\mathbb{A})$ of $G(\mathbb{A})$, then $f(g) = 0$.

(c) $G(F) \cap \tilde{G}(\mathbb{A})^D = \tilde{G}(F)^D$.

Proof. (a) and (c) are obvious.

(b) Assume $P(\mathbb{A})$ be the standard proper parabolic subgroup of $G(\mathbb{A})$ associated to the partition $(r_1, r_2, ..., r_k)$. If $g \in G(\mathbb{A})$ is conjugated to an element of $P(\mathbb{A})$, then, for every place v, the characteristic polynomial of g breaks into a product of polynomials of degrees $r_1, r_2, ..., r_k$. But there exists a place $v_0 \in V$ and there exists i such that d_{v_0} does not divide r_i (by class field theory, the least common multiple of all d_v is n). So $f(g) = 0$ by (a). $\square$

For every $o \in \tilde{O}^D_{G(F)}$ fix once and for all an element $\gamma_o \in o$. Let G_{γ_o} denote the centralizer of γ_o in G. The centralizer $G_{\gamma_o}(\mathbb{A})$ of γ_o in $G(\mathbb{A})$ is the restricted product of the local centralizers G_{v,γ_o}. These local tori are endowed with measures like in the previous section, and $G_{\gamma_o}(\mathbb{A})$ is endowed with the product measure. For $f \in H(\tilde{G}(\mathbb{A}))$ and $o \in O^D_{G(F)}$ we consider the orbital integral

$$\Phi(f; \gamma_o) = \int_{G_{\gamma_o}(\mathbb{A}) \backslash G(\mathbb{A})} f(x^{-1}\gamma_o x)\, dx$$

where dx is the quotient measure. Then $\Phi(f; \gamma_o)$ is the product of local orbital integral $\Phi(f_v; \gamma_o)$. We will also have to use orbital integrals $\Phi(f; z\gamma_o)$, where $z \in J$. As $J \subset Z(\mathbb{A})$, we have $G_{\gamma_o}(\mathbb{A}) = G_{z\gamma_o}(\mathbb{A})$.

For every $o' \in \tilde{O}_{G'(F)}$ fix once and for all an element $\gamma_{o'} \in o'$. For $f' \in H(\tilde{G}'(\mathbb{A}))$ and $\gamma_{o'}$ we define the orbital integral $\Phi(f'; \gamma_{o'})$ in the same way.

4.3. **Trace formula in characteristic** p**.** Laumon and Lafforgue developed, following ideas of Arthur, a trace formula in non zero characteristic. In this section we review the trace formula for $G(\mathbb{A})$ in characteristic p from [25] (our Theorem 4.3). This section is devoted to the definition of the ingredients of the formula (we show in the next subsection that most of them are null for suitable functions).

Let $h : G(\mathbb{A})/J \to \mathbb{C}$ be a locally constant function with compact support and P a parabolic subgroup of G. The convolution operator $\varphi \mapsto \varphi * h$ in the space of square integrable complex functions on $M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J$ is the operator $R^P_G(\check{h})$ where $\check{h}(g) = h(g^{-1})$. It is an integral operator with kernel given by:

$$K_{h,P}(x, y) = \int_{N_P(\mathbb{A})} \sum_{\gamma \in M_P(F)} h(y^{-1}\gamma n_P x) dn_P.$$

We set $K_h := K_{h,G}$.

Because the function $x \mapsto K_h(x, x)$ is not integrable in general, Arthur defined a notion of truncated trace as follows.

We define $\mathfrak{a}_\emptyset = \mathbb{Q}^r$ and for $i = 1, ..., r-1$ let α_i be the linear form on $\mathfrak{a}_\emptyset$ defined by $\alpha_i(h) = h_i - h_{i+1}$.

If $I \subset \Delta$ we denote $\mathfrak{a}_I = \{x \in \mathfrak{a}_\emptyset, \alpha_i(x) = 0, \forall i \in I\}$, so $\mathfrak{a}_\Delta = \mathbb{Q}(1, ..., 1)$. Let r_I be the partition associated to I. The projection $\mathfrak{a}_I \to \mathbb{Q}^k, h \mapsto (h_{r_1}, h_{r_1+r_2}, ..., h_{r_1+r_2+...+r_k})$ is an isomorphism. We denote λ_i for $i = 1, \cdots, r-1$,the fundamental weights, linear forms on $\mathfrak{a}_\emptyset$, vanishing on $\mathfrak{a}_\Delta$ and defined by $\lambda_i(h) = h_1+...+h_i - \frac{i}{r}(h_1+...+h_r)$.

For each $J \subset I \subset \Delta$ we have $\mathfrak{a}_J = \mathfrak{a}_I \oplus \mathfrak{a}_J^I$ where $\mathfrak{a}_J^I = \{h \in \mathfrak{a}_J, h_1 + ... + h_r = 0, \lambda_i(h) = 0, \forall i \in \Delta \setminus I\}$.

Arthur defines a function $\hat{\tau}_J^I$ from $\mathfrak{a}_\emptyset$ to $\{0, 1\}$ characteristic function of the cone

$$\mathfrak{a}_I + \{h \in \mathfrak{a}_J^I, \lambda_i(h) > 0, \forall i \in I \setminus J\} + \mathfrak{a}_\emptyset^J.$$

If $g \in G(\mathbb{A})$ we can write $g = n_\emptyset m_\emptyset k$ with $n_\emptyset \in N_\emptyset, m_\emptyset \in M_\emptyset$ and $k \in K$. $m_\emptyset$ is uniquely defined up to multiplication on the right by element of $M_\emptyset \cap K$.

Therefore one can define a map $H_\emptyset : G(\mathbb{A}) \to \mathfrak{a}_\emptyset$, with $H_\emptyset(g) = deg_{M_\emptyset}(m_\emptyset)$.

Let $T \in \mathfrak{a}_\emptyset$, one defines the Arthur truncated diagonal kernel as being the function on $G(\mathbb{A})$ defined by:

$$K_h^T(x, x) = \sum_{P \in \mathcal{P}_0^s} (-1)^{|P|-1} \sum_{\delta \in P(F)\backslash G(F)} K_{h,P}(\delta x, \delta x) \mathbf{1}_P^T(\delta x)$$

where $x \in G(\mathbb{A})$, $\mathbf{1}_P^T$ are the functions on $G(\mathbb{A})$ defined by $\mathbf{1}_P^T(g) = \hat{\tau}_I^\Delta(H_\emptyset(g) - T)$ with $P = P_I, I \subset \Delta$. This is well defined because for fixed x, the sum over δ is finite (For characteristic zero this was proved by Arthur [4] Lemma 5.1, in positive characteristic it is the lemma 11.1.1 of [28]).

The Arthur truncated diagonal kernel is a compactly supported function on $G(F)\backslash G(\mathbb{A})/J$ according to the Proposition 11 page 227 of [25]. Therefore one can define the truncated trace of $R_G(\check{h})$ denoted $Tr^T(h)$ as being

$$Tr^T(h) = \int_{G(F)\backslash G(\mathbb{A})/J} K_h^T(x, x) dx.$$

This is denoted $Tr^{\leq p}(h)$ in Lafforgue [25] where p is a polygon defined by T on page 221.

We now recall the results on the spectral side for general h.

We need some definitions.

Let $P \in \mathcal{P}_0$, one denotes Λ_P the abelian complex Lie group (of dimension $|P|-1$) of complex characters $\chi : M_P(\mathbb{A})/J \to \mathbb{C}^\times$ which factorize through the surjective homomorphism $deg_{M_P} : M_P(\mathbb{A})/J \to \mathbb{Z}^{|P|}/(r_1, ..., r_{|P|})\mathbb{Z}$.

As a result each $\lambda_P \in \Lambda_P$ can be written uniquely as $\lambda_P(m) = q^{\sum_{j=1}^k \rho_j \frac{deg(m_j)}{r_j}}$ with $(\rho_j) \in \times_{j=1}^k \mathbb{C}/\frac{2i\pi}{r_j logq}\mathbb{Z}$ and $\sum_j \frac{\rho_j}{r_j} \in \mathbb{Z}$.

We have $\Lambda_P = Im\Lambda_P \times Re\Lambda_P$ where $Im\Lambda_P$ (resp . $Re\Lambda_P$) denotes the Lie group of unitary characters (resp. of real positive characters). $Im\Lambda_P$ is a compact group, we denote $d\lambda_P$ its normalized Haar measure. If $\lambda_P \in \Lambda_P$ we can canonically extend λ_P to a function on $P(\mathbb{A})$ right $N(\mathbb{A})$ -invariant and then to a function on $G(\mathbb{A})$ right $K-$ invariant using the decomposition $G(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A})K$ ([25] p280).

If (P,π) is a discrete pair, $K' \subset K$ an open subgroup, we denote $L^2_{K'}(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)$ the space of functions $\varphi : M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J \to \mathbb{C}$ right invariant by K' and such that $\forall k \in K$ the function $\varphi_k : M_P(F)\backslash M_P(\mathbb{A})/J \to \mathbb{C}, m \mapsto \rho_P(m)^{-1}\varphi(mk)$ belongs to the isotypical component $\pi \subset L^2_\infty(M_P(F)\backslash M_P(\mathbb{A})/J,\chi_\pi)$, where we have denoted ρ_P the square root of the modular character of the group $P(\mathbb{A})$.

We let $L^2_\infty(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)$ be the inductive limit $\varinjlim L^2_{K'}(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)$.

One may endow $L^2_\infty(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)$ with a structure of $G(\mathbb{A})$ representation defined by $I_P(\pi) = ind_{P(\mathbb{A})}^{G(\mathbb{A})}(\pi \otimes 1_{N_P(\mathbb{A})})$.

At this point it is convenient to use the notation of Laumon $\pi(\lambda_P) = \pi \otimes \lambda_P$, when (P,π) is a discrete pair and $\lambda_P \in \Lambda_P$. Because it is sometimes convenient to represent $I_P(\pi(\lambda_P))$ in a vector space independent of λ_P one is led to define the multiplication operator

$$[\lambda_P] : L^2_\infty(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi) \to L^2_\infty(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi(\lambda_P)),$$

$$f \mapsto f\lambda_P,$$

which is a vector space isomorphism (here λ_P is the function defined on whole $G(\mathbb{A})$ as explained).

We denote W the Weyl group of $GL_r(F)$, it is isomorphic to the permutation group $\mathfrak{S}_r$, and we fix an inclusion $W \subset GL_r(F)$ associating to each permutation the permutation matrix. If M is a Levi subgroup containing M_0, we denote $W_M = W \cap M$. If M, M' are two Levi subgroups containing M_0 we denote $Hom(M,M')$ the set of $\sigma \in W_{M'}\backslash W/W_M$ such that $\sigma M \sigma^{-1} \subset M'$. If P, P' are two parabolic subgroups element of $\mathcal{P}_0$, we denote $Hom(P,P') = Hom(M_P, M_{P'})$.

Let (P,π) a discrete pair and $\sigma : P \to P'$ an isomorphism, each such σ is represented by an element $w \in W_{M'_P}\backslash W/W_{M_P}$, and to the representation π of $M_P(\mathbb{A})$ one can associate a representation $\sigma(\pi)$ of $M_{P'}(\mathbb{A})$ acting on the space $\{\varphi(w^{-1}.w), \varphi \in \pi\}$. Two discrete pairs (P,π) and (P',π') are said to be **equivalent** if there exists an isomorphism $\sigma : P \to P'$ and a character $\lambda_P \in \Lambda_P$ such that $\pi' = \sigma(\pi \otimes \lambda_P)$.

Let $P, P' \in \mathcal{P}_0$ satisfying the condition $M_P = M_{P'}$, and let $\varphi \in L^2_\infty(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)$. One defines the function $M_P^{P'}(\varphi,\lambda_P)$ of $g \in G(\mathbb{A})$ a usual by:

$$M_P^{P'}(\varphi,\lambda_P)(g) = \lambda_{P'}(g)^{-1}\int_{N_P(\mathbb{A})\cap N_{P'}(\mathbb{A})\backslash N_{P'}(\mathbb{A})} \frac{dn_{P'}}{dn_{P,P'}}\varphi(n_{P'}g)\lambda_P(n_{P'}g)$$

where we have denoted $dn_{P,P'}$ the normalized Haar measure on $N_P(\mathbb{A}) \cap N_{P'}(\mathbb{A})$ and $\frac{dn_{P'}}{dn_{P,P'}}$ the quotient measure on $N_P(\mathbb{A})\cap N_{P'}(\mathbb{A})\backslash N_{P'}(\mathbb{A})$, and $\lambda_{P'}$ is the function defined on $G(\mathbb{A})$ extending the character on $M_{P'}(\mathbb{A}) = M_P(\mathbb{A})$ defined by λ_P, for the precise definitions see [25].

The integral is convergent under some conditions on λ_P recalled in [25] page 285 and for fixed φ, the function $\lambda_P \mapsto M_P^{P'}(\varphi,\lambda_P)$ admits a meromorphic continuation to the whole Λ_P.

If λ_P is such that $M_P^{P'}(\varphi,\lambda_P)$ is well defined, the function $g\mapsto M_P^{P'}(\varphi,\lambda_P)(g)$ belongs to $L^2(M_{P'}(F)N_{P'}(\mathbb{A})\backslash G(\mathbb{A})/J,\pi')$ where (P',π') is the discrete pair defined by $\pi'=\sigma(\pi)$ with σ associated to an element w of the Weyl group satisfying $M_{P'}=wM_Pw^{-1}=M_P$.

The map $[\lambda_{P'}]\circ M_P^{P'}(.,\lambda_P)\circ[\lambda_P]^{-1}$:

$$L^2_\infty(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi(\lambda_P))\to L^2_\infty(M_{P'}(F)N_{P'}(\mathbb{A})\backslash G(\mathbb{A})/J,\pi'(\lambda_{P'}))$$

is an intertwining operator between the representations $I_P(\pi\otimes\lambda_P)$ and $I_{P'}(\pi'\otimes\lambda_{P'})$.

One defines $Fix(P,\pi)$ to be the finite set of couples (τ,μ_P) where τ is an isomorphism $\tau:P\to P$ and $\mu_P\in\Lambda_P$ such that π is isomorphic to $\tau(\pi\otimes\mu_P)$, μ_P is necessarily unitary. $Fix(P,\pi)$ can be endowed with a structure of finite group ([25] page 283) defined as follows:

$$(\tau',\mu'_P)(\tau,\mu_P)=(\tau'\tau,\tau^{-1}(\mu'_P)\mu_P),$$

and for each $(\tau,\lambda)\in Fix(P,\pi)$, one denotes $Fix(P,\pi,\tau,\lambda)$ the subgroup of elements of $Fix(P,\pi)$ commuting with (τ,λ). Lafforgue defines a discrete quadruplet $(P,\pi,\sigma,\lambda_\pi)$ as being a discrete pair (P,π) and a couple $(\sigma,\lambda_\pi)\in Fix(P,\pi)$. If $\sigma:P\to P'$ is an isomorphism, Lafforgue defines a generalization of the previous intertwining operator [25] page 286, $M_{P,\sigma}^{P'}(.,\lambda_P):L^2_\infty(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)\to L^2_\infty(M_{P'}(F)N_{P'}(\mathbb{A})\backslash G(\mathbb{A})/J,\sigma(\pi))$ and the operator $[\sigma(\lambda_P)]\circ M_{P,\sigma}^{\sigma(P)}(.,\lambda_P)\circ[\lambda_P]^{-1}$ is an intertwining operator between the representation $I_P(\pi\otimes\lambda_P)$ and $I_{\sigma(P)}(\sigma(\pi\otimes\lambda_P))$.

In the following, if $\phi\in L^2_\infty(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)$, $h(\phi,.)$ denotes the analytical function $\lambda_P\mapsto h(\phi,\lambda_P)=((\phi\lambda_P)\star h)\lambda_P^{-1}$.

In [25] lemma 6 page 303, Lafforgue introduces the functions $\hat{\mathbf{1}}_P^T$, which are rational functions on Λ_P and satisfy, under some condition on $\lambda_P^0\in Re\Lambda_P$:

$$(-1)^{|P|-1}\mathbf{1}_P^T(g)=\int_{Im\Lambda_P}\hat{\mathbf{1}}_P^T(\lambda_P\lambda_P^0)(\lambda_P\lambda_P^0)(g)d\lambda_P,\forall g\in(M_P(F)N_P(\mathbb{A}))\backslash G(\mathbb{A})/J.$$

.

He associates (page 299) to each permutation τ of $\mathfrak{S}_l$ two surjective maps τ^+ (resp.τ^-) from the set $\{1,..,l\}$ to $\{1,...,l^+\}$ (resp. $\{1,...,l^-\}$). Lafforgue defines (Lemma 5) a generalization of the functions $\mathbf{1}_P^T$, denoted $\mathbf{1}_{P,\tau}^T=\mathbf{1}_{\tau^-(P)}^T(1-\mathbf{1}_{\tau^+(P)}^T)$ with $\tau\in\mathfrak{S}_{|P|}$, and their Fourier transform $\hat{\mathbf{1}}_{P,\tau}^T$ which are rational functions on Λ_P satisfying the following equality on functions on $M_{\tau(P)}(F)N_{\tau(P)}(\mathbb{A})\backslash G(\mathbb{A})/J$:

$$\int_{Im\Lambda_P}d\mu_P\hat{\mathbf{1}}_{P,\tau}^T(\mu_0\mu_P)\tau(\mu_0\mu_P)(.)=(-1)^{|\tau^-(P)|-1}\mathbf{1}_{P,\tau}^T$$

where $\mu_0\in Re\Lambda_P$ is sufficiently small in the sense of Lafforgue [25] page 301.

Finally one obtains the theorem (theorem 12 page 309), where we have used the formula of the Th I.9 contained in [26] which corrects two minor misprints (the absence of $|\sigma|$ the incorrect $\tau\sigma(\lambda_\pi^\sigma)$ instead of $\tau\sigma(\lambda_\pi)$.). There is an additional misprint concerning the place of $[\tau\sigma(\lambda_\pi)]$ which should be located on the left.

Theorem 4.3. *(Lafforgue) We have*

$$Tr^T(h) = \sum_{(P,\pi,\sigma,\lambda_\pi)} Tr^T_{(P,\pi,\sigma,\lambda_\pi)}(h)$$

where the sum is taken over all good representative of equivalence classes of discrete quadruplet with π unitary and

$$Tr^T_{(P,\pi,\sigma,\lambda_\pi)}(h) = \frac{1}{|Fix(P,\pi,\sigma,\lambda_\pi)|.|\sigma|}\int_{Im_{\Lambda_{P_\sigma}}} d\lambda_\sigma$$
$$\sum_{\lambda^\sigma_\pi} Tr_{L^2(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)}(M^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(.,\lambda_\sigma,h))$$

where $M^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(.,\lambda,h)$ is a finite rank endomorphism of $L^2(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)$ defined by

$$M^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(.,\lambda,h) = \lim_{\substack{\mu_\sigma\to 1\\ \mu_\sigma\in\Lambda_{P_\sigma}}} \sum_{\tau\in\mathfrak{S}_{|P_\sigma|}} \hat{\mathbf{1}}^T_{P_\sigma,\tau}(\mu_\sigma\sigma(\lambda^\sigma_\pi)/\lambda^\sigma_\pi\sigma(\lambda_\pi))$$
$$([\tau\sigma(\lambda_\pi)]\circ M^{\tau(P)}_{P,\tau\sigma}(.,\lambda\lambda^\sigma_\pi))^{-1}\circ M^{\tau(P)}_{P,\tau}(.,\lambda^\sigma_\pi\lambda/\mu_\sigma)\circ h(.,\lambda^\sigma_\pi\lambda/\mu_\sigma).$$

In this formula we need to explain the notations P_σ, λ^σ_π.

To $(\sigma,\lambda_\pi)\in Fix(P,\pi)$, one associates a parabolic subgroup P_σ ([25] page 305). $|P_\sigma|$ is the number of cycles in the permutation σ. We denote $|\sigma|$ the integer product of the cardinal of orbits of the permutation σ. One defines $F_{(P,\pi,\sigma,\lambda_\pi)}$: $Im\Lambda_P\to Im\Lambda_P, \lambda_P\mapsto\sigma(\lambda_P)/(\lambda_P\sigma(\lambda_\pi))$, the set $X_{P,\pi,\sigma,\lambda_\pi} = F_{(P,\pi,\sigma,\lambda_\pi)}(Im\Lambda_P)\cap Im\Lambda_{P_\sigma}$ is finite and we denote $\{\lambda^\sigma_\pi\}\subset Im\Lambda_P$ a set such that the restriction $F_{(P,\pi,\sigma,\lambda_\pi)}:\{\lambda^\sigma_\pi\}\to X_{P,\pi,\sigma,\lambda_\pi}$ is a bijection. In particular we have $\sigma(\lambda^\sigma_\pi)/\lambda^\sigma_\pi\sigma(\lambda_\pi)\in Im\Lambda_{P_\sigma}$ i.e is fixed by σ. Note that the operator $M^{\tau(P)}_{P,\tau\sigma}(.,\lambda\lambda^\sigma_\pi/\mu_\sigma)$ and $[\tau\sigma(\lambda_\pi)]\circ M^{\tau(P)}_{P,\tau\sigma}(.,\lambda\lambda^\sigma_\pi)$ are vector space isomorphisms from $L^2(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)$ to $L^2(M_{\tau(P)}(F)N_{\tau(P)}(\mathbb{A})\backslash G(\mathbb{A})/J,\tau(\pi))$ because $\tau\sigma(\pi)=\tau(\pi)\otimes\tau\sigma(\lambda_\pi)^{-1}$.

4.4. **The simple spectral side.** If S is a finite set of places of F, we will write G_S for the Cartesian product $\times_{v\in S}G_v$, and G^S for the restricted product $\times'_{v\notin S}G_v$. The same, if $f\in H(\tilde{G}^D(\mathbb{A}))$, we write f_S for $\otimes_{v\in S}f_v$ (viewed as a function on G_S) and f^S for $\otimes_{v\notin S}f_v$ (view as a function on G^S). If Q is a standard parabolic subgroup of G with Levi decomposition $Q=MN$, we adopt the same notation Q_S, M_S etc.. Recall the definition of the constant term along the parabolic group Q_S of a function f_S like before: it is the function f^Q_S defined on M_S by the formula

$$f^Q_S(l) := \delta^{1/2}_{Q_S}(l)\int_{N_S}\int_{K_S} f_S(k^{-1}lnk)dkdn$$

for every $l\in M_S$, where δ_{Q_S} is the modulus function of Q_S (which plays no role here as we will show and use only that the integral vanishes under particular hypothesis). If $S=\{v\}$, i.e. is made of only one place, we replace index S simply by index v.

The subgroup J of $G(\mathbb{A})$ is not product. However, by choice of the generator a of J, we have that J is isomorphic to a subgroup of G_{T_a} which we denote

J_{T_a}, and we see $G(\mathbb{A})/J$ as the product $G_0{\times}G_V{\times}G^{T_a\cup V}$, where $G_0 = G_{T_a}/J_{T_a}$ (recall we chose T_a disjoint from V). We use the same notation for G', so that $G'(\mathbb{A})/J = G_0{\times}G'_V{\times}G^{T_a\cup V}$.

We show a simple form of the spectral side of the trace formula for functions in $H(\tilde{G}^D(\mathbb{A}))$.

Let $f \in H(\tilde{G}^D(\mathbb{A}))$, and set $h(g) := \sum_{z\in J} f(zg)$ (for each g the sum is finite as the support of f is compact). We see also h as a map from $G(\mathbb{A})/J$ to $\mathbb{C}$ locally constant with compact support. Moreover, and this is important in the sequel, h is a tensor product, namely $h = h_0 \otimes (\otimes_{v\notin T_a} h_v)$ where h_0 is a function on the quotient group G_0 and, for $v \notin T_a$, we have $h_v = f_v$.

Proposition 4.4. *We have:*

$$Tr^T(h) = \sum_\pi \mathrm{tr}\pi(h)$$

where π runs over the set of discrete series of $G(\mathbb{A})$.

Proof. We want to prove that the terms $Tr^T_{(P,\pi)}(h)$ associated to proper parabolic subgroups $P(\mathbb{A})$ in the Lafforgue's Theorem 4.3 vanish for functions h as in the proposition. This will be implied by the vanishing of $m^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(\lambda_\sigma, h) = Tr(M^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(.,\lambda_\sigma,h))$ for all $(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)$ and $\lambda_\sigma \in Im_{\Lambda_{P_\sigma}}$.

We say (P,π) is **regular** if $Fix(P,\pi)$ is reduced to one single element, the identity.

In order to simplify the argument we first explain the vanishing of this term when (P,π) is regular. This implies that we have $P_\sigma = P$ and $\{\lambda^\sigma_\pi\}$ can be chosen to be the singleton $\{1\}$; we set $\lambda := \lambda_\sigma$. Therefore $Tr^T_{(P,\pi,\sigma=1,\lambda_\pi)}(h)$ is given by the formula of proposition 4.3 and the the expression giving $\sum_{\lambda_\pi} M^T_{(P,\pi,id,\lambda_\pi,1)}(.,\lambda,h)$ is exactly the formula (11.4.10) of Laumon [28].

Let $M(\mathbb{A})$ be a proper Levi subgroup of $G(\mathbb{A})$, the proof is the same as the series of results contained in 11.5 to 11.8 in [28] which apply as soon as π is regular, based themselves on results of Arthur and particularly splitting formula for (G,M) families (see for example [2] and [3]).

Let M be a standard Levi subgroup of G. Let $(n_1,n_2,...,n_k)$ the partition of n associated to M. We say that M transfers at the place $v \in V$ if $d_v|n_i$ for all $1 \le i \le k$ (recall V is the set of places where D does not split).

Lemma 4.5. *If M is proper, then there are at least two places in V where M does not transfer.*

Proof. This comes from arithmetic consideration. For $v \in V$, we have $G'_v = GL_{r_v}(D_v)$ where $\dim_{F_v} D_v = d_v^2$ and $r_v d_v = n$. According to class field theory, we have that the Hasse invariant of D at any place $v \in V$ is of the form $\frac{r_v x_v}{n}$, with x_v positive integer and $gcd(x_v,d_v) = 1$. Moreover, $\frac{x_v}{d_v}$ is the Hasse invariant of D_v and $\sum_v \frac{r_v x_v}{n}$is an integer, which we prefer to write as: n *divides* $\sum_{v\in V} r_v x_v$. This is true in case A is a simple central algebra over F. As here A is, moreover, a division algebra, the least common multiple of $d_v, v \in V$ is n.

Let m be the greatest common divisor of the n_i. As M is proper, $m < n$, and, as the least common multiple of $d_v, v \in V$ is n, there exists at least one place, $v_0 \in V$, such that d_{v_0} does not divide m. So M does not transfer at v_0. But we also know that n divides $\sum_{v\in V} r_v x_v$.

Assume, for every $v \in V$, $v \neq v_0$, we had $d_v|m$. As $n = r_v d_v$, one has $n|mr_v$ for every $v \in V$, $v \neq v_0$. As $n|\sum_{v\in V} r_v x_v$, we have $n|mr_{v_0}x_{v_0}$. Then $d_{v_0}|mx_{v_0}$. But $gcd(d_{v_0}, x_{v_0}) = 1$, so $d_{v_0}|m$ which is not possible. □

We will use this lemma to simplify the trace formula: one of the two places, so to say, for applying the splitting formula of Arthur (Laumon for characteristic p), the other one to kill almost all the remaining terms.

Lemma 4.6. *Let M be a proper standard Levi subgroup of G and v_0 a place where M does not transfer. Set $V_\infty := V\backslash\{v_0\}$. Let $f \in H(\tilde{G}^D(\mathbb{A}))$, then we have:*

(a) For every proper parabolic subgroup Q of G containing M, $f_{V_\infty}(k^{-1}xk) = 0$, for all $k \in K_{V_\infty}$, and all $x \in Q_{V_\infty}$. In particular,

$$f_{V_\infty}^{Q_{V_\infty}} = 0.$$

(b) $Tr(I_{P^{V_\infty}}^{G^{V_\infty}}(\pi^{V_\infty}(\lambda)(f^{V_\infty}))) = 0$.

Proof. (a) Let L be the Levi component of Q containing M. According to Lemma 4.5, there are two places in V where L does not transfer, so at least one place v_L in V_∞. The support of f_{v_L} contains solely elements g such that P_g has irreducible factors of degree all divisible by d_{v_L}. Any element in Q_{v_L} has characteristic polynomial which is a product of polynomials of degrees equal to the sizes of the blocs of L. So, no element in Q_{v_L} may be conjugated to an element in the support of f_{v_L}. Now f_{V_∞} is a tensor product of functions, one of which is f_{v_L} and the result follows.

(b) The global trace is a product of local traces, and it is enough to prove that $Tr(I_{P_{v_0}}^{G_{v_0}}(\pi_{v_0})(\lambda)(f_{v_0}))) = 0$. By the same argument as before, because M does not transfer at the place v_0, the support of f_{v_0} does not meet any conjugated of M_{v_0}, and the constant term of f_{v_0} along a parabolic subgroup having M_{v_0} as Levi component vanishes. The Lemma 7.5.7 of Laumon [28](Part I page 189) shows then that the trace of the induced representation vanishes on f_{v_0}. □

We now apply the series of results contained in Laumon [28] which are based on the notion of (G, M) family. The properties of (G, M) families and of the weighted mean values have been first introduced by Arthur and their definition and properties are recalled in the review article of Arthur [1].

Let us recall that when $(c_P)_P$ is a (G, M) family of holomorphic functions on Λ_P, one can associate to it the function c_M (called in the sequel *the weighted mean value* of (c_P)) defined by the proposition 11.5.7 of Laumon [28] i.e

$$c_M(\mu) = \sum_{P\in\mathcal{P}(M)} \frac{c_P(\mu)}{\theta_P(\mu)}$$

where $\theta_P(\mu) = \prod_{\alpha\in\Delta_P}(1-\check{\alpha}(\mu))$ and $\mathcal{P}(M)$ denotes the set of parabolic subgroups having M as Levi component. The meromorphic function c_M admits a holomorphic extension on the whole Λ_P.

One can define a restriction operator on (G,M) families recalled in [1]: if Q is a parabolic subgroup of G containing M, we denote M_Q its Levi subgroup, to each (G,M) family c one associates a (M_Q,M) family denoted c^Q. One has the splitting formula of Arthur which enables to evaluate the weighted mean value of the product of two (G,M) families c,c' as:

$$(cc')_M = \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2)c_{L_1}^{P_{L_1}}{c'}_{L_2}^{P_{L_2}}.$$

where $\mathcal{L}(M)$ is the set of Levi components of parabolic subgroups containing M, P_L is a certain parabolic subgroup of G having L as Levi component and $d_M^G(L_1,L_2)$ are complex coefficients which definition are recalled in [1].

The properties of (G,M) families and of the weighted mean values are recalled in the review article of Arthur [1].

Laumon follows three major steps:

1) One first expresses $m^T_{(P,\pi,\sigma=id,\lambda_\pi,1)}(\lambda,h)$ as the value at λ_π^{-1} of the weighted mean value of a product of two (G,M) families as in example 11.5.9 of [28].

Indeed in the present notations (we denote $\mu=\mu_\sigma,$) we can define (G,M) families $c(\lambda_\pi;.),c'(.)$ as follows:

$$c_\tau(\lambda_\pi;\mu) = Tr_{L^2(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)}(([\tau(\lambda_\pi)]\circ M_{P,\tau}^{\tau(P)}(.,\lambda))^{-1}\circ M_{P,\tau}^{\tau(P)}(.,\frac{\lambda}{\lambda_\pi\mu})\circ h(.,\frac{\lambda}{\lambda_\pi\mu})).$$

This is a (G,M) family of functions of μ indexed by $\tau\in\mathfrak{S}_{|P|}$ (the set of parabolic group $\mathcal{P}(M)$ which normally indexes the (G,M) family is here indexed by τ because $\mathcal{P}(M)=\{\tau(P),\tau\in\mathfrak{S}_{|P|}\}$), and ${c'}_\tau(\mu)=\hat{\mathbf{1}}^T_{P,\tau}(\mu)\theta_\tau(\mu)$, with $\theta_\tau=\theta_P$ (where P corresponds to τ) and we have

$$m^T_{(P,\pi,\sigma=id,\lambda_\pi,1)}(\lambda,h) = \lim_{\mu\to 1}(c(\lambda_\pi;.)c'(.))_M(\mu\lambda_\pi^{-1}),$$

this is exactly the formula Example (11.5.9) of [28].

Remark: c' is a (G,M) family as soon as $T\in\mathfrak{a}_{\emptyset,\mathbb{Z}}$ where $\mathfrak{a}_{\emptyset,\mathbb{Z}}\subset\mathfrak{a}_\emptyset$ is the root lattice generated by (α_i). We assume in the sequel, as in [28], that T satisfies this integrality condition.

2) Because h satisfies the fact that its constant term h^Q vanishes for every proper parabolic subgroup containing $M(\mathbb{A})$, we have that the weighted mean value of the (M_Q,M) family $c(\lambda_\pi,.)^Q$ is equal to 0 (lemma 11.5.15 [28]), therefore using the splitting formula of Arthur we obtain that $m^T_{(P,\pi,id,\lambda_\pi,1)}(\lambda,h) = c_M(\lambda_\pi;\lambda_\pi^{-1})c'_P(\lambda_\pi^{-1})$. From the analysis of Laumon (corollary 11.5.14 [28]) if $c_M(\lambda_\pi;\lambda_\pi^{-1})$ is different from zero then λ_π is the restriction to M of a character of G. Moreover by an explicit computation involving the exact expression of c'_P given by the formula of section 10.1 of [35] or by the equivalent expression (Lemma 11.5.5 ii) of [28] one obtains that when λ_π is the restriction to M of a character of G,

$c'_P(\lambda_\pi^{-1}) = 0$ unless $\lambda_\pi = 1$ and in this case $c'_P(\lambda_\pi^{-1}) = |\Gamma_P|$ where Γ_P is the finite group appearing in [35].$|\Gamma_P|$ can be computed and is equal to $e = n/m$.

Therefore $m^T_{(P,\pi,id,\lambda_\pi,1)}(\lambda, h)$ is null unless λ_π is trivial and in this case we have

$$m^T_{(P,\pi,id,\lambda_\pi=1,1)}(\lambda, h) = |\Gamma_P| Tr(\mathcal{R}_M(\pi,\lambda) \circ h(.,\lambda))$$

where one defines the weighted mean value operator

$$\mathcal{R}_M(\pi,\lambda) = \lim_{\mu\to 1} \sum_{\tau\in\mathfrak{S}_{|P|}} \frac{1}{\theta_\tau(\mu)} (M^{\tau(P)}_{P,\tau}(.,\lambda))^{-1} \circ M^{\tau(P)}_{P,\tau}(.,\lambda/\mu).$$

3) Using this proposition, one can then express $m^T_{P,\pi,id,\lambda_\pi=1,1}(\lambda, h)$, as the value at $\mu = 1$ of the weighted mean value of the product of two (G, M) families $c_{V_\infty}, c^{V_\infty}$ given by straightforward generalization of the Lemma 11.6.6 of [28] (one has to replace ∞ in his formulas by the finite set V_∞). We assume that $h = h_{V_\infty} \otimes h^{V_\infty}$. The vanishing of the constant term of h_{V_∞} for every proper parabolic subgroup implies that by the splitting formula we have the factorization given by proposition 11.6.8 of [28]:

$$m^T_{P,\pi,id,1}(\lambda, h) = |\Gamma_P| Tr(\mathcal{R}_M(\pi_{V_\infty},\lambda) \circ Ind^{G(F_{V_\infty})}_{P(F_{V_\infty})}(\pi_{V_\infty}(\lambda)(\check{h}_{V_\infty})) Tr(I^{G^{V_\infty}}_{P^{V_\infty}}(\pi^{V_\infty}(\lambda)(\check{h}^V_\infty))),$$

and $\mathcal{R}_M(\pi_{V_\infty},\lambda)$ is the generalization of the operator given by Laumon page 194 [28] which reads in our case:

$$\mathcal{R}_M(\pi_{V_\infty},\lambda) = \lim_{\mu\to 1} \sum_{\tau\in\mathfrak{S}_{|P|}} \frac{1}{\theta_\tau(\mu)} (\widehat{M}^{\tau(P)}_{P,\tau}(.,\pi_{V_\infty},\lambda))^{-1} \circ \widehat{M}^{\tau(P)}_{P,\tau}(.,\pi_{V_\infty},\lambda/\mu),$$

where $\widehat{M}^{\tau(P)}_{P,\tau}(.,\pi_{V_\infty},\lambda) = \bigotimes_{v\in V_\infty} \widehat{M}^{\tau(P)}_{P,\tau}(.,\pi_v,\lambda)$ are tensor product of the normalized local intertwining operator of Langlands-Shahidi, see theorem 11.6.4 of [28].

After these steps which give an explicit expression for $m^T_{(P,\pi,\sigma,\lambda_\pi)}(\lambda, h)$ in term of local components, it is sufficient to show that $Tr(I^{G^{V_\infty}}_{P^{V_\infty}}(\pi^{V_\infty}(\lambda)(\check{h}^{V_\infty})))$ vanishes, which holds as a consequence of Lemma (4.6).

We therefore have shown that each of the term $Tr^T_{(P,\pi,\sigma=id,\lambda_\pi)}(h)$ vanishes when P is proper and π is regular.

When π is not regular we can generalize the previous arguments as follows.

We fix $(\sigma,\lambda_\pi) \in Fix(P,\pi)$ and we fix a choice of $\{\lambda^\sigma_\pi\}$.

Step 1. amounts to show that $m^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(\lambda_\sigma, h)$ is the evaluation at $\frac{\sigma(\lambda^\sigma_\pi)}{\lambda^\sigma_\pi\sigma(\lambda_\pi)}$ of the weighted mean value of the product of two (G, M_σ) families.

We can define (G, M_σ) families of functions, (this is proven in [35] proposition 10.8, lemma 11.9 and corollary 11.10), $c(\lambda_\pi,\lambda^\sigma_\pi;.), c'(.)$ on Λ_{P_σ} as:

$$c_\tau(\lambda_\pi,,\lambda^\sigma_\pi;\mu_\sigma) =$$

$$Tr_{L^2(M_P(F)N_P(\mathbb{A})\backslash G(\mathbb{A})/J,\pi)}([\tau\sigma(\lambda_\pi)] \circ (M^{\tau(P)}_{P,\tau\sigma}(.,\lambda_\sigma\lambda^\sigma_\pi))^{-1} \circ M^{\tau(P)}_{P,\tau}(.,\frac{\lambda_\sigma\sigma(\lambda^\sigma_\pi)}{\mu_\sigma\sigma(\lambda_\pi)}) \circ h(.,\frac{\lambda_\sigma\sigma(\lambda^\sigma_\pi)}{\mu_\sigma\sigma(\lambda_\pi)})$$

and $c'_\tau(\mu_\sigma) = \hat{\mathbf{1}}^T_{P_\sigma,\tau}(\mu_\sigma)\theta_\tau(\mu_\sigma)$, where these (G, M_σ) families are indexed by $\tau \in \mathfrak{S}_{|P_\sigma|}$.

We have

$$m^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(\lambda_\sigma,h)=\lim_{\mu_\sigma\to 1}(c(\lambda_\pi,\lambda^\sigma_\pi;.)c'(.))_{M_\sigma}(\mu_\sigma\frac{\sigma(\lambda^\sigma_\pi)}{\lambda^\sigma_\pi\sigma(\lambda_\pi)}).$$

Step 2. can be modified as follows. Because h satisfies the fact that its constant term h^Q vanishes for every proper parabolic subgroup Q containing M, we have that h^Q vanishes for every proper parabolic containing $M_\sigma\supset M$. As a result we have that the weighted mean value of the (M_Q,M_σ) family $c(\lambda_\pi,\lambda^\sigma_\pi;.)^Q$ is equal to 0. As a result, using the splitting formula we obtain that

$$m^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(\lambda_\sigma,h)=c_{M_\sigma}(\frac{\sigma(\lambda^\sigma_\pi)}{\lambda^\sigma_\pi\sigma(\lambda_\pi)})c'_{P_\sigma}(\frac{\sigma(\lambda^\sigma_\pi)}{\lambda^\sigma_\pi\sigma(\lambda_\pi)}).$$

We can do the same analysis as Laumon: the last expression is null unless $\frac{\sigma(\lambda^\sigma_\pi)}{\lambda^\sigma_\pi\sigma(\lambda_\pi)}$ is the restriction to M_σ of a character of G. In this case, $c'_{P_\sigma}(\frac{\sigma(\lambda^\sigma_\pi)}{\lambda^\sigma_\pi\sigma(\lambda_\pi)})$ is null unless $\frac{\sigma(\lambda^\sigma_\pi)}{\lambda^\sigma_\pi\sigma(\lambda_\pi)}$ is trivial.

Therefore we have the formula:

$$m^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(\lambda_\sigma,h)=|\Gamma_{P_\sigma}|Tr(\mathcal{R}^\sigma_M(\pi,\lambda_\sigma\lambda^\sigma_\pi)\circ h(\cdot,\lambda_\sigma\lambda^\sigma_\pi)),$$

where

$$\mathcal{R}^\sigma_M(\pi,\lambda_\sigma\lambda^\sigma_\pi)=\lim_{\mu_\sigma\to 1}\sum_{\tau\in\mathfrak{S}_{|P_\sigma|}}\frac{1}{\theta_\tau(\mu_\sigma)}([\tau\sigma(\lambda_\pi)]\circ M^{\tau(P)}_{P,\tau\sigma}(.,\lambda_\sigma\lambda^\sigma_\pi))^{-1}\circ M^{\tau(P)}_{P,\tau}(.,\frac{\lambda_\sigma\lambda^\sigma_\pi}{\mu_\sigma})$$

Step 3. reduces to the fact that the right hand side can be expressed as the weighted mean value at $\mu_\sigma=1$ of the product of two (G,M_σ) families defined by:

$$c_{\tau,V_\infty}(\mu_\sigma)=tr(R^\sigma_\tau(\mu_\sigma)\circ Ind^{G(F_{V_\infty})}_{P(F_{V_\infty})}(\pi_{V_\infty}(\lambda_\sigma\lambda^\sigma_\pi))(\check{h}_{V_\infty}))$$
$$c^{V_\infty}_\tau(\mu_\sigma)=tr(S^\sigma_\tau(\mu_\sigma)\circ Ind^{G^{V_\infty}}_{P^{V_\infty}}(\pi^{V_\infty}(\lambda_\sigma\lambda^\sigma_\pi))(\check{h}^{V_\infty}))$$

where

$$R^\sigma_\tau(\mu_\sigma)=\bigotimes_{v\in V_\infty}(\widehat{M}^{\tau(P)}_{P,\tau\sigma}(.,\pi_w,\lambda_\sigma\lambda^\sigma_\pi)\circ[\tau\sigma(\lambda_\pi)])^{-1}\circ\widehat{M}^{\tau(P)}_{P,\tau}(.,\pi_w,\frac{\lambda_\sigma\lambda^\sigma_\pi}{\mu_\sigma}),$$

where $\widehat{M}^{\tau(P)}_{P,\tau}(.,\pi_w,\lambda)$ are the normalized local intertwining operator of Langlands-Shahidi defined by $\widehat{M}^{\tau(P)}_{P,\tau}(.,\pi_w,\lambda)=a_\tau(\pi_w,\lambda)M^{\tau(P)}_{P,\tau}(.,\pi_w,\lambda)$, $M^{\tau(P)}_{P,\tau}(.,\pi_w,\lambda)$ is the local part at place w of $M^{\tau(P)}_{P,\tau}(.,\pi,\lambda)$ and $a_\tau(\pi_w,.)$ are rational functions of the variable $\lambda\in\Lambda_P$, whose properties are recalled in the Theorem 11.6.4 of [28].

$$S^\sigma_\tau(\mu_\sigma)=\prod_{v\in V_\infty}(a_\tau(\pi_w,\lambda_\sigma\lambda^\sigma_\pi)^{-1}a_\tau(\pi_w,\frac{\lambda_\sigma\lambda^\sigma_\pi}{\mu_\sigma}))$$
$$\times([\tau\sigma(\lambda_\pi)]\circ M^{\tau(P)}_{P,\tau\sigma}(.,\pi^{V_\infty},\lambda_\sigma\lambda^\sigma_\pi))^{-1}\circ M^{\tau(P)}_{P,\tau}(.,\pi^{V_\infty},\frac{\lambda_\sigma\lambda^\sigma_\pi}{\mu_\sigma}).$$

One has to show that $c_{V_\infty},c^{V_\infty}$ are two (G,M_σ) families. c_{V_∞} is easily shown to be a (G,M_σ) family, the only non trivial point, as in the regular case, is to show that c^{V_∞} is also a (G, M_σ) family. Proving this goes along the same line as the proof of [28] Lemma 11.6.6. We assume that $h = h_{V_\infty} \otimes h^{V_\infty}$. The vanishing of the constant term of h_{V_∞} for every proper parabolic subgroup containing M_σ implies that one obtains the exact analog of the factorization formula which reads:

$$\begin{aligned}
&\frac{1}{|\Gamma_{P_\sigma}|} m^T_{(P,\pi,\sigma,\lambda_\pi,\lambda^\sigma_\pi)}(\lambda_\sigma, h) = (c_{V_\infty}, c^{V_\infty})_{M_\sigma}(1) \\
&= (c_{V_\infty})_{M_\sigma}(1)(c^{V_\infty})_{P_\sigma}(1) \\
&= Tr(\mathcal{R}^\sigma_M(\pi_{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi) \circ Ind^{G(F_{V_\infty})}_{P(F_{V_\infty})}(\pi_{V_\infty}(\lambda_\sigma\lambda^\sigma_\pi)(\check{h}_{V_\infty})) \\
&\times Tr(\mathcal{S}^\sigma(\pi^{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi) I^{G^{V_\infty}}_{P^{V_\infty}}(\pi^{V_\infty}(\lambda_\sigma\lambda^\sigma_\pi)(\check{h}^{V_\infty}))),
\end{aligned}$$

where $\mathcal{R}^\sigma_M(\pi_{V_\infty}, \lambda)$ is the operator which reads in our case:

$$\mathcal{R}^\sigma_M(\pi_{V_\infty}, \lambda) = \lim_{\mu_\sigma \to 1} \sum_{\tau \in \mathfrak{S}_{|P_\sigma|}} \frac{1}{\theta_\tau(\mu_\sigma)} \bigotimes_{v \in V_\infty} (\widehat{M}^{\tau(P)}_{P,\tau}(., \pi_w, \lambda))^{-1} \circ \widehat{M}^{\tau(P)}_{P,\tau}(., \pi_w, \lambda/\mu_\sigma).$$

and

$$\begin{aligned}
&\mathcal{S}^\sigma(\pi^{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi) = S^\sigma_{\tau=id}(1) \\
&= ([\sigma(\lambda_\pi)] \circ M^P_{P,\sigma}(., \pi^{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi))^{-1} \circ M^P_{P,id}(., \pi^{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi) = \\
&= ([\sigma(\lambda_\pi)] \circ M^P_{P,\sigma}(., \pi^{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi))^{-1}.
\end{aligned}$$

$[\sigma(\lambda_\sigma\lambda^\sigma_\pi)] \circ M^P_{P,\sigma}(., \pi^{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi) \circ [\lambda_\sigma\lambda^\sigma_\pi]^{-1}$ is an intertwining operator between the representation $I^{G^{V_\infty}}_{P^{V_\infty}}(\pi^{V_\infty}(\lambda_\sigma\lambda^\sigma_\pi))$ and the representation $I^{G^{V_\infty}}_{P^{V_\infty}}(\sigma(\pi^{V_\infty}(\lambda_\sigma\lambda^\sigma_\pi))$. Because $\sigma(\pi \otimes \lambda_\sigma\lambda^\sigma_\pi) = \pi \otimes \lambda_\sigma\lambda^\sigma_\pi$, due to $\sigma(\lambda^\sigma_\pi) = \lambda^\sigma_\pi\sigma(\lambda_\pi)$, we therefore have that $[\lambda_\sigma\lambda^\sigma_\pi] \circ [\sigma(\lambda_\pi)] \circ M^P_{P,\sigma}(., \pi^{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi) \circ [\lambda_\sigma\lambda^\sigma_\pi]^{-1}$ is an intertwining operator of the representation $I^{G^{V_\infty}}_{P^{V_\infty}}(\pi^{V_\infty}(\lambda_\sigma\lambda^\sigma_\pi))$, which is irreducible because it is locally induced from irreducible unitary. As a result $[\sigma(\lambda_\pi)] \circ M^P_{P,\sigma}(., \pi^{V_\infty}, \lambda_\sigma\lambda^\sigma_\pi)$ is a scalar operator and it is therefore sufficient to show that

$Tr(I^{G^{V_\infty}}_{P^{V_\infty}}(\pi^{V_\infty}(\lambda_\sigma\lambda^\sigma_\pi)(\check{h}^{V_\infty}))) = 0$. But this is implied by the previous lemma. This ends the proof.

□

4.5. **The simple geometric side.** We show a simple form of the geometric side of the trace formula for functions in $H(\tilde{G}^D(\mathbb{A}))$. Like in the previous subsection, we let $f \in H(\tilde{G}^D(\mathbb{A}))$ and set $h(g) := \sum_{z \in J} f(zg)$ which we see as a map from $G(\mathbb{A})/J$ to $\mathbb{C}$ locally constant with compact support. Here again, we have to play this game between h and f for the reason that h is a function on $G(\mathbb{A})/J$ and it is not properly speaking a tensor product over places.

Proposition 4.7. *We have*

$$Tr^T(h) = \sum_{o \in \tilde{O}^D_{G(F)}} vol(G_{\gamma_o}(F) \backslash G_{\gamma_o}(\mathbb{A})/J) \sum_{z \in J} \Phi(f; z\gamma_o).$$

Proof. Recall

$$Tr^T(h) = \int_{G(F)\backslash G(\mathbb{A})/J} dg \sum_{P\in\mathcal{P}_0^s} (-1)^{|P|-1} \sum_{\delta\in P(F)\backslash G(F)} K_{h,P}(\delta g,\delta g)\mathbf{1}_P^T(\delta g)$$

where

$$K_{h,P}(x,y) = \int_{N_P(\mathbb{A})} \sum_{\gamma\in M_P(F)} h(x^{-1}\gamma n y).$$

As $f \in H(\tilde{G}(\mathbb{A})^D)$, by Proposition 4.2 (b) we have that $K_{h,P}(x,x)$ is null for proper P. So

$$Tr^T(h) = \int_{G(F)\backslash G(\mathbb{A})/J} \sum_{\gamma\in\tilde{G}(F)^D} h(g^{-1}\gamma g)\, dg.$$

Moreover, using the claim (c) of the same proposition ($\tilde{G}(\mathbb{A})^D$ is stable under conjugation), we have:

$$Tr^T(h) = \int_{G(F)\backslash G(\mathbb{A})/J} \sum_{\gamma\in\tilde{G}(F)^D} h(g^{-1}\gamma g)\, dg = \int_{G(F)\backslash G(\mathbb{A})/J} \sum_{\gamma\in\tilde{G}(F)^D} \sum_{z\in J} f(g^{-1}z\gamma g)\, dg.$$

We have

$$Tr^T(h) = \int_{G(F)\backslash G(\mathbb{A})/J} \sum_{\gamma\in\tilde{G}(F)^D} \sum_{z\in J} f(g^{-1}z\gamma g)\, dg =$$

$$= \int_{G(F)\backslash G(\mathbb{A})/J} \sum_{o\in\tilde{O}^D_{G(F)}} \sum_{\gamma\in o} \sum_{z\in J} f(g^{-1}z\gamma g)\, dg =$$

$$= \sum_{O\in\tilde{O}^D_{G(F)}} \int_{G(F)\backslash G(\mathbb{A})/J} \sum_{\gamma\in o} \sum_{z\in J} f(g^{-1}z\gamma g)\, dg =$$

$$= \sum_{o\in\tilde{O}^D_{G(F)}} \int_{G(F)\backslash G(\mathbb{A})/J} \sum_{t\in G_{\gamma_o}(F)\backslash G(F)} \sum_{z\in J} f(g^{-1}t^{-1}z\gamma_o t g)\, dg =$$

$$= \sum_{o\in\tilde{O}^D_{G(F)}} \int_{G_{\gamma_o}(F)\backslash G(\mathbb{A})/J} \sum_{z\in J} f(g^{-1}z\gamma_o g)\, dg =$$

$$= \sum_{o\in\tilde{O}^D_{G(F)}} vol(G_{\gamma_o}(F)\backslash G_{\gamma_o}(\mathbb{A})/J) \sum_{z\in J} \Phi(f; z\gamma_o).$$

□

As in the proof of Deligne-Kazhdan simple trace formula, manipulations are allowed as for these elements γ_o everything converges.

4.6. **Comparison with $G'(\mathbb{A})$.** Let $f \in H(\tilde{G}^D(\mathbb{A}))$, $f' \in H(\tilde{G}'(\mathbb{A}))$ such that $f \overset{\mathbb{A}}{\leftrightarrow} f'$. Let DS be the set of irreducible subrepresentations of R_G and DS' the set of irreducible subrepresentations of $R_{G'}$.

Proposition 4.8. *We have:*

$$\sum_{\pi \in DS} \mathrm{tr}\pi(f) = \sum_{\pi' \in DS'} \mathrm{tr}\pi'(f').$$

Proof. Set $h(g) := \sum_{z \in J} f(zg)$, $h'(g) := \sum_{z \in J} f'(zg)$ and consider h and h' as maps from $G(\mathbb{A})/J$ and respectively $G'(\mathbb{A})/J$ to $\mathbb{C}$, locally constant with compact support. It is enough to prove $\sum_{\pi \in DS} \mathrm{tr}\pi(h) = \sum_{\pi' \in DS'} \mathrm{tr}\pi'(h')$, as $\mathrm{tr}\pi(f) = \mathrm{tr}\pi(h)$ (by definition, $\pi(f) = \int_{G(\mathbb{A})} f\pi$ while $\pi(h) = \int_{G(\mathbb{A})/J} h\pi$ and the central character of π is trivial on J). Due to the hypothesis on f, the Propositions 4.4 and 4.7 imply:

$$\sum_{\pi \in DS} \mathrm{tr}\pi(h) = \sum_{o \in \tilde{O}^D_{G(F)}} vol(G_{\gamma_o}(F) \backslash G_{\gamma_o}(\mathbb{A})/J) \sum_{z \in J} \Phi(f; z\gamma_o).$$

The group $G'(F)\backslash G'(\mathbb{A})/J$ is compact ([25] III.6 Lemme 5 (ii)). So we have a similar formula:

$$\sum_{\pi' \in DS'} \mathrm{tr}\pi'(h') = \sum_{o \in \tilde{O}_{G'(F)}} vol(G'_{\gamma_o}(F) \backslash G'_{\gamma_o}(\mathbb{A})/J) \sum_{z \in J} \Phi(f'; z\gamma_o).$$

where $\{\gamma'_o\}$ is a system of representatives for $\tilde{O}_{G'(F)}$ such that $\gamma'_o \in O$ for all $o \in \tilde{O}_{G'(F)}$.

The Lemma 4.1 establishes the unique characteristic polynomial preserving bijection between $\tilde{O}^D_{G(F)}$ and $\tilde{O}_{G'(F)}$. We have then equality term by term between the right hand member of these two equalities due to choices of measures and functions compatible with the local transfer. $\square$

4.7. **End of the proof.** Now the proof goes the standard way, following ideas of Langlands. As this was usually applied in zero characteristic, we recall briefly the steps giving when needed the argument in non zero characteristic.

Let $\pi \in DS$ be D-compatible. Let U be the set of places v of F such that $v \notin T_a$, G'_v splits (i.e. $v \notin V$) and π_v is spherical. Let U^c be the set of places of F not in U, which is known to be a finite set. Let DS_π be the subset of DS made of representations τ such that $\tau_v \simeq \pi_v$ for all $v \in U$. Let DS'_π be the subset of DS' made of representations τ' such that $\tau'_v \simeq \pi_v$ for all $v \in U$. Then we have, for f, f' as before:

$$\sum_{\tau \in DS_\pi} \mathrm{tr}\tau(f) = \sum_{\tau' \in DS'_\pi} \mathrm{tr}\tau'(f'). \tag{4.1}$$

This relation 4.1 is known to be a consequence of the Proposition 4.8 and the beautiful proof due to Langlands is now "standard" (it is detailed in the paper of Flath [18] for example). The proof comes from the fact that an absolutely convergent sum of characters of non-isomorphic unitary spherical representations of G^{U^c}

is null if and only if the sum is void. This is based on the Satake isomorphism and abstract functional analysis and do not require zero characteristic.

By multiplicity one theorem ([40], [37]), $DS_\pi = \{\pi\}$. Now we take $f_v = f'_v = 1_{K_v}$ for all $v \in U$. Then $\mathrm{tr}\pi_v(f_v) = 1$ for $v \in U$. So the relation 4.1 becomes:

$$\prod_{v \in U^c} \mathrm{tr}\pi_v(f_v) = \sum_{\tau' \in DS'_\pi} \prod_{v \in U^c} \mathrm{tr}\tau'_v(f'_v).$$

We know ([8], [9] Theorem 3.2) that the number of non isomorphic representations in DS'_π is finite. As representations in $R_{G'}$ appear with finite multiplicity, the number of elements of DS'_π is finite. As the number of representations involved in the equality is finite, we may switch from traces to characters:

$$\prod_{v \in U^c} \chi_{\pi_v}(g_v) = \sum_{\tau' \in DS'_\pi} \prod_{v \in U^c} \chi_{\tau'_v}(g'_v)$$

whenever, for every $v \in U^c$, $g_v \leftrightarrow g'_v$. By Theorem 2.5, and the hypothesis that π is D-compatible, we may "transfer" characters from left to right. Writing $\mathbf{LJ}_v$ for the Jacquet-Langlands local correspondence for unitary representations at the place v:

$$0 = \epsilon \prod_{v \in U^c} \chi_{\mathbf{LJ}_v(\pi_v)}(g'_v) + \sum_{\tau' \in DS'_\pi} \prod_{v \in U^c} \chi_{\tau'_v}(g'_v)$$

where ϵ is a sign (which appears from the local transfer Theorem 2.5). If we assume the linear independence of characters on groups G'_v we have linear independence of characters for their product and we find there is just one τ' in DS'_π and it verifies $\tau' = \mathbf{LJ}_v(\pi_v)$ which is what we want. So let us give references for the linear independence in non zero characteristic. In [24] lemma 7.1 the linear independence of traces is proved, and the proof is independent of the characteristic. To pass from this result to the linear independence of characters it is enough to know the local integrability of characters. For groups like G'_v (i.e. local inner forms of GL_n in non zero characteristic) this is proved in [10] and [31].

On the other direction, to show the surjectivity, we start with $\pi' \in DS'$, let U' be the set of places v of F such that G'_v splits and π'_v is spherical. We shortly come to a relation of the same type as 4.1

$$\sum_{\tau \in DS_{\pi'}} \mathrm{tr}\tau(f) = \sum_{\tau' \in DS'_{\pi'}} \mathrm{tr}\tau'(f').$$

where now $DS_{\pi'}$ and $DS'_{\pi'}$ are made of representations which have the same local component as π' at places in U'. By the multiplicity one theorem, $DS_{\pi'}$ is void or contain a single representation. Again, the local independence of characters will show that, as $DS'_{\pi'}$ is not void, $DS_{\pi'}$ is not void neither and that the unique representation it contains is D-compatible. Then everything goes the same until the end of the proof. □

5. Answer to two questions in [29]

Here we answer two questions from [29]. Only the second one is directly related to the main result of this paper. But the same question is related in [29] also to the first, so we take the opportunity to answer it here too.

1. Let F be a local field of non zero characteristic and set $G := GL_n(F)$. Let π be a square integrable representation of G. Denote $z(\pi)$ the Zelevinsky dual of π. In [29], section 13.8, the authors ask the following question. Is there a function $f \in H(G)$ such that

(i) the orbital integrals of f are null on regular semisimple elements which are not elliptic,

(ii) if u is an irreducible unitary representation then $\mathrm{tr}u(f) \neq 0$ if and only if u is isomorphic to π or $z(\pi)$?

Such a function is known to exist if the characteristic of F is zero. We give here the proof in non zero characteristic. It is known that the Paley-Wiener theorem ([14]) allows one to construct a pseudocoefficient for π, i.e. $f \in H(G)$ such that

- $\mathrm{tr}\tau(f) = 0$ for all fully induced representation τ from any proper parabolic subgroup of G,
- $\mathrm{tr}\tau(f) = 0$ for all tempered representation τ of G such that τ is not isomorphic to π.
- $\mathrm{tr}\pi(f) = 1$.

A detailed proof of the existence may be found in [7] theorem 2.2. It is proved (op. cit. Lemme 2.4) that f satisfies then the property (i). Let us explain why f satisfies (ii). Let u be an irreducible unitary representation of G such that $\mathrm{tr}u(f) \neq 0$. Then, in the Grothendieck group of smooth representations of finite length of G, u is a sum $u = \sum_{i=1}^k s_i$ of standard representations s_i all of which have the same cuspidal support as u. A standard representation is always tempered or fully induced from a proper parabolic subgroup. The reader will find definitions and proofs in [16], A.4.f. Now $\mathrm{tr}u(f) = \sum_{i=1}^k \mathrm{tr}s_i(f)$ so there is some s_i which verifies $\mathrm{tr}s_i(f) \neq 0$. So one of the representations s_i has to be π. So u has the same cuspidal support as π, i.e. a Zelevinsky segment. According to the Tadić classification of unitary representations of GL_n ([41], any characteristic), u is fully induced from a product of Speh representations twisted with some characters. As $\mathrm{tr}\tau(f) = 0$ for any fully induced representation τ from any proper parabolic subgroup of G, the product contains only one term and u is a Speh representation. The cuspidal support of a Speh representation is easy to describe directly from its very definition (see [41] for the definition), in particular it is easy to see that it has multiplicities unless u is isomorphic to π or $z(\pi)$. This finishes the proof.

Remark that the question of [29] is asked in the Aubert (and non Zelevinsky) dual setting. But in [6] it is proved that the two duals differ by the sign $(-1)^k$ where k is the number of cuspidal representations in the cuspidal support of π.

A formula for the orbital integrals of f on the elliptic set is also conjectured in [29] 13.8, which follows, in characteristic p, from Theorems 4.3 (ii) and 5.1 of [7].

2. The second question asked in [29] is their Hypothesis 14.23. The authors explain in 14.24 that this Hypothesis would follow from the global Jacquet-Langlands correspondence. We confirm that the global Jacquet-Langlands correspondence, as stated and proved in our Theorem 3.2, implies the Hypothesis in the way described in [29] 14.23. As remarked by the authors, together with the results proved here in **1**, the Hypothesis implies then their Conjecture 14.21. Also the global Jacquet-Langlands correspondence simplifies their proof of the Theorem 14.12, as they do remark in the Remark 14.12. Indeed, let D be a central global division algebra of degree n^2 over F and π' any discrete series of $D^\times$ which is Steinberg at one split place. Then π' corresponds by Jacquet-Langlands to a discrete series π of GL_n which is Steinberg at the same place. Then π is cuspidal because it has a local component which is square integrable. So π is generic at every place. So π' is generic at every place where D splits.

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