Higher order operator splitting methods for the bidomain model

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Outline

Overview

- ischaemic heart disease leading cause of death (7.4M)
- affects 1 in 4 adults in the U.S.
- >$320B annually spent on treatment in the U.S.
- implicate abnormalities in electrical activity

Mathematical Model

- Human heart has $\sim 10^{10}$ muscle cells.
- Cell has outside and inside, separated by membrane.
- Cell interiors connected via gap junctions.
- Bidomain model: homogenization of cardiac tissue.
- Multi-scale reaction-diffusion PDE system.

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Cell Models

Cell models take the form

$$\frac{dv}{dt} = -\frac{1}{C_m}\sum_{i=1}^{n_{ion}} I_i(t, v, \mathbf{c}, \mathbf{m}),$$

$$\frac{dc_j}{dt} = g_j(t, c_j, \mathbf{m}, v), \qquad j = 1, 2, \ldots, n_c,$$

$$\frac{dm_k}{dt} = \frac{m_{\infty,k} - m_k}{\tau_{m_k}}, \qquad k = 1, 2, \ldots, n_m.$$

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Let

$$\mathbf{s} = (\mathbf{c}, \mathbf{m}).$$

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\begin{aligned}
\frac{\partial \mathbf{s}}{\partial t} &= \mathbf{f}(v, \mathbf{s}), \\
\chi C_m \frac{\partial v}{\partial t} + \chi I_{ion}(v, \mathbf{s}) &= \nabla \cdot (\sigma_i \nabla v) + \nabla \cdot (\sigma_i \nabla u_e), \\
0 &= \nabla \cdot (\sigma_i \nabla v) + \nabla \cdot ((\sigma_i + \sigma_e) \nabla u_e),
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subject to

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0 &= \hat{n} \cdot (\sigma_i \nabla v + \sigma_i \nabla u_e), \\
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and initial conditions.

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- σ_i and σ_e are conductivity tensors.
- χ is the area of the cell membrane per unit volume.
- C_m is the cell membrane capacitance per unit area.

Monodomain Model

$$\begin{aligned}
\frac{\partial \mathbf{s}}{\partial t} &= \mathbf{f}(v, \mathbf{s}),\\
\frac{\partial v}{\partial t} &= \frac{\lambda}{1+\lambda}\frac{1}{\chi C_m}\nabla \cdot \sigma_i \nabla v - \frac{1}{C_m}(\mathbf{s}, v, t),
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subject to boundary

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- Rewrite u_e in terms of v
- Set $\lambda = \sigma_e / \sigma_i$

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and initial conditions.

- Often solved via operator splitting.
- Traditionally employed for large, strongly non-linear systems

Operator Splitting

$$\begin{aligned} \frac{\partial \mathbf{s}}{\partial t} &= \mathbf{f}(v, \mathbf{s}), \\ \chi C_m \frac{\partial v}{\partial t} + \chi I_{ion}(v, \mathbf{s}) &= 0, \end{aligned}$$

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Operator Splitting

- Specialized knowledge for each sub-problem.
- Tissue PDEs are linear; preconditioned, solved in parallel.
- Cell PDEs are independent; solver tunable to cell model.

Godunov Splitting

Consider the initial-value problem

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}) := \mathbf{f}_1(t, \mathbf{y}) + \mathbf{f}_2(t, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0.$$

One step of Godunov splitting (first-order accurate) is

$$\begin{aligned}
\text{Step } \Delta t: \qquad & \frac{d\mathbf{y}^*}{dt} = \mathbf{f}_1(t, \mathbf{y}^*), \qquad && \mathbf{y}^*(t_{n-1}) = \mathbf{y}_{n-1}. \\
\text{Step } \Delta t: \qquad & \frac{d\mathbf{y}^{**}}{dt} = \mathbf{f}_2(t, \mathbf{y}^{**}), \qquad && \mathbf{y}^{**}(t_{n-1}) = \mathbf{y}^*_n. \\
\text{Set:} \qquad & \mathbf{y}_n = \mathbf{y}^{**}_n.
\end{aligned}$$

Godunov Splitting

Two systems:

$$\frac{\mathbf{s}_n - \mathbf{s}_{n-1}}{\Delta t} = \mathbf{f}(v_{n-1}, \mathbf{s}_{n-1}, t_{n-1}),$$
$$\frac{\hat{v}_n - v_{n-1}}{\Delta t} = -\frac{1}{C_m} I_{\text{ion}}(v_{n-1}, \mathbf{s}_{n-1}, t_{n-1}),$$

and

$$\chi C_m \frac{v_n - \hat{v}_n}{\Delta t} = \nabla \cdot (\sigma_i \nabla v_n) + \nabla \cdot (\sigma_i \nabla u_{e,n}),$$
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Operator-Splitting methods

Successful First- and Second-Order OS implementation

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What about Higher Orders?

Background

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- Sheng–Suzuki (SS) Theorem : Higher Order OS Method require backward time integration
- Ill-posed for deterministic parabolic equation
  - In the bidomain model we have parabolic PDEs (reaction-diffusion)

A different class of OS methods

Consider the initial-value problem

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(t, \mathbf{y}) := \mathbf{f}_1(t, \mathbf{y}) + \mathbf{f}_2(t, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{y}_0.$$

- Idea: express $\exp(\sum_{j=1}^{n} \mathbf{f}_j \Delta t)$ as $\left(e^{\alpha \mathbf{f}_1} e^{\alpha \mathbf{f}_2}, \ldots, e^{\alpha \mathbf{f}_n}\right)^b$

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  - $b = \pm 1 \rightarrow$ allows for backward time integration
  - $\alpha \rightarrow$ chosen using the Campbell–Baker–Hausdorff (CBH) formula

The CBH formula

The CBH formula, up to 5^{th} order reads

$$\begin{aligned}\exp(\alpha \mathbf{f}_1)\exp(\alpha \mathbf{f}_2) = \exp[&\alpha(\mathbf{f}_1 + \mathbf{f}_2) + \frac{1}{2}\alpha^2 \mathbf{f}_{12} \\ &+ \frac{1}{12}\alpha^3(\mathbf{f}_{112} + \mathbf{f}_{221}) + \frac{1}{24}\alpha^4 \mathbf{f}_{1221} \\ &+ \frac{1}{720}\alpha^5(\mathbf{f}_{11112} - 2\mathbf{f}_{21112} \\ &- 6\mathbf{f}_{11221} - 6\mathbf{f}_{22111} - 2\mathbf{f}_{12221} + \mathbf{f}_{22221}) + \mathcal{O}(\alpha^6)]\end{aligned}$$

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- Here $\mathbf{f}_{kl\ldots mn}$ is a commutator:

$$\mathbf{f}_{kl\ldots mn} \doteq [\mathbf{f}_k,[\mathbf{f}_l,\ldots,[\mathbf{f}_m,\mathbf{f}_n]\ldots]]$$
$$\text{where } [\mathbf{f}_m,\mathbf{f}_n] \doteq \mathbf{f}_m\mathbf{f}_n-\mathbf{f}_n\mathbf{f}_m$$

New class of OS methods

Defining $\beta = \left(e^{\Delta t \mathbf{f}_1} e^{\Delta t \mathbf{f}_2}\right)$ and $(\beta)^T = \left(e^{\Delta t \mathbf{f}_2} e^{\Delta t \mathbf{f}_1}\right)$

- First-order $\rightarrow \beta$
  - $\left(e^{\Delta t \mathbf{f}_1} e^{\Delta t \mathbf{f}_2}\right) = e^{(\alpha(\mathbf{f}_1 + \mathbf{f}_2))} \rightarrow \alpha = \Delta t$
- Second-order $\rightarrow (\beta)(\beta)^T$
  - $\alpha = \frac{\Delta t}{2} \rightarrow$ Strang Splitting
- Third-order $\rightarrow (\beta)^T(\beta)(\beta)(\beta)(\beta)^T(-2\beta)^T(\beta)(\beta)(\beta)$
  - $\alpha = \frac{\Delta t}{6}$

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As expected, the third order OS has backward time integration

Numerical Results

- *Mixed root-mean-square error* (MRMS) error of v:

$$e_{MRMS} := \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\hat{v}_n - v_n}{1+|\hat{v}_n|}\right)^2}$$

- Order $p = \frac{\log(e_1/e_2)}{\log(\Delta t_1/\Delta t_2)}$

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Numerical Results:Order

Monodomain model

- *Mixed root-mean-square error* (MRMS)
- Let $p_1 = 1e-6/1e-5$ and $p_2 = 1e-5/1e-4$
- 1D model, Chebyshev nodes, ODE:RK4, PDE:SDIRK3O4

Cell Model	p_1	p_2
FitzHugh-Nagumo	3.12	3.04
TenTusscher (Epi, 2006)	3.05	2.94

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Cell Model	p_1	p_2
FitzHugh-Nagumo	2.88	2.95
TenTusscher (Epi, 2006)	3.01	2.91

Numerical Results:Order

Bidomain model

- *Mixed root-mean-square error* (MRMS)
- Let $p_1 = 1e-6/1e-5$ and $p_2 = 1e-5/1e-4$
- 3D model, Uniform grid, ODE: RK4, PDE: SDIRK2O3

Cell Model	p_1	p_2
Luo Rudy (1991)	2.98	3.08
TenTusscher (Epi, 2006)	2.91	2.94

Numerical Results: Efficiency

- *Mixed root-mean-square error* (MRMS) error of v:

$$e_{MRMS} := \sqrt{\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\hat{v}_n - v_n}{1 + |\hat{v}_n|}\right)^2}$$

  over $N = 50$ equally spaced points.
- Find largest time step such that $e_{MRMS} \lesssim 5\%$.
- Record CPU time (minimum of 100 runs).

Numerical Results: Efficiency

Monodomain model

- OS2: $\rightarrow$ Heun (ODE), Crank–Nicolson (PDE)
- $OS3_1$: $\rightarrow$ RK3 (ODE), SDIRK2O3 (PDE)
- $OS3_2$: $\rightarrow$ SDIRK3O4 (ODE), SDIRK3O4 (PDE)

Cell Model	OS2		$OS3_1$		$OS3_2$	
	Δt	Time	Δt	Time	Δt	Time
LR	5E-05	4.27s	2.1E-05	5.11s	1.2E-04	2.38s
TT	2E-05	3.46s	1.3E-05	6.18s	5.5E-05	2.91s
P	2.2E-05	3.82s	1.4E-05	4.16s	5E-05	3.28s

Where:

- LR $\rightarrow$ Luo-Rudy (1991)
- TT $\rightarrow$ TenTusscher (Epi, 2006)
- P $\rightarrow$ Pandit (2003)

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Numerical Results: Efficiency

Bidomain model

- OS2: $\rightarrow$ Heun (ODE), SDIRK202 (PDE)
- OS3: $\rightarrow$ RK3 (ODE), SDIRK2O3 (PDE)

Cell Model	Δt	OS2		OS3	
		Time	MRMS	Time	MRMS
LR	2E-03	3402.11s	0.046	3544.91s	0.035
LR	5E-03	2043.45s	0.056	2108.61s	0.044
LR	8E-03	1639.28s	0.063	1679.75s	0.048
TT	2E-03	6583.68s	0.042	7443.42s	0.038
TT	5E-03	3307.87s	0.061	3657.02s	0.043
TT	8E-03	2282.25s	0.064	2443.11s	0.046

Where:

- LR $\rightarrow$ Luo-Rudy (1991)
- TT $\rightarrow$ TenTusscher (Epi, 2006)

Numerical Results: Efficiency

Bidomain model

- OS2: $\rightarrow$ Heun (ODE), SDIRK202 (PDE)
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Conclusions

- Hopes of success: High Order Operator Splitting can be applied to the monodomain and bidomain model
- The ODE solver plays an important role when we are considering efficiency
- Third-Order OS methods seems to be faster than the Second-Order methods for the type of cell models we have tested

Future Work

- Perform stability analysis Higher-Order methods
- Perform optimization on the Third-Order method we have
- Study and implement a Fourth-Order OS method

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