Recent progress in the development of parameter free continuous finite element methods for compressible fluids

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Overview

What is the problem ?

Integration of

$$\frac{\partial U}{\partial t} + \text{ div } \mathbf{F}(U) = 0$$

or

$$\frac{\partial U}{\partial t} + \text{ div } \mathbf{F}(U) = \text{div } \mathbf{F}_v(U, \nabla U)$$

with initial and boundary condition on $\Omega \subset \mathbb{R}^d$.

Target and problems

- Target: Euler, Navier Stokes, accoustics, waves, etc
- Complex domains: use of unstructured (possibly hybrid) meshes
- look for high order in space (and time) methods: integration over long periods (i.e. many time steps). Need to minimize dissipation and dispersion.
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- Unstructured meshes:
- Numerical method : compactness of the numerical stencil for ease of implementation. Finite element like methods seem to be method of choice
- Lots of efforts in approximating div F terms: reuse this with as little as possible modifications.

How-to and problems

- Classical framework: one starts by a variational formulation, choose test and trial space, develop. This leads to form:

$$M\frac{dU}{dt}+F=0\longrightarrow\frac{dU}{dt}=-M^{-1}F$$

and use of ODE solvers
- Problems:
  1. invert the mass matrix (DG), write the mass matrix,
  2. is the mass matrix invertible ?

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About Mass matrix

- Discontinuous Galerkin methods: OK from this point of view (invertible and block diagonal)
  but DG methods have a large number of DOF, and the stabilization of discontinuities is not fully understood
- Continuous Finite Element: OK from this point of view, but the mass matrix is only sparse
  smaller number of DOF, stabilization of discontinuities : artificial viscosity which is parameter dependent
- Residual distribution methods: same number of DOFs as continuous FEM, good stabilization of discontinuities
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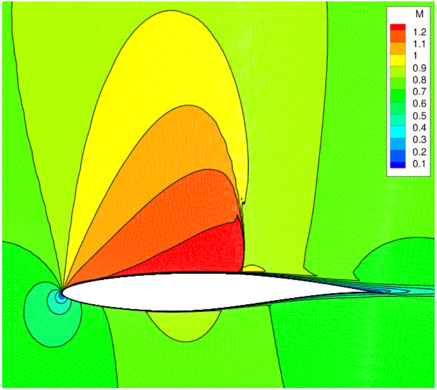
Residual distribution schemes

RAE2822 airfoil, turbulent, M=0.734, Re=6.5 10^6, AoA=2.79°, third order accurate



Mach

References:
Abgrall, Ricchiuto, de Santis, SIAM J. Scientific Computing, 2014, vol 36(3), pp A955-A983
R. Abgrall and D. de Santis, Journal of Computationnal Physics, 2015, vol 283, pp 329-359.

An example, div $(\mathbf{a}\ u) = 0$, u given on inflow boundary of Ω

- conformal $\mathcal{T}_h$ triangulation of Ω. Take $\mathbb{P}^1$ element, DOF (σ) are vertices of triangles.
- Scheme:

$$\begin{cases} \text{for } \sigma \in \Omega & \sum_{K \ni \sigma} \Phi_\sigma^K(u^h) = 0 \\ u_\sigma^h & \text{given for } \sigma \text{ inflow boundary} \end{cases}$$

- $\Phi_\sigma^K(u^h) = \beta_\sigma^K \int_K \text{div}\ (\mathbf{a}u^h) d\mathbf{x}$, $\{\beta_\sigma^K\}$ sum to unity

In the $\mathbb{P}^1$ case:

- SUPG:

$$\beta_\sigma^K = \frac{1}{3} + h_K \tau \mathbf{a} \cdot \nabla \varphi_\sigma$$

- Non linear RD

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Possible variational formulations

Write $\beta_\sigma^K = \dfrac{1}{3} + \gamma_\sigma^K$, $\sum_{\sigma\in K}\gamma_\sigma^K = 0$. [$\gamma_\sigma^K = 0$ or $h_K \tau \mathbf{a}\cdot\nabla\varphi_\sigma$, or...]

On can write:

$$\beta_\sigma^K \int_K \mathrm{div}\,(\mathbf{a}u)d\mathbf{x} = \int_K \Big(\varphi_\sigma + \gamma_\sigma^K\Big)\ \mathrm{div}\,(\mathbf{a}\ u)$$
$$= \int_K \Big(\varphi_\sigma + \gamma_\sigma^K b_K\Big)\ \mathrm{div}\,(\mathbf{a}u)d\mathbf{x}$$

$$\int_K \varphi_\sigma d\mathbf{x} = \frac{|K|}{3}$$

with b_K a bubble function of mass unity.

So the scheme can be interpreted as find $u^h \in V^h$ such as for all

- Petrov Galerkin: $v_h \in$ span $\{\varphi_\sigma + \gamma_\sigma^K, \forall\sigma \text{ DOF}\}$
- With bubble functions: or $v_h \in$ span $\{\varphi_\sigma + \gamma_\sigma^K b_K, \forall\sigma \text{ DOF and K}\}$,

$$a(u,v) = \sum_K \int_K v\ (\ \mathrm{div}\,(au)d\mathbf{x}.$$

Note: β_σ^K may depend on u^h...

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$$= \int_K \left(\varphi_\sigma + \gamma_\sigma^K b_K\right) \mathrm{div}\ (\mathbf{a}u) d\mathbf{x} \qquad \int_K b_K d\mathbf{x} = 1$$

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Partial conclusion

- There is a real need to develop finite element (like) methods for unsteady problem where there is no need of a mass matrix inversion.
- How to do this? this is the purpose of this talk.

Overview

Warming up: the $\mathbb{P}^1$ case, second order in time

$$\frac{\partial u}{\partial t} + \mathbf{a}\nabla u = 0 \qquad + \text{ initial and boundary conditions}$$

Take one's favorite FEM for the operator $\mathbf{a}\nabla u$:

$$\forall \sigma, \sum_K \int_K \psi_\sigma \ \mathbf{a}\nabla u^h = 0$$

where

$$\psi_\sigma = \varphi_\sigma, \text{ or } \varphi_\sigma + h_k \tau \mathbf{a} \cdot \nabla \varphi_\sigma, \text{ or } \varphi_\sigma + \gamma_\sigma^K, \text{or...}$$

In all cases:

- $\psi_\sigma = \varphi_\sigma + \theta_\sigma^K$ and
- $\sum_{\sigma \in K} \theta_\sigma^K = 0$ and
- $|\theta_\sigma^K| \leq C$

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Ricchiuto & Abgrall, *Explicit Runge-Kutta residual distribution schemes for time dependent problems: Second order case*, JCP 2010, v 229, pp 5653-5691

Take RK2: $\dfrac{du}{dt} = L(u)$:

$$u^{(0)} = u^n$$

$$\frac{u^{(1)} - u^{(0)}}{\Delta t} = L(u^{(0)})$$

$$\frac{u^{(2)} - u^{(0)}}{\Delta t} = \frac{1}{2}\left(L(u^{(0)}) + L(u^{(1)}\right)$$

$$u^{n+1} = u^{(2)}$$

Generic step:

$$\frac{u^{(k+1)} - u^{(0)}}{\Delta t} = \mathcal{L}(u^{(k)}, u^{(0)}).$$

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apply the variational form:

$$\sum_{K \ni \sigma} \int_K \psi_\sigma \left(\frac{u^{(k+1)} - u^{(0)}}{\Delta t} - \mathcal{L}(u^{(k)}, u^{(0)}) \right) d\mathbf{x}$$

Leads to mass matrix problem and implicit scheme.

Trick: slightly modify the scheme $[\psi_\sigma = \varphi_\sigma + \theta_\sigma]$

Goal: we want to keep the space approximation because we are happy with it.

$$\int_K \psi_\sigma \left(\frac{u^{(k+1)} - u^{(0)}}{\Delta t} - \mathcal{L}(u^{(k)}, u^{(0)}) \right) d\mathbf{x} \int_K \varphi_\sigma \left(\frac{u^{(k+1)} - u^{(0)}}{\Delta t} - \mathcal{L}(u^{(k)}, u^{(0)}) \right) d\mathbf{x}$$
$$+ \int_K \theta_\sigma \left(- \mathcal{L}(u^{(k)}, u^{(0)}) \right) d\mathbf{x}$$

Idea: Choose $\frac{\widetilde{u^{(k+1)} - u^{(0)}}}{\Delta t}$ so that we do not spoil the accuracy.

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Modified scheme

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with:

- First step: $\frac{\widetilde{u^{(1)} - u^{(0)}}}{\Delta t} = 0$
- Second step: $\frac{\widetilde{u^{(2)} - u^{(0)}}}{\Delta t} = \frac{u^{(1)} - u^{(0)}}{\Delta t}$

Proof: see *M. Ricchiuto, R. Abgrall, Explicit Runge Kutta schemes for time dependent problems: second order case, J. Comput. Phys., 229(16), pp 5653-5691, 2010.*

Idea: Analysis of the truncation error, see *R. Abgrall, Toward the ultimate conservative scheme: Following the quest. J. Comput. Phys., 167(2):277-315, 2001*

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- First step: $\frac{\widetilde{u^{(1)} - u^{(0)}}}{\Delta t} = 0 = \frac{u^{(1)} - u^{(0)}}{\Delta t} + O(\Delta t)$
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After some simple algebra:

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becomes:

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and thus:

$$\int_\Omega \varphi_\sigma \frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \sum_{K \ni \sigma} \int_K \psi_\sigma \left(\frac{u^{(k)} - u^{(0)}}{\Delta t} + \mathcal{L}(u^{(k)}, u^{(0)}) \right) = 0$$

So one can apply mass-lumping without spoiling the accuracy (for regular enough meshes)

One gets a second order scheme, oscillation free if we start from an oscillation free scheme, explicit, no mass matrix to invert

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$$\int_K \varphi_\sigma \left(\frac{u^{(k+1)} - u^{(0)}}{\Delta t} - \mathcal{L}(u^{(k)}, u^{(0)}) \right) d\mathbf{x} + \int_K \theta_\sigma \left(\frac{u^{(k+1)} - u^{(0)}}{\Delta t} - \mathcal{L}(u^{(k)}, u^{(0)}) \right) d\mathbf{x}$$

becomes:

$$\int_K \varphi_\sigma \frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \int_K \psi_\sigma \left(\frac{u^{(k)} - u^{(0)}}{\Delta t} + \mathcal{L}(u^{(k)}, u^{(0)}) \right)$$

and thus:

$$\int_\Omega \varphi_\sigma \frac{u^{(k+1)} - u^{(k)}}{\Delta t} + \sum_{K \ni \sigma} \int_K \psi_\sigma \left(\frac{u^{(k)} - u^{(0)}}{\Delta t} + \mathcal{L}(u^{(k)}, u^{(0)}) \right) = 0$$

So one can apply mass-lumping without spoiling the accuracy (for regular enough meshes)

One gets a second order scheme, oscillation free if we start from an oscillation free scheme, explicit, no mass matrix to invert

Two illustrations

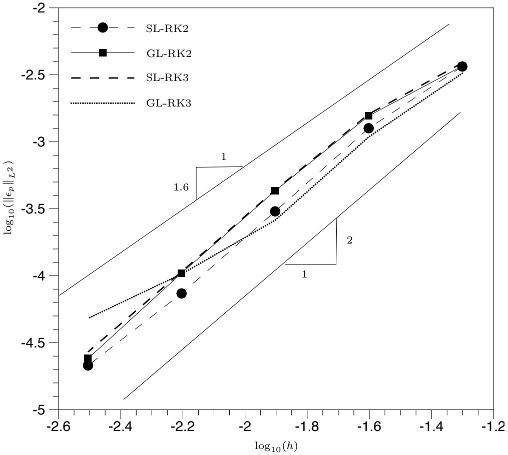

Figure 1: Vortex advection : grid convergence for the LDA scheme with F1. : convergence history.

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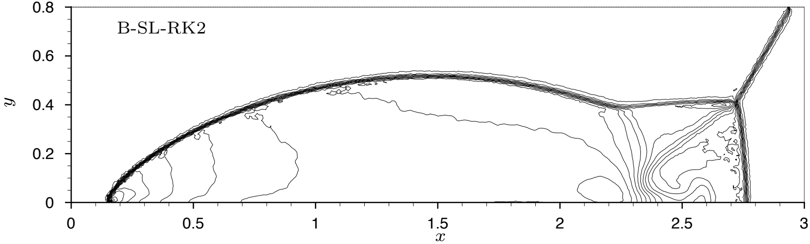

Overview

What is the essence of these manipulations ?

We have two pieces in our toolkit:

- A first order (in time scheme): From U^n, compute V such that $L^1(V, U^n) = 0$ where
$$L^1(V, U^n)_\sigma = |C_\sigma| \frac{V_\sigma - U^n_\sigma}{\Delta t} + \sum_{K \ni \sigma} \int_K \psi_\sigma \mathcal{L}(U^{(n)}, U^{(n)})$$
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- A (formaly) second order scheme: From U^n, compute W such that $L^2(W, U^n) = 0$ where
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$$\text{compute } U^{(1)} \text{ such that } L^1(U^{(1)}, U^n) = L^1(U^{(0)}, U^n) - L^2(U^{(0)}, U^n)$$
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Defect correction for ODEs $\frac{du}{dt} = L(u), u(t=0) = u_0$

[Dutt, Greegard, Rokhlin, BIT, vol 40(2), 2000]

Idea: mimic Picard iteration. in $[t_n, t_{n+1}]$, Intermediate times:$t_{n,0} = t_n < t_{n,1} < \ldots < t_{n,m} \ldots < t_{n,m} = t_{n+1}$

- Picard: $u^{n+1} = u^n + \int_{t_n}^{t_{n+1}} f(u(s))ds \approx u^n + \int_{t_n}^{t_{n+1}} I_\ell(f(u,s)\, ds$
- Define L^1 as the Euler forward method:
$$L^1(U, u^n) = \Big(U^m - u^n + \Delta t \int_{t_{n,0}}^{t_{n,m}} I_0(s)ds, \ldots$$
$$U^p - u^n + \Delta t \int_{t_{n,0}}^{t_{n,p}} I_0(s)ds, \ldots, U^0 - u^n + \Delta t \int_{t_{n,0}}^{t_{n,0}} I_0(s)ds \Big)^T$$
- Define L^2 as the high order method
$$L^2(U, u^n) = \Big(U^m - u^n + \Delta \int_{t_{n,0}}^{t_{n,m}} I_m(s)ds, \ldots$$
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- Clearly $L^1(U, u^n) - L^2(U, u^n) = O(\Delta t)$
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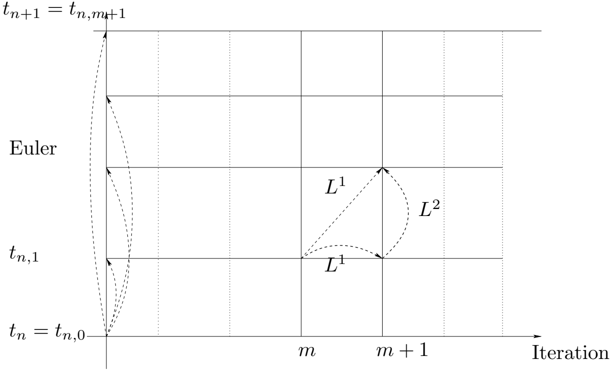
Defect correction: principle



Application to finite elements

$$\frac{\partial u}{\partial t} + \text{ div } f(u) = 0 \quad t \in [0, T],$$
$$u(0) = u_0$$

Operators

- L^2 operatior defined from:

$$\int_\Omega \psi_\sigma \frac{u^{n+1,m} - u^n}{\Delta t} + \int_{t_n}^{t_{n,k}} \psi_\sigma \text{div } I_m(f(u_l^n); s) ds = 0$$

- L^1 operator defined from:

$$|C_\sigma| \frac{u_\sigma^{n+1,m} - u_\sigma^n}{\Delta t} + \int_{t_n}^{t_{n,k}} \psi_\sigma \text{div } I_0(f(u_l^n); s) ds = 0$$

Questions:

- What is $|C_\sigma|$?
- Can we have a condition like $L^1 - L^2 = O(\Delta t) + O(h)$? If so, under which condition(s) ?

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Construction of the L^1 operator

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- $|C_\sigma| > 0$
- Under which conditions under $||L^1 - L^2|| = O(\Delta t) + O(h)$?
  We write, for any σ, $L_\sigma^\ell = (L_{\sigma,0}^\ell, L_{\sigma,1}^\ell, \dots, L_{\sigma,m}^\ell)^T$ and look for:

$$\max_{k=0,m} ||L_k^1 - L_k^2||_2.$$

We have

$$||L_k^1 - L_k^2||_2 = \sup_{v_\sigma} \frac{\sum_\sigma v_\sigma (L_\sigma^1 - L_\sigma^2)}{||v_h||_2}.$$

$$\sum_\sigma v_\sigma (L_\sigma^1 - L_\sigma^2) = \sum_K \sum_{\sigma \in K} v_\sigma (L_\sigma^1 - L_\sigma^2)$$

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$$L^1_{\sigma,p} = |C_\sigma \cap K|(U^m_\sigma - u^n_\sigma) + \int_{t_n}^{t_{n+1}} \int_K \psi_\sigma \text{ div } I_0(f(u^{n,l}; s) ds,$$

$$L^2_{\sigma,p} = \int_K \psi_\sigma (U^m - u^n) + \int_{t_n}^{t_{n+1}} \psi_\sigma \text{ div } I_m(f(u^{n,l}; s) ds$$

so that

$$\begin{aligned} L^1_{\sigma,p} - L^2_{\sigma,p} &= |C_\sigma \cap K|(U^m_\sigma - u^n_\sigma) - \int_K \psi_\sigma (U^m - u^n) \\ &\quad + \int_{t_n}^{t_{n+1}} \text{ div } \big(I_0(f(u^{n,l}; s) - I_m(f(u^{n,l}; s)\big)\psi_\sigma \end{aligned}$$

I_0 piecewise constant, I_m Lagrange interpolation of degree m

Structural conditions

If $\sum_{\sigma\in K} |C_\sigma \cap K|(U_\sigma^m - u_\sigma^n) = \int_K (U^m - u^n) d\mathbf{x}$

i.e.

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$$\sum_{\sigma \in K} |C_\sigma \cap K| (U_\sigma^m - u_\sigma^n) = \int_K (U^m - u^n) d\mathbf{x}$$

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- On simplices: Bezier polynomials, Nurbs:
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  "Proof" One can see the scheme as a perturbation of the original scheme:

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  Result from [Richtmyer-Morton] ends the proof.
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Overview

Linear case

Problem

$$\frac{\partial u}{\partial t} + \text{ div } \mathbf{f}(u, \mathbf{x}) = 0$$
$$u(\mathbf{x}, 0) = u_0(\mathbf{x})$$

with

$$\mathbf{f}(u, \mathbf{x}) = (-2\pi y, 2\pi x)\ u, \qquad u_0(\mathbf{x}) = e^{-\frac{||\mathbf{x}-\mathbf{x}_0||^2}{40}}$$

Schemes

- SUPG:

$$\Phi_\sigma^K(u^h) = \int_K \left(\varphi_\sigma + h_K \nabla_u \mathbf{f} \cdot \nabla \varphi_\sigma\right) \tau \left(\frac{\Delta u}{\Delta t} + \text{ div } \mathbf{f}(x, \mathbf{x})\right) dx,$$

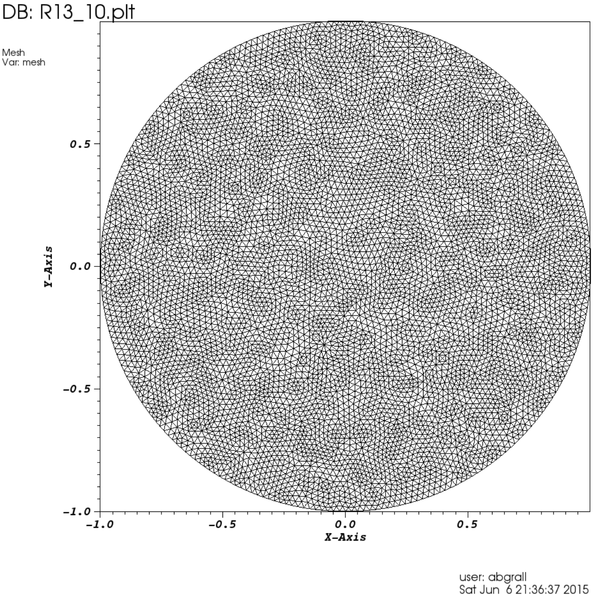
- Galerkin with jump stabilization (Burman et al)

$$\Phi_\sigma^K = \int_K \varphi_\sigma \left(\frac{\Delta u}{\Delta t} + \text{ div } \mathbf{f}(x, \mathbf{x})\right) dx + \sum_{\text{edges of K}} \Gamma h_e^2 \int_e \left(\nabla u_K^h - \nabla u_{K^+}^h\right) \cdot \nabla(\varphi_\sigma)_K.$$

1D case, $u_0(x) = e^{-80(x-0.4)^2}$

$\log_{10} h$	$\log_{10} \text{error}_{L^1}$	slope	$\log_{10} \text{error}_{L^2}$	slope	$\log_{10} \text{error}_{L^\infty}$	slope
			B_2 approximation			
-0.8239	-2.2530	-		-	-2.131	-
-1.1250	-3.2430	3.287	-3.2670	3.214	-3.088	3.178
-1.4260	-4.1820	3.119	-4.1920	3.073	-4.003	3.039
-1.7270	-5.0970	3.039	-5.1000	3.016	-4.932	3.086
-2.0280	-5.9860	2.953	-6.0010	2.993	-5.825	2.966
-2.3290	-6.9010	3.039	-6.9070	3.009	-6.746	3.059
			B_3 approximation			
-1.0000	-2.8890	-	-2.310		-1.6170	-
-1.301	-4.0040	3.704	-3.4980	3.946	-2.8430	4.073
-1.602	-5.2370	4.096	-4.6810	3.930	-3.9430	3.654
-1.903	-6.4250	3.946	-5.8790	3.980	-5.1050	3.860
-2.204	-7.6320	4.009	-7.0820	3.996	-6.2990	3.966
-2.505	-8.8350	3.996	-8.2860	4.000	-7.4990	3.986
			B_4 approximation			
-1.000	-3.5230	-	-3.0500	-	-2.3970	-
-1.3010	-5.0080	4.933	-4.4400	4.617	-3.6410	4.132
-1.6020	-6.4360	4.744	-5.9260	4.936	-5.1260	4.933
-1.9030	-7.9440	5.009	-7.4270	4.986	-6.6220	4.970
-2.2040	-9.4440	4.983	-8.9290	4.990	-8.1180	4.970
-2.5050	-10.610	3.873	-10.1900	4.189	-9.5060	4.611

Test case: 8 rotations



Test case: 8 rotations

SUPG, 2 iters

Test case: 8 rotations

SUPG, 6 iters

Test case: 8 rotations

Galerkin+jump stabilization, 2 iters

Error analysis

Rotation, $T = 1$

Initial condition: $u_0(\mathbf{x}) = e^{-\frac{||\mathbf{x}-\mathbf{x}_0||^2}{40}}$, $\qquad h \approx \sqrt{N_{dofs}}$

N_{dofs}	L^1	slope	L^2	slope	L^∞	slope
1236	$1.351\ 10^{-1}$	—	$1.335\ 10^{-1}$	—	$5.217\ 10^{-1}$	—
4821	$2.997\ 10^{-2}$	2.21	$4.207\ 10^{-2}$	1.69	$1.967\ 10^{-1}$	1.43
19041	$3.976\ 10^{-3}$	2.94	$7.133\ 10^{-3}$	2.58	$4.149\ 10^{-2}$	2.26
75681	$6.710\ 10^{-4}$	2.57	$1.217\ 10^{-3}$	2.56	$7.063\ 10^{-3}$	2.56

Second order

N_{dofs}	L^1	slope	L^2	slope	L^∞	slope
4825	$2.508\ 10^{-2}$	—	$3.056\ 10^{-2}$	—	$1.161\ 10^{-1}$	—
19041	$1.354\ 10^{-3}$	4.24	$2.592\ 10^{-3}$	3.54	$1.347\ 10^{-2}$	3.13
75297	$1.094\ 10^{-4}$	3.24	$2.003\ 10^{-4}$	3.72	$1.137\ 10^{-3}$	3.59
300993	$1.547\ 10^{-5}$	2.82	$2.653\ 10^{-5}$	2.91	$1.742\ 10^{-4}$	2.70

Third order

Non linear case

Non convex case: KPP case

$$\frac{\partial u}{\partial t} + \text{ div } \mathbf{f}(u) = 0$$
$$u(\mathbf{x}, 0) = u_0(\mathbf{x})$$

$\mathbf{f} = (\cos u, \sin u)$ and

$$u_0(\mathbf{x}) = \begin{cases} \frac{7}{2}\pi & \text{if } ||\mathbf{x}|| < 1 \\ \frac{\pi}{4} & \text{else.} \end{cases}$$

Difficulties:

- Composite waves (shock attached to fans)
- Existence of sonic points on $||x|| = 1$.

Schemes with jump filtering

Residuals

$$\Phi_\sigma(u^h) = \beta_\sigma(u^h)\Phi_{tot} + \sum_{\text{edges of } \partial K} h_e^2 ||\overline{\nabla_u f}|| \int_{\partial K} [\nabla u^h]\cdot[\nabla \varphi_\sigma] d\ell$$

Choice of $\beta_\sigma(u^h)$

- Choice 1:

$$\beta_\sigma^{PSI}(u^h) = \frac{\left(\frac{\Phi_\sigma^{LxF}}{\Phi_{tot}}\right)^+}{\sum_{\sigma'\in K}\left(\frac{\Phi_{\sigma'}^{LxF}}{\Phi_{tot}}\right)^+},$$

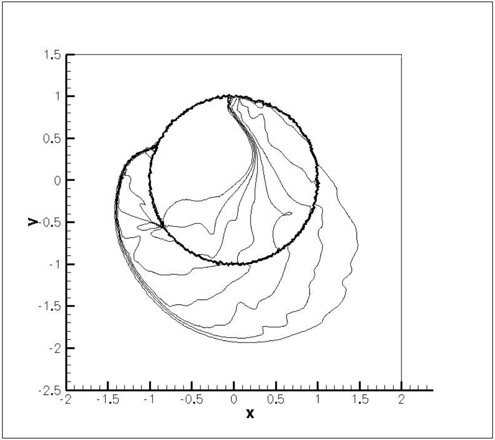
- Choice 2:

$$\beta_\sigma(u^h)\Phi_{tot} = (1-\theta)\beta_\sigma^{PSI}(u^h)\Phi_{tot} + \theta\Phi_\sigma^{LxF}$$

$$\theta = \frac{|\Phi_{tot}|}{\sum_{\sigma'\in K}|\Phi_{LxF}^\sigma|}$$

Results, $t = 1$, $CFL = 0.4$ for $O3$ and 0.8 for $O2$

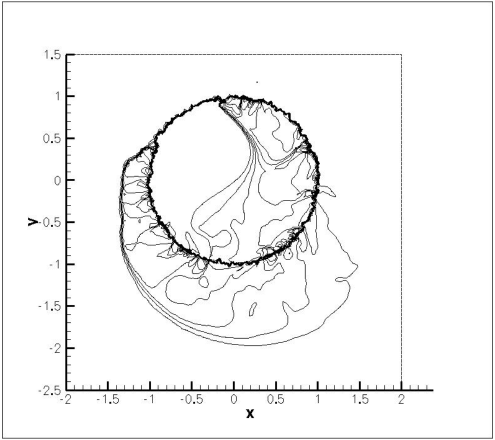
Meshes O2 and O3: same number of DOFs



O2, residual with choice 1

Results, $t = 1$, $CFL = 0.4$ for $O3$ and 0.8 for $O2$

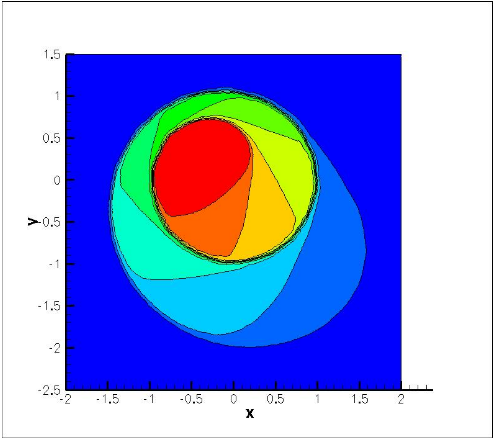
Meshes O2 and O3: same number of DOFs



O3, residual with choice 1

Results, $t = 1$, $CFL = 0.3$ for $O3$ and 0.6 for $O2$

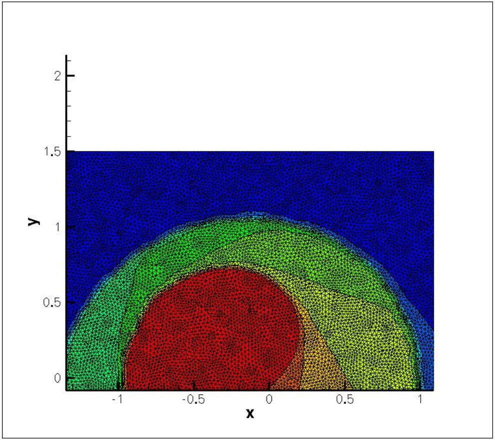
Meshes O2 and O3: same number of DOFs



O2, Choice2

Results, $t = 1$, $CFL = 0.3$ for $O3$ and 0.6 for $O2$

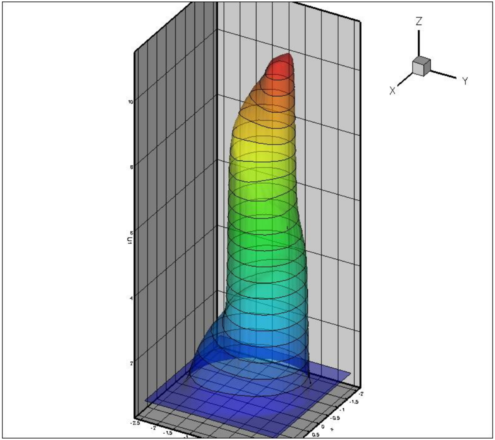
Meshes O2 and O3: same number of DOFs



O2, Choice2

Results, $t = 1$, $CFL = 0.3$ for $O3$ and 0.6 for $O2$

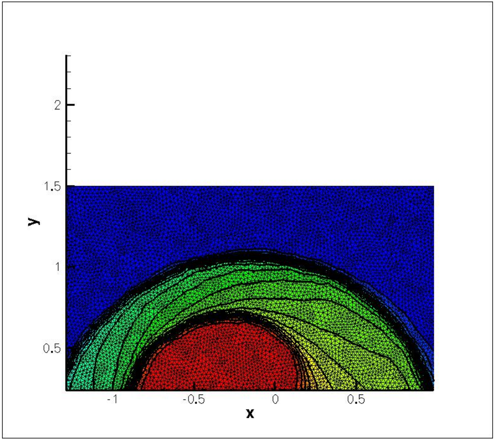
Meshes O2 and O3: same number of DOFs



O3, Choice2

Results, $t = 1$, $CFL = 0.3$ for $O3$ and 0.6 for $O2$

Meshes O2 and O3: same number of DOFs



O3, Choice2

Overview

Conclusions

- High order schemes for finite element methods *without* mass matrix: possible
- Seems to work
- Still some issues to be understood: this are preliminary results
- Go to systems and implicit (Viscous problems)

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