Well-balanced schemes for Friedrichs systems and related problems

B. Després (LJLL-UPMC) and IUF 2016, C. Buet (CEA)

thanks to E. Franck (Inria/Strasbourg), T. Leroy (CEA), X. Valentin (CEA), G. Morel (PhD CEA/LJLL)

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$\mathbf{dim}(\mathcal{V}) < n$
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Section 1

Introduction

Model reduction of kinetic equations

• Example : the Discrete Ordinates Method for approximation of transfer yields

$$\partial_t I + \mu \partial_x I = \sigma \left(< I > - I \right), \quad I = I(x,t,\mu), \quad -1 \le \mu < 1.$$

Approximate
$I(x,t,\mu) = \sum_{i=1}^n w_i f_i(x,t)\delta(\mu-\mu_i) + \sum_{i=1}^n w_i g_i(x,t)\delta(\mu+\mu_i)$.
Normalize $U = \left(\sqrt{\mathrm{diag}(w_i)}\ f, \sqrt{\mathrm{diag}(w_i)}\ g\right)^t \in \mathbb{R}^{2n}$, with
$\mathbf{w} = (\sqrt{w_1}, \ldots, \sqrt{w_n}, \sqrt{w_1}, \ldots, \sqrt{w_n}) \in \mathbb{R}^{2n}$.

> One gets the $\mathbf{S}_n$ **model**
>
> $$\partial_t U + A\partial_x U = -\sigma RU, \qquad A = A^t, \quad R = -\mathbf{w}\otimes\mathbf{w} + I_d = R^t \ge 0.$$
>
> It is a Friedrichs systems with large size and with relaxation.

Linearization of Fluid equations with relaxation (gravity, friction, ...) yields similar systems with non constant coefficients.

- Model problem= hyperbolic heat equation
  It is representative of many radiation/neutron transport problems.

$$P^\varepsilon : \qquad \begin{cases} \partial_t u + \frac{1}{\varepsilon} \operatorname{div} \mathbf{v} = 0, \\ \\ \partial_t \mathbf{v} + \frac{1}{\varepsilon} \nabla u = -\frac{\sigma}{\varepsilon^2} \mathbf{v}. \end{cases}$$

  The unknown is $(u, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^2$.
  The coefficient is $\sigma > 0$. The small parameter is $0 < \varepsilon \leq 1$.

- For $\varepsilon \to 0^+$, it admits the limit diffusion equation :

$$P^0 : \qquad \partial_t u - \frac{1}{\sigma} \Delta u = 0.$$

- This problem is mixed hyperbolic/parabolic.

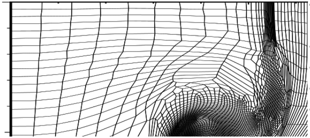
Summary of the challenges

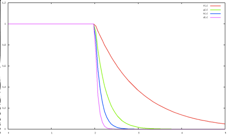
Address Friedrichs systems with relaxation

- Large size
- Change of type from hyperbolic to parabolic
- Obtain Well-Balanced (WB) discrete schemes which respect stationary solutions

$$\mathcal{U} = \{\mathbf{x} \mapsto U(\mathbf{x});\ \ \partial_x(AU) = -RU\}\,.$$

- Obtain Asymptotic-Preserving (AP) discrete schemes with uniform accuracy hyperbolic/parabolic.
- Should work for highly distorted meshes with robust methods of low degree : FV, P0, ...(part I).
- Discontinuous coefficients yield boundary layers (part II).





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Goal of this talk : for linear equations, address the structure of some of these methods.

Usual idea : plug sources in solvers+FV

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Basic example : hyperbolic heat equation with FV scheme on regular grid

$$\begin{cases} \partial_t p + \partial_x u = 0, \\ \partial_t u + \partial_x p = -\sigma u \end{cases}$$

The main point is to modify the fluxes (Riemann solvers)

$$\begin{cases} p^*_{j+\frac12} &= \dfrac{p_j + p_{j+1}}{2} &+ \dfrac{u_j - u_{j+1}}{2}, \\ u^*_{j+\frac12} &= \dfrac{u_j + u_{j+1}}{2} &+ \dfrac{p_j - p_{j+1}}{2}. \end{cases}$$

- Steady state approximation of Huang-Liu 1986, or Jin-Levermore 1996 :

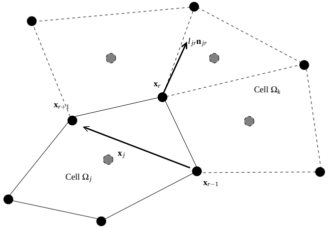
$$\begin{cases} p^*_{j+\frac12} &= \dfrac{p_j + p_{j+1}}{2} &+ (1-\sigma\Delta x/2)\dfrac{u_j - u_{j+1}}{2}, \\ u^*_{j+\frac12} &= \dfrac{u_j + u_{j+1}}{2} &+ (1-\sigma\Delta x/2)\dfrac{p_j - p_{j+1}}{2}. \end{cases}$$

- Gosse-Toscani scheme 2002 :

$$\begin{cases} p^{**}_{j+\frac12} = \dfrac{p_j + p_{j+1}}{2} + \dfrac{u_j - u_{j+1}}{2}, \\ u^{**}_{j+\frac12} = M\left(\frac{u_j + u_{j+1}}{2} + \dfrac{p_j - u_{j+1}}{2}\right), \quad M = \frac{1}{1+\sigma\Delta x/2}. \end{cases}$$

MultiD extension : $\mathbf{x}_r$ is for nodes

Denote the **corner normal** $\mathbf{n}_{jr} = (\cos\theta_{jr}, \sin\theta_{jr})$.



$$\begin{cases} p_{jr} - p_j + (\mathbf{n}_{jr}, \mathbf{u}_r - \mathbf{u}_j) + \frac{\sigma}{\varepsilon}(\mathbf{x}_r - \mathbf{x}_j, \mathbf{u}_r) = 0, \\ \sum_j l_{jr}\mathbf{n}_{jr}p_{jr} = 0. \end{cases}$$

$$\begin{cases} s_j \frac{d}{dt} p_j + \frac{1}{\varepsilon}\sum_r \mathbf{C}_{jr}\cdot\mathbf{u}_r = 0, & \mathbf{C}_{jr} = l_{jr}\mathbf{n}_{jr}, \\ s_j \frac{d}{dt}\mathbf{u}_j + \frac{1}{\varepsilon}\sum_r \mathbf{C}_{jr}p_{jr} = -\frac{1}{\varepsilon^2}\sigma\sum_r \mathbf{C}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j)\mathbf{u}_r. \end{cases}$$

Consistency of the RHS is from the identity $\sum_r \mathbf{C}_{jr}\otimes(\mathbf{x}_r - \mathbf{x}_j) = V_j I_d$.

- in 1D, equal to the Gosse-Toscani scheme.

- equal to the scheme in Buet, D. Franck, Numerish Mathematik, 2012.

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- Performed on the problem

$$\begin{cases} \partial_t p + \frac{1}{\varepsilon}\nabla\cdot\mathbf{u} &= 0, \\ \partial_t \mathbf{u} + \frac{1}{\varepsilon}\nabla p &= -\frac{1}{\varepsilon}^2\sigma\mathbf{u}, \end{cases}$$

where $\sigma > 0$ is given and $\varepsilon > 0$ is a small parameter.
Next is the first 2D result of convergence.

Theorem (2D on general grids).
Additionally to being **WB**, the implicit scheme is **AP** with the error estimate

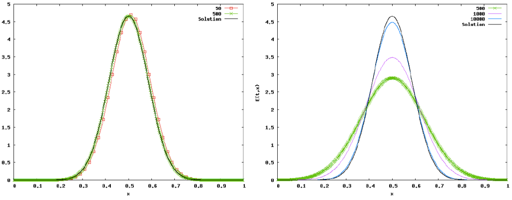
$$\|p_h - p, \mathbf{u}_h - \mathbf{u}\|_{L^2([0,T]\times\Omega)} \leq C\left(\Delta t^{\frac{1}{2}} + h^{\frac{1}{4}}\right),$$

uniformly with respect to the small parameter $\varepsilon \in (0,1]$.

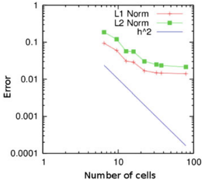
- Buet, D., Franck and Leroy, Proof of uniform convergence for a cell-centered AP discretization of the hyperbolic heat equation on general meshes, Math. of Comp. 2016 online.

Illustration

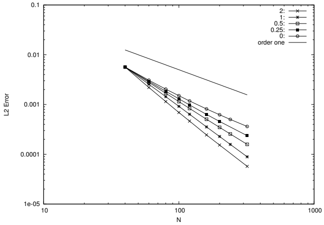
1D : AP versus non AP.



2D non AP



2D : AP ($\varepsilon = h^{\tau}$) versus



Section 2

Reformulation

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Question

Even if the method to obtain WB-AP-FV based on modifications of fluxes has huge successes, it is an a posteriori approach where one one designs (dirty ?) fixes.
The same for many other works in the literature.

Can we understand the structure a priori ?

- Take $\varepsilon = 1$ for a while

$$\partial_t U + A\partial_x + B\partial_y U = -RU, \quad U(t, \mathbf{x}) \in \mathbb{R}^n,$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$, and $R = R^t \geq 0$.

- The vectorial space of stationary states is

$$\mathcal{U} = \{\mathbf{x} \mapsto U(\mathbf{x}); \;\; A\partial_x U + B\partial_y U = -RU\}.$$

- Buet, D., The structure of well-balanced schemes for Friedrichs systems with linear relaxation. Appl. Math. Comput. 272 (2016).

Main idea : the dual equation

- The vectorial space of dual stationary states is

$$\mathcal{V} = \left\{\mathbf{x} \mapsto V(\mathbf{x}); \quad A^t \partial_x V + B^t \partial_y V = R^t U\right\}.$$

$$\boxed{\textbf{Property} \ : \quad \partial_t(U,V) + \partial_x(AU,V) + \partial_y(BU,V) = 0.}$$

- Pick up $V_p \in \mathcal{V}$ and define $\alpha_p = (U, V_p) \in \mathbb{R}$

$$\partial_t \alpha_p + \partial_x(AU, V_p) + \partial_y(BU, V_p) = 0$$

One gets a conservation law ! !

---

a) Assemble the system of conservation laws for $\alpha = (\alpha_p)$.
b) Discretize the new system of conservation laws with a standard FV scheme.
c) Rewrite the new scheme for the original primal variable.

Basic 1D example : $\det(A) \neq 0$

Here $\dim(\mathcal{V}) = n$: one has

$$V \in \mathcal{V} \Longrightarrow \partial_x V = A^{-1}RV \Longrightarrow V(x) = e^{A^{-1}Rx}V(0)$$

where $e^{A^{-1}Rx}$ is a **matrix exponential**. One gets

$$\partial_t \left(U, e^{A^{-1}Rx}V(0)\right) + \partial_x \left(AU, e^{A^{-1}Rx}V(0)\right) = 0, \quad \forall V(0).$$

and

$$\partial_t \left(\underbrace{P(x)U}_{\text{new unknown}}, V(0) \right) + \partial_x \left(P(x)AU, V(0)\right) = 0, \quad \forall V(0)$$

Proposition : Define the change of unknown

$$\alpha = P(x)U \Longleftrightarrow U = P^{-1}(x)\alpha, \qquad P(x) = e^{RA^{-1}x}.$$

The new conservative system rewrites

$$\partial_t \alpha + \partial_x (Q(x)\alpha) = 0, \quad Q(x) = P(x)AP^{-1}(x).$$

Discretization with standard FV

- FV discretization of $\partial_t \alpha + \partial_x \beta = 0$

$$\frac{\alpha_j^{n+1} - \alpha_j^n}{\Delta t} + \frac{\beta_{j+\frac{1}{2}}^n - \beta_{j-\frac{1}{2}}^n}{\Delta x_j} = 0, \qquad \beta = Q(x)\alpha,$$

where $\beta_{j+\frac{1}{2}}^n$ is the flux at time step $t_n = n\Delta t$.
The matrix $Q(x)$ is similar to A

$$Q(x) = P(x) A P(x)^{-1}.$$

- From the spectral decomposition

$$Au_p = \lambda_p u_p, \quad \lambda_p \neq 0,$$

the **right** and **left** spectral decompositions $Q^* = Q(x^*) = P(x^*)AP(x^*)^{-1}$ are

$$\begin{cases} Q(x^*) r_p^* = \lambda_p r_p^*, & r_p^* = P(x^*) u_p, \\ Q^t(x^*) s_p^* = \lambda_p s_p^*, & s_p^* = P(x^*)^{-t} u_p. \end{cases}$$

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Standard Riemann solvers write

$$\begin{cases} \left(u_p, U^*_{j+\frac{1}{2}} - U_j\right) = 0, & \lambda_p > 0, \\ \left(u_p, U^*_{j+\frac{1}{2}} - U_{j+1}\right) = 0, & \lambda_p < 0. \end{cases}$$

One-state solvers : given by the well-posed linear system

$$\begin{cases} \left(s_p^*, \beta^* - \beta_L\right) = 0, & \lambda_p > 0, \\ \left(s_p^*, \beta^* - \beta_R\right) = 0, & \lambda_p < 0. \end{cases}$$

It defines a first Riemann solver $\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$

$$\varphi(\beta_L, \beta_R, x^*) = \beta^*.$$

The scheme with the one-state solver

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The scheme writes

$$\frac{\alpha_j^{n+1} - \alpha_j^n}{\Delta t} + \frac{\varphi(\beta_j^n, \beta_{j+1}^n, x_{j+\frac{1}{2}}) - \varphi(\beta_{j-1}^n, \beta_j^n, x_{j-\frac{1}{2}})}{\Delta x_j} = 0$$

or

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + P(x_j)^{-1} \frac{\varphi(\beta_j^n, \beta_{j+1}^n, x_{j+\frac{1}{2}}) - \varphi(\beta_{j-1}^n, \beta_j^n, x_{j-\frac{1}{2}})}{\Delta x_j} = 0$$

Property : the scheme is WB.

Proof : If $\beta_j = \beta$ for all j, the solution is stationary.

Local formulation

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + A\frac{U^*_{j+\frac12} - U^*_{j-\frac12}}{\Delta x_j}$$

$$+\frac{P(x_j)^{-1}P(x_{j+\frac12}) - I}{\Delta x_j} AU^*_{j+\frac12} + \frac{I - P(x_j)^{-1}P(x_{j-\frac12})}{\Delta x_j} AU^*_{j-\frac12} = 0.$$

where the "flux" $U^*_{j+\frac12}$ is solution of the linear system

$$\begin{cases} \left(u_p, U^*_{j+\frac12} - e^{-A^{-1}R\Delta x_j^+} U_j\right) = 0, & \lambda_p > 0, \\ \left(u_p, U^*_{j+\frac12} - e^{-A^{-1}R\Delta x_{j+1}^-} U_{j+1}\right) = 0, & \lambda_p < 0, \end{cases}$$

with $\Delta x_j^\pm = x_{j\pm\frac12} - x_j$.
Notice the compatible discretization of the sink.

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Same principle but with upwinding of the eigenvectors

$$\begin{cases} \left(s_p^L, \beta^{**} - \beta_L\right) = 0, & \lambda_p > 0, \\ \left(s_p^R, \beta^{**} - \beta_R\right) = 0, & \lambda_p < 0, \end{cases}$$

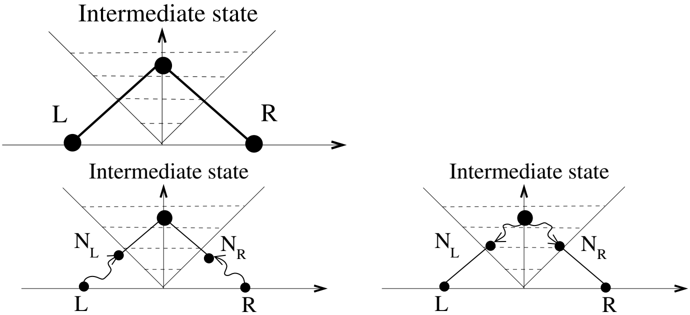
Theorem : Assume $R \geq 0$. Then the family $\left\{s_p^L\right\}_{\lambda_p > 0} \cup \left\{s_p^R\right\}_{\lambda_p < 0}$ is linearly independent.

It defines a second Riemann solver $\psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^n$

$$\psi(\beta_L, \beta_R, x_L, x_R) = \beta^{**}$$

with similar properties.

Schematic



The New-left and New-right states are modification of the initial Left and Right states of the standard Riemann solver (top).

One-state=Jin-Levermore Two-states=Gosse-Toscani.

When applied to S_n, our method gives back the Gosse scheme (2013), but with a standard FV construction.

MultiD : hyperbolic heat equation

- Assume $\sigma > 0$ is constant

$$\begin{cases} \partial_t p & +\partial_x u & +\partial_y v & = 0, \\ \partial_t u & +\partial_x p & & = -\sigma u, \\ \partial_t v & & +\partial_y p & = -\sigma v, \end{cases}$$

Set $\alpha_1 = p$, $\alpha_2 = \sigma x p + u$ and $\alpha_3 = \sigma y p + v$.
The system writes

$$\partial_t \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} + \partial_x \begin{pmatrix} m_1 \\ \alpha_1 + \sigma x m_1 \\ \sigma y m_1 \end{pmatrix} + \partial_y \begin{pmatrix} m_2 \\ \sigma x m_2 \\ 1 + \sigma y m_2 \end{pmatrix} = 0, \qquad (1)$$

where $m_1 = -\sigma x \alpha_1 + \alpha_2$ and $m_2 = -\sigma y \alpha_1 + \alpha_3$ ...

Using a 2D corner-based FV scheme, one gets back the AP 2D scheme discussed in the introduction.

Degenerate case $\det(A) = 0$

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• Example : P^1 radiation model coupled a linear temperature equation

$$\left\{ \begin{array}{lll} \partial_t p & +\partial_x u & = \tau(T-p), \\ \partial_t u & +\partial_x p & = -\sigma u, \\ \partial_t T & & = \tau(p-T), \end{array} \right. \qquad (2)$$

for which

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } R = \begin{pmatrix} \tau & 0 & -\tau \\ 0 & \sigma & 0 \\ -\tau & 0 & \tau \end{pmatrix}.$$

Here $\dim(\mathcal{V}) = 2 < 3$ which indicates a degeneracy. One can write **only two** conservation laws for

$$\alpha_1 = (U, V_1) = p + T \text{ and } \alpha_2 = (U, V_2) = \sigma x(p+T) + u.$$

The previous theory does not make sense
and something must be done ...

Trefftz methods

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Who is Trefftz ?

Erich Trefftz

Seminal Trefftz contributions in numerical analysis (1926) have been neglected in the numerical analysis community.

In modern language :
Trefftz = Special mixed formulations
= DG with non polynomial special basis functions = UWVF = ...

Who is Trefftz ?

Proceedings of the 2nd International Congress of Applied Mechanics

Zurich, 1926

pp 131 to 137

A counterpart to Ritz's Method

by E. Trefftz, Dresden

The Ritz method solves partial differential equation boundary value problems using the minimum potential energy theorem. Obviously, the error of the approximate solution is the difference between the strain energy of the exact mathematical model and that obtained using the Ritz trial functions.

It is important to note that the Ritz method does not allow an error estimate or some sort of error bounds since it yields only an upper bound to the true potential energy of the system. The objective of this paper is to present an analogue to the Ritz method that produces a lower bound to the potential energy. A combination of the Ritz method and the novel approach thus yields the desired error bounds.

First ever proved error estimate for a clamped plate problem (similar to a posteriori estimate) ! ! ! ! !

Comparing with finite and boundary elements, in 1997 Zienkiewicz [121] stated: *"... it seems without doubt that in the future Trefftz type elements will frequently be encountered in general finite element codes.... It is the author's belief that the simple Trefftz approach will in the future displace much of the boundary type analysis with singular kernels."* While this prediction has not yet come true, in the last years more and more work has been devoted to the formulation, the analysis and the validation of these methods and substantial progress has been accomplished.

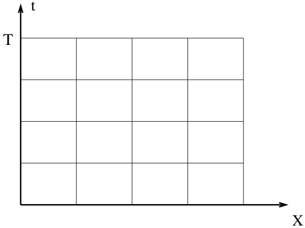
From Hiptmair-Moiola-Perugia (15')

Formalization

Start from the space-time equations

$$\partial_t U + \sum_{i=1}^{d} A_i \partial_{x_i} U + RU = f$$

in a domain $\Omega = \mathcal{D} \times [0, T]$. Split the domain $\overline{\Omega} = \overline{\cup_k \Omega_k}$



$$A(n) = I_p n_t + \sum_{i=1}^{d} A_i n_i = A(n)^t \in \mathbb{R}^{p\times p}, \quad n = (n_t, n_x) \in \mathbb{R}^{d+1}, \quad n_x \in \mathbb{R}^d$$

Decompose in positive and negative parts $A(n) = A^+(n) + A^-(n)$ with

$$A^+(n) = \sum_{\lambda_j>0} \lambda_j(n) r_j(n) \otimes r_j(n) \quad A^-(n) = \sum_{\lambda_j<0} \lambda_j(n) r_j(n) \otimes r_j(n).$$

Idea : space-time DG formulations

- Consider a discontinuous test function $V = (V_k) \in \sum_k H^1(\Omega_k)^p$ with V_k local solution of the adjoint equation

$$\partial_t V_k + \sum_{i=1}^{d} A_i \partial_{x_i} V_k - R V_k = 0, \quad (x,t) \in \Omega_k.$$

- Consider continuity of sol. U at interfaces : ex. $U \in H^1(\Omega)^p$.

- One can check : $\forall V \in \oplus_k H^1(\Omega_k)^p$ one has that

$$\begin{aligned} &\sum_k \sum_{j<k} \int_{\Sigma_{kj}} (V_k - V_j) \cdot (A^+_{kj} U_k + A^-_{kj} U_j) d\sigma \qquad \left(A^{\pm}_{kj} = A^{\pm}(n_{kj})\right) \\ &- \cancel{\sum_k \int_{\Omega_k} \left(\partial_t V_k + \sum_{i=1}^{d} A_i \partial_{x_i} V_k - R V_k \right) \cdot U dx} \\ &+ \sum_k \int_{\Sigma_{kk}} V \cdot A U d\sigma = \sum_k \int_{\Omega_k} V \cdot f d\Omega. \end{aligned} \tag{3}$$

- In standard DG, one uses $[V] = V_j - V_k$ and $\{V\} = \frac{1}{2}(V_j + V_k)$: we will not do that.

– huge literature on space-time DG : ..., Claes Johnson, Thomée, ...

Rewriting as UWVF

- Instead set $B(n) = \frac{1}{\sqrt{2}}\left(\sqrt{A^+(n)} - \sqrt{-A^-(n)}\right)$ and $C(n) = \frac{1}{\sqrt{2}}\left(\sqrt{A^+(n)} + \sqrt{-A^-(n)}\right)$.

- Define $X = \oplus_k (B(n_k) + C(n_k))U_k \in V$, $V = \oplus_k L^2(\partial\Omega_k)^p$ and a test function $Y = -\oplus_k (B(n_k) + C(n_k))V_k \in V$.

Then one has the variational formulation posed on the skeleton

$$(X, Y)_V - (\Pi X, FY) = (b, Y) \quad \forall Y \in V \tag{4}$$

with Π the exchange operator on interfaces and F defined by $FY = \oplus_k (B(n_k) - C(n_k))V_k$.

- Discrete formulations write
$(X_h, Y_h)_V - (\Pi X_h, FY_h) = (b, Y_h) \quad \forall Y_h \in V_h \subset V$.

- quite popular approach for time harmonic wave eq. : D. CRAS 1994, D.-Cessenat 1998, D.-Imbert-Gérard 2012, Hiptmair and al, Monk and al, Moiola 2016, ...

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• One takes

$$V_h = \oplus_k \ \mathrm{Span}\{Y_k^i\}_{1\le i\le m}, \qquad Y_k^i = (B(n_k) - C(n_k))\, W_k^i,$$

with W_k^i an analytical solution of

$$\partial_t W_k + \sum_{i=1}^{d} A_i \partial_{x_i} W_k - RW_k = 0 \quad (x,t) \in \Omega_k.$$

That is

$$Y_h(x,t) = \sum_k \alpha_k^i Y_k^i(x,t) \mathbf{1}_{\partial\Omega_k}(x,t).$$

• It is at first sight kind of a paradox that such basis functions represent correctly

$$X_h = \sum_k (B(n_k) - C(n_k))\, U_{|\partial\Omega_k} \mathbf{1}_{\partial\Omega_k}(x,t)$$

with

$$\partial_t U + \sum_{i=1}^{d} A_i \partial_{x_i} W_k + RW_k = 0, \quad (x,t) \in \Omega.$$

- - here the stiffness reflects in the design of the basis functions.

Interest of UWVF for numerical analysis

UWVF formulations are endowed with good *a priori* estimates. In a nutshell

- Set the norm $\| \cdot \| = \| \cdot \|_{L^2(Skeleton)}$
- One has $F \in \mathcal{L}(V)$ with the estimate $\|F\| \leq 1$
- Set $A = F^*\Pi$ and $Z = X - X_h$

$$\|(I-A)Z\|^2 + \left(\|Z\|^2 - \|\Pi Z\|^2\right)$$

$$= \|Z\|^2 - 2(AZ, Z) + \|F^*\Pi Z\|^2 + \left(\|Z\|^2 - \|\Pi Z\|^2\right)$$

$$\leq 2\,((1-A)Z, Z)$$

$$\leq 2\,((1-A)Z, X - X_h)$$

$$\leq 2\,((1-A)Z, X - Y_h) \quad \forall Y_h \in V_h.$$

- Therefore $\|(I-A)Z\| \leq 2 \inf_{Y_h \in V_h} \|X - Y_h\|$ and

$$\|Z\|^2 - \|\Pi Z\|^2 \leq 4 \inf_{Y_h \in V_h} \|X - Y_h\|^2.$$

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Back to our problem

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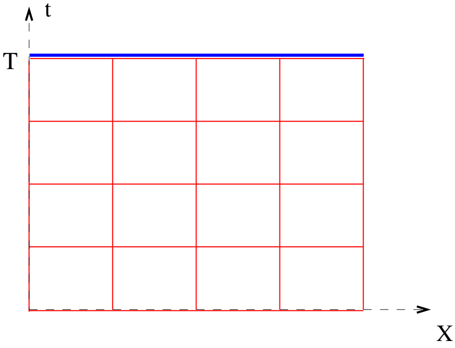
More precisely considering our space-time problem

$$\|X - X_h\|_{L^2(\Gamma_{\mathrm{top}})} \leq 2 \inf_{Y_h \in V_h} \|X - Y_h\|_{L^2(\mathrm{Skeleton})}$$



- need to compare with the literature : Moiola and al, Ern and Guermond, Hesthaven and Warburton, ...

Application to the telegraph equation

$$\begin{cases} \partial_t p + \partial_x u = 0, \\ \partial_t u + \partial_x p = -\sigma u, \end{cases}$$

$$\mathcal{V} : \quad \begin{cases} \partial_x \overline{u} = 0, \\ \partial_x \overline{p} = +\sigma \overline{u}, \end{cases} \iff \overline{p} = \alpha + \beta(x - x_G), \quad \overline{u} = \frac{1}{\sigma}\beta.$$

It yields two basis functions par cell : $(\alpha_k, \beta_k) \in \mathbb{R}^2$

$$\dim(V_h) = 2\#N_{\text{cell}} \approx \frac{2}{h^2}.$$

Assume the solution (p, u) is smooth. Then

$$\begin{aligned} \|X - X_h\|_{L^2(\Gamma_{\text{top}})} &\leq 2\sqrt{\#N_{\text{cell}} \max_k \|X^\varepsilon - Y_h^\varepsilon\|^2_{L^2(\partial\Omega_k)}} \\ &\leq 2\sqrt{\tfrac{2}{h^2} C h^3} \\ &\leq \widehat{C}\sqrt{h}. \end{aligned}$$

The constant $\widehat{C}$ is uniform with respect to h.

Numerics : $\partial_t p + \frac{1}{\varepsilon}\partial_x u = 0,\ \partial_t u + \frac{1}{\varepsilon}\partial_x p = -\frac{\sigma}{\varepsilon^2} u$

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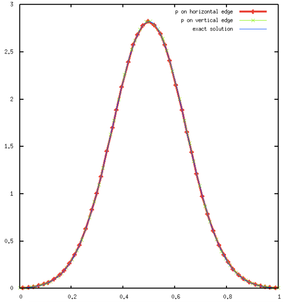
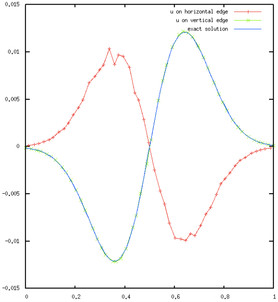
Set $a = \sigma\Delta x/(2\varepsilon)$ and

$$\begin{cases} \frac{p_k^{n+1}-p_k^n}{\Delta t} + \frac{1}{2\varepsilon\Delta x}\left((1+a)(u_{k+1}-u_{k-1}) - p_{k+1} + 2p_k - p_{k-1}\right)^{n+1} = 0, \\ \frac{u_k^{n+1}-u_k^n}{\Delta t} + \frac{1}{2\varepsilon\Delta x}\left((1-a)(u_{k+1}-u_{k-1}) + (a^2-1)(p_{k+1}+p_{k-1}) + 2(1+a)^2 p_k\right)^{n+1} = 0 \end{cases}$$



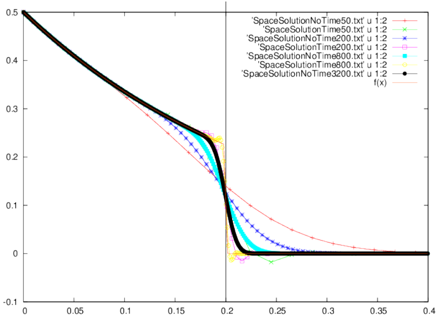


2 stationary basis functions. Plots of $x \mapsto (p^\varepsilon, u^\varepsilon)(x, T_{\text{end}})$

The scheme is AP (ask proof to G. Morel).

2 versus 4 basis functions

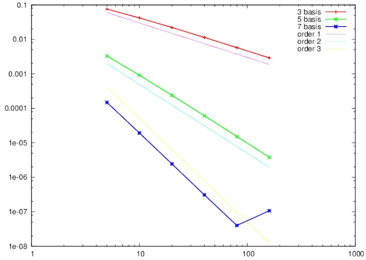
Propagative solution : 2 basis functions (stationary in space) versus 4 basis functions (2 additional time dependent functions)



The Trefftz-DG-UWVF (4 basis functions) is much more accurate than the standard low order method.

Numerical illustrations : 2D

Here we use Petrov-Galerkin : 3, 5, 7 basis functions

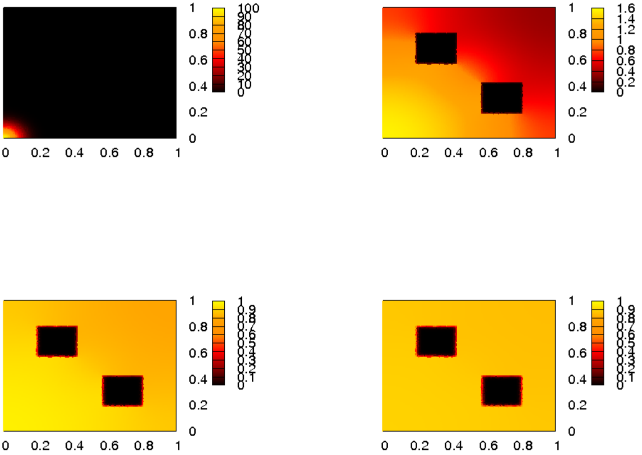


Order of convergence in dimension 2 in space (and 1 in time) : this is characteristic of the gain of accuracy of UWVF.

$$\text{order} \approx \frac{N-1}{2}, \qquad N = \text{number of basis functions}$$

A more physical example (E. F.)

Neutrons propagate in a transparent/opaque medium $\sigma = 1$ or 10000, and $\varepsilon = 1$. Implicit discretization (9 points stencil, good conditioning).



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Conclusion

- Dual equation techniques yield a powerful approach for discretization of relaxation operators in Friedrichs systems.
- First approach is rewriting equations as a new system of conservation laws : it is well adapted to FV
- Second approach is based on Trefftz/UWVF/DG philosophy : it yields new AP schemes : new means $\neq$ G.T. or J.L.

  This is on going work.