Weierstrass Institute for
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Towards pressure-robust mixed methods for the incompressible Navier–Stokes equations

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The object of interest

Question

How to discretize the incompressible Navier–Stokes equations (iNSE)?

Content

Part 1:

- *review of classical mixed methods for iNSE*
- *starting point: nearly all flow solvers relax the divergence constraint*
- *celebrated mixed methods for iNSE: only a partial solution !*
- *they replace volume locking by another locking phenomenon !*
- *work well for divergence-free forces, but not for irrotational forces*

Part 2:

- *these problems in inf-sup stable mixed methods can be efficiently fixed*
- *idea: relaxing the divergence constraint requires a slightly modified $\mathbf{L}^2$ scalar product for vector fields !*
- $\Rightarrow$ *pressure-robust mixed methods: dramatic accuracy improvements — at least for low-order methods — possible!*
- *some examples*

Model (iNSE in primal variables)

$$
\begin{aligned}
\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, &\qquad \mathbf{x}\in D, t\in(0,T],\\
\nabla\cdot\mathbf{u} = 0, &\qquad \mathbf{x}\in D, t\in(0,T],\\
\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}), &\qquad \mathbf{x}\in D,\\
\mathbf{u}(t,\mathbf{x}) = \mathbf{u}_b(t,\mathbf{x}), &\qquad \mathbf{x}\in\partial D, t\in(0,T].
\end{aligned}
$$

Focus of the talk

Remark (Focus on)

- *discretization on regular unstructured grids*
- *finite element methods*
- *however: results quite universal, also valid for DG methods, FVM, . . .*

Classical mixed methods (I)

Definition

Classical mixed methods:

inf-sup stable mixed methods for the iNSE that relax the divergence constraint.

Remarks (Main ingredients)

- *discrete velocity space X_h with norm $\|\cdot\|_{X_h}$*
- *discrete pressure space Q_h with norm $\|\cdot\|_{Q_h}$*
- *surjective discrete divergence operator*

$$\mathrm{div}_h \colon X_h \to Q_h,$$

$$\forall q_h \in Q_h \exists \mathbf{v}_h \in X_h : \quad \mathrm{div}_h \mathbf{v}_h = q_h \quad \wedge \quad \|\mathbf{v}_h\|_{X_h} \leq \frac{1}{\bar{\beta}} \|q_h\|_{Q_h}.$$

- *space of discretely divergence-free vector fields:*

$$V_h^0 = \{\mathbf{v}_h \in X_h : \quad \mathrm{div}_h \mathbf{v}_h = 0\}.$$

Remarks (Some classical mixed methods)

- *conforming mixed finite elements* *with* $X_h \subset \mathbf{H}^1_0$, $Q_h \subset L^2_0(D)$
  - $\mathrm{div}_h \mathbf{v}_h = \pi^{Q_h}_{L^2}(\nabla \cdot \mathbf{v}_h)$
  - *mini element* P_1^+-P_1 *(order 1, continuous pressures)*
  - *Taylor-Hood family* ($k \geq 2$): P_k-P_{k-1} *(order k, continuous pressures)*
  - *Bernardi–Raugel element* P_1^{FB}-P_0 *(order 1, discontinuous pressures)*
  - P_2^+-P_1^{disc} *(order 2, discontinuous pressures)*
- *nonconforming mixed FEMs like Crouzeix–Raviart element* P_1^{nc}-P_0 *(order 1, discontinuous pressures)*
- *staggered discretizations on unstructured grids*
- *...*

Remark

A strange observation:

In CFD many observers claim that at high Reynolds numbers high order methods are

- *more robust*
- *more accurate*

than low order methods.

This is at least strange

- *from a pure viewpoint of approximation theory*
- *and contradicts experience with low order methods for singularly perturbed scalar advection-diffusion equations.*

Conjecture

However, we conjecture a possible explanation for classical mixed methods.

Remark

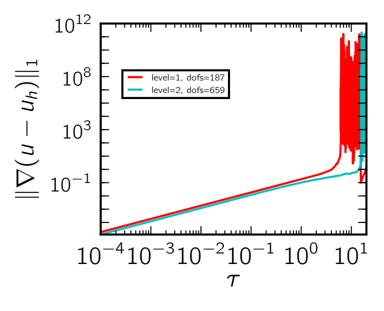
- *Since I believe that there is a lot of confusion in CFD about basic concepts, I will make a little hide-and-seek game about it.*
- *I will report on simulations of an unknown flow.*

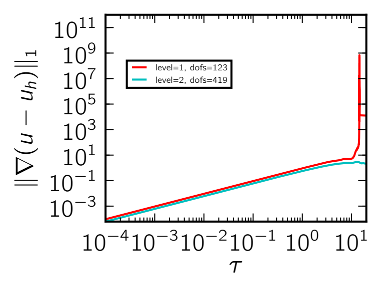
Example

- *perform fully time-dependent iNSE simulations of a smooth laminar flow*
  - $D = [0,1]^2$, $t \in (0,20]$
  - $\nu \in 10^{-4}$
  - $\mathbf{u}_0 = \mathbf{0}$
  - $\mathbf{u}_b = \mathbf{0}$
  - $$\mathbf{f} = \begin{pmatrix} 6\sqrt{\frac{105}{2609}}x^2y + 4\sqrt{\frac{105}{2609}}xy^2 \\ 2\sqrt{\frac{105}{2609}}x^3 + 4\sqrt{\frac{105}{2609}}x^2y + 8\sqrt{\frac{105}{2609}}y^3 \end{pmatrix}$$
  - $||\mathbf{f}||_{\mathbf{L}^2} = 1$
- *space discretization by:*
  - P_2-P_1 *(second order)*
  - P_3-P_2 *(third order)*
  - P_4-P_3 *(fourth order)*
- *time discretization: fully implicit backward Euler,* $\Delta t = 10^{-4}$

Motivation (III)

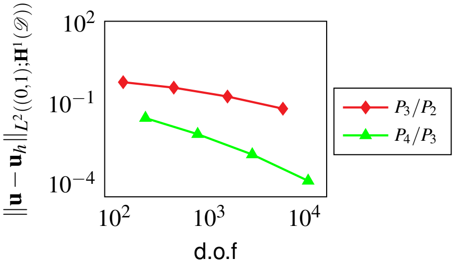
Observations (Report on robustness)





- *left*: P_2-P_1, *right*: P_3-P_2
- *third order more robust than second order!*
- *remove P_2-P_1 from the competition !*

Motivation (IV)



Observations (Reports on accuracy)

- *P_3-P_2 error quite large*
- *P_4-P_3 error much better (at least three refinement levels !)*

Question

Are high order methods for iNSE (at high Reynolds numbers) really better than low order methods? If yes, why?

Hint (First one)

- *given flow problem is quite special !*
- *for Helmholtz projector* $\mathbb{P}(\mathbf{f})$ *holds*

$$\|\mathbb{P}(\mathbf{f})\|_{\mathbf{L}^2} = 0!$$

Question

What is the Helmholtz projector (Leray projector) ?

Definition (L^2 divergence-free vector fields with zero normal traces)

$$\mathbf{L}^2_\sigma(D) := \overline{\{\boldsymbol{\chi} \in C_0^\infty(D)^d : \quad \nabla \cdot \boldsymbol{\chi} = 0\}}^{\mathbf{L}^2}.$$

Remark (L^2-orthogonality of gradient fields & divergence-free vector fields)

$$\forall \psi \in C^\infty(D), \forall \boldsymbol{\chi} \in C_0^\infty(D)^d \text{ with } \quad \nabla \cdot \boldsymbol{\chi} = 0 : \qquad (\nabla \psi, \boldsymbol{\chi}) = -(\psi, \nabla \cdot \boldsymbol{\chi}) = 0!$$

Lemma ($\mathbf{L}^2$-orthogonality of $\mathbf{L}^2_\sigma(D)$ and gradient fields)

$$\forall \psi \in H^1(D), \quad \forall \boldsymbol{\chi} \in \mathbf{L}^2_\sigma(D) \qquad (\nabla \psi, \boldsymbol{\chi}) = 0!$$

Definition

Helmholtz projector $\mathbb{P}(\mathbf{f})$: $\mathbf{L}^2$-projection of $\mathbf{f}$ onto $\mathbf{L}^2_\sigma$!

Lemma

$$\mathbb{P}(\nabla \psi) = \mathbf{0}, \qquad \forall \psi \in H^1(D)!$$

Remark

For $D = [0,1]^2$ it follows also:

$$\mathbb{P}(\mathbf{f}) = \mathbf{0} \quad \Rightarrow \quad \mathbf{f} = \nabla \psi \quad \textit{for some } \psi \in H^1(D).$$

Remarks

- *'difficult' time-dependent benchmark from above is solved by*

$$\mathbf{u} = \mathbf{0},$$
$$p = 2\sqrt{\frac{105}{2609}}\left(-\frac{-157}{360} + x^3y + x^2y^2 + y^4\right).$$

- *example shows: low order classical mixed methods fail for large irrotational forces !*
- *example can be solved exactly on every mesh by the P_5-P_4 element (fifth order).*
- *large irrotational forces: high order classical mixed methods more robust & more accurate !*

Remark

However:

- *space discretization with the (first order) Bernardi–Raugel element*
- *with slightly modified scalar product*

$$(\mathbf{f}, \pi_F^{\mathrm{BDM}_1} \mathbf{v}_h)$$

- *delivers on every mesh the discrete solution*

$$\mathbf{u}_h = \mathbf{0},$$
$$p_h = \pi_{L^2}^{Q_h} \left(2\sqrt{\frac{105}{2609}} \left(-\frac{-157}{360} + x^3 y + x^2 y^2 + y^4 \right) \right).$$

- *this novel pressure-robust discretization treats large irrotational forces better !*
- *indeed here: best approximations in X_h & Q_h !*
- *perfectly robust, perfectly accurate though first order !*

Motivation (XI)

Question

Why is first order Bernardi-Raugel element with

$$(\mathbf{f}, \pi_F^{\mathrm{BDM}_1} \mathbf{v}_h)$$

perfectly robust and perfectly accurate (as P_5-P_4)?

Idea

Assume now $\forall t \in (0,T] : \mathbf{u}(t) \in H^2(D)$

$\Rightarrow$ *testing with* $\mathbf{v} \in \mathbf{H}_0^1(D)$ *with* $\nabla \cdot \mathbf{v} = 0$ *yields*

$$(\mathbf{u}_t, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \qquad \Leftrightarrow$$

$$(\mathbb{P}(\mathbf{u}_t), \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + (\mathbb{P}\left[(\mathbf{u} \cdot \nabla)\mathbf{u}\right], \mathbf{v}) = (\mathbb{P}(\mathbf{f}), \mathbf{v})$$

Conclusion

- *in iNSE (& in every vector field equation !) there are two momentum balances (one for divergence-free and one irrotational forces) !*
- *separated by the* $\mathbf{L}^2$ *scalar product for vector fields*
- *relaxing divergence constraint in velocity test functions destroys this separation !*
- $\Rightarrow$ *divergence-free and irrotational forces interact wrongly!*
- *sometimes called poor mass conservation !*

Definition

Every classical mixed method has a discrete Helmholtz projector $\mathbb{P}_h(\mathbf{f})$*:*

$$\mathbf{L}^2\text{-projection of } \mathbf{f} \text{ onto } V_h^0 \text{ !}$$

Reference

A. L., C. Merdon: Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier-Stokes equations. CMAME, 2016.

Lemma

It holds for $\psi \in C^\infty(D)$

$$\sup_{\mathbf{v} \in V_h^0} \left| \frac{(\mathbb{P}_h(\nabla \psi), \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{L^2}} \right| = \sup_{\mathbf{v} \in V_h^0} \left| \frac{(\psi, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{L^2}} \right|$$

$$= \sup_{\mathbf{v} \in V_h^0} \left| \frac{(\psi - \pi_{L^2}^{Q_h} \psi, \nabla \cdot \mathbf{v}_h)}{\|\nabla \mathbf{v}_h\|_{L^2}} \right|$$

$$\leq \|\psi - \pi_{L^2}^{Q_h} \psi\|_{L^2}$$

$$\approx C h^{l+1} |\psi|_{l+1}$$

$$\neq \mathbf{0} \qquad !!!$$

Conclusion

- *classical mixed methods:* $\mathbb{P}_h(\nabla \psi) \neq \mathbf{0}$ *!!!*
- *higher order mixed methods have a more accurate discrete Helmholtz projector !*

Model

$$\begin{aligned} -\nu \Delta \mathbf{u} + \nabla p &= \mathbf{f}, \qquad \mathbf{x} \in \Omega \\ -\nabla \cdot \mathbf{u} &= 0, \qquad \mathbf{x} \in \Omega \\ \mathbf{u} &= \mathbf{v}_D, \qquad \mathbf{x} \in \partial\Omega \end{aligned}$$

Theorem (Stokes error estimate, conforming mixed FEMs)

$$\|\nabla \mathbf{u} - \nabla \mathbf{u}_h\|_{L^2} \leq 2\,(1 + C_F) \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\nabla \mathbf{u} - \nabla \mathbf{w}_h\|_{L^2} + \frac{1}{\nu} \inf_{q_h \in Q_h} \|p - q_h\|_{L^2}$$

Remark

- *Locking for $\nu \ll 1$ (in a phenomenological sense) !!!*
- *Velocity error small only, when $\mathbf{u}$ and $\frac{1}{\nu} p$ well-resolved simultaneously !*
- *Tragedy for mixed FEMs in Navier–Stokes: not the best possible estimate !*

Model

Steady incompressible Stokes equations:

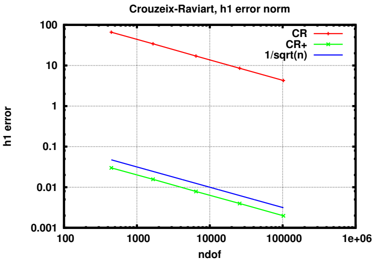
$$
\begin{aligned}
-\nu\Delta\mathbf{v} + \nabla p &= \underline{\mathbf{f}}, \qquad \mathbf{x} \in \Omega \\
\nabla \cdot \mathbf{v} &= 0, \qquad \mathbf{x} \in \Omega \\
\mathbf{v} &= 0 \qquad \mathbf{x} \in \partial\Omega.
\end{aligned}
$$

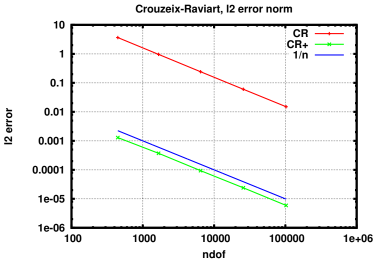
- $\nu = 10^{-3}$
-

$$
\begin{aligned}
\xi &= x^2(1-x)^2y^2(1-y)^2, \\
\mathbf{v} &= \nabla \times \xi, \\
p &= x^3 + y^3 - \frac{1}{2}.
\end{aligned}
$$

- $\mathbf{f} := -\nu\Delta\mathbf{v} + \nabla p$ $\qquad$ ν *small* $\Rightarrow$ $\mathbf{f}$ *nearly a gradient !*

Two Crouzeix–Raviart mixed FEMs (I)

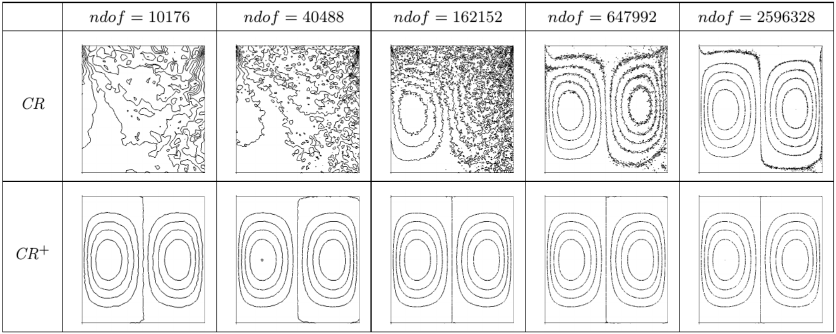




Remark

- *classical Crouzeix–Raviart element:*
  - *first-order convergent*
  - *right hand side:* $(\mathbf{f}, \mathbf{v}_h)$
- *(modified) pressure-robust Crouzeix–Raviart element (A. L.: CMAME 2014):*
  - *first-order convergent*
  - *right hand side:* $(\mathbf{f}, \Pi_F \mathbf{v}_h)$, $\quad \Pi_F : \mathrm{RT}_0$ *standard Fortin interpolator.*

Isolines of the vertical velocity component:



Remark

Example: Pressure-robust mixed method is 10 refinement levels more accurate.
2D: reduction of numerical effort: $4^{10} \approx 10^6$ *!*

Model

$$-\nu\Delta\mathbf{u}+\nabla p=\mathbf{f},\qquad \mathbf{x}\in\Omega$$
$$-\nabla\cdot\mathbf{u}=g,\qquad \mathbf{x}\in\Omega$$
$$\mathbf{u}=\mathbf{v}_D,\qquad \mathbf{x}\in\partial\Omega.$$

Theorem (Stokes velocity error estimate)

Classical: $$\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|_{L^2}\le 2(1+C_F)\inf_{\mathbf{w}_h\in\mathbf{X}_h}\|\nabla\mathbf{u}-\nabla\mathbf{w}_h\|_{L^2}+\frac{1}{\nu}\inf_{q_h\in Q_h}\|p-q_h\|_{L^2}$$

Pressure-robust: $$\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|_{L^2}\le 2(1+C_F)\inf_{\mathbf{w}_h\in\mathbf{X}_h}\|\nabla\mathbf{u}-\nabla\mathbf{w}_h\|_{L^2}+C_\# h^k|\mathbf{u}|_{k+1}$$

Remark

- *Pressure-robust mixed method* $\approx$ *velocity error is pressure-independent!*
- *velocity error small, when only* $\mathbf{u}$ *is well-resolved !*

Promise

- *you give me an inf-sup stable Stokes discretization*
- *I give you back a pressure-robust Stokes discretization !*
- *does not add any artificial viscosity !*

Pressure-robust sibling methods (II)

References

- *A. L.: On the role of the Helmholtz decomposition in mixed methods for incompressible flows and a new variational crime. CMAME, 2014.*
- *A. L., G. Matthies, L. Tobiska: Robust arbitrary order mixed finite element methods for the incompressible Stokes equations. M2AN, 2016.*
- *P. Lederer, A. L., C. Merdon, J. Schöberl: Divergence-free reconstruction operators for pressure-robust Stokes discretizations with continuous pressure finite elements. WIAS Preprint 2288, 2016.*

Idea

- L^2*-orthogonality:* $\mathbf{f}$, $(\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, \dot{\mathbf{u}}_h, \dots$ *must be improved !*
- *mixed methods &relaxation of divergence condition:*
  - *good idea in trial functions*
  - *bad idea in some test functions !*

Scheme (Pressure-robust sibling method)

For all $\mathbf{v}_h \in X_h$ *holds*

$$\nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h \mathrm{div}_h \mathbf{v}_h) = (\mathbf{f}, \Pi_F \mathbf{v}_h), \qquad \mathrm{div}_h \mathbf{u}_h = 0.$$

- Π_F *: appropriate* $H(\mathrm{div})$*-conforming standard interpolator*
- *main properties for pressure-robustness & optimal convergence:*

$$\Pi_F(V_h^0) \in \mathbf{L}^2_\sigma(D)!$$

- $$\nabla \cdot (\Pi_F(X_h)) \supset Q_h$$
- $$\sup_{\mathbf{v}_h \in V_0^h} \left| \frac{(\Delta \mathbf{u}, \Pi_F \mathbf{v}_h) + (\nabla \mathbf{u}, \nabla \mathbf{v})}{\|\nabla \mathbf{v}_h\|_{L^2}} \right| \leq C h^k |\mathbf{u}|_{k+1}$$

Example (Bernardi–Raugel element)

- $Q_h = P_0 \quad \Rightarrow \quad \Pi_F = \Pi_F^{\mathrm{BDM}_1}$ *possible !*
- *discrete divergence:* *elementwise average divergence*

$$(\mathrm{div}_h \mathbf{v}_h)_{|T} = \frac{1}{|T|} \int_T \nabla \cdot \mathbf{v}_h \, dx = \frac{1}{|T|} \int_{\partial T} \mathbf{v}_h \cdot \mathbf{n}_F \, dS$$

- *discretely divergence-free* $=$ *elementwise divergence-free !*
- X_h *is* $H_0^1(D)^d$*-conforming* $\Rightarrow$ *for all interior faces* F $\int_F \mathbf{v}_h \cdot \mathbf{n}_F \, dS$ *continuous !*
- *this moment preserved by* $\Pi_F^{\mathrm{BDM}_1} \mathbf{v}_h$ *!*
- $\Pi_F^{\mathrm{BDM}_1}(V_h^0) \subset \mathbf{L}_\sigma^2$ *!*
-

$$(\Delta \mathbf{u}, \Pi_F^{\mathrm{BDM}_1} \mathbf{v}_h) + (\nabla \mathbf{u}, \nabla \mathbf{v}_h) = (\Delta \mathbf{u}, \Pi_F^{\mathrm{BDM}_1} \mathbf{v}_h - \mathbf{v}_h) = \mathcal{O}(h) \|\nabla \mathbf{v}_h\|_{L^2} !$$

Scheme (Pressure-robust sibling method)

For all $\mathbf{v}_h \in X_h$ *holds*

$$\nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \mathrm{div}_h \mathbf{v}_h) = (\mathbf{f}, \Pi_F \mathbf{v}_h), \qquad \mathrm{div}_h \mathbf{u}_h = 0.$$

Remark

$$\|\nabla \mathbf{u} - \nabla \mathbf{u}_h\|_{L^2} \leq 2\,(1 + C_F) \inf_{\mathbf{w}_h \in \mathbf{X}_h} \|\nabla \mathbf{u} - \nabla \mathbf{w}_h\|_{L^2} + C_{\#}\, h^k |\mathbf{u}|_{k+1}$$

Questions

- *what can we gain for* $\mathbf{f} = \mathbf{0}$ *?*
- *arise large irrotational forces also in non-academic benchmarks ?*

Answer

Reference:
A. L., C. Merdon: Pressure-robustness and discrete Helmholtz projectors in mixed finite element methods for the incompressible Navier-Stokes equations. CMAME, 2016.

Remark

- *Potential flows:* $(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}^2)$
- $$((\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, \mathbf{v}_h) \qquad \textit{vs.} \qquad ((\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, \Pi_F \mathbf{v}_h)$$

Remark

- *Compare* P_2^+-P_1^{disc} *with pressure-robust sibling (with* RT_1*)*
- *2D:* $h = y^5 - 5x^4y - 10x^2y^3$
-

ν	*ndof*						
	304	*1200*	*4529*	*18175*	*71847*	*287593*	*1146124*
$1e+00$	1.00	1.00	1.00	1.00	1.00	1.00	1.00
$1e-01$	1.01	1.00	1.00	1.00	1.01	1.01	1.01
$1e-02$	1.31	1.34	1.21	1.41	1.44	1.45	1.47
$2e-03$	-	3.67	3.28	4.72	5.03	5.31	5.48
$1e-03$	-	-	5.12	8.48	9.39	10.20	10.72
$2e-04$	-	-	-	22.46	33.79	41.26	47.57
$1e-04$	-	-	-	-	49.01	68.30	84.17
$5e-05$	-	-	-	-	-	-	139.19

Table: Reduction of L^2 gradient error

- *Speedup: 3 refinement levels* $\approx$ *factor* $4^3 = 64$ *!*

Remark

- *Compare Bernardi–Raugel element with pressure-robust sibling (with RT_0)*
- *3D:* $h = xyz$
ν	*ndof*			
	884	*5124*	*36555*	*277056*
$1e+00$	1.01	1.01	1.02	1.03
$1e-01$	1.63	2.19	2.42	2.58
$1e-02$	12.31	19.46	21.95	23.84
$2e-03$	35.69	71.82	97.94	114.61
$1e-03$	40.19	93.73	156.28	208.62
$5e-04$	38.57	102.71	203.99	328.12
$2e-04$	-	-	133.33	441.19

  Table: Reduction of L^2 gradient error
- *Speedup: 8 refinement levels* $\approx$ *factor* $8^8 \approx 16$ *millions !*

Conclusion

- *relaxing divergence constraint: dangerous in velocity test functions !*
- *a conjecture: large gradients are everywhere !*
- L^2*-orthogonality of discretely divergence-free and gradients fields can be efficiently repaired* $\Rightarrow$ *pressure-robustness !*
- *low order classical mixed methods can be accelerated significantly in some benchmarks by pressure-robust modifications !*

Reference

V. John, A. L., C. Merdon, M. Neilan, L. Rebholz: On the divergence constraint in mixed finite element methods for incompressible flows. SIAM Review, accepted 2016 (WIAS Preprint 2177).