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Discontinuous Galerkin methods for the elastodynamics problem on polygonal and polyhedral meshes

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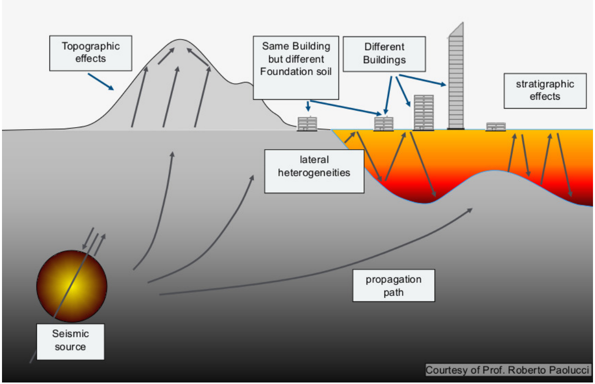
Outline

Aims and motivation



Requirements on the Numerical Scheme

- Flexibility
- Accuracy
- Efficiency

Non-standard numerical methods (DG, VEM, MFD):

- Geometrical flexibility offered by polyhedral elements
- High order polynomials
- Natively parallel

The Mathematical Model

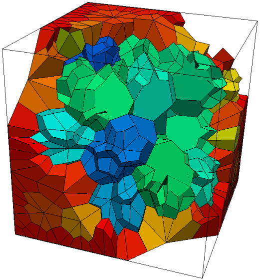
Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ with boundary $\partial\Omega = \Gamma$ regular enough. The mathematical model of linear elastodynamics reads:

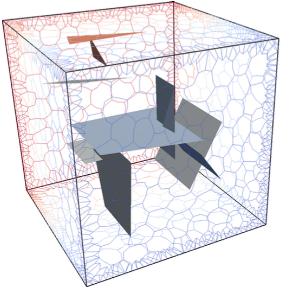
$$
\begin{aligned}
\rho(\mathbf{x})\mathbf{u}_{tt}(\mathbf{x},t) - \nabla \cdot \boldsymbol{\sigma}(\mathbf{x},t) &= \mathbf{f}(\mathbf{x},t), && \text{in } \Omega \times (0,T],\\
\boldsymbol{\sigma}(\mathbf{x},t) - \mathcal{D}\boldsymbol{\varepsilon}(\mathbf{u}(\mathbf{x},t)) &= \mathbf{0} && \text{in } \Omega \times (0,T],\\
\mathbf{u}(\mathbf{x},t) &= \mathbf{0}, && \text{on } \Gamma \times (0,T],\\
\mathbf{u}_t(\mathbf{x},0) &= \mathbf{u}_1(\mathbf{x}), && \text{in } \Omega \times \{0\},\\
\mathbf{u}(\mathbf{x},0) &= \mathbf{u}_0(\mathbf{x}), && \text{in } \Omega \times \{0\},
\end{aligned}
$$

- $\mathbf{u}$ displacement of the medium
- ρ material density s.t. $0 < \rho_* \leq \rho(\mathbf{x}) \leq \rho^* \quad \forall \mathbf{x} \in \Omega$
- $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^\top)$, strain tensor
- $\mathcal{D} : \mathbb{S} \to \mathbb{S}$, $\mathcal{D}\tau = 2\mu\tau + \lambda\mathrm{tr}(\tau)\mathbb{I}$: stiffness tensor
- λ, $\mu \in L^\infty(\Omega)$: Lamé coefficients.

Mesh setting and discontinuous finite element space

- $\mathcal{T}_h = \bigcup \kappa$ grid of the computational domain Ω (κ polygon/polyhedron)
- $\mathcal{F}_h$ union of all open interfaces, $\mathcal{F}_h = \mathcal{F}_h^I \cup \mathcal{F}_h^D$
- Element interfaces $\gamma \in \mathcal{F}_h$ with **arbitarily small measure**
- **Arbitrary** number of interfaces of each polyhedron





Discrete space

$$\mathbf{V}_h^p = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathbf{v}|_\kappa \in [\mathbb{P}_p(\kappa)]^d \quad \forall \kappa \in \mathcal{T}_h\}$$

Trace operators

$\kappa^+, \kappa^- \in \mathcal{T}_h$, $\gamma \in \mathcal{F}_h^I$ such that $\overline{\gamma} = \partial\overline{\kappa}^+ \cap \partial\overline{\kappa}^-$

On $\gamma \in \mathcal{F}_h^I$ ($\mathbf{n}_\kappa^\pm$ outward unit normal vector on γ relative to $\kappa^\pm$):

$$\{\mathbf{v}\} = \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-) \qquad [\![\mathbf{v}]\!] = \mathbf{v}^+ \otimes \mathbf{n}_\kappa^+ + \mathbf{v}^- \otimes \mathbf{n}_\kappa^-$$

$$\{\sigma\} = \frac{1}{2}(\boldsymbol{\sigma}^+ + \boldsymbol{\sigma}^-) \qquad [\![\boldsymbol{\sigma}]\!] = \boldsymbol{\sigma}^+\mathbf{n}_\kappa^+ + \boldsymbol{\sigma}^-\mathbf{n}_\kappa^-$$

On $\gamma \in \mathcal{F}_h^D$ ($\mathbf{n}_\kappa$ outward unit normal vector on Γ):

$$\{\mathbf{v}\} = \mathbf{v}^+ \qquad [\![\mathbf{v}]\!] = \mathbf{v}^+ \otimes \mathbf{n}_\kappa$$

$$\{\boldsymbol{\sigma}\} = \boldsymbol{\sigma}^+ \qquad [\![\boldsymbol{\sigma}]\!] = \boldsymbol{\sigma}^+\mathbf{n}_\kappa$$

DG formulation

For any $t \in (0, T]$ *find* $\mathbf{u}_h = \mathbf{u}_h(t) \in \mathbf{V}_h^p$ *such that*

$$(\rho\, \ddot{\mathbf{u}}_h, \mathbf{v})_\Omega + \mathcal{B}(\mathbf{u}_h, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h^p,$$

where $\mathbf{u}_h^0, \dot{\mathbf{u}}_h^0$ denote suitable approximations of $\mathbf{u}_0$ and $\mathbf{u}_1$, respectively.

The bilinear form associated to the interior penalty DG method reads

$$\begin{aligned}\mathcal{B}(\mathbf{u}, \mathbf{v}) = (\mathcal{D}\, \varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{T}_h} &- (\{\mathcal{D}\, \varepsilon(\mathbf{u})\}, [\![\mathbf{v}]\!])_{\mathcal{F}_h} \\ &- ([\![\mathbf{u}]\!], \{\mathcal{D}\, \varepsilon(\mathbf{v})\})_{\mathcal{F}_h} + (\eta\, [\![\mathbf{u}]\!], [\![\mathbf{v}]\!])_{\mathcal{F}_h}.\end{aligned}$$

[Bassi *et al.* 2012], [Antonietti, Giani, Houston, 2013], [Cangiani, Georgoulis, Houston, 2014], [Antonietti, Houston, Sarti, Verani, 2014], [Cangiani, Dong, Georgoulis, Houston, 2015], [Antonietti, Cangiani, Collis, Dong, Georgoulis, Giani, Houston, 2016],[Cangiani, Dong, Geourgolis, 2016]

Penalization function η

Let $\eta : \mathcal{F}_h \to \mathbb{R}_+$ be defined facewise by

$$\eta(\mathbf{x}) = \begin{cases} C_\eta \{\mathcal{D}^{\frac{1}{2}}\} C_{INV} \max_{\kappa \in \{\kappa^+, \kappa^-\}} \{ \dfrac{p^2|\gamma|}{|\kappa|} \}, & \mathbf{x} \in \gamma, \, \gamma \in \mathcal{F}_h^I, \\ C_\eta \{\mathcal{D}^{\frac{1}{2}}\} C_{INV} \dfrac{p^2|\gamma|}{|\kappa|}, & \mathbf{x} \in \gamma, \, \gamma \in \mathcal{F}_h^D. \end{cases}$$

with $C_\eta > 0$ large enough, depending on C_F and independent of p, $|\gamma|$ and $|\kappa|$. C_{INV} is the constant of the inverse-trace inequality.

[Cangiani, Georgoulis, Houston, 2014], [Antonietti, Cangiani, Collis, Dong, Georgoulis, Giani, Houston, 2016]

Extended DG formulation

Given $\mathbf{u}_h^0, \dot{\mathbf{u}}_h^0 \in \mathbf{V}_h^p$, $\forall t \in (0,T]$ *find* $\mathbf{u}_h \equiv \mathbf{u}_h(t) \in \mathbf{V}_h^p$ *such that*

$$(\rho\, \ddot{\mathbf{u}}_h, \mathbf{v})_\Omega + \widetilde{\mathcal{B}}(\mathbf{u}_h, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h^p,$$

where $\mathbf{u}_h^0, \dot{\mathbf{u}}_h^0$ denote suitable approximations of $\mathbf{u}_0$ and $\mathbf{u}_1$, respectively.

Let $\boldsymbol{Y} = \mathbf{H}_0^1(\Omega) \oplus \mathbf{V}_h^p$. The extended DG bilinear form $\widetilde{\mathcal{B}} : \boldsymbol{Y} \times \boldsymbol{Y} \to \mathbb{R}$ reads

$$\begin{aligned}\widetilde{\mathcal{B}}(\mathbf{u}, \mathbf{v}) = (\mathcal{D}\,\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{T}_h} &- (\{\boldsymbol{\Pi}(\mathcal{D}\,\varepsilon(\mathbf{u}))\}, [\![\mathbf{v}]\!])_{\mathcal{F}_h} \\ &- ([\![\mathbf{u}]\!], \{\boldsymbol{\Pi}(\mathcal{D}\,\varepsilon(\mathbf{v}))\})_{\mathcal{F}_h} + (\eta\, [\![\mathbf{u}]\!], [\![\mathbf{v}]\!])_{\mathcal{F}_h},\end{aligned}$$

where $\boldsymbol{\Pi}$ is the L^2-projection operator.

Well posedness

Given $\mathbf{u}_h^0, \dot{\mathbf{u}}_h^0 \in \mathbf{V}_h^p$, $\forall t \in (0,T]$ *find* $\mathbf{u}_h = \mathbf{u}_h(t) \in \mathbf{V}_h^p$ *such that*

$$(\rho\, \ddot{\mathbf{u}}_h, \mathbf{v})_\Omega + \widetilde{\mathcal{B}}(\mathbf{u}_h, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h^p.$$

$$\|\mathbf{v}\|_{\mathrm{DG}}^2 = \|\mathcal{D}^{\frac{1}{2}} \varepsilon(\mathbf{v})\|_{0,\Omega}^2 + \|\eta^{\frac{1}{2}} [\![\mathbf{v}]\!] \|_{0,\mathcal{F}_h}^2 \quad \forall \mathbf{v} \in \boldsymbol{Y}$$

- $\widetilde{\mathcal{B}}(\cdot,\cdot)$ is ***continuous*** and ***coercive*** (provided C_η is large enough) on $\boldsymbol{Y}$ with respect to the DG-norm, i.e.,

$$\begin{aligned} \widetilde{\mathcal{B}}(\mathbf{u}, \mathbf{v}) &\lesssim \|\mathbf{u}\|_{\mathrm{DG}} \|\mathbf{v}\|_{\mathrm{DG}} && \forall \mathbf{u}, \mathbf{v} \in \boldsymbol{Y}, \\ \widetilde{\mathcal{B}}(\mathbf{u}, \mathbf{u}) &\gtrsim \|\mathbf{u}\|_{\mathrm{DG}}^2 && \forall \mathbf{u} \in \boldsymbol{Y} \end{aligned}$$

Stability

$$\|\mathbf{v}_h\|_{\mathcal{E}}^2 = \|\rho^{\frac{1}{2}}\dot{\mathbf{v}}_h\|_{0,\Omega}^2 + \|\mathbf{v}_h\|_{\mathrm{DG}}^2$$

Let $\mathbf{u}_h(t)$ be the approximate solution of the semidiscrete DG formulation obtained with C_η large enough. Then,

(i) in the absence of external forces, i.e., $\mathbf{f} = \mathbf{0}$,

$$\|\mathbf{u}_h(t)\|_{\mathcal{E}} \lesssim \|\mathbf{u}_h^0\|_{\mathcal{E}}, \qquad 0 < t \leq T;$$

(ii) if $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$, then

$$\|\mathbf{u}_h(t)\|_{\mathcal{E}} \lesssim \|\mathbf{u}_h^0\|_{\mathcal{E}} + \int_0^t \rho_0^{-1}\|\mathbf{f}(\tau)\|_{0,\Omega}\, d\tau, \qquad 0 < t \leq T.$$

[Antonietti, Ayuso de Dios, M., Quarteroni, 2016],[Antonietti, M., Niccolò, submitted]

Stability (sketch of the proof)

$$\mathbf{v} = \dot{\mathbf{u}}_h \quad \Rightarrow \quad \frac{1}{2}\frac{d}{dt}\Big(\|\mathbf{u}_h\|_{\mathcal{E}}^2 - 2(\{\mathbf{\Pi}(\mathcal{D}\,\varepsilon(\mathbf{u}_h))\}, [\![\mathbf{u}_h]\!])_{\mathcal{F}_h} \Big) = (\mathbf{f}, \dot{\mathbf{u}}_h)_\Omega,$$

Integrating in time between 0 and t

$$\|\mathbf{u}_h\|_{\mathcal{E}}^2 - 2(\{\mathbf{\Pi}(\mathcal{D}\,\varepsilon(\mathbf{u}_h))\}, [\![\mathbf{u}_h]\!])_{\mathcal{F}_h} =$$
$$\|\mathbf{u}_h^0\|_{\mathcal{E}}^2 - 2(\{\mathbf{\Pi}(\mathcal{D}\,\varepsilon(\mathbf{u}_h^0))\}, [\![\,\mathbf{u}_h^0\,]\!])_{\mathcal{F}_h} + 2\int_0^t (\mathbf{f}, \dot{\mathbf{u}}_h)_\Omega \, d\tau,$$

It can be shown

$$\|\mathbf{u}_h\|_{\mathcal{E}}^2 - 2(\{\mathbf{\Pi}(\mathcal{D}\,\varepsilon(\mathbf{u}_h))\}, [\![\mathbf{u}_h]\!])_{\mathcal{F}_h} \gtrsim \|\mathbf{u}_h\|_{\mathcal{E}}^2,$$
$$\|\mathbf{u}_h^0\|_{\mathcal{E}}^2 - 2(\{\mathbf{\Pi}(\mathcal{D}\,\varepsilon(\mathbf{u}_h^0))\}, [\![\,\mathbf{u}_h^0\,]\!])_{\mathcal{F}_h} \lesssim \|\mathbf{u}_h^0\|_{\mathcal{E}}^2,$$

From which it follows

$$\|\mathbf{u}_h\|_{\mathcal{E}}^2 \lesssim \|\mathbf{u}_h^0\|_{\mathcal{E}}^2 + \int_0^t \rho_0^{-1} \|\mathbf{f}(\tau)\|_{0,\Omega} \, \|\mathbf{u}_h\|_{\mathcal{E}} \, d\tau.$$

Error analysis

Consistency error: Let $R_h(\cdot,\cdot) : \boldsymbol{Y} \times \mathbf{V}_h^p \to \mathbb{R}$ be the residual defined as

$$R_h(\mathbf{v},\mathbf{w}) = \mathcal{B}(\mathbf{v},\mathbf{w}) - \widetilde{\mathcal{B}}(\mathbf{v},\mathbf{w}) \qquad \forall \mathbf{v} \in \boldsymbol{Y},\, \forall \mathbf{w} \in \mathbf{V}_h^p,\, \mathbf{w} \neq \mathbf{0}.$$

In particular,

$$R_h(\mathbf{v},\mathbf{w}) = (\{\mathcal{D}\,\varepsilon(\mathbf{v}) - \mathbf{\Pi}(\mathcal{D}\,\varepsilon(\mathbf{v}))\}, [\![\mathbf{w}]\!])_{\mathcal{F}_h}.$$

It holds that

$$|R_h(\mathbf{v},\mathbf{w})| \lesssim \|\mathbf{v}\|_{\mathcal{E}}\, \|\mathbf{w}\|_{\mathcal{E}}$$

Equation for the approximation error $\mathbf{e}_h(t) = \mathbf{u}(t) - \mathbf{u}_h(t)$

$$\int_\Omega \rho\, \ddot{\mathbf{e}}_h \cdot \mathbf{v}_h \, d\mathbf{x} + \widetilde{\mathcal{B}}(\mathbf{e}_h,\mathbf{v}_h) + R_h(\mathbf{e}_h,\mathbf{v}_h) = 0 \qquad \forall \mathbf{v}_h \in \mathbf{V}_h^p$$

Error analysis (cont'd)

$$\|\mathbf{v}(t)\|_{\mathcal{E}}^2 = \|\rho^{\frac{1}{2}}\dot{\mathbf{v}}(t)\|_{0,\Omega}^2 + \|\mathbf{v}(t)\|_{\mathrm{DG}}^2 \qquad \forall \mathbf{v} \in \mathcal{C}^2(0,T;\mathbf{V}_h^p),\ \forall t \in [0,T]$$

A-priori error estimates

Let $\mathbf{u}_h$ be the approximated solution of $\mathbf{u}$, assumed sufficiently regular. Assume that p is uniform over $\mathcal{T}$, that h is quasi uniform and that C_η is large enough. Then

$$\sup_{0<t\leq T} \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{\mathcal{E}} \leq C(T,\mathbf{u},\dot{\mathbf{u}},\ddot{\mathbf{u}}) \frac{h^{m-1}}{p^{k-3/2}}$$

where $m = \min(p+1,k)$ with $k > 1 + d/2$.

Error analysis (sketch of the proof)

- Split the error as:
$$\|\mathbf{e}_h\|_{\mathcal{E}} \leq \|\underbrace{(\mathbf{u}-\widetilde{\mathbf{\Pi}}\mathbf{u})}_{\boldsymbol{\omega}_I}\|_{\mathcal{E}} + \|\underbrace{(\mathbf{u}_h-\widetilde{\mathbf{\Pi}}\mathbf{u})}_{\boldsymbol{\omega}_h}\|_{\mathcal{E}}$$
- From the variational formulation for the approximation error
$$(\rho\,\ddot{\boldsymbol{\omega}}_h,\mathbf{v}_h)_\Omega + \widetilde{\mathcal{B}}(\boldsymbol{\omega}_h,\mathbf{v}_h) = (\rho\,\ddot{\boldsymbol{\omega}}_I,\mathbf{v}_h)_\Omega + \widetilde{\mathcal{B}}(\boldsymbol{\omega}_I,\mathbf{v}_h) - R_h(\boldsymbol{\omega}_I,\mathbf{v}_h),$$
- Take $\mathbf{v}_h = \dot{\boldsymbol{\omega}}_h$
$$\frac{1}{2}\frac{d}{dt}\Big(\|\boldsymbol{\omega}_h\|^2_{\mathcal{E}} - 2(\llbracket\boldsymbol{\omega}_h\rrbracket, \{\mathbf{\Pi}(\mathcal{D}\,\varepsilon(\boldsymbol{\omega}_h))\})_{\mathcal{F}_h}\Big)$$
$$= (\rho\,\ddot{\boldsymbol{\omega}}_I,\dot{\boldsymbol{\omega}}_h)_\Omega + \widetilde{\mathcal{B}}(\boldsymbol{\omega}_I,\dot{\boldsymbol{\omega}}_h) - R_h(\boldsymbol{\omega}_I,\dot{\boldsymbol{\omega}}_h).$$
- Use stability
$$\|\boldsymbol{\omega}_h\|_{\mathcal{E}} \lesssim \|\boldsymbol{\omega}_I\|_{\mathcal{E}} + \int_0^t \|\dot{\boldsymbol{\omega}}_I\|_{\mathcal{E}}\,d\tau,$$
- The thesis follows from approximation bounds
$$\|\boldsymbol{\omega}_I\|_{\mathcal{E}} \lesssim \frac{h^{m-1}}{p^{k-3/2}}\Big(\|\mathbf{u}\|_{k,\Omega} + \frac{h}{p^{3/2}}\|\dot{\mathbf{u}}\|_{k,\Omega}\Big),$$

Algebraic formulation

- $\Omega \subset \mathbb{R}^2$ partitioned in N disjoint polygonal elements
- $n_p = \frac{1}{2}(p+1)(p+2)$, $D = \sum_{k=1}^{N} \frac{1}{2}(p+1)(p+2) = N\, n_p$
- Basis for $\mathbf{V}_h^p$: $\{\mathbf{\Phi}_i^1,\ \mathbf{\Phi}_i^2\}_{i=1}^{D}$

Given $\mathbf{u}_h^0, \dot{\mathbf{u}}_h^0 \in \mathbf{V}_h^p$, $\forall t \in (0,T]$ *find* $\mathbf{u}_h = \mathbf{u}_h(t) \in \mathbf{V}_h^p$ *such that*

$$(\rho\, \ddot{\mathbf{u}}_h, \mathbf{v})_\Omega + \mathcal{B}(\mathbf{u}_h, \mathbf{v}) = \mathcal{L}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h^p,$$

where $\mathbf{u}_h^0, \dot{\mathbf{u}}_h^0$ suitable approximations of $\mathbf{u}_0$ and $\mathbf{u}_1$, respectively.

$$M\ddot{\mathbf{U}} + B\mathbf{U} = \mathbf{F}$$

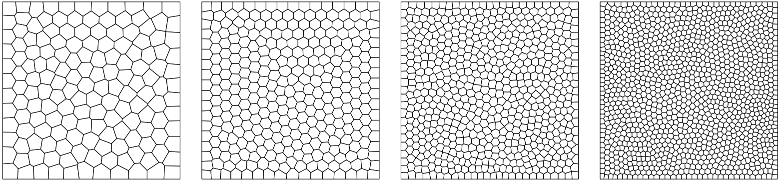
$$V \leftrightarrow (\mathcal{D}\,\varepsilon(\boldsymbol{v}_h), \varepsilon(\boldsymbol{w}_h))_\Omega \qquad I \leftrightarrow (\{\mathcal{D}\,\varepsilon(\boldsymbol{v}_h)\}, [\![\boldsymbol{w}_h]\!])_{\mathcal{F}_h}$$

$$I^T \leftrightarrow ([\![\boldsymbol{v}_h]\!], \{\mathcal{D}\,\varepsilon(\boldsymbol{w}_h)\})_{\mathcal{F}_h} \qquad S \leftrightarrow (\eta\, [\![\mathbf{v}_h]\!], [\![\boldsymbol{w}_h]\!])_{\mathcal{F}_h}$$

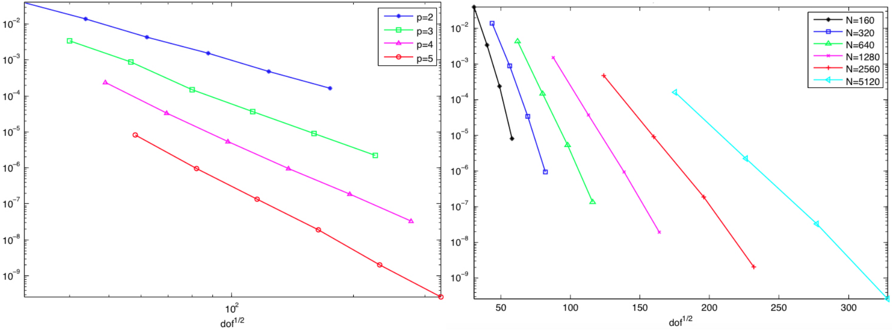
$$B = V - I - I^T + S$$

Numerical results (elastostatic case)

- *Domain*: $\Omega = (0,1)^2$
- *Exact solution*: $\mathbf{u}(\mathbf{x}) = \begin{bmatrix} \mathrm{e}^x \cos(2\pi x)\ \sin(2\pi y) \\ \mathrm{e}^y \sin(2\pi x)\ \cos(2\pi y) \end{bmatrix}$
- $\rho = 1,\ \lambda = 1,\ \mu = 1$
- *Polynomial degree*: $p = 2, 3, 4, 5$



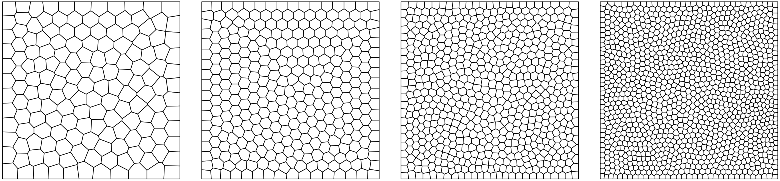
Numerical results (elastostatic case)



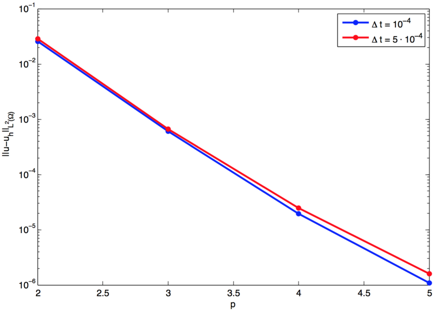
Computed errors in the DG norm vs $1/h$ (left) and p (right).

Numerical results

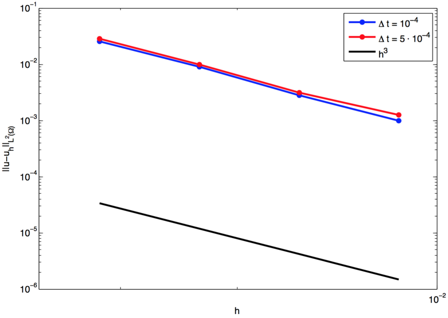
- *Domain*: $\Omega = (0,1)^2$
- *Exact solution*: $\mathbf{u}(\mathbf{x},t) = \sin(\sqrt{2}\pi t)\begin{bmatrix} -\sin(\pi x)^2 \sin(2\pi y) \\ \sin(2\pi x) \sin(\pi y)^2 \end{bmatrix}$
- $\rho = 1,\ \lambda = 1,\ \mu = 1$
- *Polynomial degree*: $p = 2, 3, 4, 5$
- Time integration using the leap-frog scheme



Numerical results



Computed errors versus the polynomial degree p, $T = 0.5$. $\Delta t = 10^{-4}$ (blue) and $\Delta t = 5 \cdot 10^{-4}$ (red)



Computed errors versus $1/h$, $p = 2$, $T = 0.5$. $\Delta t = 10^{-4}$ (blue) and $\Delta t = 5 \cdot 10^{-4}$ (red)

Conclusions and ongoing work

- DG methods seem to be a promising tool for the approximation of the elastodynamics equation on polytopic grids.

- Ongoing:
  - 3d implementation in SPEED
  - Dissipation/dispersion analysis
  - Real earthquake scenarios

http://speed.mox.polimi.it

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