A well-balanced SHP scheme

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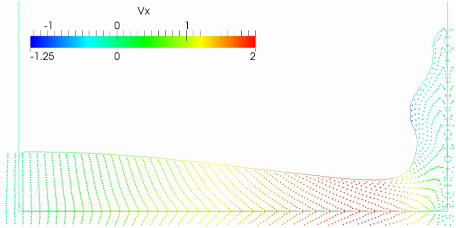
HydrOcean and NEXTFLOW Software

- Created in 2007 by Erwan Jacquin
- Main objectives : naval, offshore, sailing and marine energies
- Strong collaboration with LHEEA
- About twenty employees, essentially engineers and PhD
- Working with DCNS, Total, Michelin (for instance)

SPH-Flow code

- Based on the SPH method
- Developped in collaboration with LHEEA
- Meshless Lagrangian method
- Explicit scheme, weakly compressible flows
- free surface flows



Navier-Stokes equations

$$\left\{\begin{array}{lcl}
\dfrac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) & = & 0 \\
\dfrac{\partial \rho \boldsymbol{u}}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{u} \otimes \boldsymbol{u} + p \bar{\bar{I}}\right) - \boldsymbol{\nabla} \cdot \bar{\bar{\tau}} & = & \rho \boldsymbol{g} \\
\dfrac{\partial \rho E}{\partial t} + \boldsymbol{\nabla} \cdot ((\rho E + p)\, \boldsymbol{u}) - \boldsymbol{\nabla} \cdot (\bar{\bar{\tau}} \cdot \boldsymbol{u}) & = & \rho \boldsymbol{g} \cdot \boldsymbol{u}
\end{array}\right.$$

with

- gravity source term
- viscosity tensor $\bar{\bar{\tau}}$

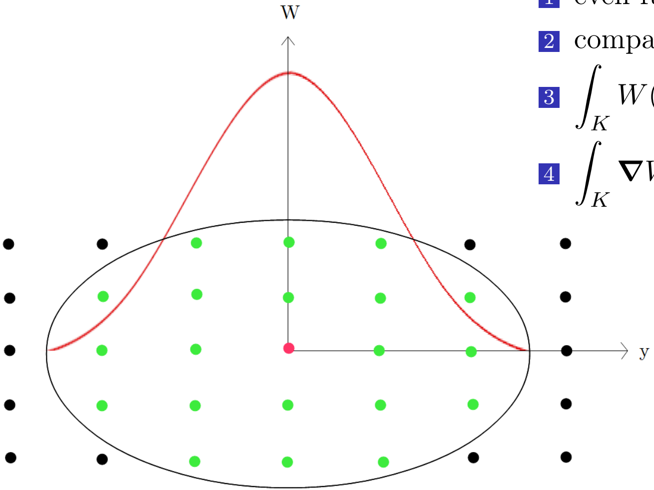
Pressure law in the form: $p = \kappa \rho^{\gamma} - B$

Particule reformulations (basic idea)

Reformulation of a function

$$\Pi(f)(\boldsymbol{x}) = (f * \delta)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y})\delta(\boldsymbol{x} - \boldsymbol{y})\ d\boldsymbol{y} = f(\boldsymbol{x})$$

The kernel



1. even function
2. compact support
3. $\displaystyle\int_K W(\boldsymbol{x}-\boldsymbol{y},h)\; d\boldsymbol{y} = 1$
4. $\displaystyle\int_K \boldsymbol{\nabla} W(\boldsymbol{x}-\boldsymbol{y},h)\, d\boldsymbol{y} = 0$

Approximate particule reformulation

Approximate reformulation of a fonction

$$\Pi(f)(\boldsymbol{x}) = (f * \delta)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y})\ d\boldsymbol{y} = f(\boldsymbol{x})$$

$$\Pi^h(f)(\boldsymbol{x}) = (f * W)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y})W(\boldsymbol{x}-\boldsymbol{y},h)\ d\boldsymbol{y}$$

Approximate reformulation of a gradient

$$\int_K \boldsymbol{\nabla} f(\boldsymbol{y})W(\boldsymbol{x}-\boldsymbol{y},h)\ d\boldsymbol{y} = \int_K f(\boldsymbol{y})\boldsymbol{\nabla} W(\boldsymbol{x}-\boldsymbol{y},h)\ d\boldsymbol{y}$$

$$\boldsymbol{\nabla}\Pi^h(f)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y})\boldsymbol{\nabla} W(\boldsymbol{x}-\boldsymbol{y},h)\ d\boldsymbol{y}$$

Particule approximation method

Based on quadrature formulae

$$\int_{\mathbb{R}^d} f(\boldsymbol{y})\ d\boldsymbol{y} \simeq \sum_{j\in\mathcal{P}} \omega(\boldsymbol{x}_j) f(\boldsymbol{x}_j) = \sum_{j\in\mathcal{P}} \omega_j f_j$$

$$\Pi^h(f)(\boldsymbol{x}) = \int_K f(\boldsymbol{y}) W(\boldsymbol{x}-\boldsymbol{y},h)\ d\boldsymbol{y} \longrightarrow \Pi^h(f)_i = \sum_{j\in\mathcal{P}} \omega_j f_j W_{ij}$$

Main difficulties

- $\displaystyle\sum_{j\in\mathcal{P}} \omega_j W_{ij} \neq 1$
- $\displaystyle\sum_{j\in\mathcal{P}} \omega_j \boldsymbol{\nabla} W_{ij} \neq 0$ and $\displaystyle\sum_{j\in\mathcal{P}} \omega_j x_j^k \boldsymbol{\nabla} W_{ij} \neq 1^k$, $k \in [\![1,d]\!]$

Improvements

Derivation improvement: exact for constant functions

$$\begin{aligned} D_h(f)_i &= \boldsymbol{\nabla}\Pi^h(f)_i - f_i \boldsymbol{\nabla}\Pi^h(1)_i \\ &= \sum_{j\in\mathcal{P}} \omega_j (f_j - f_i) \boldsymbol{\nabla} W_{ij,i} \end{aligned}$$

Variable supports of the kernel

$$h = h(\boldsymbol{x}) \rightsquigarrow W(\boldsymbol{x}_i - \boldsymbol{x}_j, h) = W(\boldsymbol{x}_i - \boldsymbol{x}_j, h_i) = W_{ij,i}$$

$$\Pi^h(f)_i = \sum_{j\in\mathcal{P}} \omega_j f_j W_{ij,i}$$

$$\boldsymbol{\nabla}\Pi^h(f)_i = \sum_{j\in\mathcal{P}} \omega_j f_j \boldsymbol{\nabla} W_{ij,i}$$

Associated weak formulation

Discret scalar product: $\langle f, g \rangle_h = \sum_{j \in \mathcal{P}} \omega_j f_j g_j$

Adjoint of D_h

$\langle D_h(f), g \rangle_h = - \langle f, D_h^*(g) \rangle_h$ to get

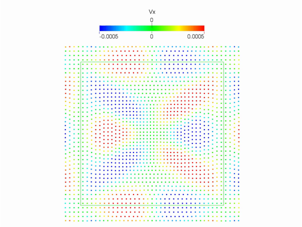
$$D_h^*(f)_i = \sum_{j \in \mathcal{P}} \omega_j \left(f_j \boldsymbol{\nabla} W_{ij,j} + f_i \boldsymbol{\nabla} W_{ij,i} \right)$$

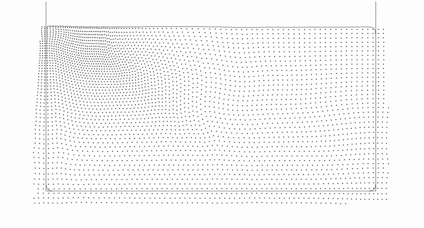
$\rightsquigarrow$ this formulation is conservative: $\sum_{i \in \mathcal{P}} \omega_i D_h^*(f)_i = 0$

Problèmes de cette formulation

Global consistance, but lost of the local consistance with the strong operator

Consequences





PEPS AMIES

SPHINX
SPH Improvement for Numerical approXimations

Objectives

- To get a consistant weak operateur useful for industrial applications
- obtention d'un opérateur faible localement consistant

Suggested approach

To modify the weak operator to be consistant and to preserve the duality relation strong / weak operator

A new weak operator

Based on the Lacôme approach: weak formulation with a variable h

$$\Pi^h_{weak}(f)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y}) \left(1 + \frac{\boldsymbol{x}-\boldsymbol{y}}{h(\boldsymbol{y})}.\boldsymbol{\nabla} h(\boldsymbol{y})\right) W\left(\boldsymbol{x}-\boldsymbol{y}, h(\boldsymbol{y})\right) d\boldsymbol{y}$$

From weak to strong to get a approximate gradient

$$\boldsymbol{\nabla}\Pi^h_{strong}(f)(\boldsymbol{x}) = \int_{\mathbb{R}^d} f(\boldsymbol{y})\boldsymbol{\xi}^{strong}\left(\boldsymbol{x}-\boldsymbol{y}, h(\boldsymbol{x})\right) d\boldsymbol{y}$$

where ξ^{strong} is consistent with ∇W

The definition of ξ^{strong} is long and contains ∇h

Modified formulation

Modification of the strong formulation to preserve the derivation of constant functions

$$
\begin{aligned}
D^h_{strong}(\boldsymbol{\nabla} f)(\boldsymbol{x}) &= \boldsymbol{\nabla}\Pi^h_g(f)(\boldsymbol{x}) - f(\boldsymbol{x})\boldsymbol{\nabla}\Pi^h_g(1)(\boldsymbol{x}) \\
&= \int_{\mathbb{R}^d} \left(f(\boldsymbol{y}) - f(\boldsymbol{x})\right) \boldsymbol{\xi}^{strong}\left(\boldsymbol{x}-\boldsymbol{y}, h(\boldsymbol{x})\right) d\boldsymbol{y}
\end{aligned}
$$

From strong to weak

$$
D^h_{weak}(\boldsymbol{\nabla} f)(\boldsymbol{x}) = \int_{\mathbb{R}^d} \Big[f(\boldsymbol{y})\boldsymbol{\xi}^{weak}\left(\boldsymbol{x}-\boldsymbol{y}, h(\boldsymbol{y})\right) + f(\boldsymbol{x})\boldsymbol{\xi}^{strong}\left(\boldsymbol{x}-\boldsymbol{y}, h(\boldsymbol{x})\right)
$$

où $\boldsymbol{\xi}^{weak}\left(\boldsymbol{x}-\boldsymbol{y}, h(\boldsymbol{y})\right) = -\boldsymbol{\xi}^{strong}\left(\boldsymbol{y}-\boldsymbol{x}, h(\boldsymbol{y})\right)$

Collaboration continuation: Well-balanced SPH-scheme for shallow-water

1D model

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x(hu^2 + \frac{g}{2}h^2) = -hg\partial_x Z, \end{cases}$$

SPH-discretization gives

$$\begin{cases} \dfrac{h_i^{n+1} - h_i^n}{\Delta t} + \displaystyle\sum_{j \in \mathcal{P}} 2\omega_j (hu)_{ij} W'_{ij} = 0 \\ \dfrac{h_i^{n+1}u_i^{n+1} - h_i^n u_i^n}{\Delta t} + \displaystyle\sum_{j \in \mathcal{P}} 2\omega_j \left(hu^2 + \frac{1}{2} g h^2 \right)_{ij} W'_{ij} = s_i. \end{cases}$$

- s_i is a discretization of the source term
- F_{ij} finite volume hybridation

Reformulation of the model

Main idea

- Notations: $H = h + Z$ and $X = h/H$
- Consider the free surface H instead of h in the numerical flux function
- Initial data for the lake at rest: $H = \text{cste}$ and $u = 0$

Main reformulation

$$\begin{cases} \partial_t h + \partial_x XHu = 0 \\ \partial_t hu + \partial_x \left(X \left(Hu^2 + gH^2/2 \right) - \frac{g}{2} hZ \right) + gh\partial_x Z = 0 \end{cases}$$

Set $W = \begin{pmatrix} H \\ Hu \end{pmatrix}$ to write

$$\partial_t w + \partial_x \left(Xf(W) - \begin{pmatrix} 0 \\ \frac{g}{2} hZ \end{pmatrix} \right) + \begin{pmatrix} 0 \\ gh\partial_x Z \end{pmatrix}$$

SPH discretization

To rewrite the scheme for the shallow-water equations using the reformulation

$$\left\{\begin{array}{l} \dfrac{h_i^{n+1}-h_i^n}{\Delta t}+\displaystyle\sum_{j\in\mathcal{P}} 2\omega_j X_{ij}(Hu)_{ij}W'_{ij}=0 \\ \dfrac{h_i^{n+1}u_i^{n+1}-h_i^n u_i^n}{\Delta t}+\displaystyle\sum_{j\in\mathcal{P}} 2\omega_j X_{ij}\left(Hu^2+\dfrac{1}{2}gH^2\right)_{ij}W'_{ij}= \\ \qquad\qquad\qquad\qquad\qquad\qquad \left(\dfrac{g}{2}\partial_x(hZ)-gh\partial_x Z\right)_i \end{array}\right.$$

Source term discretization

To get a *well-balanced* and *conservative* scheme, we adopt the following consistent source term discretization:

$$\begin{aligned}\left(\frac{g}{2}\partial_x(hZ) - gh\partial_x Z\right)_i = \quad & \frac{g}{2}\sum_{j\in\mathcal{P}} 2\omega_j X_{ij} H_{ij}^2 \left(1 - X_{ij}\right) W'_{ij} - \\ & g\bar{X}_i\bar{H}_i \sum_{j\in\mathcal{P}} 2\omega_j \left(1 - X_{ij}\right) W'_{ij} + \\ & 2g\bar{H}_i^2\bar{X}_i(1-\tilde{X}_i)\sum_{j\in\mathcal{P}}\omega_j W'_{ij}\end{aligned}$$

averages H_{ij}, $\bar{H}_i$, $\bar{X}_i$, $\tilde{X}_i$ and X_{ij} still need to be determined

Main result

- Assume

$$H_{ij} = \bar{H}_i = H, \qquad \text{as soon as } H_i = H_j = H$$

$$\bar{X}_i = \frac{1}{2} \frac{\sum_{j\in\mathcal{P}} \omega_j X_{ij}^2 W'_{ij}}{\sum_{j\in\mathcal{P}} \omega_j \left(X_{ij} - 1\right) W'_{ij} + \left(\tilde{X}_i - 1\right) \sum_{j\in\mathcal{P}} \omega_j W'_{ij}}$$

  The scheme defined by the SPH hybridization and source term discretization *preserves the lake at rest*, for any expression of X_{ij} and $\tilde{X}_i$ consistent with X.

- The scheme is conservative

$\bar{X}_i$ is consistent with $X = h/H$

Thank you for your attention