Computation of Maxwell's eigenvalues

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Maxwell's eigenvalues

The problem we are solving: find $\lambda \neq 0$ such that

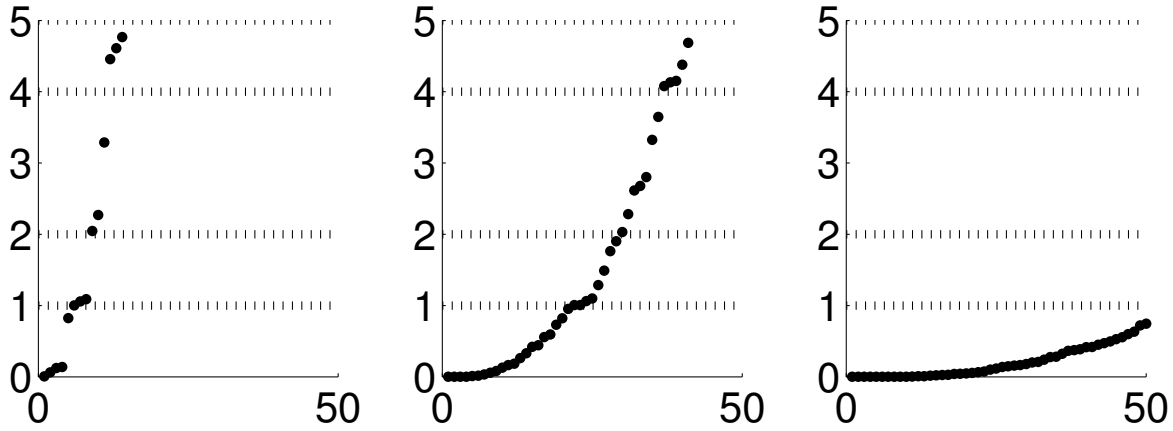
$$\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) :$$
$$(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl})$$

We take $\Omega =]0, \pi[\times]0, \pi[$ so that the exact solutions are

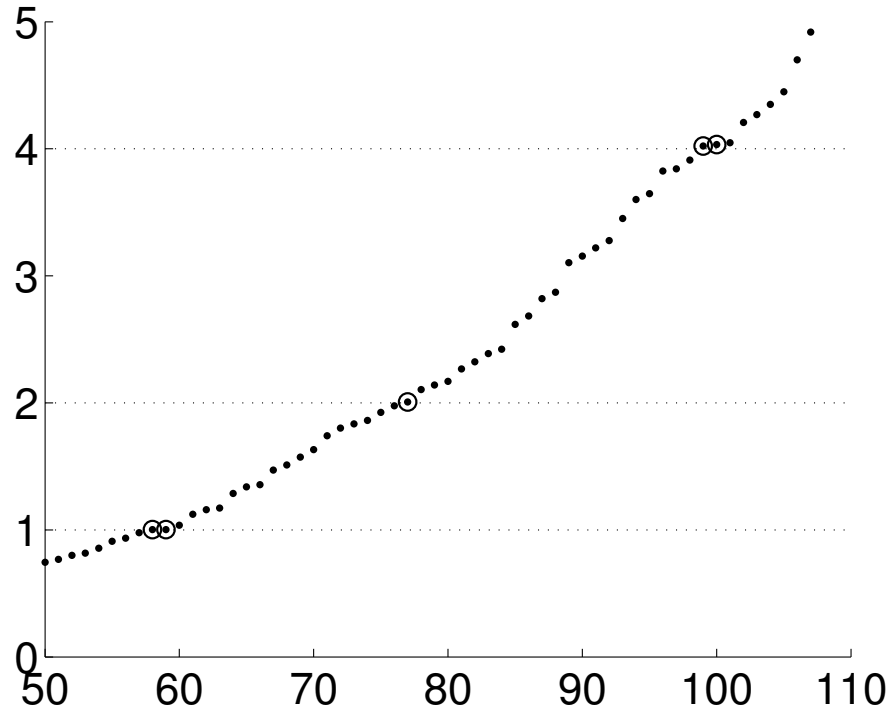
$$\lambda_{mn} = m^2 + n^2 = 1, 1, 2, 4, 4, 5, 5, \ldots$$
$$\mathbf{u}_{mn}(x, y) = \mathbf{curl}(\cos(mx)\cos(ny))$$

Standard P1 elements

Unstructured mesh ($N = 4, 8, 16$)



Zoom for $N = 16$



Standard P1 elements (cont'ed)

Some eigenfunctions

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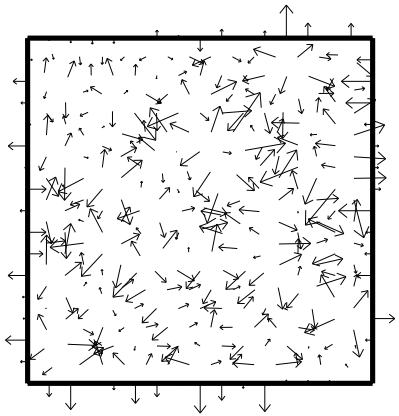

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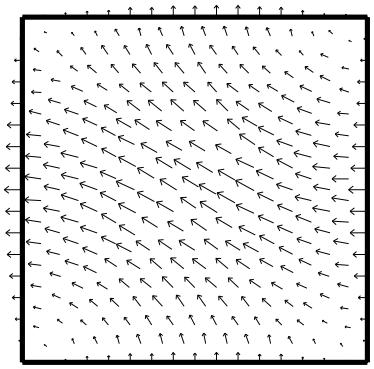

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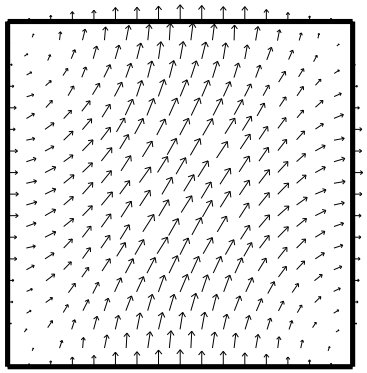

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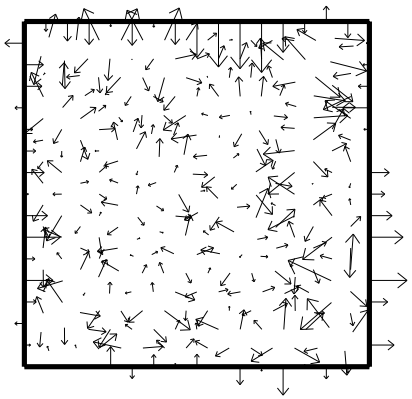
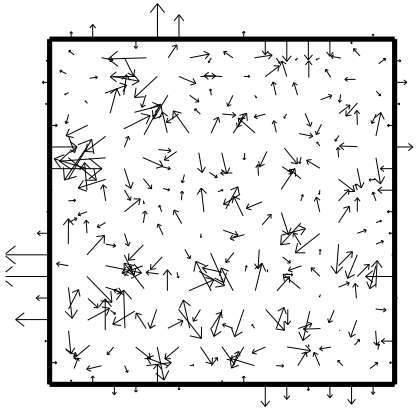


Standard P1 elements (cont'ed)

Some more eigenfunctions

76



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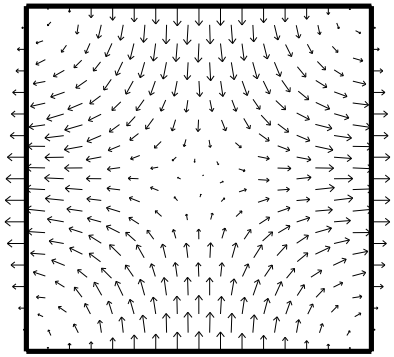

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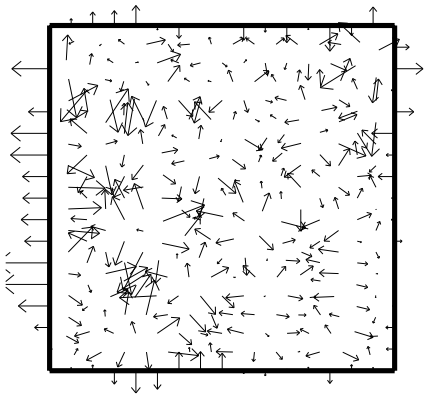
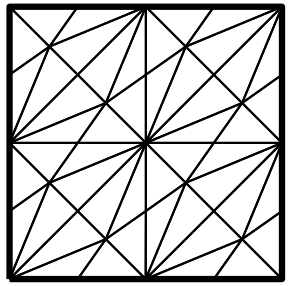
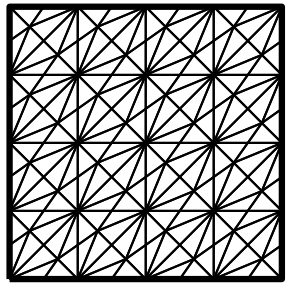
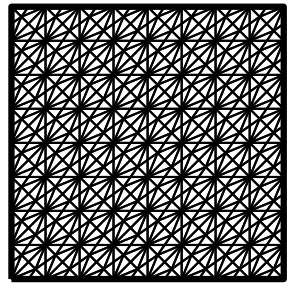


Wong and Cendes meshes

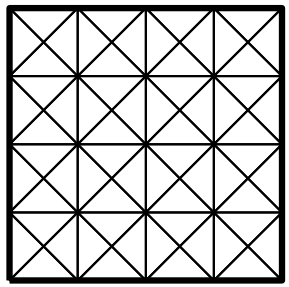
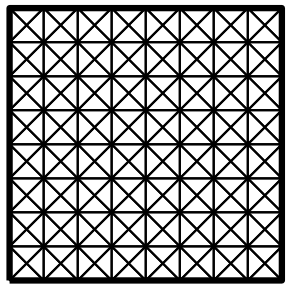
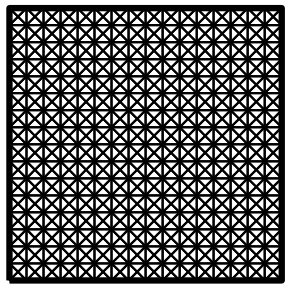
Compatible meshes and good results

Exact	Computed (rate)				
	$N=2$	$N=4$	$N=8$	$N=16$	$N=32$
1	1.0163	1.0045 (1.9)	1.0011 (2.0)	1.0003 (2.0)	1.0001 (2.0)
1	1.0445	1.0113 (2.0)	1.0028 (2.0)	1.0007 (2.0)	1.0002 (2.0)
2	2.0830	2.0300 (1.5)	2.0079 (1.9)	2.0020 (2.0)	2.0005 (2.0)
4	4.2664	4.1212 (1.1)	4.0315 (1.9)	4.0079 (2.0)	4.0020 (2.0)
4	4.2752	4.1224 (1.2)	4.0316 (2.0)	4.0079 (2.0)	4.0020 (2.0)
5	5.2244	5.1094 (1.0)	5.0326 (1.7)	5.0084 (2.0)	5.0021 (2.0)
5	5.5224	5.2373 (1.1)	5.0647 (1.9)	5.0164 (2.0)	5.0041 (2.0)
8	5.8945	8.3376 (2.6)	8.1198 (1.5)	8.0314 (1.9)	8.0079 (2.0)
9	6.3737	9.5272 (2.3)	9.1498 (1.8)	9.0382 (2.0)	9.0096 (2.0)
9	6.8812	9.5911 (1.8)	9.1654 (1.8)	9.0420 (2.0)	9.0105 (2.0)
zeros	7	39	175	735	3007
DOF	46	190	766	3070	12286

The crisscross mesh

Another compatible mesh?

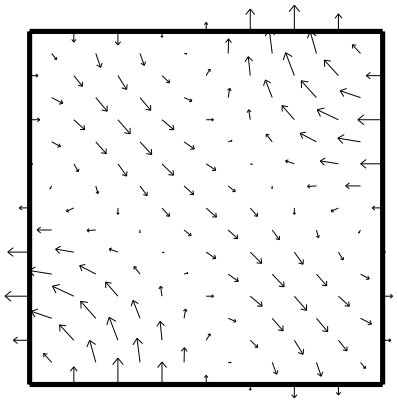




Let's look at the results...

	Computed (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
2	2.2035	2.0678 (1.6)	2.0171 (2.0)	2.0043 (2.0)	2.0011 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
5	6.1338	5.3971 (1.5)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
5	6.4846	5.3971 (1.9)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
6	6.4846	5.6712 (0.6)	5.9229 (2.1)	5.9807 (2.0)	5.9952 (2.0)
8	11.0924	8.8141 (1.9)	8.2713 (1.6)	8.0685 (2.0)	8.0171 (2.0)
9	11.0924	10.2540 (0.7)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
9	11.1164	10.2540 (0.8)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
15		9.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
15		19.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
17		20.9907	18.1813 (1.8)	17.3073 (1.9)	17.0773 (2.0)
zero	3	15	63	255	1023
dof	14	62	254	1022	4094

Spurious modes

70



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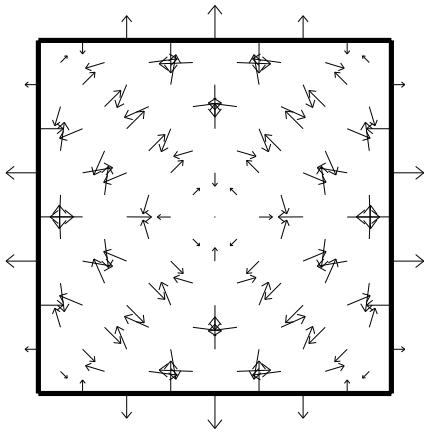

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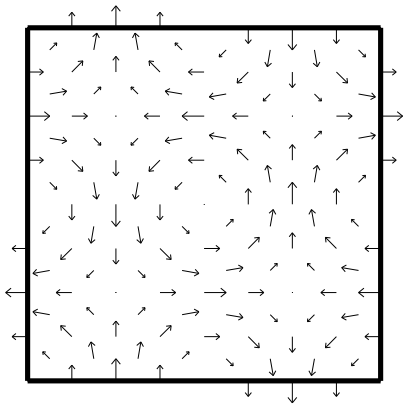
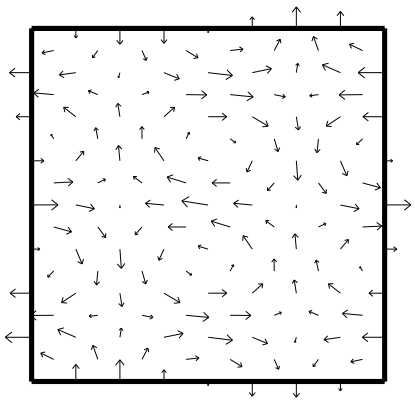


More spurious modes

78



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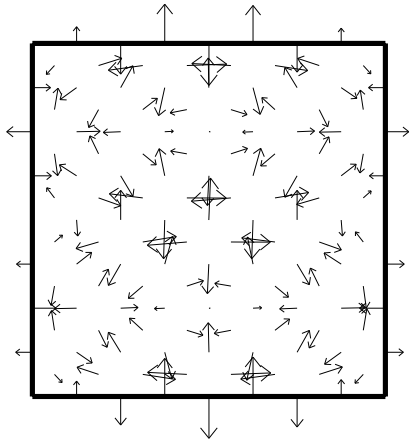

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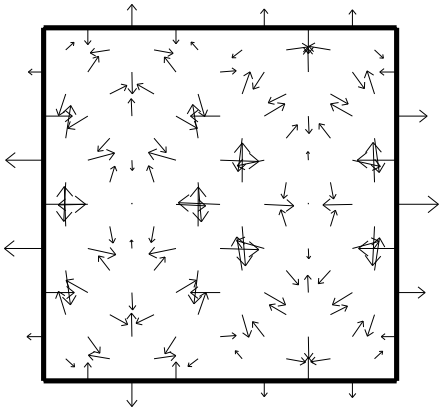

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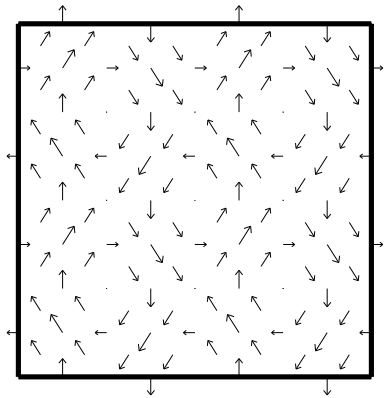
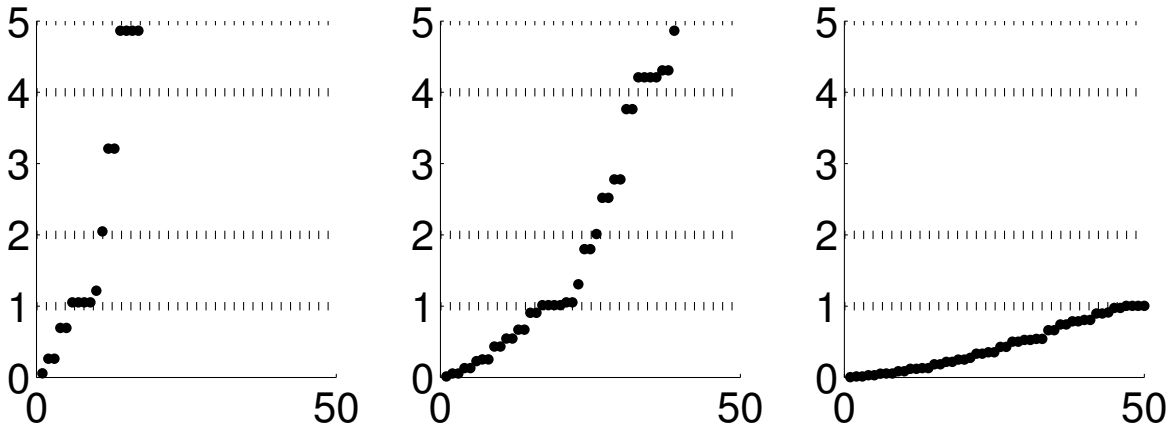


Mesh of quadrilaterals

Bilinear elements (Q1): same results as for the P1 element



Projected Q1

The Q1-P0 scheme: the discrete eigenvalues can be explicitely computed:

$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

$$\mathbf{u}_h^{(mn)} = (u^{(mn)}, v^{(mn)})$$

$$u^{(m,n)}(x_i, y_j) = \frac{2}{h}\sin\left(\frac{mh}{2}\right)\cos\left(\frac{nh}{2}\right)\sin(mx_i)\cos(ny_j)$$

$$v^{(m,n)}(x_i, y_j) = -\frac{2}{h}\cos\left(\frac{mh}{2}\right)\sin\left(\frac{nh}{2}\right)\cos(mx_i)\sin(ny_j)$$

Does it converge?

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	1.9909	1.9995 (4.1)	2.0000 (4.0)	2.0000 (4.0)	2.0000 (4.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
8	7.2951	7.9636 (4.3)	7.9978 (4.1)	7.9999 (4.0)	8.0000 (4.0)
9	8.7285	10.0803 (-2.0)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
10	11.2850	10.8308 (0.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
13	12.5059	13.1992 (1.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
13	12.8431	13.1992 (-0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
16	12.8431	14.7608 (1.3)	16.8382 (0.6)	16.2067 (2.0)	16.0515 (2.0)
16		17.5489	16.8382 (0.9)	16.2067 (2.0)	16.0515 (2.0)
17		19.4537	17.1062 (4.5)	17.1814 (-0.8)	17.0452 (2.0)
17		19.4537	17.7329 (1.7)	17.1814 (2.0)	17.0452 (2.0)
18		19.9601	17.7329 (2.9)	17.7707 (0.2)	17.9423 (2.0)
18		19.9601	17.9749 (6.3)	17.9985 (4.0)	17.9999 (4.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
#	15	63	255	1023	4095
#	30	126	510	2046	8190

Wrong proof?

$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

Indeed, if $h = \pi/N$, we have:

$$\lim_{N\to\infty} \lambda_h^{(N-1,N-1)} = 18,$$

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	1.9909	1.9995 (4.1)	2.0000 (4.0)	2.0000 (4.0)	2.0000 (4.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
8	7.2951	7.9636 (4.3)	7.9978 (4.1)	7.9999 (4.0)	8.0000 (4.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
13	12.8431	13.1992 (-0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
13	12.8431	13.1992 (-0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
16		17.5489	16.8382 (0.9)	16.2067 (2.0)	16.0515 (2.0)
16		19.4537	16.8382 (2.0)	16.2067 (2.0)	16.0515 (2.0)
17		19.4537	17.7329 (1.7)	17.1814 (2.0)	17.0452 (2.0)
17		19.9601	17.7329 (2.0)	17.1814 (2.0)	17.0452 (2.0)
18	8.7285	14.7608 (1.5)	17.1062 (1.9)	17.7707 (2.0)	17.9423 (2.0)
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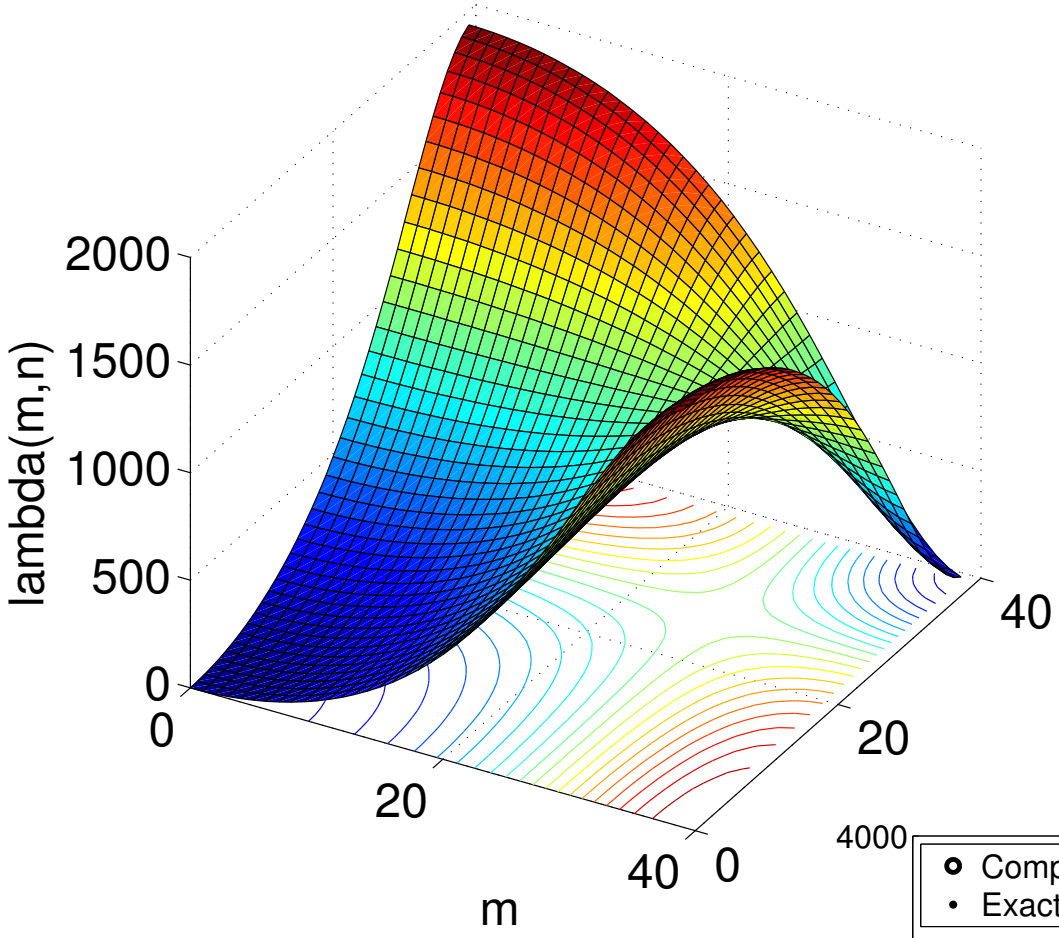
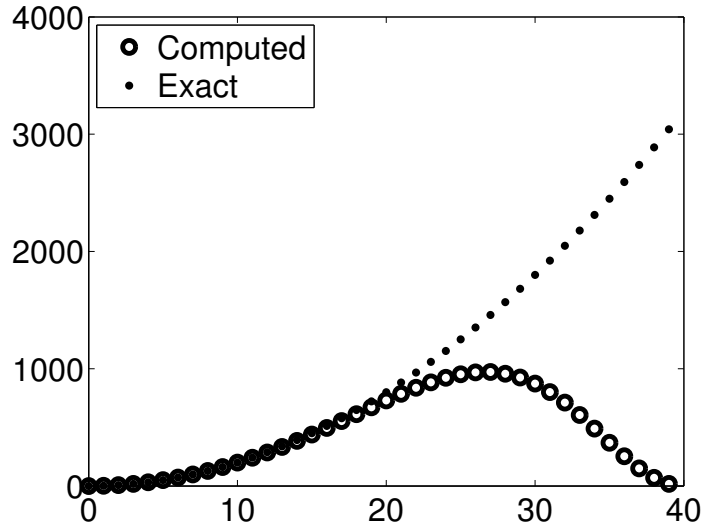
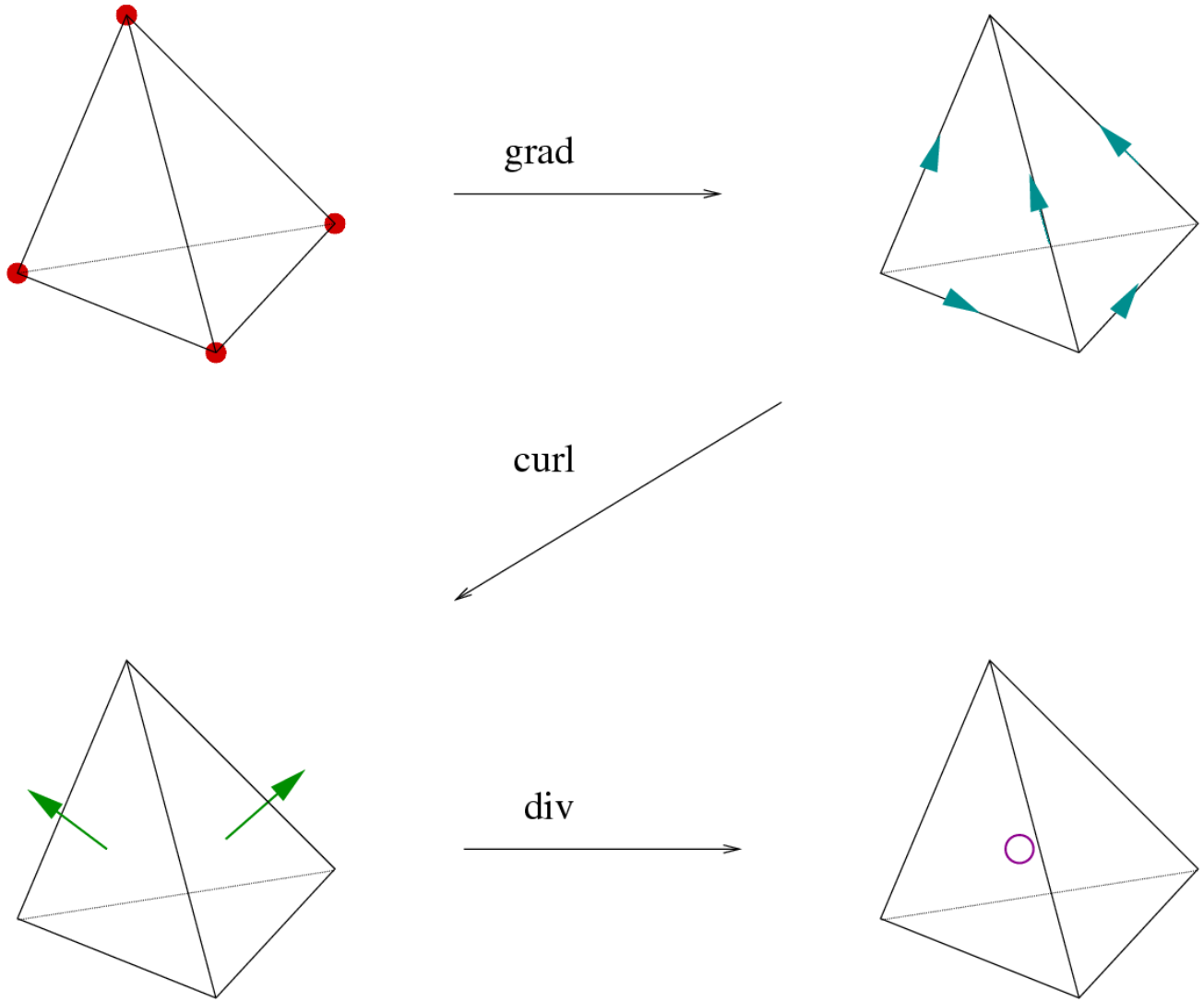
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
#	15	63	255	1023	4095
#	30	126	510	2046	8190


lambda(m,n)
2000
1500
1000
500
0
0
20
40
m
0
20
40

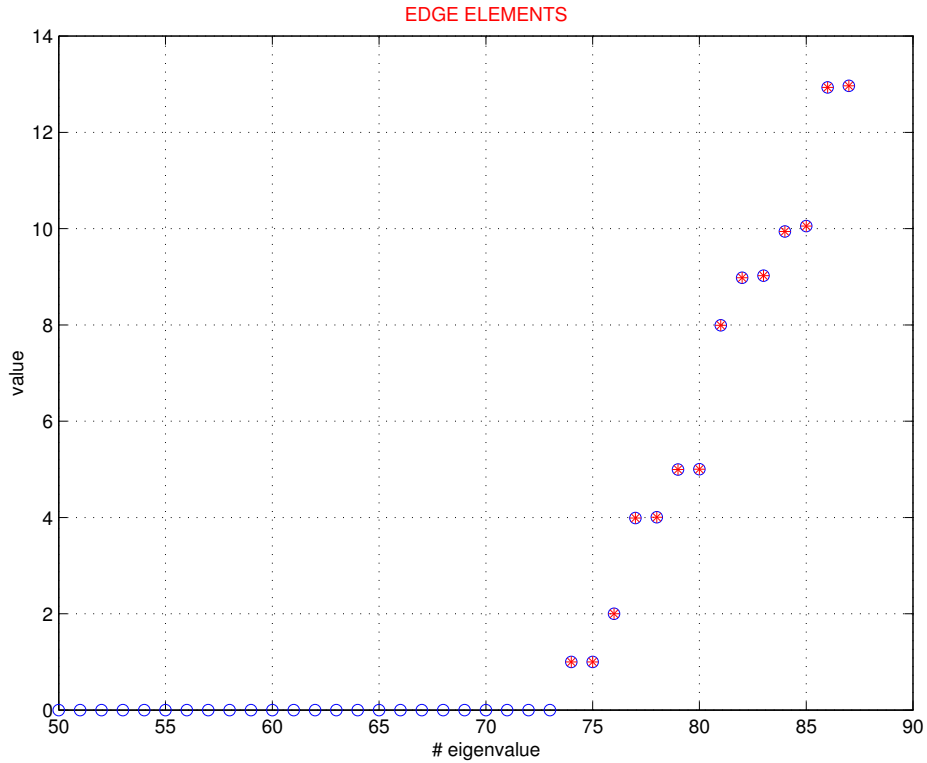

Computed
Exact
4000
3000
2000
1000
0
0
10
20
30
40

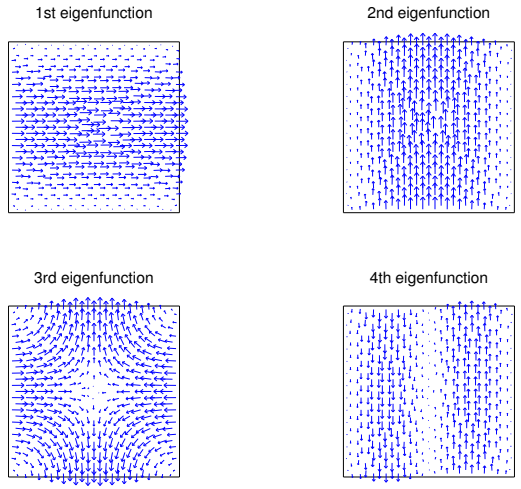
Plot for $m = n$

Edge elements



Nice convergence on general meshes





	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	0.9702	0.9923 (2.0)	0.9981 (2.0)	0.9995 (2.0)	0.9999 (2.0)
1	0.9960	0.9991 (2.2)	0.9998 (2.1)	0.9999 (2.0)	1.0000 (2.0)
2	2.0288	2.0082 (1.8)	2.0021 (2.0)	2.0005 (2.0)	2.0001 (2.0)
4	3.7227	3.9316 (2.0)	3.9829 (2.0)	3.9957 (2.0)	3.9989 (2.0)
4	3.7339	3.9325 (2.0)	3.9829 (2.0)	3.9957 (2.0)	3.9989 (2.0)
5	4.7339	4.9312 (2.0)	4.9826 (2.0)	4.9956 (2.0)	4.9989 (2.0)
5	5.1702	5.0576 (1.6)	5.0151 (1.9)	5.0038 (2.0)	5.0010 (2.0)
8	7.4306	8.1016 (2.5)	8.0322 (1.7)	8.0084 (1.9)	8.0021 (2.0)
9	7.5231	8.6292 (2.0)	8.9061 (2.0)	8.9764 (2.0)	8.9941 (2.0)
9	7.9586	8.6824 (1.7)	8.9211 (2.0)	8.9803 (2.0)	8.9951 (2.0)
zero	9	49	225	961	3969
dof	40	176	736	3008	12160