Eigenvalue approximation in mixed form and the *hp* version of edge finite elements

Daniele Boffi

Dipartimento di Matematica "F. Casorati", Università di Pavia
http://www-dimat.unipv.it/boffi

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Contents

1. Maxwell's eigenvalue problem.
    - Recall the analysis for the h version of edge finite elements and equivalence with mixed formulations.
    - Discrete compactness property.
    - Analysis does not extend trivially to p and hp version.

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1. Maxwell's eigenvalue problem.
   - Recall the analysis for the h version of edge finite elements and equivalence with mixed formulations.
   - Discrete compactness property.
   - Analysis does not extend trivially to p and hp version.
2. Exterior calculus.
   - Discrete compactness property in the framework of differential forms.
   - Recent results on Poincaré map give discrete compactness for the p version of discrete differential forms.

Maxwell eigenvalues

Ampère and Faraday's laws: find resonance frequencies $\omega \in \mathbb{R}$ (with $\omega \neq 0$) and electromagnetic fields $(\mathbf{E}, \mathbf{H}) \neq (0,0)$ such that

$$
\begin{aligned}
&\mathbf{curl}\,\mathbf{E} = i\omega\mu\mathbf{H} && \text{in } \Omega\\
&\mathbf{curl}\,\mathbf{H} = -i\omega\varepsilon\mathbf{E} && \text{in } \Omega\\
&\mathbf{E} \times \mathbf{n} = 0 && \text{on } \partial\Omega\\
&\mathbf{H} \cdot \mathbf{n} = 0 && \text{on } \partial\Omega
\end{aligned}
$$

$\omega \neq 0$ gives divergence conditions

$$
\begin{aligned}
&\operatorname{div} \varepsilon\mathbf{E} = 0 && \text{in } \Omega\\
&\operatorname{div} \mu\mathbf{H} = 0 && \text{on } \Omega
\end{aligned}
$$

It is then standard to eliminate one field and to obtain the **curl curl** problem

Eliminate $\mathbf{H}$ and take $\mathbf{u} = \mathbf{E}$ $(\lambda = \omega^2)$

$$\begin{cases} \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{u}) = \lambda\varepsilon\mathbf{u} & \text{in } \Omega \\ \mathrm{div}(\varepsilon\mathbf{u}) = 0 & \text{in } \Omega \\ \mathbf{u}\times\mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Well-known and intensively studied problem. Special (*edge*) finite elements required for its approximation. We review classical analysis for the *h* version which covers basically all known families of edge finite elements.
The ultimate goal of more recent work is to analyze the convergence for the *p* and *hp* version of FEM.

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Well-known and intensively studied problem. Special (*edge*) finite elements required for its approximation. We review classical analysis for the *h* version which covers basically all known families of edge finite elements.
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For ease of presentation, we take $\mu = \varepsilon = 1$ and simple topology from now on.

Standard formulation

The standard variational formulation reads

$$
\begin{aligned}
&\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) : \\
&(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \qquad && \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\
&(\mathbf{u}, \mathbf{grad}\,\phi) = 0 && \forall \phi \in H^1_0
\end{aligned}
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The most commonly used variational formulation in based on the replacement of the divergence free constraint by the condition $\lambda \neq 0$

$$\begin{aligned} &\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}) : \\ &(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \end{aligned}$$

The kernel $\lambda = 0$ corresponds to the infinite dimensional space $\mathbf{grad}\, H^1_0$.

Mixed formulations

<Kikuchi '89>

Divergence free constraint imposed via Lagrange multiplier ψ

$\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}),\ \psi \in H^1_0:$

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,\psi, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{grad}\,\phi, \mathbf{u}) = 0 & \forall \phi \in H^1_0 \end{cases}$$

Mixed formulations

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$$\mathbf{u} \in \mathbf{H}_0(\mathbf{curl}),\ \psi \in H_0^1 :$$
$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,\psi, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{grad}\,\phi, \mathbf{u}) = 0 & \forall \phi \in H_0^1 \end{cases}$$

<B–Fernandes–Gastaldi–Perugia '99>

Second mixed formulation ($\mathbf{H}_0(\mathrm{div}^0) = \mathbf{curl}(\mathbf{H}_0(\mathbf{curl}))$)

$$\boldsymbol{\sigma} \in \mathbf{H}_0(\mathbf{curl}),\ \mathbf{z} \in \mathbf{H}_0(\mathrm{div}^0) :$$
$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{z}) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{curl}\,\boldsymbol{\sigma}, \mathbf{w}) = -\lambda(\mathbf{z}, \mathbf{w}) & \forall \mathbf{w} \in \mathbf{H}_0(\mathrm{div}^0) \end{cases}$$

Eigenvalues in mixed form

The equivalence with mixed formulations allowed us to apply general theory of eigenvalue approximation in mixed form.

<B.–Brezzi–Gastaldi '97>

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The main tool for the analysis (exploited for the *h* version) is the construction of a Fortin operator that converges to the identity in norm: *Fortid* property.

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Discrete Compactness Property may also be used.

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<Monk–Demkowicz ’00>
<Caorsi–Fernandes–Raffetto ’00>

The two approaches are indeed equivalent

<B. ’07>

Mixed conditions for Kikuchi formulation

[ELKER] Ellipticity in the discrete kernel

There exists $\alpha > 0$ such that

$$(\mathrm{curl}\,\mathbf{v}_k, \mathrm{curl}\,\mathbf{v}_k) \geq \alpha \|\mathbf{v}_k\|^2_{L^2} \quad \forall \mathbf{v}_k \in K^d_k$$

[WA1] Weak approximability of $Q = H^{1+s}_0$

There exists $\omega_1(k)$ tending to zero such that

$$\sup_{\mathbf{v}_k \in K^d_k} \frac{(\mathbf{v}_k, \mathbf{grad}\,\psi)}{\|\mathbf{v}_k\|_{\mathrm{curl}}} \leq \omega_1(k)\|\psi\|_{H^1} \quad \forall \psi \in Q$$

[SA1] Strong approximability of $V_0 = \mathbf{H}^s_0(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}^0)$

There exists $\omega_2(k)$ tending to zero such that for every $\mathbf{u} \in V_0$ there exists $\mathbf{u}^I \in K^d_k$ such that

$$\|\mathbf{u} - \mathbf{u}^I\|_{\mathrm{curl}} \leq \omega_2(k)\|\mathbf{u}\|_{V_0}$$

Kikuchi resolvent operators: continuous...

$$\begin{cases} (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})+(\mathbf{grad}\,p,\mathbf{v})=(\mathbf{f},\mathbf{v}) & \forall\mathbf{v}\in\mathbf{H}_0(\mathbf{curl})\\ (\mathbf{grad}\,q,\mathbf{u})=0 & \forall q\in H^1_0 \end{cases}$$

$T^{Ki}\in\mathcal{L}(L^2)$: $T^{Ki}(\mathbf{f})=\mathbf{u}$

...and discrete one

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}_k,\mathbf{curl}\,\mathbf{v})+(\mathbf{grad}\,p_k,\mathbf{v})=(\mathbf{f},\mathbf{v}) & \forall\mathbf{v}\in V_k\\ (\mathbf{grad}\,q,\mathbf{u}_k)=0 & \forall q\in Q_k \end{cases}$$

$T^{Ki}_k\in\mathcal{L}(L^2)$: $T^{Ki}_k(\mathbf{f})=\mathbf{u}_k$

<B.–Brezzi–Gastaldi '97>

Theorem

If the ellipticity in the discrete kernel [ELKER], the weak approximability of Q [WA1], and the strong approximability of V_0 [SA1] are satisfied, then the following convergence in norm holds true

$$\|T^{Ki} - T_k^{Ki}\|_{\mathcal{L}(L^2)} \to 0$$

Remark

Convergence in norm allows us to use the classical Babuška–Osborn theory for eigenmode convergence

Mixed conditions for second formulation

[WA2] Weak approximability of $Z^0 = \mathbf{H}_0^s(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}^0)$

There exists $\omega_3(k)$ tending to zero such that

$$(\mathrm{curl}\,\boldsymbol{\tau}_k, \mathbf{z}) \le \omega_3(k) \|\boldsymbol{\tau}_k\|_{L^2} \|\mathbf{z}\|_{Z^0} \quad \forall \boldsymbol{\tau}_k \in K_k^c,\ \forall \mathbf{z} \in Z^0$$

[SA2] Strong approximability of $Z^0 = \mathbf{H}_0^s(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}^0)$

There exists $\omega_4(k)$ tending to zero such that for every $\mathbf{z} \in Z^0$ there exists $\mathbf{z}^I \in K_k^c$ such that

$$\|\mathbf{z} - \mathbf{z}^I\|_{L^2} \le \omega_4(k) \|\mathbf{z}\|_{Z^0}$$

Fortin operator

$\Pi_k : V^0 \to V_k$ such that $\forall \boldsymbol{\sigma} \in V^0$

$$\begin{cases} (\mathbf{curl}(\boldsymbol{\sigma} - \Pi_k \boldsymbol{\sigma}), \mathbf{w}_k) = 0 & \forall \mathbf{w}_k \in Z_k \\ \|\Pi_k \boldsymbol{\sigma}\|_{\mathrm{curl}} \le C \|\boldsymbol{\sigma}\|_{V^0} \end{cases}$$

[FORTID] Fortid property

There exists $\omega_5(k)$ tending to zero such that

$$\|\boldsymbol{\sigma} - \Pi_k \boldsymbol{\sigma}\|_{L^2} \le \omega_5(k) \|\boldsymbol{\sigma}\|_{V^0} \quad \forall \boldsymbol{\sigma} \in V^0$$

Alternative resolvent operators: continuous...

$$\begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau},\mathbf{z}) = 0 & \forall\boldsymbol{\tau}\in\mathbf{H}_0(\mathbf{curl})\\ (\mathbf{curl}\,\boldsymbol{\sigma},\mathbf{w}) = -(\mathbf{g},\mathbf{w}) & \forall\mathbf{w}\in\mathbf{curl}(\mathbf{H}_0(\mathbf{curl})) \end{cases}$$

$T^{M2}\in\mathcal{L}(L^2)$: $T^{M2}(\mathbf{g})=\mathbf{z}$

...and discrete one

$$\begin{cases} (\boldsymbol{\sigma}_k,\boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau},\mathbf{z}_k) = 0 & \forall\boldsymbol{\tau}\in V_k\\ (\mathbf{curl}\,\boldsymbol{\sigma}_k,\mathbf{w}) = -(\mathbf{g},\mathbf{w}) & \forall\mathbf{w}\in Z_k \end{cases}$$

$T_k^{M2}\in\mathcal{L}(L^2)$: $T_k^{M2}(\mathbf{g})=\mathbf{z}_k$

<B.–Brezzi–Gastaldi '97>

Theorem

If the weak approximability of Z^0 [WA2] and the strong approximability of Z^0 [SA2] are satisfied, and if there exists a Fortin operator satisfying the Fortid property [FORTID], then the following convergence in norm holds true

$$\|T^{M2} - T_k^{M2}\|_{\mathcal{L}(L^2)} \to 0$$

Compactness properties

The space $\mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div}^0)$ is compactly embedded in L^2

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Compactness can be rephrased as

Given a sequence $\{\mathbf{u}_n\} \subset \mathbf{H}_0(\mathbf{curl})$ such that

$$(\mathbf{u}_n, \mathbf{grad}\,\phi) = 0 \quad \forall \phi \in H^1_0,\ \forall n$$

If $\{\mathbf{u}_n\}$ is uniformly bounded in $\mathbf{H}_0(\mathbf{curl})$, $\|\mathbf{curl}\,\mathbf{u}_n\|_{L^2} \le 1$, then there exits a subsequence (still denoted $\{\mathbf{u}_n\}$) and $\mathbf{u} \in L^2$ such that

$$\|\mathbf{u}_n - \mathbf{u}\|_{L^2} \to 0$$

Discrete compactness property

Discrete analogue for the spaces $V_k \subset \mathbf{H}_0(\mathbf{curl})$ and $Q_k \subset H^1_0$.

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Given a sequence $\{\mathbf{u}_k\} \subset V_k$ *discretely divergence free*, i.e.,

$$(\mathbf{u}_k, \mathbf{grad}\,\phi_k) = 0 \quad \forall \phi_k \in Q_k,\ \forall k$$

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If $\{\mathbf{u}_k\}$ is uniformly bounded in $\mathbf{H}_0(\mathbf{curl})$, $\|\mathbf{curl}\,\mathbf{u}_k\|_{L^2} \le 1$, then there exits a subsequence (still denoted $\{\mathbf{u}_k\}$) and $\mathbf{u} \in L^2$ such that

$$\|\mathbf{u}_k - \mathbf{u}\|_{L^2} \to 0$$

Strong DCP

We say that the SDCP is satisfied if $\mathbf{u}$ is divergence free $\operatorname{div}\mathbf{u} = 0$. This is true, for instance, if Q_k is a good approximation to H^1_0.

Commuting diagram property

<Douglas–Roberts '82>
<Bossavit '88>
<Arnold '02>

$Q \subset H^1_0,\ V \subset \mathbf{H}_0(\mathbf{curl}),\ U \subset \mathbf{H}_0(\mathrm{div}),\ S \subset L^2/\mathbb{R}$

$$
\begin{array}{ccccccccc}
0 \to & Q & \xrightarrow{\mathbf{grad}} & V & \xrightarrow{\mathbf{curl}} & U & \xrightarrow{\mathrm{div}} & S & \to 0 \\
 & \downarrow \Pi_k^Q & & \downarrow \Pi_k^V & & \downarrow \Pi_k^U & & \downarrow \Pi_k^S & \\
0 \to & Q_k & \xrightarrow{\mathbf{grad}} & V_k & \xrightarrow{\mathbf{curl}} & U_k & \xrightarrow{\mathrm{div}} & S_k & \to 0
\end{array}
$$

- Kikuchi formulation uses Q and V
- Alternative formulation uses V and U
- U and S are used for Darcy flow or mixed Laplacian

Equivalence

<B. '07>

Given $V_k \subset \mathbf{H}_0(\mathbf{curl})$, construct Q_k and Z_k such that
$\mathbf{grad}\, Q_k \subset V_k, \quad \mathbf{curl}\, V_k \subset Z_k$

- $Z_k = \mathbf{curl}\, V_k$
- The kernel of **curl** in V_k consists of gradient. Take Q_k as set of potentials vanishing on the boundary $\partial\Omega$

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- The kernel of **curl** in V_k consists of gradient. Take Q_k as set of potentials vanishing on the boundary $\partial\Omega$

Theorem

The following three sets of conditions are equivalent

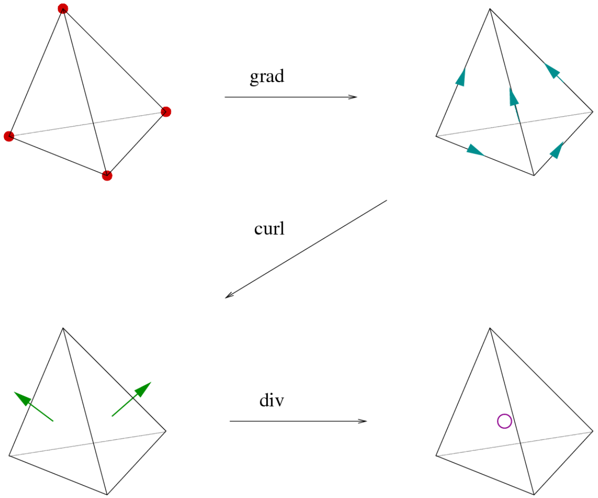
i) *ELKER, WA1, SA1*

ii) *WA2, SA2, FORTID*

iii) *SDCP and standard approximation property: for any* $\mathbf{v} \in V_0$ *there exists* $\mathbf{v}_k^I \in V_k$ *such that*

$$\|\mathbf{v} - \mathbf{v}_k^I\|_{\mathbf{curl}} \to 0$$

Finite elements



h version (2D)

<Falk–Osborn '80>

The analysis for the *h* version of edge elements is fairly easy in the two dimensional case.

h version (2D)

<Falk–Osborn '80>

The analysis for the *h* version of edge elements is fairly easy in the two dimensional case.

- The two dimensional curl operator is isomorphic to the div operator (and **curl** corresponds to **grad**)
- Edge elements are isomorphic to Raviart–Thomas elements
- The RT interpolant is a Fortin operator

$$
\int_K w_h \operatorname{div}(\sigma - \Pi_h \Sigma) = -\int_K \mathbf{grad}\, w_h \cdot (\sigma - \Pi_h\Sigma) + \int_{\partial K} w_h (\sigma - \Pi_h \Sigma)\cdot \mathbf{n} = 0
$$

h version (3D)

While the RT interpolant is still a Fortin operator, the edge interpolant is not.

h version (3D)

While the RT interpolant is still a Fortin operator, the edge interpolant is not.

Moreover, standard estimates for mixed approximations don't help (we need uniform convergence!)

$$\boldsymbol{\sigma} \in \mathbf{H}_0(\mathbf{curl}),\ \mathbf{z} \in \mathbf{H}_0(\mathrm{div}^0):$$
$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{z}) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl}) \\ (\mathbf{curl}\,\boldsymbol{\sigma}, \mathbf{w}) = -(\mathbf{g}, \mathbf{w}) & \forall \mathbf{w} \in \mathbf{H}_0(\mathrm{div}^0) \end{cases}$$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{H}(\mathbf{curl})} + \|\mathbf{z} - \mathbf{z}_h\|_{L^2} \le C \inf_{\boldsymbol{\tau}_h, \mathbf{w}_h} (\underbrace{\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|}_{O(1)} + \underbrace{\|\mathbf{z} - \mathbf{w}_h\|}_{O(h)})$$

Estimate for $\|\mathbf{z} - \mathbf{z}_h\|_{L^2}$ not involving $\mathbf{curl}\,\boldsymbol{\sigma}$ needed.

A better estimate can be obtained, for instance, with the help of Fortin operator

$$
\begin{aligned}
\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2} &\le C\left(\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L^2}+(1/\sqrt{\alpha})\inf_{\mathbf{w}_h}\|\mathbf{z}-\mathbf{w}_h\|_{L^2}\right)\\
\|\mathbf{z}-\mathbf{z}_h\|_{L^2} &\le C\left(\inf_{\mathbf{w}_h}\|\mathbf{z}-\mathbf{w}_h\|_{L^2}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2}\right)
\end{aligned}
$$

Proof

A better estimate can be obtained, for instance, with the help of Fortin operator

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2} &\le C\left(\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L^2}+(1/\sqrt{\alpha})\inf_{\mathbf{w}_h}\|\mathbf{z}-\mathbf{w}_h\|_{L^2}\right)\\ \|\mathbf{z}-\mathbf{z}_h\|_{L^2} &\le C\left(\inf_{\mathbf{w}_h}\|\mathbf{z}-\mathbf{w}_h\|_{L^2}+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2}\right)\end{aligned}$$

▸ Proof

The result then follows from the Fortid property

$$\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L^2}\le Ch^s\|\boldsymbol{\sigma}\|_{H^s}$$

Fortid for edge elements

<B. '00>

A Fortin operator can be easily constructed by using the inf-sup condition for edge elements.
The uniform estimate follows from the commuting diagram and a particular bound for the edge interpolant

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^I\|_{L^2} \le Ch^s \|\boldsymbol{\sigma}\|_{H^s} \quad \text{when } \mathbf{curl}\,\boldsymbol{\sigma} \text{ is discrete}$$

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Remark

The last estimate needs to be generalized to p and hp versions

p and *hp* versions

Some preliminary steps towards *hp* DCP

- Numerical evidence of *p* convergence <Monk '94>

p and *hp* versions

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- Numerical evidence of *p* convergence <Monk '94>
- Convergence proof of *hp* DCP for 2D triangular meshes modulo a conjectured L^2 estimate <B.–Costabel–Demkowicz '03>

p and *hp* versions

Some preliminary steps towards *hp* DCP

- Numerical evidence of *p* convergence <Monk '94>
- Convergence proof of *hp* DCP for 2D triangular meshes modulo a conjectured L^2 estimate
  <B.–Costabel–Demkowicz '03>
- Rigorous proof of *hp* DCP for 2D rectangular meshes (allowing for 1-irregular hanging nodes)
  <B.–Costabel–Dauge–Demkowicz '06>

Existing proof does not extend to more general situations (triangles or 3D)

Differential forms

<Arnold–Falk–Winther '06-'09>

We consider a complex of differential forms, $\Omega \subset \mathbb{R}^n$

$$0 \longrightarrow \Lambda^0(\Omega) \xrightarrow{d_0} \Lambda^1(\Omega) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(\Omega) \longrightarrow 0$$

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$$0 \to \Lambda^0(\Omega) \xrightarrow{d_0} \Lambda^1(\Omega) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(\Omega) \to 0$$

We define $V^\ell = H_0(d_\ell, \Omega)$, so that we have the complex

$$0 \to V^0 \xrightarrow{d_0} V^1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} V^n(\Omega) \to 0$$

and, given finite element approximation spaces $V^\ell_p \subset V^\ell$, we consider the discrete differential complex

$$0 \to V^0_p \xrightarrow{d_0} V^1_p \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} V^n_p(\Omega) \to 0$$

Identification table

Differential form		Proxy representation	
		$d = 2$	$d = 3$
$\ell = 0$	d_0	$\mathbf{grad}$	$\mathbf{grad}$
	$tr_{\partial\Omega}\phi$	$\phi_{\vert\partial\Omega}$	$\phi_{\vert\partial\Omega}$
	$H_0(d_0, \Omega)$	$H_0^1(\Omega)$	$H_0^1(\Omega)$
$\ell = 1$	d_1	curl	$\mathbf{curl}$
	$tr_{\partial\Omega}\mathbf{u}$	$(\mathbf{u} \times \mathbf{n})_{\vert\partial\Omega}$	$(\mathbf{u} \times \mathbf{n})_{\vert\partial\Omega}$
	$H_0(d_1, \Omega)$	$\mathbf{H}_0(\mathrm{curl})$	$\mathbf{H}_0(\mathbf{curl})$
	δ_1	div	div
$\ell = 2$	d_2	0	div
	$tr_{\partial\Omega}\mathbf{q}$	0	$(\mathbf{q} \cdot \mathbf{n})_{\vert\partial\Omega}$
	$H_0(d_2, \Omega)$	$L_0^2(\Omega)$	$\mathbf{H}_0(\mathrm{div})$
	δ_2	$\overrightarrow{\mathrm{curl}}$	$\mathbf{curl}$

Eigenvalue problem

The abstract counterpart of Maxwell's eigenvalue problem is the following general formulation related to Hodge–Laplace problem

$$
\begin{aligned}
&u \in H_0(d_\ell,\Omega):\\
&(d_\ell u, d_\ell v) = \lambda(u,v) \quad \forall v \in H_0(d_\ell,\Omega)
\end{aligned}
$$

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The corresponding discretization reads

$$u_p \in V_p^\ell :$$
$$(d_\ell u_p, d_\ell v) = \lambda_p(u_p, v) \quad \forall v \in V_p^\ell$$

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\end{aligned}
$$

Remark

For $\ell = 0$ we have the standard eigenvalue problem for Laplace operator.

Kikuchi mixed formulation

Kikuchi-like formulation for differential forms

$$u \in H_0(d_\ell, \Omega),\ \psi \in H_0(d_{\ell-1}, \Omega):$$

$$\begin{cases} (d_\ell u, d_\ell v) + (d_{\ell-1}\psi, v) = \lambda(u, v) & \forall v \in H_0(d_\ell, \Omega) \\ (d_{\ell-1}\phi, u) = 0 & \forall \phi \in H_0(d_{\ell-1}, \Omega) \end{cases}$$

Inclusion $d_{\ell-1}(H_0(d_{\ell-1}, \Omega)) \subset H_0(d_\ell, \Omega)$ implies $\psi = 0$.

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We need to write the appropriate discrete compactness property in order to prove eigenvalue convergence.

<B. '07>

DCP for differential forms

We start from the commuting diagram (for a fixed ℓ)

$$\begin{array}{ccc} S(\Omega,\Lambda^{\ell-1}) & \xrightarrow{d_{\ell-1}} & X(\Omega,\Lambda^{\ell}) \\ \pi_p^{\ell-1}\Big\downarrow & & \Big\downarrow\pi_p^{\ell} \\ V_p^{\ell-1} & \xrightarrow{d_{\ell-1}} & V_p^{\ell} \end{array}$$

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Given $\{u_p\}$ with $u_p \in V_p^\ell$ and satisfying

$$(u_p, d_{\ell-1}\phi) = 0 \quad \forall \phi \in V_p^{\ell-1} \ \forall p,$$

if $\{u_p\}$ is bounded uniformly in $H_0(d_\ell, \Omega)$, $\|d_\ell u_p\|_{L^2} \leq 1$, then there exists a subsequence (denoted by $\{u_p\}$) and $u \in L^2(\Omega, \Lambda^\ell)$ such that

$$\|u_p - u\|_{L^2} \to 0$$

DCP and eigenvalue convergence

Using the arguments of <B. – [CMAME] '07 >, it is possible to show that SDCP and standard approximation properties imply the eigenvalue convergence.

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Analysis relies on:

- compactness result <Picard '84>
- equivalence with Kikuchi formulation
- eigenvalue convergence for mixed problems <B.–Brezzi–Gastaldi '00>

Proof for the *p* version of DCP

<B.–Costabel–Dauge–Demkowicz–Hiptmair '09>

The proof relies on a Poincaré map which *respects* polynomials

<Costabel–McIntosh '08>

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The main assumptions are the following ones

1. Regularity and compactness
   $H_0(d_\ell,\Omega)\cap H(\delta_\ell 0,\Omega)\hookrightarrow X(\Omega,\Lambda^\ell)$
2. Locality of projectors $\pi_p^{\ell-1}$ and π_p^ℓ and commuting diagram
3. Local approximation property
   $\|d_{\ell-1}(\phi-\pi_{p,K}^{\ell-1}\phi)\|_{L^2(K,\Lambda^{\ell-1})}\le\varepsilon_{\ell-1}(p)\|\phi\|_{S(K,\Lambda^{\ell-1})}$
4. Poincaré map $\kappa_j: C^\infty(K,\Lambda^j)\to C^\infty(K,\Lambda^{j-1})$ for $j=\ell,\ell+1$ such that $d_{\ell-1}\circ\kappa_\ell+\kappa_{\ell+1}\circ d_\ell=Id_\ell$ and $\kappa_{\ell+1}\circ d_\ell: V_p^\ell(K)\to V_p^\ell(K)$ with $\kappa_\ell\in\mathcal{L}(X(K,\Lambda^\ell),S(K,\Lambda^{\ell-1}))$ and $\kappa_{\ell+1}\in\mathcal{L}(L^2(K,\Lambda^{\ell+1}),X(K,\Lambda^\ell))$

Main theorem

Theorem (B.–Costabel–Dauge–Demkowicz–Hiptmair '09)

If hypotheses 1+2+3+4 are satisfied, then the Discrete Compactness Property holds true.

Remark

Approximation properties indeed imply that the Strong Discrete Compactness Property is valid.

Sketch of the proof I

We are given $\{u_p\}$, with $u_p \in V_p^\ell$, such that $(u_p, d_{l-1}\phi) = 0\ \forall\phi$, $\|d_\ell u_p\|_{L^2(\Omega,\Lambda^\ell)} \leq 1$.
We perform the *continuous* Hodge decomposition of $\{u_p\}$

$$\begin{aligned} &\tilde{u}_p = u_p + d_{\ell-1}\tilde{\psi}_p && \tilde{\psi}_p \in H_0(d_{\ell-1},\Omega) \\ &(\tilde{u}_p, d_{\ell-1}\phi) = 0 && \forall\phi \in H_0(d_{\ell-1},\Omega) \end{aligned}$$

Hence $\tilde{u}_p \in X(\Omega,\Lambda^\ell)$. From the compactness of $X(\Omega,\Lambda^\ell)$ in $L^2(\Omega,\Lambda^\ell)$, $\{\tilde{u}_p\}$ has a subsequence strongly convergent to $u \in L^2(\Omega,\Lambda^\ell)$. We will show that the same subsequence of $\{u_p\}$ converges to u in $L^2(\Omega,\Lambda^\ell)$.

We use Nédélec trick

$$\begin{aligned} \|\tilde{u}_p - u_p\|^2_{L^2(\Omega,\Lambda^\ell)} &= (\tilde{u}_p - u_p, \tilde{u}_p - \pi_p^\ell \tilde{u}_p + \pi_p^\ell \tilde{u}_p - u_p) \\ &= (\tilde{u}_p - u_p, \tilde{u}_p - \pi_p^\ell \tilde{u}_p + d_{\ell-1}\pi_p^{\ell-1}\tilde{\psi}_p) \\ &= (\tilde{u}_p - u_p, \tilde{u}_p - \pi_p^\ell \tilde{u}_p) \end{aligned}$$

Sketch of the proof II

$$\|\tilde{u}_p - u_p\|_{L^2(\Omega,\Lambda^\ell)} \le \|\tilde{u}_p - \pi_p^\ell \tilde{u}_p\|_{L^2(\Omega,\Lambda^\ell)}$$

The final result follows from the approximation assumption and the Poincaré map

Lemma

If $u \in X(\Omega, \Lambda^\ell)$ *satisfies* $d_\ell u \in d_\ell V_p^\ell$, *then*

$$\|u - \pi_p^\ell u\|_{L^2(\Omega,\Lambda^\ell)} \le C\epsilon_{\ell-1}(p)\|u\|_{X(\Omega,\Lambda^\ell)}$$

In order to prove the last estimate, we can work on a single element K (locality assumption).

Sketch of the proof III

Poincaré map gives $u = d_{\ell-1}\kappa_\ell u + \kappa_{\ell+1} d_\ell u$ and $\|\kappa_{\ell+1} d_\ell u\|_{X(K,\Lambda^\ell)} \le C\|d_\ell u\|_{L^2(K,\Lambda^\ell)}$

We set $\psi = \kappa_\ell u$, so that $\|\psi\|_{S(K,\Lambda^{\ell-1})} \le C\|u\|_{X(K,\Lambda^\ell)}$

From $u = d_{\ell-1}\psi + \kappa_{\ell+1} d_\ell u$ we obtain

$$\begin{aligned}(Id - \pi^\ell_{p,K})u &= d_{\ell-1}(Id - \pi^{\ell-1}_{p,K})\psi + (Id - \pi^\ell_{p,K})\kappa_{\ell+1} d_\ell u \\ &= d_{\ell-1}(Id - \pi^{\ell-1}_{p,K})\psi\end{aligned}$$

Hence

$$(Id - \pi^\ell_{p,K})u \le \epsilon_{\ell-1}(p)\|\psi\|_{S(K,\Lambda^{\ell-1})} \le C\epsilon_{\ell-1}(p)\|u\|_{X(K,\Lambda^\ell)}$$

Summary and additional results

- The numerical analysis of edge finite element approximation of Maxwell's eigenvalues has been a challenging problem for more than a decade
- The use of nodal finite element is known to produce unreliable results
- Enforcing the divergence free condition with nodal elements and by a penalty procedure may be problematic
- Analysis for the h version of edge elements is complete
- Exterior calculus is a powerful tool for the analysis of our problem
- Analysis for the p version of edge elements is covered by the much more general theory of DCP for differential forms
- Extension to nonconstant coefficients and nontrivial topologies

Conclusions

What is covered by our theory...

- Basically all known edge element families for Maxwell's equations in two and three space dimensions (simplices, parallelepipeds, prisms,...)
- Raviart–Thomas elements for mixed Laplacian
- Standard Laplacian

...and what is not

- General quadrilateral and hexahedral elements

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$P = L^2$-projection

$$
\begin{aligned}
\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{L^2} &= (\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}, \Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\\
&= (\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}, \Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - (\mathbf{curl}(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), \mathbf{z} - P\mathbf{z})\\
&\le \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\|\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\mathbf{curl}(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|\|\mathbf{z} - P\mathbf{z}\|\\
&\le \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| \left(\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\| + (1/\sqrt{\alpha})\|\mathbf{z} - P\mathbf{z}\|\right)
\end{aligned}
$$

$$
\begin{aligned}
\|P\mathbf{z} - \mathbf{z}_h\|_{L^2} &\le C \sup_{\boldsymbol{\tau}_h} \frac{(P\mathbf{z} - \mathbf{z}_h, \mathbf{curl}\,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{curl}}}\\
&\le C \sup_{\boldsymbol{\tau}_h} \frac{(P\mathbf{z} - \mathbf{z}, \mathbf{curl}\,\boldsymbol{\tau}_h) + (\mathbf{z} - \mathbf{z}_h, \mathbf{curl}\,\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{curl}}}\\
&\le C \left(\|P\mathbf{z} - \mathbf{z}\| + \sup_{\boldsymbol{\tau}_h} \frac{-(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{curl}}}\right)\\
&\le C \left(\|P\mathbf{z} - \mathbf{z}\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|\right)
\end{aligned}
$$

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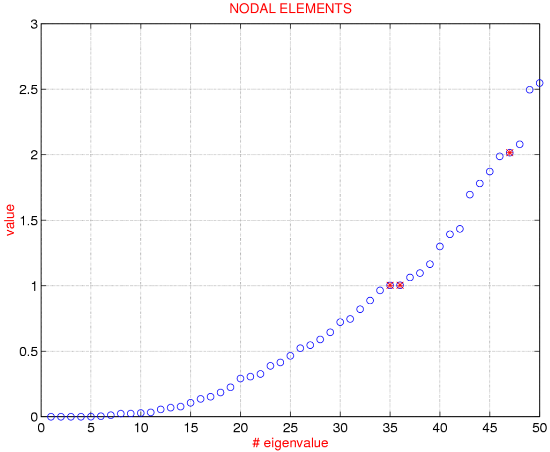
Nodal finite elements

<B.–Fernandes–Gastaldi–Perugia '99>

Nodal elements on unstructured meshes produce awful results



<B.–Brezzi–Gastaldi '97>

Nodal elements on structured meshes produce dangerous results

Mode		$n = 8$	$n = 16$	$n = 32$
(1,0)	1	1.00428	1.00107	1.00027
(0,1)	1	1.00428	1.00107	1.00027
(1,1)	2	2.01711	2.00428	2.00107
(2,0)	4	4.06804	4.01710	4.00428
(0,2)	4	4.06804	4.01710	4.00428
(2,1)	5	5.10634	5.02674	5.00669
(1,2)	5	5.10634	5.02674	5.00669
??	6	5.92293	5.98074	5.99518
(2,2)	8	8.27128	8.06845	8.01713
(3,0)	9	9.34085	9.08640	9.02166
(0,3)	9	9.34085	9.08640	9.02166
# zeros		63	255	1023

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Penalty formulation

Penalty formulation works on convex domains only!

$$(\mathbf{curl}\, \mathbf{u}, \mathbf{curl}\, \mathbf{v}) + s(\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{v}$$

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Due to the fact that $\mathbf{H}^1 \cap \mathbf{H}_0(\mathbf{curl})$ is a *closed* subspace of $\mathbf{H}_0(\mathbf{curl}) \cap \mathbf{H}(\mathrm{div})$

It is possible to use a weighted formulation which weakens the constraint in the proximity of reentrant corners

<Costabel-Dauge '02>

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More realistic situations

The case of variable coefficients (different materials) can be handled with the tools of <Caorsi–Fernandes–Raffetto '01>

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The case of nontrivial topologies gives rise to a de Rham complex which is no longer exact. The cohomology is however finite (a finite dimensional space of harmonic forms shows up), so that the DCP proof still remains valid

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Quadrilateral finite elements

<Arnold–B.–Falk '02-'05>

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Three-dimensional analysis still in progress

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