Laplace eigenvalue problem

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Rayleigh quotient

$$\lambda^{(1)} \le \lambda^{(2)} \le \cdots \le \lambda^{(k)} \le \cdots$$
$$a(u^{(m)}, u^{(n)}) = b(u^{(m)}, u^{(n)}) = 0 \quad \text{if } m \ne n$$

$$\lambda^{(1)} = \min_{v \in V} \frac{a(v,v)}{b(v,v)} \qquad u^{(1)} = \arg\min_{v \in V} \frac{a(v,v)}{b(v,v)}$$

$$\lambda^{(k)} = \min_{v \in \left(\bigoplus_{i=1}^{k-1} E^{(i)}\right)^{\perp}} \frac{a(v,v)}{b(v,v)} \qquad u^{(k)} = \arg\min_{v \in \left(\bigoplus_{i=1}^{k-1} E^{(i)}\right)^{\perp}} \frac{a(v,v)}{b(v,v)}$$

Rayleigh quotient (discrete)

$$\lambda_h^{(1)} \le \lambda_h^{(2)} \le \cdots \le \lambda_h^{(k)} \le \lambda_h^{(N(h))}$$

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$$\boxed{\Rightarrow \lambda^{(1)} \le \lambda_h^{(1)}}$$

Minmax characterization

The k-th eigenvalue $\lambda^{(k)}$ satisfies

$$\lambda^{(k)} = \min_{E \in V^{(k)}} \max_{v \in E} \frac{a(v,v)}{b(v,v)}$$

$V^{(k)}$ set of all subspaces of V with $\dim(V^{(k)}) = k$

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Proof.

$\boxed{\lambda^{(k)} \geq \min \max}$: take $E = \bigoplus_{i=1}^{k} E^{(i)}$, so that $v = \sum_{i=1}^{k} \alpha_i u^{(i)}$

Then $a(v,v)/b(v,v) \leq \lambda^{(k)}$ thanks to orthogonalities

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$\boxed{\lambda^{(k)} \leq \min\max}$: minimum $\lambda^{(k)}$ attained for $E = \bigoplus_{i=1}^{k} E^{(i)}$ and the choice $v = u^{(k)}$. Otherwise, there exists $v \in E$ orthogonal to $u^{(i)}$ for all $i \leq k$ and hence $a(v,v)/b(v,v) \geq \lambda^{(k)}$

$\square$

Minmax characterization (continuous and discrete)

$$\lambda^{(k)} = \min_{E \in V^{(k)}} \max_{v \in E} \frac{a(v,v)}{b(v,v)}$$

$V^{(k)}$ set of all subspaces of V with $\dim(V^{(k)}) = k$

$$\lambda_h^{(k)} = \min_{E_h \in V_h^{(k)}} \max_{v \in E_h} \frac{a(v,v)}{b(v,v)}$$

$V_h^{(k)}$ set of all subspaces of V_h with $\dim(V_h^{(k)}) = k$

$$\boxed{\Rightarrow \lambda^{(k)} \le \lambda_h^{(k)} \quad \forall k}$$

Laplace's eigenvalues

We need the upper bound

$$\lambda_h^{(k)} \leq \lambda^{(k)} + \varepsilon(h)$$

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We are going to use

$$E_h = \Pi_\mathrm{h}\, V^{(k)}$$

in the minmax characterization of the discrete eigenvalues, where

$$V^{(k)} = \bigoplus_{i=1}^{k} E^{(i)}$$

and $\Pi_\mathrm{h} : V \to V_h$ denotes the elliptic projection

$$(\nabla(u - \Pi_\mathrm{h}\, u), \nabla\, v_h) = 0 \quad \forall v_h \in V_h.$$

We need to check that the dimension of E_h is equal to k

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Take h small enough so that
$\|v - \Pi_\mathrm{h} v\|_{L^2(\Omega)} \le \frac{1}{2}\|v\|_{L^2(\Omega)} \quad \forall v \in V^{(k)}$

Then $\|\Pi_\mathrm{h} v\|_{L^2(\Omega)} \ge \|v\|_{L^2(\Omega)} - \|v - \Pi_\mathrm{h} v\|_{L^2(\Omega)} \quad \forall v \in V$
implies that Π_h is injective from $V^{(k)}$ to E_h

Taking E_h in the discrete minmax equation gives

$$
\begin{aligned}
\lambda_h^{(k)} &\le \max_{w\in E_h} \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}} = \max_{v\in V^{(k)}} \frac{\|\nabla(\Pi_\mathrm{h} v)\|^2_{L^2(\Omega)}}{\|\Pi_\mathrm{h} v\|^2_{L^2(\Omega)}} \\
&\le \max_{v\in V^{(k)}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|\Pi_\mathrm{h} v\|^2_{L^2(\Omega)}} = \max_{v\in V^{(k)}} \frac{\|\nabla v\|^2_{L^2(\Omega)}}{\|v\|^2_{L^2(\Omega)}} \frac{\|v\|^2_{L^2(\Omega)}}{\|\Pi_\mathrm{h} v\|^2_{L^2(\Omega)}} \\
&\le \lambda^{(k)} \max_{v\in V^{(k)}} \frac{\|v\|^2_{L^2(\Omega)}}{\|\Pi_\mathrm{h} v\|^2_{L^2(\Omega)}}.
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&\le \lambda^{(k)} \max_{v\in V^{(k)}} \frac{\|v\|^2_{L^2(\Omega)}}{\|\Pi_h v\|^2_{L^2(\Omega)}}.
\end{aligned}
$$

Take Ω convex (for simplicity). Then

$$
\begin{aligned}
\|v - \Pi_h v\|_{L^2(\Omega)} \le Ch^2\|\Delta v\|_{L^2(\Omega)} &\le C\lambda^{(k)}h^2\|v\|_{L^2(\Omega)} \\
&= C(k)h^2\|v\|_{L^2(\Omega)}
\end{aligned}
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Hence

$$\|\Pi_h v\|_{L^2(\Omega)} \ge \|v\|_{L^2(\Omega)}(1-C(k)h^2)$$

Eigenvalue estimate

Finally

$$\lambda_h^{(k)} \le \lambda^{(k)} \left(\frac{1}{1 - C(k)h^2} \right)^2 \simeq \lambda^{(k)} \big(1 + C(k)h^2\big)^2$$
$$\simeq \lambda^{(k)} (1 + 2C(k)h^2)$$

In general

$$\lambda_h^{(k)} \le \lambda^{(k)} \Big(1 + C(k) \sup_{\substack{v \in V^{(k)} \\ \|v\|=1}} \|v - \Pi_h v\|^2_{H^1(\Omega)} \Big)$$

Estimate for the eigenfunctions

Let's start with a simple eigenvalue $\lambda^{(k)}$

$$\rho_h^{(k)} = \max_{i \neq k} \frac{\lambda^{(k)}}{|\lambda^{(k)} - \lambda_h^{(i)}|},$$

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$$w_h^{(k)} = (\Pi_h u^{(k)}, u_h^{(k)}) u_h^{(k)}$$

$$\begin{aligned} \|u^{(k)} - u_h^{(k)}\|_{L^2(\Omega)} \leq \; & \|u^{(k)} - \Pi_h u^{(k)}\| \\ & + \|\Pi_h u^{(k)} - w_h^{(k)}\| \\ & + \|w_h^{(k)} - u_h^{(k)}\| \end{aligned}$$

Second term

$$\Pi_\mathrm{h}\, u^{(k)} - w_h^{(k)} = \sum_{i\neq k}(\Pi_\mathrm{h}\, u^{(k)}, u_h^{(i)})u_h^{(i)}$$

$$\|\, \Pi_\mathrm{h}\, u^{(k)} - w_h^{(k)}\|^2 = \sum_{i\neq k}(\Pi_\mathrm{h}\, u^{(k)}, u_h^{(i)})^2$$

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$$\begin{aligned}(\Pi_h u^{(k)}, u_h^{(i)}) &= \frac{1}{\lambda_h^{(i)}} (\nabla(\Pi_h u^{(k)}), \nabla u_h^{(i)}) \\ &= \frac{1}{\lambda_h^{(i)}} (\nabla u^{(k)}, \nabla u_h^{(i)}) = \frac{\lambda^{(k)}}{\lambda_h^{(i)}} (u^{(k)}, u_h^{(i)})\end{aligned}$$

$$\boxed{\lambda_h^{(i)} (\Pi_h u^{(k)}, u_h^{(i)}) = \lambda^{(k)} (u^{(k)}, u_h^{(i)})}$$

$$(\lambda_h^{(i)} - \lambda^{(k)})(\Pi_h u^{(k)}, u_h^{(i)}) = \lambda^{(k)}(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)})$$

$$|(\Pi_h u^{(k)}, u_h^{(i)})| \le \rho_h^{(k)} |(u^{(k)} - \Pi_h u^{(k)}, u_h^{(i)})|$$

$$\begin{aligned}\| \Pi_h u^{(k)} - w_h^{(k)} \|^2 &\le \left(\rho_h^{(k)}\right)^2 \sum_{i \ne k} (u^{(k)} - \Pi_h u^{(k)}), u_h^{(i)})^2 \\ &\le \left(\rho_h^{(k)}\right)^2 \| u^{(k)} - \Pi_h u^{(k)} \|^2\end{aligned}$$

Third term

We are going to show that

$$\|u_h^{(k)} - w_h^{(k)}\| \le \|u^{(k)} - w_h^{(k)}\|$$

so that

$$\|u_h^{(k)} - w_h^{(k)}\| \le \|u^{(k)} - \Pi_\mathrm{h}\, u^{(k)}\| + \|\Pi_\mathrm{h}\, u^{(k)} - w_h^{(k)}\|$$

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$$u_h^{(k)} - w_h^{(k)} = u_h^{(k)}(1 - ((\Pi_h\, u^{(k)}, u_h^{(k)})).$$

$$\|u^{(k)}\| - \|u^{(k)} - w_h^{(k)}\| \le \|w_h^{(k)}\| \le \|u^{(k)}\| + \|u^{(k)} - w_h^{(k)}\|$$

$$1 - \|u^{(k)} - w_h^{(k)}\| \le |(\Pi_h\, u^{(k)}, u_h^{(k)})| \le 1 + \|u^{(k)} - w_h^{(k)}\|,$$

$$\boxed{\Big| |(\Pi_h\, u^{(k)}, u_h^{(k)})| - 1 \Big| \le \|u^{(k)} - w_h^{(k)}\|}$$

Simple eigenfunction estimate

Sign choice for $u_h^{(k)}$ such that

$$(\Pi_h u^{(k)}, u_h^{(k)}) \geq 0$$

Then $\left| |(\Pi_h u^{(k)}, u_h^{(k)})| - 1 \right| = \|w_h^{(k)} - u_h^{(k)}\|$

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Final estimate

$$\| u^{(k)} - u_h^{(k)} \|_{L^2(\Omega)} \leq 2(1 + \rho_h^{(k)}) \| u^{(k)} - \Pi_\mathrm{h}\, u^{(k)} \|_{L^2(\Omega)}$$

Eigenfunction estimate in H^1

$$
\begin{aligned}
C\|u^{(k)} - u_h^{(k)}\|^2_{H^1(\Omega)} &\le \|\nabla(u^{(k)} - u_h^{(k)})\|^2_{L^2(\Omega)} \\
&= \|\nabla u^{(k)}\|^2 - 2(\nabla u^{(k)}, \nabla u_h^{(k)}) + \|\nabla u_h^{(k)}\|^2 \\
&= \lambda^{(k)} - 2\lambda^{(k)}(u^{(k)}, u_h^{(k)}) + \lambda_h^{(k)} \\
&= \lambda^{(k)} - 2\lambda^{(k)}(u^{(k)}, u_h^{(k)}) + \lambda^{(k)} - (\lambda^{(k)} - \lambda_h^{(k)}) \\
&= \lambda^{(k)}\|u^{(k)} - u_h^{(k)}\|^2_{L^2(\Omega)} - (\lambda^{(k)} - \lambda_h^{(k)})
\end{aligned}
$$

$$
\boxed{\|u^{(k)} - u_h^{(k)}\|_{H^1(\Omega)} \le C(k) \sup_{\substack{v\in V^{(k)}\\ \|v\|=1}} \|v - \Pi_{\mathrm{h}}\, v\|_{H^1(\Omega)}}
$$

Multiple eigenfunctions

$$\lambda^{(k)} = \lambda^{(k+1)}$$
$$\lambda^{(i)} \neq \lambda^{(k)} \text{ for } i \neq k, k+1$$

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$$w_h^{(k)} = \alpha_h u_h^{(k)} + \beta_h u_h^{(k+1)}$$

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$$\alpha_h = (\Pi_h u^{(k)}, u_h^{(k)}), \quad \beta_h = (\Pi_h u^{(k)}, u_h^{(k+1)})$$

$$\|u^{(k)} - w_h^{(k)}\|_{L^2(\Omega)} \leq (1 + \rho_h^{(k)})\|u^{(k)} - \Pi_h u^{(k)}\|_{L^2(\Omega)}$$

$$\|u^{(k)} - w_h^{(k)}\|_{H^1(\Omega)} \leq C(k) \sup_{\substack{v\in V^{(k+1)}\\ \|v\|=1}} \|v - \Pi_h v\|_{H^1(\Omega)}$$