# Babuška–Osborn theory

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The dimension of $\ker(\lambda - T)^{\alpha}$ is called algebraic multiplicity of $\lambda$ and the element of $\ker(\lambda - T)^{\alpha}$ are the generalized eigenvectors of $T$ associated with $\lambda$

A generalized vector is of order $k$ if it is in $\ker(\lambda - T)^{k}$ and not in $\ker(\lambda - T)^{k-1}$

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The generalized eigenvectors of order 1 are called eigenvectors of $T$ associated with $\lambda$ and are the elements of $\ker(\lambda - T)$

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If $T$ is selfadjoint, which will be the case for all examples discussed here, then the ascent multiplicity of each eigenvalue is equal to one; this implies that all generalized eigenvectors are eigenvectors and that the geometric and the algebraic multiplicities coincide

# Babuška–Osborn theory

**Theorem**

*Let us assume that the convergence in norm is satisfied. For any compact set* $K \subset \rho(T)$*, there exists* $h_0 > 0$ *such that for all* $h < h_0$ *it holds* $K \subset \rho(T_h)$ *(absence of spurious modes). If* $\lambda$ *is a nonzero eigenvalue of* $T$ *with algebraic multiplicity equal to* $m$*, then there are* $m$ *eigenvalues* $\lambda_{1,h}, \lambda_{2,h}, \dots, \lambda_{m,h}$ *of* $T_h$*, repeated according to their algebraic multiplicities, such that each* $\lambda_{i,h}$ *converges to* $\lambda$ *as* $h$ *goes to* $0$*.*

*Moreover, the gap between the direct sum of the generalized eigenspaces associated with* $\lambda_{1,h}, \lambda_{2,h}, \dots, \lambda_{m,h}$ *and the generalized eigenspace associated to* $\lambda$ *tends to zero as* $h$ *goes to* $0$*.*

# Estimating the rate of convergence

The first theorem concerns the approximation of eigenvectors.

**Theorem**

*Let $\mu$ be a nonzero eigenvalue of $T$, $E = E(\mu)X$ its generalized eigenspace, and $E_h = E_h(\mu)X$. Then*

$$\hat{\delta}(E, E_h) \leq C\|(T - T_h)_{|E}\|_{\mathcal{L}(X)}.$$

**Corollary (standard elliptic case)**

*Let $\lambda$ be an eigenvalue, $E = E(\lambda^{-1})V_1$ its generalized eigenspace and $E_h = E_h(\lambda^{-1})V_1$. Then*

$$\hat{\delta}(E, E_h) \leq C \sup_{\substack{u \in E \\ \|u\|=1}} \inf_{v \in V_{1,h}} \|u - v\|_{V_1}.$$

The same theorem written without the gap.

### Theorem

*Let $u^{(k)}$ be a unit eigenfunction associated with an eigenvalue $\lambda^{(k)}$ of multiplicity $m$, such that $\lambda^{(k)} = \cdots = \lambda^{(k+m-1)}$ and denote by $u_h^{(k)}, \dots, u_h^{(k+m-1)}$ the eigenfunctions associated with the $m$ discrete eigenvalues converging to $\lambda^{(k)}$. Then, there exists $w_h^{(k)} \in \mathrm{span}\{u_h^{(k)}, \dots, u_h^{(k+m-1)}\}$ such that*

$$\|u^{(k)} - w_h^{(k)}\|_V \le C \sup_{\substack{u \in E \\ \|u\|=1}} \inf_{v \in V_h} \|u - v\|_V,$$

*where $E$ denotes the eigenspace associated with $\lambda^{(k)}$.*

In the case of multiple eigenvalues it has been observed that is is convenient to introduce the arithmetic mean of the approximating eigenvalues

### Theorem

*Let $\mu$ be a nonzero eigenvalue of $T$ with algebraic multiplicity equal to $m$ and denote by $\widehat{\mu}_h$ the arithmetic mean of the $m$ discrete eigenvalues of $T_h$ converging towards $\mu$. Denote by $\phi_1, \ldots \phi_m$ a basis of generalized eigenvectors in $E = E(\mu)X$ a and by $\phi_1^*, \ldots \phi_m^*$ a dual basis of generalized eigenvectors in $E^* = E^*(\mu)X$. Then*

$$\begin{aligned} |\mu - \widehat{\mu}_h| \leq & \frac{1}{m} \sum_{i=1}^{m} |((T - T_h)\phi_i, \phi_i^*)| \\ & + C \|(T - T_h)_{|E}\|_{\mathcal{L}(X)} \|(T^* - T_h^*)_{|E^*}\|_{\mathcal{L}(X)}. \end{aligned}$$

## Corollary

*Let $\lambda$ be an eigenvalue and denote by $\widehat{\lambda}_h$ the arithmetic mean of the $m$ discrete eigenvalues converging towards $\lambda$. Then*

$$|\lambda - \widehat{\lambda}_h| \le C \sup_{\substack{u \in E \\ \|u\|=1}} \inf_{v \in V_{1,h}} \|u - v\|_{V_1} \sup_{\substack{u \in E^* \\ \|u\|=1}} \inf_{v \in V_{2,h}} \|u - v\|_{V_2},$$

*where $E$ is the space of generalized eigenfunctions associated with $\lambda$ and $E^*$ is the space of generalized adjoint eigenfunctions associated with $\lambda$ (see the adjoint problem).*

The estimate of the error in the eigenvalues involves the ascent multiplicity $\alpha$.

Theorem

*Let $\phi_1, \dots, \phi_m$ be a basis of the generalized eigenspace $E = E(\mu)X$ of $T$ and $\phi_1^*, \dots, \phi_m^*$ a dual basis. Then, for $i = 1, \dots m$,*

$$|\mu - \mu_{i,h}|^\alpha \leq C\Big\{ \sum_{j,k=1}^{m} |((T - T_h)\phi_j, \phi_k^*)| + \|(T - T_h)_{|E}\|_{\mathcal{L}(X)} \|(T^* - T_h^*)_{|E^*}\|_{\mathcal{L}(X)} \Big\},$$

*where $\mu_{1,h}, \dots, \mu_{m,h}$ denote the $m$ discrete eigenvalues (repeated according to their algebraic multiplicity) converging to $\mu$ and $E^*$ is the space of generalized eigenvectors of $T^*$ associated with $\overline{\mu}$.*

### Corollary

*With the analogous notation as in the previous theorem, for* $i = 1, \dots, m$ *we have*

$$|\lambda - \lambda_{i,h}|^{\alpha} \leq C \sup_{\substack{u \in E \\ \|u\|=1}} \inf_{v \in V_{1,h}} \|u - v\|_{V_1} \sup_{\substack{u \in E^* \\ \|u\|=1}} \inf_{v \in V_{2,h}} \|u - v\|_{V_2},$$

*where* $E$ *is the space of generalized eigenfunctions associated with* $\lambda$ *and* $E^*$ *is the space of generalized adjoint eigenfunctions associated with* $\lambda$ *(see the adjoint problem).*

# Application to non-conforming Crouzeix–Raviart

1. Prove the convergence
2. Estimate error in the eigenvalues
3. Estimate error in the eigenfunctions ($L^2$ and energy norm)

# Estimating the eigenfunctions in $L^2$

Immediate from uniform convergence in $\mathcal{L}(L^2)$

# Estimating the eigenvalues

Need to estimate $((T - T_h)u, v)$ where $u$ and $v$ are eigenfunctions associated with $\lambda$

$$\begin{aligned}((T - T_h)u, v) &= a_h((T - T_h)u, Tv) + a_h(T_h u, (T - T_h)v) \\ &= a_h((T - T_h)u, (T - T_h)v) \\ &\quad + a_h((T - T_h)u, T_h v) + a_h(T_h u, (T - T_h)v)\end{aligned}$$

First term is clearly $O(h^2)$
Second and third terms are similar to each other

$$
\begin{aligned}
a_h((T - T_h)u, T_h v) &= a_h(Tu, T_h v) - (u, T_h v) \\
&= \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\nabla Tu \cdot \mathbf{n}) T_h v
\end{aligned}
$$

$$
\sum_{e \in \mathcal{E}_h} \int_e \big((\nabla Tu \cdot \mathbf{n}_e) - P_e(\nabla Tu \cdot \mathbf{n}_e)\big)(T_h v - P_e T_h v) =
$$

$$
\sum_{e \in \mathcal{E}_h} \int_e \big((\nabla Tu \cdot \mathbf{n}_e) - P_e(\nabla Tu \cdot \mathbf{n}_e)\big)\big((T_h v - P_e T_h v) - (Tv - P_e Tv)\big)
$$

$$
\begin{aligned}
&|a_h((T - T_h)u, T_h v)| \\
&\qquad \le \sum_{e\in\mathcal{E}_h} \|(I - P_e)(\nabla Tu \cdot \mathbf{n}_e)\|_{L^2(e)} \|(I - P_e)(Tv - T_h v)\|_{L^2(e)}
\end{aligned}
$$

$$
\sum_{e\in\mathcal{E}_h} \|(I - P_e)(\nabla Tu \cdot \mathbf{n}_e)\|_{L^2(e)} \le Ch^{1/2} \sum_{K\in\mathcal{T}_h} \|Tu\|_{H^2(K)}
$$

$$
\begin{aligned}
\sum_{e\in\mathcal{E}_h} \|(I - P_e)(Tv - T_h v)\|_{L^2(e)} &\le Ch^{1/2} \|(T - T_h)v\|_h \\
&\le Ch^{3/2} \|v\|_{L^2(\Omega)}
\end{aligned}
$$

# Estimating the eigenfunctions in the energy norm

$$\begin{aligned} u - u_h &= \lambda T u - \lambda_h T_h u_h \\ &= (\lambda - \lambda_h) T u + \lambda_h (T - T_h) u + \lambda_h T_h (u - u_h), \end{aligned}$$

$$\|u - u_h\|_h \leq |\lambda - \lambda_h| \|Tu\|_{H^1(\Omega)} + \lambda_h \|(T - T_h)u\|_h + \lambda_h \|T_h(u - u_h)\|_h$$

Third term can be bounded by:

$$C\|T_h(u-u_h)\|_h^2 \leq a_h(T_h(u-u_h), T_h(u-u_h)) = (u-u_h, T_h(u-u_h))$$

# Improved theory for multiple eigenvalues
# General elliptic case
# (Knyazev–Osborn)

# Improved estimates (notation)

$\{u_i, \dots, u_j\} \in H_0^1(\Omega)$ eigenfunctions of the continuous problem

$\boxed{E_{i,\dots,j}} = \mathrm{span}\{u_i, \dots, u_j\}$

$\{u_{i,h}, \dots, u_{j,h}\} \in V_h$ eigenfunctions of the discrete problem

$\boxed{E_{i,\dots,j,h}} = \mathrm{span}\{u_{i,h}, \dots, u_{j,h}\}$

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Elliptic projection onto $E_{i,\dots,j}$

$$a(u - P_{i,\dots,j}u, v) = 0 \quad \forall u \in H^1_0(\Omega),\ \forall v \in E_{i,\dots,j}$$

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$$a(u - P_{i,\dots,j}u, v) = 0 \quad \forall u \in H^1_0(\Omega),\ \forall v \in E_{i,\dots,j}$$

Discrete elliptic projection onto $E_{i,\dots,j,h}$

$$a_h(u - P_{i,\dots,j,h}u, v) = 0 \quad \forall u \in H^1_0(\Omega),\ \forall v \in E_{i,\dots,j,h}$$

Discrete elliptic projection onto $V_h$

$$a_h(u - P_h u, v) = 0 \quad \forall u \in H^1_0(\Omega),\ \forall v \in V_h$$

# Improved estimates

## Theorem (Osborn–Knyazev)

*Assume that* $\lambda_p$ *(*$p>1$*) has multiplicity* $m>1$ *so that*

$$\lambda_{p-1} < \lambda_p = \cdots = \lambda_{p+m-1} < \lambda_{p+m}$$

*Then, for* $i = p, \dots, p+m-1$ *we have*

$$0 \le \frac{\lambda_{i,h} - \lambda_p}{\lambda_{i,h}} \le \|(I - P_h + P_{1,\dots,p-1,h})P_{p,\dots,i}\|^2_{\mathcal{L}(V)}$$

$$\le \left(1 + \max_{j=1,\dots,p-1} \frac{\lambda_{j,h}^2\lambda_p^2}{|\lambda_{j,h} - \lambda_p|^2} \|(I - P_h)TP_{1,\dots,p-1,h}\|^2_{\mathcal{L}(V)}\right)$$

$$\|(I - P_h)P_{p,\dots,i}\|^2_{\mathcal{L}(V)}$$

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- $P_{p,\dots,i}$*: projection onto any (*$i-p+1$*)-dimensional subspace of the eigenspace associated with* $\lambda_p$

# The theory explained

$\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4$ $(p = m = 2)$
For $i = 2$

$$\frac{\lambda_{2,h} - \lambda_2}{\lambda_{2,h}} \le \left(1 + \frac{\lambda_{1,h}^2 \lambda_2^2}{|\lambda_{1,h} - \lambda_2|^2} \|(I - P_h) T P_{1,h}\|_{\mathcal{L}(V)}^2\right) \|(I - P_h) P_2\|_{\mathcal{L}(V)}^2$$

For $i = 3$

$$\frac{\lambda_{3,h} - \lambda_3}{\lambda_{3,h}} \le \left(1 + \frac{\lambda_{1,h}^2 \lambda_2^2}{|\lambda_{1,h} - \lambda_2|^2} \|(I - P_h) T P_{1,h}\|_{\mathcal{L}(V)}^2\right) \|(I - P_h) P_{2,3}\|_{\mathcal{L}(V)}^2$$