Approximation of variationally posed eigenvalue problems

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Plan of the lecture

- Standard finite element approximation of Laplace eigenproblem

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☛ Standard finite element approximation of Laplace eigenproblem

☛ Approximation theory of compact operators

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Variationally posed eigenproblem (Laplace operator)

Eigenvalue problem

Find $\lambda \in \mathbb{R}$ such that for some $u \in V$ with $u \not\equiv 0$ it holds

$a(u, v) = \lambda b(u, v) \quad \forall v \in V$

Variationally posed eigenproblem (Laplace operator)

Eigenvalue problem

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Assumptions

$H \quad (= L^2(\Omega)) \quad , V \quad (= H_0^1(\Omega)) \quad \subset H$

Hilbert spaces, V compactly embedded in H

$a(u, v) \quad (= \int_\Omega \nabla u \cdot \nabla v \, d\mathbf{x}) \qquad V \times V \to \mathbb{R}$

bilinear, continuous, symmetric, coercive

$b(u, v) \quad (= (u, v)) \qquad H \times H \to \mathbb{R}$

bilinear, continuous, symmetric

Laplace eigenproblem

Strong form

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Laplace eigenproblem

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Weak form

$\lambda \in \mathbb{R}$, $u \in V$, $u \not\equiv 0$:

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Laplace eigenproblem

Strong form

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Weak form

$\lambda \in \mathbb{R}$, $u \in V$, $u \not\equiv 0$:

$a(u, v) = \lambda b(u, v) \quad \forall v \in V$

Resolvent operator

$T : H \to H$, $\quad T(H) \subset V$ implies T is compact

$a(Tf, v) = b(f, v) \quad \forall v \in V$

Laplace eigenproblem

Strong form

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Weak form

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Resolvent operator

$T : H \to H$, $\quad T(H) \subset V$ implies T is compact

$$a(Tf, v) = b(f, v) \quad \forall v \in V$$

$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \leq \cdots$ (numbered s. t. all have multiplicity one)

$E_i = \mathrm{span}(u_i)$, normalization $b(u_i, u_i) = 1$

$V = \bigoplus_{i=1}^{\infty} E_i$ (orthonormal basis)

Laplace eigenproblem: approximation

$V_h \subset V$, $\dim V_h = N(h)$

Discrete problem

Find $\lambda_h \in \mathbb{R}$ such that for some $u_h \in V_h$ with $u_h \not\equiv 0$ it holds

$a(u_h, v) = \lambda_h b(u_h, v) \quad \forall v \in V_h$

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Discrete (compact) resolvent operator

$T_h : H \longrightarrow H$

$a(T_h f, v) = b(f, v) \quad \forall v \in V_h$

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$\lambda_{1,h} \le \lambda_{2,h} \le \cdots \le \lambda_{i,h} \le \cdots \le \lambda_{N(h),h}$

$E_{i,h} = \mathrm{span}(u_{i,h})$, normalization $b(u_{i,h}, u_{i,h}) = 1$

$$V_h = \bigoplus_{i=1}^{N(h)} E_{i,h}$$

Rayleigh quotient

Continuous problem

$$\lambda_1 = \min_{v \in V} \frac{a(v,v)}{b(v,v)} = R(v), \quad u_1 = \arg\min_{v \in V} R(v)$$

Rayleigh quotient

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$$\lambda_i = \min_{v \in \left(\bigoplus_{k=1}^{i-1} E_k \right)^{\perp}} R(v), \qquad u_i = \arg \min_{v \in \left(\bigoplus_{k=1}^{i-1} E_k \right)^{\perp}} R(v)$$

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Discrete problem

$$\lambda_{1,h} = \min_{v\in V_h} = R(v), \quad u_{1,h} = \arg\min_{v\in V_h} R(v)$$

$$\lambda_{i,h} = \min_{v\in\left(\bigoplus_{k=1}^{i-1} E_{k,h}\right)^\perp} R(v), \quad u_{i,h} = \arg \min_{v\in\left(\bigoplus_{k=1}^{i-1} E_{k,h}\right)^\perp} R(v)$$

Rayleigh quotient (cont'd)

A first easy consequence

$\lambda_1 = \min\limits_{v \in V} R(v)$, $\lambda_{1,h} = \min\limits_{v \in V_h} R(v)$

$V_h \subset V$ implies $\boxed{\lambda_{1,h} \geq \lambda_1}$

Rayleigh quotient (cont'ed)

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$V_h \subset V$ implies $\boxed{\lambda_{1,h} \ge \lambda_1}$

Same argument does not apply to the next eigenvalues

$$\lambda_i = \min_{v \in \left(\bigoplus_{k=1}^{i-1} E_k\right)^{\perp}} R(v), \quad u_i = \arg \min_{v \in \left(\bigoplus_{k=1}^{i-1} E_k\right)^{\perp}} R(v)$$

$$\lambda_{i,h} = \min_{v \in \left(\bigoplus_{k=1}^{i-1} E_{k,h}\right)^{\perp}} R(v), \quad u_{i,h} = \arg \min_{v \in \left(\bigoplus_{k=1}^{i-1} E_{k,h}\right)^{\perp}} R(v)$$

Rayleigh quotient (cont'ed)

An alternative representation

$$\lambda_i = \min_{E \in V_i} \max_{v \in E} R(v)$$

where V_i denotes the set of all subspaces E of V with $\dim E = i$

Rayleigh quotient (cont'ed)

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Proof

$\min\max \le \lambda_i$ Take $E = \bigoplus_{k=1}^{i} E_k$, $v = \sum_{k=1}^{i} \alpha_k u_k$, use orthogonalities and $\lambda_k \le \lambda_i$ for all $k \le i$ to get $R(v) \le \lambda_i$

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$\min\max \ge \lambda_i$ If $E \ne \bigoplus_{k=1}^{i} E_k$ then there exists $v \in E$ with $v \perp u_k$ for all $k \le i$, hence $R(v) > \lambda_i$

Rayleigh quotient (cont'ed)

Continuous min-max

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Discrete min-max

$$\lambda_{i,h} = \min_{E \in V_{i,h}} \max_{v \in E} R(v)$$

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Consequence

For all $i = 1, \dots, N(h)$ one has $\boxed{\lambda_{i,h} \geq \lambda_i}$

Convergence of eigenvalues/eigenfunctions

Some notation

$m : \mathbb{N} \to \mathbb{N}$ such that

$\lambda_{m(1)} < \lambda_{m(2)} < \cdots < \lambda_{m(N)} < \dots$

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$\hat{\delta}(E, F) = \max(\delta(E, F), \delta(F, E))$, where E, F subspaces of H

$$\delta(E, F) = \sup_{u \in E,\ ||u||_H = 1} \inf_{v \in F} ||u - v||_H$$

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Definition of convergence

<Comments>

$\forall \varepsilon > 0,\ \forall N \in \mathbb{N},\ \exists h_0 > 0$ such that $\forall h \leq h_0$

- $\displaystyle \max_{i=1,\ldots,m(N)} |\lambda_i - \lambda_{i,h}| \leq \varepsilon$
- $\displaystyle \hat{\delta}\left(\bigoplus_{i=1}^{m(N)} E_i, \bigoplus_{i=1}^{m(N)} E_{i,h} \right) \leq \varepsilon$

Convergence of eigenmodes (cont'ed)

Remark: from the monotonicity property, the problem we are studying cannot present spurious eigenvalues

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Uniform convergence

$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

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Theorem

If T is selfadjoint and compact

$$\boxed{\text{Uniform convergence}} \iff \boxed{\text{Eigenmodes convergence}}$$

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Theorem

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Strategy

1) prove uniform convergence,

2) estimate the order of convergence

Galerkin approximation of compact operators

Bramble–Osborn '73
Osborn '75
Kolata '78

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Céa's Lemma

$T_h = P_h T$, with P_h projection w.r.t. bilinear form a

Galerkin approximation of compact operators

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Céa's Lemma

$T_h = P_h T$, with P_h projection w.r.t. bilinear form a

$$T - T_h = (I - P_h)T$$

If $I - P_h$ converges to zero pointwise and T is compact, then $T - T_h$ converges to zero uniformly (consequence of Banach–Steinhaus uniform boundedness theorem)

<Proof>

Estimating the order of convergence

Kolata '78
Babuška–Osborn '91

Estimating the order of convergence

If

Kolata '78
Babuška–Osborn '91

$$||T - T_h||_{\mathcal{L}(V,V)} = \varepsilon(h)$$
$$||T^* - T_h^*||_{\mathcal{L}(V,V)} = \varepsilon^*(h)$$

then

$$|\lambda_i - \lambda_{i,h}| \le C(\varepsilon(h)\varepsilon^*(h))^{1/\alpha_i}$$ α_i ascent of $\lambda_i I - T$

$$\hat{\delta}(E_i, E_{i,h}) \le C\varepsilon(h)$$

Estimating the order of convergence

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then

$$|\lambda_i - \lambda_{i,h}| \le C(\varepsilon(h)\varepsilon^*(h))^{1/\alpha_i} \quad \alpha_i \text{ ascent of } \lambda_i I - T$$
$$\hat{\delta}(E_i, E_{i,h}) \le C\varepsilon(h)$$

If T symmetric then

$$|\lambda_i - \lambda_{i,h}| \le C\varepsilon(h)^2$$
$$\hat{\delta}(E_i, E_{i,h}) \le C\varepsilon(h)$$

Piecewise linear FEs for Laplace eigenproblem

Simple proof that for $h < h_0$

$$\lambda_i \leq \lambda_{i,h} \leq \lambda_i + Ch^2$$

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Piecewise linear FEs for Laplace eigenproblem

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Take $E = \bigoplus_{k=1}^{i} E_k$, P_h elliptic projection, and $E_h = P_h E$.
If h small enough, then $\dim E_h = i$

$$\lambda_{i,h} \le \sup_{v \in E_h} \frac{\|\nabla v\|^2}{\|v\|^2}$$

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Piecewise linear FEs for Laplace eigenproblem

Simple proof that for $h < h_0$

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Piecewise linears for Laplace (cont'd)

$$\lambda_{i,h} \le \sup_{v \in E} \frac{\|\nabla v\|^2}{\|P_h v\|^2}$$

Piecewise linears for Laplace (cont'ed)

$$\lambda_{i,h} \le \sup_{v \in E} \frac{\| \nabla v\|^2}{\|P_h v\|^2}$$

$$\|P_h v\| \ge \|v\| - \|P_h v - v\|$$

Piecewise linears for Laplace (cont'ed)

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$$\|P_h v\| \ge \|v\| - \|P_h v - v\|$$
$$\|P_h v - v\| \le C h^2 \|\Delta v\|$$

Piecewise linears for Laplace (cont'ed)

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$$\|P_h v\| \ge \|v\| - \|P_h v - v\|$$
$$\|P_h v - v\| \le Ch^2 \|\Delta v\| \le Ch^2 \lambda_i \|v\|$$

Piecewise linears for Laplace (cont'ed)

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$$\|P_h v\| \ge \|v\| (1 - C h^2)$$

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$$\|P_h v - v\| \le C h^2 \|\Delta v\| \le C h^2 \lambda_i \|v\|$$
$$\|P_h v\| \ge \|v\|(1 - C h^2)$$

$$\lambda_{i,h} \le \sup_{v \in E} \frac{\| \nabla v\|^2}{\|v\|^2}(1 + C h^2) \le \lambda_i + C h^2$$

Computed eigenvalues: $[0, \pi]$ 1D interval, piecewise linear FEs

	$n = 8$	$n = 16$	$n = 32$
1	1.0129160450588	1.0032168743567	1.0008034482562
4	4.2095474481529	4.0516641802355	4.0128674974272
9	10.0802909335883	9.2631305555446	9.0652448637285
16	19.4536672593288	16.8381897926118	16.2066567209423
25	33.2628304890884	27.0649225609802	25.5059230069702

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	$n = 64$	$n = 128$	$n = 256$
1	1.0002008137390	1.0000502004122	1.0000125499161
4	4.0032137930241	4.0008032549556	4.0002008016414
9	9.0162763381719	9.0040668861371	9.0010165838380
16	16.0514699897078	16.0128551720960	16.0032130198251
25	25.1257489536113	25.0313903532369	25.0078446408520

Conclusions on standard Galerkin approximation

Any finite element choice which provides a (pointwise) convergent scheme for the approximation of a problem with compact resolvent can be successfully applied to the approximation of the corresponding eigenvalue problem (uniform convergence).

No extra compatibility!

The curl curl operator

Let's consider the time harmonic model for Maxwell's system

$$\left(\varepsilon\frac{\partial}{\partial t}+\sigma\right)\mathcal{E}-\operatorname{curl}\mathcal{H}=\mathcal{J},\qquad \mu\frac{\partial}{\partial t}\mathcal{H}+\operatorname{curl}\mathcal{E}=0.$$

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Time-harmonic model

$$\mathcal{E}(x,t)=e^{-i\omega t}E(x),\qquad \mathcal{H}(x,t)=e^{-i\omega t}H(x)$$

$$-i\omega\left(\varepsilon+i\frac{\sigma}{\omega}\right)E-\operatorname{curl}H=J,\quad -i\omega\mu H+\operatorname{curl}E=0.$$

Eliminating H

$$\operatorname{curl}\left(\mu^{-1}\operatorname{curl}E\right)-\omega^2\left(\varepsilon+i\frac{\sigma}{\omega}\right)E=-i\omega J$$

The curl curl operator (cont'ed)

Change of notation $\mathbf{u} = E$, $\mathbf{f} = -i\omega J$

$+ \ \sigma = 0$

$+$ B.C.

$$\begin{cases} \operatorname{curl}\left(\mu^{-1}\operatorname{curl}\mathbf{u}\right) - \omega^2\varepsilon\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} \times \mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

The curl curl operator (cont'ed)

Change of notation $\mathbf{u} = E$, $\mathbf{f} = -i\omega J$
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The problem is well-posed if ω is not an interior Maxwell eigenvalue (otherwise $LHS = 0$ for $\mathbf{u}$ eigenfunction)

The curl curl operator (cont'ed)

Variational formulation...

$V = H_0(\mathrm{curl};\Omega)$, find $\mathbf{u} \in V$ s.t.

$$\big(\mu^{-1}\,\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v}\big) - \omega^2(\varepsilon\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in V$$

The curl curl operator (cont'ed)

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Note: $\mathrm{div}\,\mathbf{f} = 0$ implies $\mathrm{div}\,\varepsilon\mathbf{u} = 0$

The curl curl operator (cont'ed)

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...and its numerical approximation

$V_h \subset V$, find $\mathbf{u}_h \in V_h$ s.t.

$$\big(\mu^{-1}\,\mathrm{curl}\,\mathbf{u}_h, \mathrm{curl}\,\mathbf{v}\big) - \omega^2(\varepsilon\mathbf{u}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in V_h$$

The curl curl operator (cont'd)

Time-harmonic Maxwell:

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The curl curl operator (cont'ed)

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$$\left(\mu^{-1}\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v}\right)-\omega^2(\varepsilon\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v})\quad\forall\mathbf{v}\in V$$

Maxwell interior eigenvalues:

$$\left(\mu^{-1}\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v}\right)=\lambda^2(\varepsilon\mathbf{u},\mathbf{v})\quad\forall\mathbf{v}\in V$$

The curl curl operator (cont'd)

Time-harmonic Maxwell:

$$\big(\mu^{-1}\operatorname{curl}\mathbf{u},\operatorname{curl}\mathbf{v}\big)-\omega^2(\varepsilon\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v})\quad\forall\mathbf{v}\in V$$

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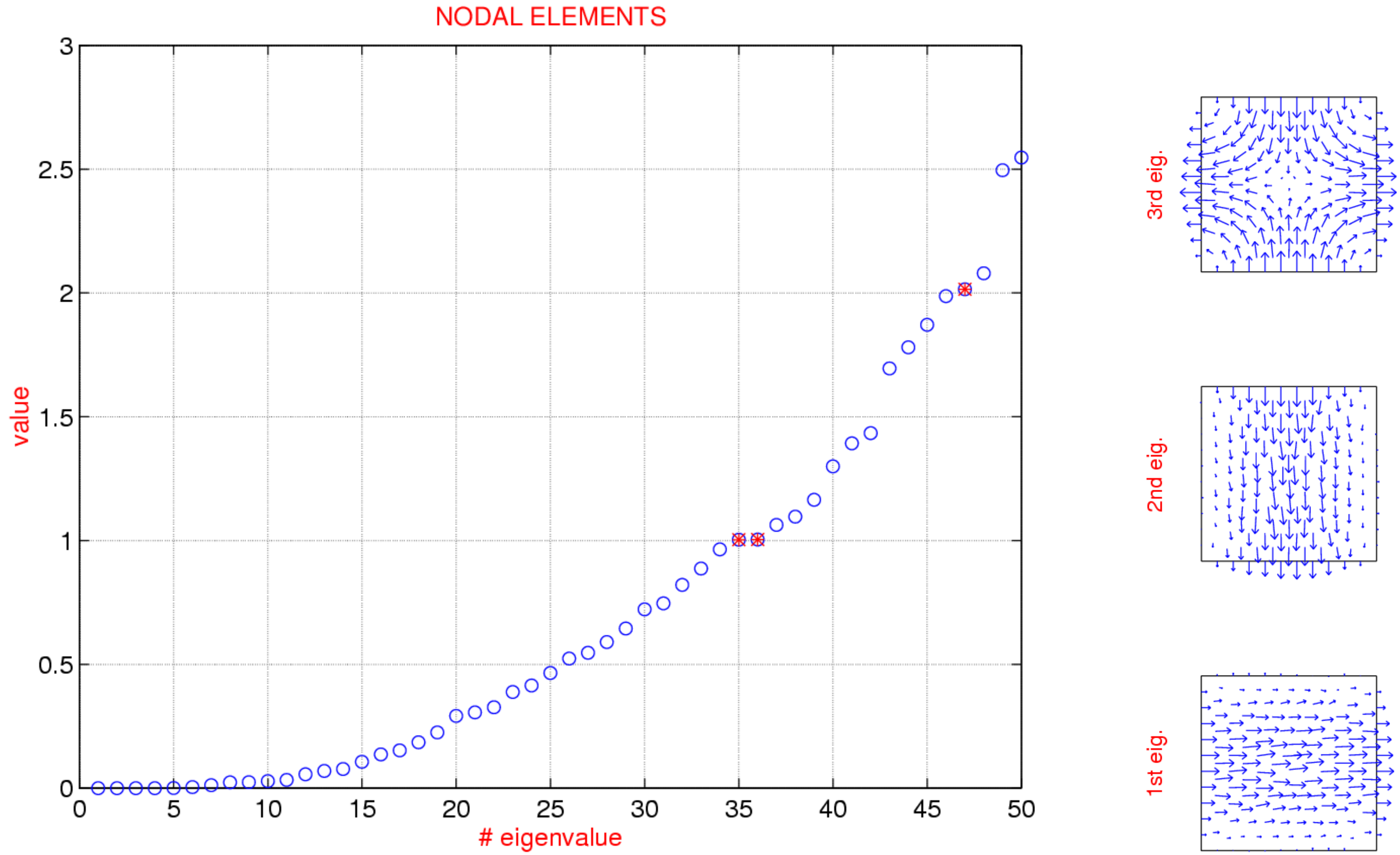
N.B.: from now on we take $\mu=\varepsilon=1$

2D numerical tests (curl curl operator)

Piecewise linear FEs on general meshes



The curl curl operator (cont'd)

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At least our theory was not wrong. . .

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Let's drop the divergence free constraint in the definition of our problem ans see what happens

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$V = H_0(\mathrm{curl})$

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Remark: positive eigenvalues are the same as before. Eigenvalue $\lambda = 0$ is added to the spectrum with infinite dimensional eigenspace $\nabla\, H_0^1$. Consequence of Helmholz decomposition which implies

$$V = [H_0(\mathrm{curl}) \cap H(\mathrm{div}^0)] \oplus \nabla\, H_0^1$$

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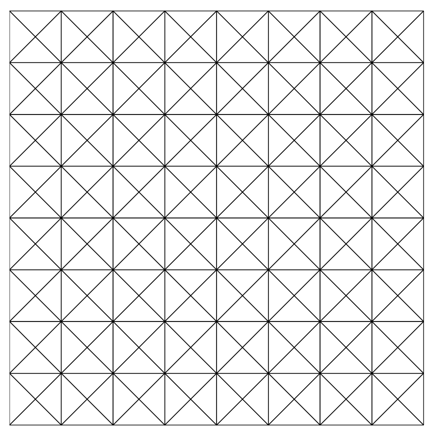
Piecewise linear elements on criss-cross mesh have this property.

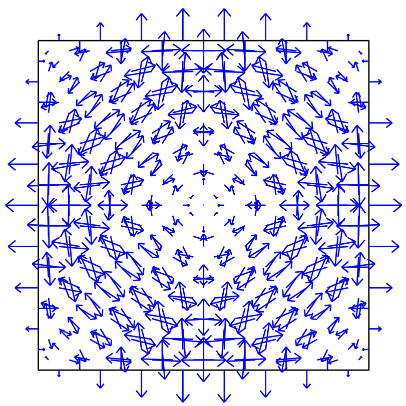


Results on criss-cross mesh

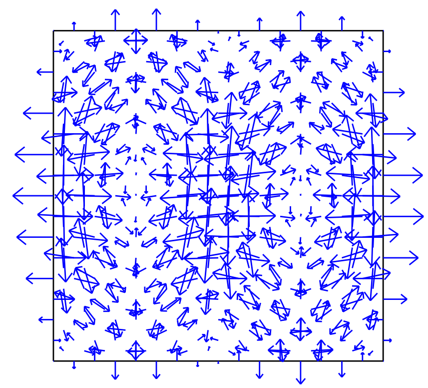
Mode		$n = 8$	$n = 16$	$n = 32$
(1,0)	1	1.00428	1.00107	1.00027
(0,1)	1	1.00428	1.00107	1.00027
(1,1)	2	2.01711	2.00428	2.00107
(2,0)	4	4.06804	4.01710	4.00428
(0,2)	4	4.06804	4.01710	4.00428
(2,1)	5	5.10634	5.02674	5.00669
(1,2)	5	5.10634	5.02674	5.00669
??	6	5.92293	5.98074	5.99518
(2,2)	8	8.27128	8.06845	8.01713
(3,0)	9	9.34085	9.08640	9.02166
(0,3)	9	9.34085	9.08640	9.02166
# zeros		63	255	1023

Spurious eigenmodes

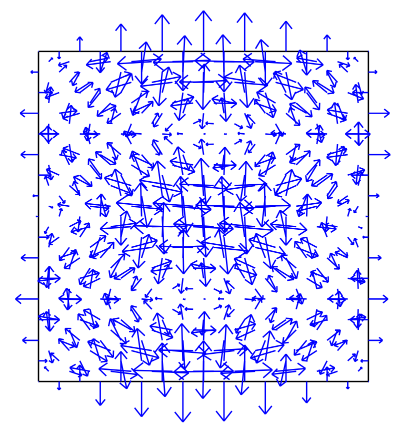
1st spurious eigenfunction



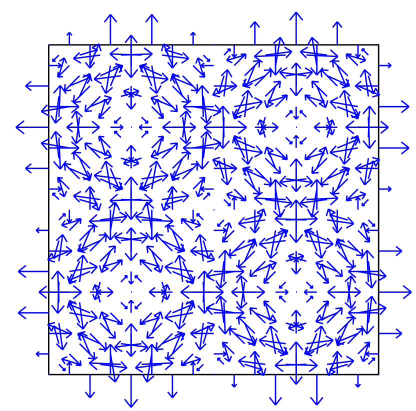
2nd spurious eigenfunction



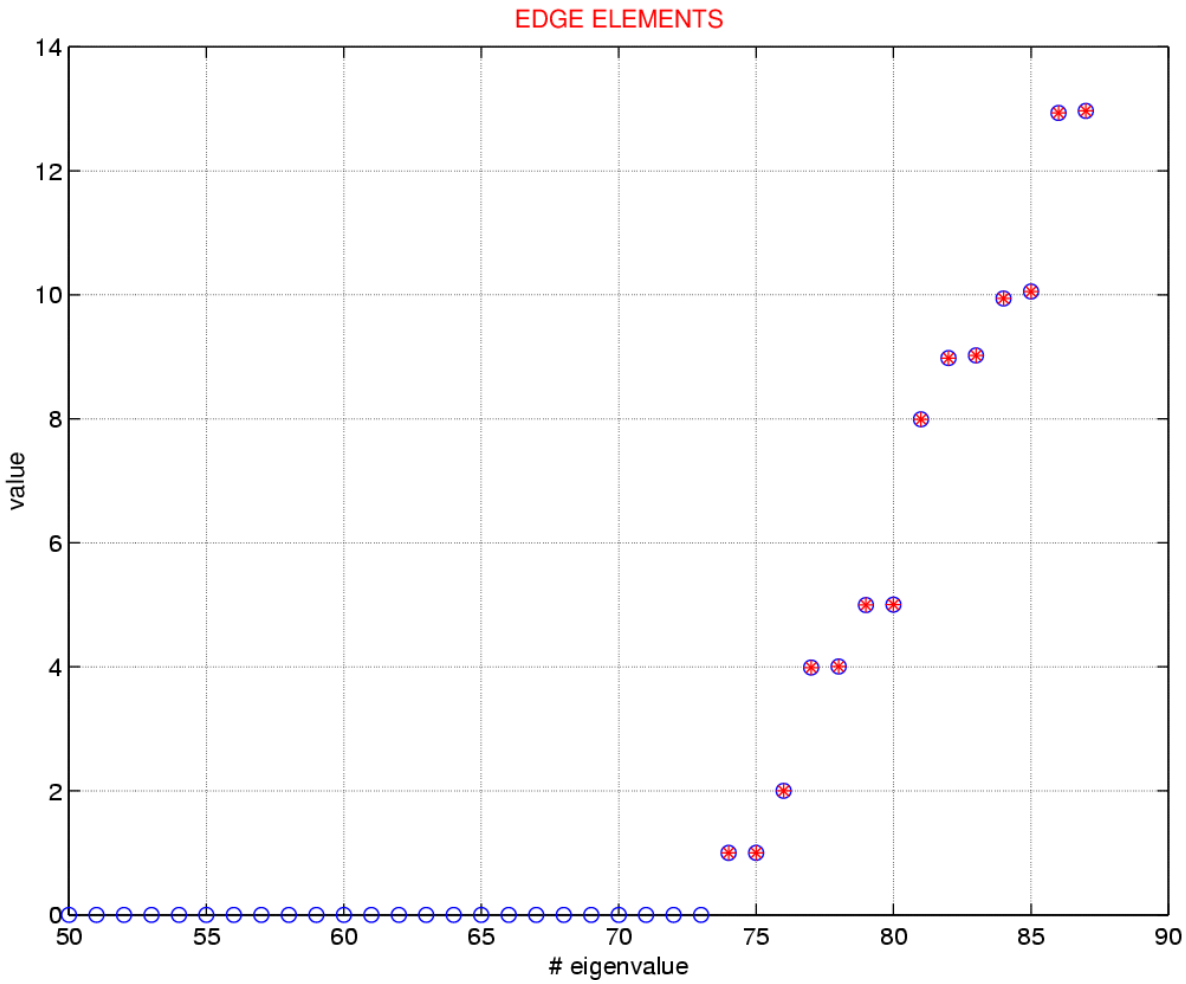
3rd spurious eigenfunction

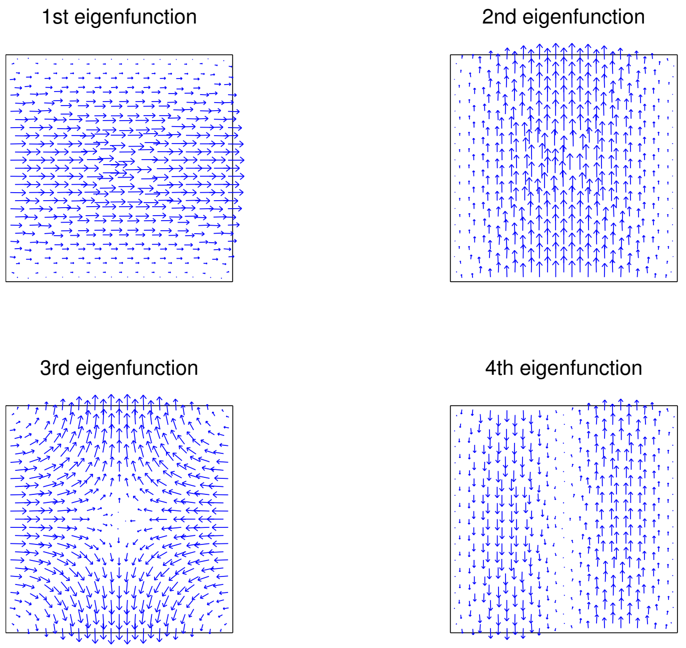


4th spurious eigenfunction



Numerical results: edge elements





Numerical results: edge elements (cont'd)

Mode		$n = 8$	$n = 16$	$n = 32$
(1,0)	1	0.99232	0.99806	0.99951
(0,1)	1	0.99914	0.99979	0.99994
(1,1)	2	2.00823	2.00212	2.00053
(2,0)	4	3.93162	3.98288	3.99572
(0,2)	4	3.93250	3.98294	3.99572
(2,1)	5	4.93116	4.98260	4.99564
(1,2)	5	5.05757	5.01511	5.00382
(2,2)	8	8.10159	8.03218	8.00844
(3,0)	9	8.62920	8.90607	8.97640
(0,3)	9	8.68245	8.92111	8.98027
# zeros		49	225	1023

Mixed formulations for curl curl eigenproblem

Kikuchi's formulation Kikuchi '89

Find $\lambda \in \mathbb{R}$ s.t. for some $\mathbf{u} \in V$ with $\mathbf{u} \not\equiv 0$ and $p \in H^1_0(\Omega) = Q$

$$(M_1) \quad \begin{cases} (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}) + (\mathbf{v}, \nabla p) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in V \\ (\mathbf{u}, \nabla q) = 0 & \forall q \in Q \end{cases}$$

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Alternative mixed formulation B.–Fernandes–Gastaldi–Perugia '99

Find $\lambda \in \mathbb{R}$ s.t. for some $u \in \operatorname{curl} V = U$ with $u \not\equiv 0$ and $\sigma \in V$

$$(M_2) \quad \begin{cases} (\sigma, \tau) + (\operatorname{curl}\tau, u) = 0 & \forall \tau \in V \qquad \sigma = -\operatorname{curl} u \\ (\operatorname{curl}\sigma, v) = -\lambda(u, v) & \forall v \in U \qquad \operatorname{curl}\sigma = -\lambda u \end{cases}$$

Discrete mixed formulations

Kikuchi's formulation

Find $\lambda_h \in \mathbb{R}$ s.t. for some $\mathbf{u}_h \in V_h$ with $\mathbf{u}_h \not\equiv 0$ and $p_h \in Q_h$

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Theorem

$M_{1,h} \iff M_{2,h}$ and equiv. to original problem $(\lambda_h \neq 0)$ provided

$$\boxed{\nabla Q_h \subset V_h \text{ and } \mathrm{curl}\, V_h \subset U_h}$$

In 2D this is simply *rotated* mixed Laplace

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Eigenproblems in mixed form

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B.–Brezzi–Gastaldi '97-'00

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Let's start with Laplace problem $-\Delta u = g$

Source problem

$\sigma \in H(\mathrm{div};\Omega) = \Sigma,\ \ u \in L^2(\Omega) = U$

$$\begin{cases} (\sigma,\tau) + (\operatorname{div}\tau, u) = 0 & \forall \tau \in \Sigma \quad (\sigma = \nabla u) \\ (\operatorname{div}\sigma, v) = -(g, v) & \forall v \in U \quad (-\operatorname{div}\sigma = g) \end{cases}$$

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Matrix form $(\Sigma_h \subset \Sigma,\ U_h \subset U)$

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} 0 \\ -\tilde{g} \end{pmatrix}$$

Laplace eigenproblem in mixed form

Find $\lambda \in \mathbb{R}$ such that for some $(\sigma, u) \in \Sigma \times U$ with $u \not\equiv 0$

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Remark. Similarly, one can deal with problems of the type

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix} = \lambda \begin{pmatrix} M_\Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix}$$

Definition of the resolvent operator

A first natural (but wrong) definition

$T_1 : U \to \Sigma \times U$

$T_1(g) = (\sigma, u)$

One would like to compute eigenvalues. . .

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$$T_{\Sigma U} \boxed{\begin{array}{c} (f, g) \xmapsto{\text{cutoff}} (0, g) \xmapsto{T_2} (\sigma, u) \\ L^2 \times L^2 \longrightarrow L^2 \times L^2 \end{array}} \text{ is compact}$$

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Let's try to follow Kolata's argument

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Remark: standard mixed estimates don't help

$$||\sigma - \sigma_h||_\Sigma + ||u - u_h||_U \le C \inf_{\tau_h, v_h} (\underbrace{||\sigma - \tau_h||_\Sigma}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)})$$

Definition of the resolvent operator (cont'ed)

A better definition

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$\sigma \in \Sigma$, $T_U g \in U$ such that

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Remark: we need an estimate for u_h which does not involve $\operatorname{div} \sigma$

Uniform convergence $||T_U - T_{U,h}|| \to 0$

- Ellipticity in the kernel
  $||\tau_h||^2_{L^2} \geq \alpha ||\tau_h||^2_\Sigma$ for all $\tau_h \in \Sigma_h$ s.t. $\{(\mathrm{div}\,\tau_h, v) = 0,\ \forall v \in U_h\}$

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- Fortin operator $\Pi_h : \Sigma^+ \to \Sigma_h$ s.t.
  $(\operatorname{div}(\sigma - \Pi_h \sigma), v) = 0 \quad \forall v \in U_h$
  $||\Pi_h \sigma||_\Sigma \leq C ||\sigma||_{\Sigma^+}$

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  $||\Pi_h\sigma||_\Sigma \le C||\sigma||_{\Sigma^+}$

$$||\sigma - \sigma_h||_{L^2} \le C\left(||\sigma - \Pi_h\sigma||_{L^2} + (1/\sqrt{\alpha}) \inf_{v_h \in U_h} ||u - v_h||_U\right)$$

$$||u - u_h||_U \ \le C\left(\inf_{v_h \in U_h} ||u - v_h||_U + ||\sigma - \sigma_h||_{L^2}\right)$$

<Proof>

Fortid condition

Definition

The spaces Σ_h, U_h satisfy the Fortid condition if there exists a Fortin operator which converges strongly to the identity operator, namely

$\Pi_h : \Sigma^+ \to \Sigma_h$ s.t.

$$(\mathrm{div}(\sigma - \Pi_h\sigma), v) = 0 \quad \forall v \in U_h$$
$$||\Pi_h\sigma||_\Sigma \le C||\sigma||_{\Sigma^+}$$

$$\boxed{\;||I - \Pi_h||_{\mathcal{L}(\Sigma^+,L^2)} \to 0\;}$$

Final convergence result

Theorem

Assume ellipticity in the kernel and Fortid condition

For any $N \in \mathbb{N}$ define $\rho_N(h) :]0,1] \to \mathbb{R}$ as

$$\rho_N(h) = \sup_{u \in \bigoplus_{i=1}^{m(N)} E_i} \left(\inf_{v_h} ||u - v_h||_U + || \nabla u - \Pi_h \nabla u||_{L^2} \right)$$

Then $||T_U - T_{U,h}||_{\mathcal{L}(U,U)} \to 0$ and the following estimates hold true

$$\sum_{i=1}^{m(N)} |\lambda_i - \lambda_{i,h}| \le C(\rho_N(h))^2$$

$$\hat{\delta}\Big(\bigoplus_{i=1}^{m(N)} E_i, \bigoplus_{i=1}^{m(N)} E_{i,h} \Big) \le C\rho_N(h)$$

Back to the criss-cross example

Crisscross mesh

$\Sigma_h = \{$continuous p.w. linears (componentwise)$\}$

$U_h = \operatorname{div} \Sigma_h \subset \{$p.w. constants$\}$

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Theorem

With the above choice of spaces, there exist a sequence $\{g_h\} \subset U$ with $\|g_h\|_0 = 1$ s.t.

$$\boxed{||u - u_h||_U \not\to 0}$$

that is $\|T_U - T_{U,h}\|_{\mathcal{L}(U,U)} \not\to 0$

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Proof

Use inequality by Qin '94 based on idea of Boland–Nicolaides '85

$<$More detail$>$

A positive example

General mesh (triangles, parallelograms, tetrahedrons, parallelepipeds)
Σ_h : Raviart–Thomas space of order k
U_h : $\mathcal{P}_{k-1}$ or tensor product polynomials $\mathcal{Q}_{k-1}$

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See also Falk–Osborn '80

Convergence: $O(h^{2k})$ eigenvalues, $O(h^k)$ eigenfunctions.

Remark: in 2D edge elements can be chosen as *rotated* Raviart–Thomas spaces

General theory

$\begin{pmatrix}\mathbf{0}\\ \mathbf{g}\end{pmatrix}$-type problems

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix}\begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix} = -\lambda \begin{pmatrix} 0 & 0 \\ 0 & M_U \end{pmatrix}\begin{pmatrix} \tilde{\sigma} \\ \tilde{u} \end{pmatrix}$$

Laplace, biharmonic, Maxwell (1), . . .

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$\begin{pmatrix}\mathbf{f}\\ \mathbf{0}\end{pmatrix}$-type problems

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} \tilde\sigma \\ \tilde u \end{pmatrix} = \lambda \begin{pmatrix} M_\Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde\sigma \\ \tilde u \end{pmatrix}$$

Stokes, Laplace with Lagrange multipliers, Maxwell (2), . . .

<More detail>

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Conclusions

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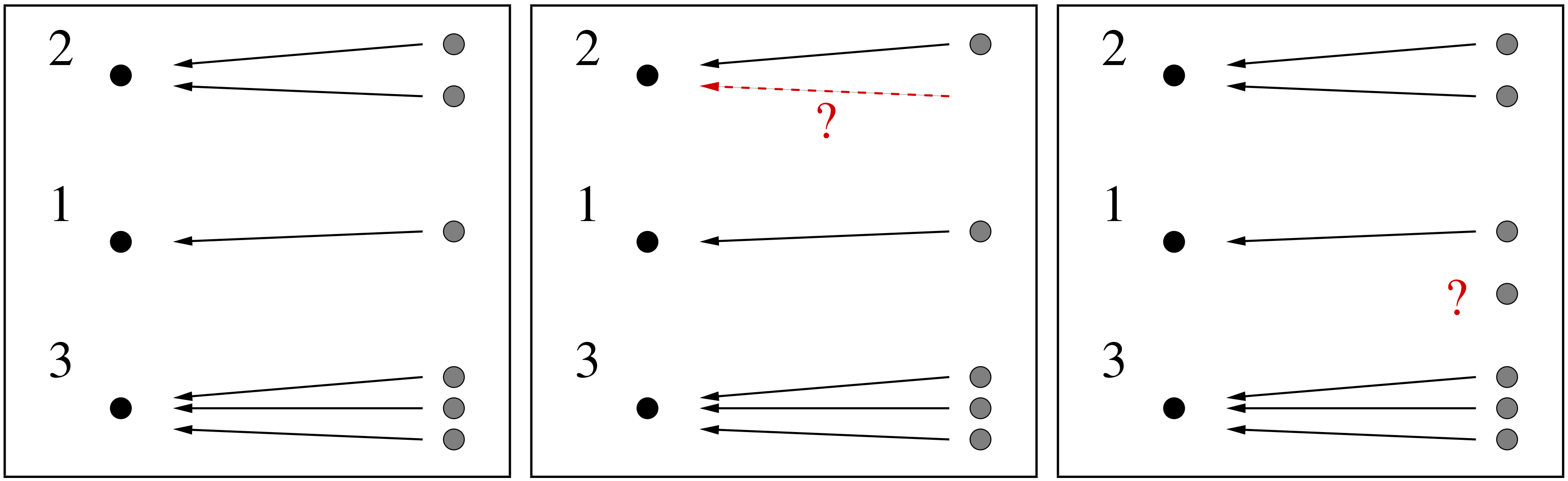
The end

Examples of good or bad convergence



Good | Bad (missing) | Bad (spurious)

<Back>

$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

$||(I - P_h)(u)||_H \to 0$ for any $u \in V$
Banach-Steinhaus thm. implies $||I - P_h||_{\mathcal{L}(V,H)} \leq C$

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By contradiction $||(I - P_h)T||_{\mathcal{L}(H,H)} \not\to 0$
Then (up to subsequences) $||(I - P_h)Tf_h||_H \not\to 0$ with $||f_h||_H \leq 1$

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If $T : H \to V$ is compact then $Tf_h \to w$ in V

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If $T : H \rightarrow V$ is compact then $Tf_h \rightarrow w$ in V

Take h small enough s.t. $||Tf_h - w||_V + ||(I - P_h)w||_H \leq \varepsilon$

$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

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$||(I-P_h)Tf_h||_H \leq ||(I-P_h)(Tf_h-w)||_H+||(I-P_h)w||_H \leq (C+1)\varepsilon$

$$||T - T_h||_{\mathcal{L}(H,H)} \to 0$$

$||(I - P_h)(u)||_H \to 0$ for any $u \in V$
Banach-Steinhaus thm. implies $||I - P_h||_{\mathcal{L}(V,H)} \leq C$

By contradiction $||(I - P_h)T||_{\mathcal{L}(H,H)} \not\to 0$
Then (up to subsequences) $||(I - P_h)Tf_h||_H \not\to 0$ with $||f_h||_H \leq 1$

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Remark. Same argument shows $||T - T_h||_{\mathcal{L}(V,V)} \to 0$ under the hypotheses $T : V \to V$ compact and $||I - P_h||_{\mathcal{L}(V,V)} \leq C$

$||T - T_h||_{\mathcal{L}(H,H)} \to 0$ (cont'd)

A direct proof using a priori estimates

$$||(T - T_h)f||_H = ||u - u_h||_H \leq Ch^\alpha ||f||_H$$

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$P = L^2$-projection onto U_h

$$
\begin{aligned}
||\Pi_h\sigma - \sigma_h||^2_{L^2} &= (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) + (\sigma - \sigma_h, \Pi_h\sigma - \sigma_h)\\
&= (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) - (\mathrm{div}(\Pi_h\sigma - \sigma_h), u - Pu)\\
&\le ||\Pi_h\sigma - \sigma||_{L^2}||\Pi_h\sigma - \sigma_h||_{L^2} + ||\,\mathrm{div}(\Pi_h\sigma - \sigma_h)||_{L^2}||u - Pu||_U\\
&\le ||\Pi_h\sigma - \sigma_h||_{L^2}\left(||\Pi_h\sigma - \sigma||_{L^2} + (1/\sqrt{\alpha})||u - Pu||_U\right)
\end{aligned}
$$

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&= (\Pi_h\sigma - \sigma, \Pi_h\sigma - \sigma_h) - (\operatorname{div}(\Pi_h\sigma - \sigma_h), u - Pu) \\
&\le ||\Pi_h\sigma - \sigma||_{L^2}||\Pi_h\sigma - \sigma_h||_{L^2} + ||\operatorname{div}(\Pi_h\sigma - \sigma_h)||_{L^2}||u - Pu||_U \\
&\le ||\Pi_h\sigma - \sigma_h||_{L^2}\left(||\Pi_h\sigma - \sigma||_{L^2} + (1/\sqrt{\alpha})||u - Pu||_U\right)
\end{aligned}
$$

$$
\begin{aligned}
||Pu - u_h||_U &\le C \sup_{\tau_h} \frac{(Pu - u_h, \operatorname{div}\tau_h)}{||\tau_h||_\Sigma} \\
&\le C \sup_{\tau_h} \frac{(Pu - u, \operatorname{div}\tau_h) + (u - u_h, \operatorname{div}\tau_h)}{||\tau_h||_\Sigma} \\
&\le C\left(||Pu - u||_U + \sup_{\tau_h} \frac{-(\sigma - \sigma_h, \tau_h)}{||\tau_h||_\Sigma}\right) \\
&\le C\left(||Pu - u||_U + ||\sigma - \sigma_h||_{L^2}\right)
\end{aligned}
$$

<Back>

Inf-sup condition for our problem

$$\inf_{v \in U_h} \sup_{\tau \in \Sigma_h} \frac{(\operatorname{div} \tau, v)}{\|\tau\|_{H(\operatorname{div})} \|v\|_{L^2}} \geq \beta > 0$$

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cannot be improved: there exists e sequence (properly chosen linear combination of checkerboards on the macroelements) $\{\tilde v_h\}$ such that

$$|(\operatorname{div}\tau,\tilde v_h)|\le C\|\tau\|_{L^2}\|\tilde v_h\|_{L^2}$$

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$$|(\operatorname{div} \tau, \tilde{v}_h)| \leq C \|\tau\|_{L^2} \|\tilde{v}_h\|_{L^2}$$

Take $g_h = \tilde{v}_h / \|\tilde{v}_h\|_{L^2}$, then $g_h \rightharpoonup 0$ weakly in L^2 and $T g_h \to 0$ strongly in L^2.

Let's prove that $\|T_h g_h\|_{L^2}$ is bounded below away from zero (uniformly).

$$
\begin{aligned}
&(\sigma_h, \tau) + (\operatorname{div} \tau, u_h) = 0 && \forall \tau \in \Sigma_h \\
&(\operatorname{div} \sigma_h, v) = -(g_h, v) && \forall v \in U_h
\end{aligned}
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|(\operatorname{div} \sigma_h, u_h)| = |(g_h, u_h)| \le \|u_h\|_{L^2}
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$$\|g_h\|_{L^2} \|\sigma_h\|_{L^2} \geq \frac{1}{C} |(\operatorname{div} \sigma_h, g_h)|$$

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$$\|g_h\|_{L^2}\|\sigma_h\|_{L^2} \ge \frac{1}{C}|(\operatorname{div}\sigma_h, g_h)| = \frac{1}{C}(g_h, g_h) = \frac{1}{C}$$

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$$\|g_h\|_{L^2} \|\sigma_h\|_{L^2} \geq \frac{1}{C} |(\operatorname{div} \sigma_h, g_h)| = \frac{1}{C}(g_h, g_h) = \frac{1}{C}$$

$$\|u_h\|_{L^2} \geq \frac{1}{C^2}$$

<Back>

$\binom{\mathbf{0}}{\mathbf{g}}$ case

$U \subset H \simeq H' \subset U'$

$\binom{\mathbf{0}}{\mathbf{g}}$ case

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Weak approximability of U_H^0 with respect to $a(\cdot, \cdot)$.

$$b(\tau_h, u) \leq \rho(h) \|u\|_{U_H^0} \|\tau_h\|_a \quad \forall u \in U_H^0,\ \forall \tau_h \in \ker_h(B)$$

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Strong approximability of U_H^0.
For each $u \in U_H^0$ there exists $u^I \in U_h$ such that

$$\|u - u^I\| \le \rho(h)\|u\|_{U_H^0}$$

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$+$ Fortid

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Weak approximability of U_0^H.

$$\sup_{\tau_h \in \ker_h(B)} \frac{b(\tau_h, u)}{\|\tau_h\|_\Sigma} \le \rho(h) \|u\|_{U_0^H}$$

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$+$ DEK

<Back>