

Preliminary computations

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1D Laplace: P1

1D Laplace

P1

P2

Spectral method

2D Laplace

1D mixed

Laplace

2D mixed

Laplace

Multiple
eigenvalues

A posteriori
analysis

	$n = 8$	$n = 16$	$n = 32$
1	1.0129160450588	1.0032168743567	1.0008034482562
4	4.2095474481529	4.0516641802355	4.0128674974272
9	10.0802909335883	9.2631305555446	9.0652448637285
16	19.4536672593288	16.8381897926118	16.2066567209423
25	33.2628304890884	27.0649225609802	25.5059230069702

1D Laplace: P1

1D Laplace

P1

P2

Spectral method

2D Laplace

1D mixed

Laplace

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16	19.4536672593288	16.8381897926118	16.2066567209423
25	33.2628304890884	27.0649225609802	25.5059230069702

	$n = 64$	$n = 128$	$n = 256$
1	1.0002008137390	1.0000502004122	1.0000125499161
4	4.0032137930241	4.0008032549556	4.0002008016414
9	9.0162763381719	9.0040668861371	9.0010165838380
16	16.0514699897078	16.0128551720960	16.0032130198251
25	25.1257489536113	25.0313903532369	25.0078446408520

1D Laplace

P1

P2

Spectral method

2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

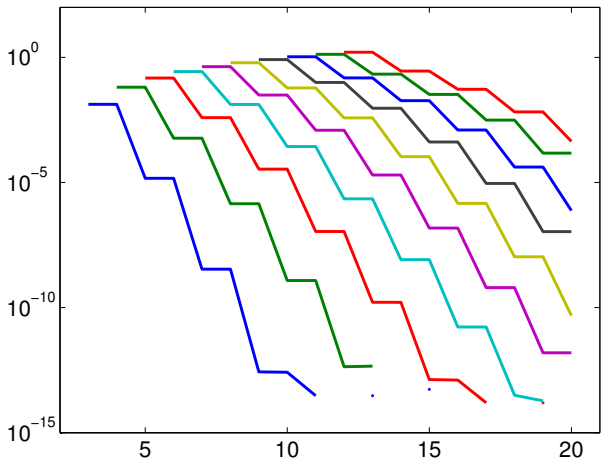
	Computed (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	1.0000	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	4.0020	4.0001 (4.0)	4.0000 (4.0)	4.0000 (4.0)	4.0000 (4.0)
9	9.0225	9.0015 (3.9)	9.0001 (4.0)	9.0000 (4.0)	9.0000 (4.0)
16	16.1204	16.0082 (3.9)	16.0005 (4.0)	16.0000 (4.0)	16.0000 (4.0)
25	25.4327	25.0307 (3.8)	25.0020 (3.9)	25.0001 (4.0)	25.0000 (4.0)
36	37.1989	36.0899 (3.7)	36.0059 (3.9)	36.0004 (4.0)	36.0000 (4.0)
49	51.6607	49.2217 (3.6)	49.0148 (3.9)	49.0009 (4.0)	49.0001 (4.0)
64	64.8456	64.4814 (0.8)	64.0328 (3.9)	64.0021 (4.0)	64.0001 (4.0)
81	95.7798	81.9488 (4.0)	81.0659 (3.8)	81.0042 (4.0)	81.0003 (4.0)
100	124.9301	101.7308 (3.8)	100.1229 (3.8)	100.0080 (3.9)	100.0005 (4.0)
#	15	31	63	127	255

Convergence of fifth eigenvalue

p	DOF	Computed
7	5	35.5593555378041
8	6	35.5593555378041
9	7	25.7779168651921
10	8	25.7779168651921
11	9	25.0306605127133
12	10	25.0306605127132
13	11	25.0004945052929
14	12	25.0004945052929
15	13	25.0000037734250
16	14	25.0000037734250
17	15	25.0000000156754
18	16	25.0000000156756
19	17	25.00000000000389
20	18	25.00000000000389

Spectral method (cont'ed)

Exponential convergence



2D Laplace: P1

1D Laplace

2D Laplace

P1

L-shaped domain
non-conf. P11D mixed
Laplace2D mixed
LaplaceMultiple
eigenvaluesA posteriori
analysis

Unstructured mesh

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.2468	2.0463 (2.4)	2.0106 (2.1)	2.0025 (2.1)	2.0006 (2.0)
5	6.5866	5.2732 (2.5)	5.0638 (2.1)	5.0154 (2.0)	5.0038 (2.0)
5	6.6230	5.2859 (2.5)	5.0643 (2.2)	5.0156 (2.0)	5.0038 (2.0)
8	10.2738	8.7064 (1.7)	8.1686 (2.1)	8.0402 (2.1)	8.0099 (2.0)
10	12.7165	11.0903 (1.3)	10.2550 (2.1)	10.0610 (2.1)	10.0152 (2.0)
10	14.3630	11.1308 (1.9)	10.2595 (2.1)	10.0622 (2.1)	10.0153 (2.0)
13	19.7789	14.8941 (1.8)	13.4370 (2.1)	13.1046 (2.1)	13.0258 (2.0)
13	24.2262	14.9689 (2.5)	13.4435 (2.2)	13.1053 (2.1)	13.0258 (2.0)
17	34.0569	20.1284 (2.4)	17.7468 (2.1)	17.1771 (2.1)	17.0440 (2.0)
17		20.2113	17.7528 (2.1)	17.1798 (2.1)	17.0443 (2.0)
#	9	56	257	1106	4573

Multiple eigenfunctions

1D Laplace

2D Laplace

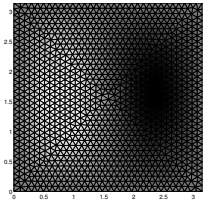
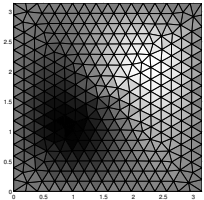
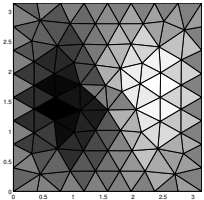
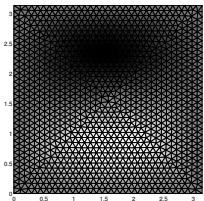
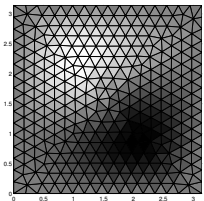
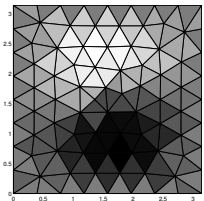
P1
L-shaped domain
non-conf. P1

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis



Multiple eigenfunctions (cont'ed)

1D Laplace

2D Laplace

P1

L-shaped domain
non-conf. P1

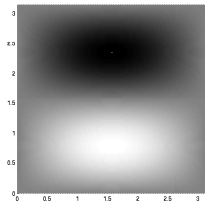
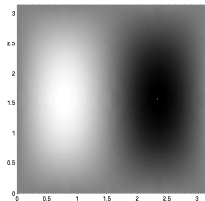
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

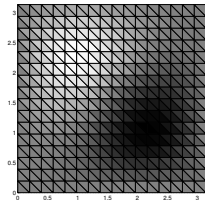
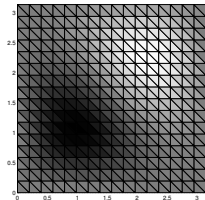
A posteriori
analysis

Exact solutions

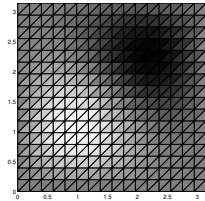
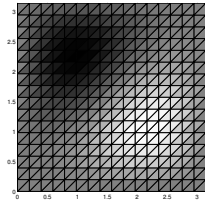


Multiple eigenfunctions (cont'ed)

Uniform mesh



Uniform mesh (reversed)



Uniform mesh

1D Laplace

2D Laplace

P1

L-shaped domain
non-conf. P11D mixed
Laplace2D mixed
LaplaceMultiple
eigenvaluesA posteriori
analysis

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.3168	2.0776 (2.0)	2.0193 (2.0)	2.0048 (2.0)	2.0012 (2.0)
	5	6.3387	5.3325 (2.0)	5.0829 (2.0)	5.0207 (2.0)
5	7.2502	5.5325 (2.1)	5.1302 (2.0)	5.0324 (2.0)	5.0081 (2.0)
	8	12.2145	9.1826 (1.8)	8.3054 (2.0)	8.0769 (2.0)
10	15.5629	11.5492 (1.8)	10.3814 (2.0)	10.0949 (2.0)	10.0237 (2.0)
	10	16.7643	11.6879 (2.0)	10.3900 (2.1)	10.0955 (2.0)
13	20.8965	15.2270 (1.8)	13.5716 (2.0)	13.1443 (2.0)	13.0362 (2.0)
	13	26.0989	17.0125 (1.7)	13.9825 (2.0)	13.2432 (2.0)
17	32.4184	21.3374 (1.8)	18.0416 (2.1)	17.2562 (2.0)	17.0638 (2.0)
	17	21.5751	18.0705 (2.1)	17.2626 (2.0)	17.0653 (2.0)
#	9	49	225	961	3969

Crisscross mesh

1D Laplace

2D Laplace

P1

L-shaped domain
non-conf. P11D mixed
Laplace2D mixed
LaplaceMultiple
eigenvaluesA posteriori
analysis

		Computed (rate)				
		$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1D mixed Laplace	2	2.0880	2.0216 (2.0)	2.0054 (2.0)	2.0013 (2.0)	2.0003 (2.0)
	5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
2D mixed Laplace	5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
	8	9.4962	8.3521 (2.1)	8.0863 (2.0)	8.0215 (2.0)	8.0054 (2.0)
Multiple eigenvalues	10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
	10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
A posteriori analysis	13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
	13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
	17	25.1471	19.3348 (1.8)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
	17	38.9073	19.3348 (3.2)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
	18	38.9073	19.8363 (3.5)	18.4405 (2.1)	18.1089 (2.0)	18.0271 (2.0)
	20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
	20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
	25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
	25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
#		25	113	481	1985	8065

L-shaped domain

1D Laplace

2D Laplace

P1

L-shaped domain

non-conf. P1

1D mixed

Laplace

2D mixed

Laplace

Multiple
eigenvalues

A posteriori
analysis

P1 elements (Neumann boundary conditions)

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0	-0.0000	0.0000	-0.0000	-0.0000	-0.0000
1.48	1.6786	1.5311 (1.9)	1.4946 (1.5)	1.4827 (1.4)	1.4783 (1.4)
3.53	3.8050	3.5904 (2.3)	3.5472 (2.1)	3.5373 (2.0)	3.5348 (2.0)
9.87	12.2108	10.2773 (2.5)	9.9692 (2.0)	9.8935 (2.1)	9.8755 (2.0)
9.87	12.5089	10.3264 (2.5)	9.9823 (2.0)	9.8979 (2.0)	9.8767 (2.0)
11.39	13.9526	12.0175 (2.0)	11.5303 (2.2)	11.4233 (2.1)	11.3976 (2.1)
#	20	65	245	922	3626

Nonconforming P1

1D Laplace

2D Laplace

P1

L-shaped domain

non-conf. P1

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	1.9674	1.9850 (1.1)	1.9966 (2.1)	1.9992 (2.0)	1.9998 (2.0)
	5	4.4508	4.9127 (2.7)	4.9787 (2.0)	4.9987 (2.0)
5	4.7270	4.9159 (1.7)	4.9790 (2.0)	4.9949 (2.0)	4.9987 (2.0)
	8	7.2367	7.7958 (1.9)	7.9434 (1.9)	7.9870 (2.1)
10	8.5792	9.6553 (2.0)	9.9125 (2.0)	9.9792 (2.1)	9.9949 (2.0)
	10	9.0237	9.6663 (1.5)	9.9197 (2.1)	9.9796 (2.0)
13	9.8284	12.4011 (2.4)	12.8534 (2.0)	12.9654 (2.1)	12.9914 (2.0)
	13	9.9107	12.4637 (2.5)	12.8561 (1.9)	12.9655 (2.1)
17	10.4013	15.9559 (2.7)	16.7485 (2.1)	16.9407 (2.1)	16.9853 (2.0)
	17	11.2153	16.0012 (2.5)	16.7618 (2.1)	16.9409 (2.0)
#	40	197	832	3443	13972

Mixed Laplace

1D Laplace

2D Laplace

1D mixed
Laplace

P1-P1
P1-P0
P2-P0

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

1D mixed Laplace: P1-P1 scheme

		Computed (rate)				
		$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
		0.0000	-0.0000	-0.0000	-0.0000	-0.0000
	1	1.0001	1.0000 (4.1)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
	4	3.9660	3.9981 (4.2)	3.9999 (4.0)	4.0000 (4.0)	4.0000 (4.0)
		7.4257	8.5541	8.8854	8.9711	8.9928
	9	8.7603	8.9873 (4.2)	8.9992 (4.1)	9.0000 (4.0)	9.0000 (4.0)
	16	14.8408	15.9501 (4.5)	15.9971 (4.1)	15.9998 (4.0)	16.0000 (4.0)
	25	16.7900	24.5524 (4.2)	24.9780 (4.3)	24.9987 (4.1)	24.9999 (4.0)
		38.7154	29.7390	34.2165	35.5415	35.8846
	36	39.0906	35.0393 (1.7)	35.9492 (4.2)	35.9970 (4.1)	35.9998 (4.0)
	49		46.7793	48.8925 (4.4)	48.9937 (4.1)	48.9996 (4.0)

P1-P1 (cont'ed)

1D Laplace

2D Laplace

1D mixed
Laplace

P1-P1

P1-P0

P2-P0

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Results for $N = 1000$

Exact	Computed
	$N = 1000$
	-0.0000000000
1	1.0000000000
4	3.9999999999
	8.9998815658
9	8.9999999992
16	15.9999999971
25	24.9999999784
	35.9981051039
36	35.9999999495
49	48.9999998977

P1-P0 scheme

1D Laplace

2D Laplace

1D mixed
Laplace

P1-P1
P1-P0
P2-P0

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

	Computed (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	1.0129	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)	1.0001 (2.0)
4	4.2095	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)	4.0008 (2.0)
9	10.0803	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)	9.0041 (2.0)
16	19.4537	16.8382 (2.0)	16.2067 (2.0)	16.0515 (2.0)	16.0129 (2.0)
25	33.2628	27.0649 (2.0)	25.5059 (2.0)	25.1257 (2.0)	25.0314 (2.0)
36	51.3724	40.3212 (1.8)	37.0525 (2.0)	36.2610 (2.0)	36.0651 (2.0)
49	69.5582	57.0672 (1.3)	50.9572 (2.0)	49.4840 (2.0)	49.1206 (2.0)
64	77.8147	77.8147 (0.0)	67.3528 (2.0)	64.8266 (2.0)	64.2059 (2.0)
81		103.0473	86.3943 (2.0)	82.3258 (2.0)	81.3299 (2.0)
100		133.0513	108.2597 (2.0)	102.0237 (2.0)	100.5030 (2.0)
#	8	16	32	64	128

P2-P0 scheme

1D Laplace

2D Laplace

1D mixed
LaplaceP1-P1
P1-P0
P2-P02D mixed
LaplaceMultiple
eigenvaluesA posteriori
analysis

	Computed (rate with respect to 6λ)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	5.7061	5.9238 (1.9)	5.9808 (2.0)	5.9952 (2.0)	5.9988 (2.0)
4	19.8800	22.8245 (1.8)	23.6953 (1.9)	23.9231 (2.0)	23.9807 (2.0)
9	36.7065	48.3798 (1.6)	52.4809 (1.9)	53.6123 (2.0)	53.9026 (2.0)
16	51.8764	79.5201 (1.4)	91.2978 (1.8)	94.7814 (1.9)	95.6925 (2.0)
25	63.6140	113.1819 (1.2)	138.8165 (1.7)	147.0451 (1.9)	149.2506 (2.0)
36	71.6666	146.8261 (1.1)	193.5192 (1.6)	209.9235 (1.9)	214.4494 (2.0)
49	76.3051	178.6404 (0.9)	253.8044 (1.5)	282.8515 (1.9)	291.1344 (2.0)
64	77.8147	207.5058 (0.8)	318.0804 (1.4)	365.1912 (1.8)	379.1255 (1.9)
81		232.8461	384.8425 (1.3)	456.2445 (1.8)	478.2172 (1.9)
100		254.4561	452.7277 (1.2)	555.2659 (1.7)	588.1806 (1.9)
#	8	16	32	64	128

Raviart–Thomas element

1D Laplace

2D Laplace

1D mixed
Laplace2D mixed
Laplace

Raviart–Thomas

Multiple
eigenvaluesA posteriori
analysis

Unstructured mesh

		Computed (rate)				
		$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
Multiple eigenvalues	2	2.0138	1.9989 (3.6)	1.9997 (1.7)	1.9999 (2.7)	2.0000 (2.8)
	5	4.8696	4.9920 (4.0)	5.0000 (8.0)	4.9999 (-2.1)	5.0000 (3.7)
	5	4.8868	4.9952 (4.5)	5.0006 (3.0)	5.0000 (5.8)	5.0000 (2.6)
	8	8.6905	7.9962 (7.5)	7.9974 (0.6)	7.9995 (2.5)	7.9999 (2.2)
	10	9.7590	9.9725 (3.1)	9.9980 (3.8)	9.9992 (1.3)	9.9999 (3.2)
	10	11.4906	9.9911 (7.4)	10.0007 (3.7)	10.0005 (0.4)	10.0001 (2.4)
	13	11.9051	12.9250 (3.9)	12.9917 (3.2)	12.9998 (5.4)	12.9999 (1.8)
	13	12.7210	12.9631 (2.9)	12.9950 (2.9)	13.0000 (7.5)	13.0000 (1.1)
	17	13.5604	16.8450 (4.5)	16.9848 (3.4)	16.9992 (4.3)	16.9999 (2.5)
	17	14.1813	16.9659 (6.4)	16.9946 (2.7)	17.0009 (2.6)	17.0000 (5.5)
#		32	142	576	2338	9400

Raviart–Thomas element (cont'ed)

1D Laplace

2D Laplace

1D mixed
Laplace2D mixed
Laplace

Raviart–Thomas

Multiple
eigenvaluesA posteriori
analysis

Uniform mesh

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.1048	2.0258 (2.0)	2.0064 (2.0)	2.0016 (2.0)	2.0004 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
8	9.7268	8.4191 (2.0)	8.1033 (2.0)	8.0257 (2.0)	8.0064 (2.0)
10	13.8955	11.0932 (1.8)	10.2663 (2.0)	10.0660 (2.0)	10.0165 (2.0)
10	13.8955	11.0932 (1.8)	10.2663 (2.0)	10.0660 (2.0)	10.0165 (2.0)
13	17.7065	14.2898 (1.9)	13.3148 (2.0)	13.0781 (2.0)	13.0195 (2.0)
13	17.7065	14.2898 (1.9)	13.3148 (2.0)	13.0781 (2.0)	13.0195 (2.0)
17	20.5061	20.1606 (0.1)	17.8414 (1.9)	17.2075 (2.0)	17.0517 (2.0)
17	20.5061	20.4666 (0.0)	17.8414 (2.0)	17.2075 (2.0)	17.0517 (2.0)
#	16	64	256	1024	4096

Babuška–Osborn example

1D Laplace

2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

$$-\left(\frac{1}{\varphi'(x)}u'(x)\right)' = \lambda\varphi'(x)u(x) \quad \text{in } (-\pi, \pi)$$

$$u(-\pi) = u(\pi)$$

$$\frac{1}{\varphi'(-\pi)}u'(-\pi) = \frac{1}{\varphi'(\pi)}u'(\pi)$$

$$\varphi(x) = \pi^{-\alpha}|x|^{1+\alpha}\text{sign}(x), \quad 0 < \alpha < 1$$

Exact solution

1D Laplace

2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

$$\lambda = 0 \quad (u(x) \equiv 1)$$

$$\lambda = k^2 \quad (k = 1, 2, \dots) \text{ double eigenvalues}$$

$$u(x) = \sin(k\varphi(x)) \quad \text{regularity } (3 + \alpha)/2$$

$$u(x) = \cos(k\varphi(x)) \quad \text{regularity } (5 + 3\alpha)/2$$

$$\alpha = .9$$

1D Laplace

2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	1.7e-03	4.2e-04 (2.00)	1.0e-04 (2.00)	2.6e-05 (2.00)	6.6e-06 (1.98)
1	4.6e-03	1.3e-03 (1.88)	3.4e-04 (1.88)	9.2e-05 (1.88)	2.5e-05 (1.88)
4	6.2e-03	1.5e-03 (2.00)	3.9e-04 (2.00)	9.7e-05 (2.00)	2.4e-05 (2.00)
4	9.0e-03	2.4e-03 (1.94)	6.2e-04 (1.94)	1.6e-04 (1.93)	4.2e-05 (1.93)
9	1.4e-02	3.4e-03 (2.00)	8.5e-04 (2.00)	2.1e-04 (2.00)	5.3e-05 (2.00)
9	1.7e-02	4.2e-03 (1.97)	1.1e-03 (1.96)	2.8e-04 (1.96)	7.1e-05 (1.96)
16	2.4e-02	6.0e-03 (2.00)	1.5e-03 (2.00)	3.8e-04 (2.00)	9.4e-05 (2.00)
16	2.7e-02	6.8e-03 (1.98)	1.7e-03 (1.98)	4.4e-04 (1.98)	1.1e-04 (1.97)
25	3.7e-02	9.4e-03 (2.00)	2.3e-03 (2.00)	5.9e-04 (2.00)	1.5e-04 (2.00)
25	4.0e-02	1.0e-02 (1.99)	2.6e-03 (1.99)	6.5e-04 (1.98)	1.6e-04 (1.98)
DOF	64	128	256	512	1024

Table: Error in the eigenvalues computed with linear elements and $\alpha = 0.9$.

$$\alpha = .5$$

1D Laplace

2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	1.2e-03	3.0e-04 (2.00)	7.4e-05 (2.00)	1.9e-05 (2.00)	4.7e-06 (2.00)
1	4.3e-03	1.4e-03 (1.59)	4.8e-04 (1.57)	1.7e-04 (1.55)	5.7e-05 (1.53)
4	4.5e-03	1.1e-03 (2.00)	2.8e-04 (2.00)	7.0e-05 (2.00)	1.8e-05 (2.00)
4	7.5e-03	2.2e-03 (1.75)	6.8e-04 (1.71)	2.2e-04 (1.67)	7.0e-05 (1.63)
9	1.0e-02	2.5e-03 (2.00)	6.2e-04 (2.00)	1.6e-04 (2.00)	3.9e-05 (2.00)
9	1.3e-02	3.6e-03 (1.86)	1.0e-03 (1.81)	3.0e-04 (1.77)	9.1e-05 (1.72)
16	1.8e-02	4.4e-03 (2.00)	1.1e-03 (2.00)	2.7e-04 (2.00)	6.9e-05 (2.00)
16	2.1e-02	5.5e-03 (1.91)	1.5e-03 (1.88)	4.2e-04 (1.84)	1.2e-04 (1.80)
25	2.7e-02	6.8e-03 (2.00)	1.7e-03 (2.00)	4.3e-04 (2.00)	1.1e-04 (2.00)
25	3.0e-02	7.9e-03 (1.94)	2.1e-03 (1.91)	5.7e-04 (1.88)	1.6e-04 (1.85)
DOF	64	128	256	512	1024

Table: Error in the eigenvalues computed with linear elements and $\alpha = 0.5$.

$$\alpha = .1$$

1D Laplace

2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	8.5e-04	2.1e-04 (2.00)	5.3e-05 (2.00)	1.3e-05 (2.00)	3.3e-06 (2.00)
1	1.3e-03	4.2e-04 (1.59)	1.6e-04 (1.45)	6.2e-05 (1.33)	2.6e-05 (1.24)
4	3.4e-03	8.4e-04 (2.00)	2.1e-04 (2.00)	5.2e-05 (2.00)	1.3e-05 (2.00)
4	3.7e-03	1.0e-03 (1.85)	3.1e-04 (1.75)	1.0e-04 (1.62)	3.6e-05 (1.49)
9	7.5e-03	1.9e-03 (2.00)	4.7e-04 (2.00)	1.2e-04 (2.00)	2.9e-05 (2.00)
9	7.8e-03	2.1e-03 (1.93)	5.6e-04 (1.87)	1.6e-04 (1.78)	5.2e-05 (1.67)
16	1.3e-02	3.3e-03 (2.01)	8.3e-04 (2.00)	2.1e-04 (2.00)	5.2e-05 (2.00)
16	1.4e-02	3.5e-03 (1.96)	9.2e-04 (1.92)	2.5e-04 (1.86)	7.4e-05 (1.78)
25	2.1e-02	5.2e-03 (2.01)	1.3e-03 (2.00)	3.2e-04 (2.00)	8.1e-05 (2.00)
25	2.1e-02	5.3e-03 (1.98)	1.4e-03 (1.95)	3.7e-04 (1.91)	1.0e-04 (1.84)
DOF	64	128	256	512	1024

Table: Error in the eigenvalues computed with linear elements and $\alpha = 0.1$.

Convergence of adaptive scheme

1D Laplace

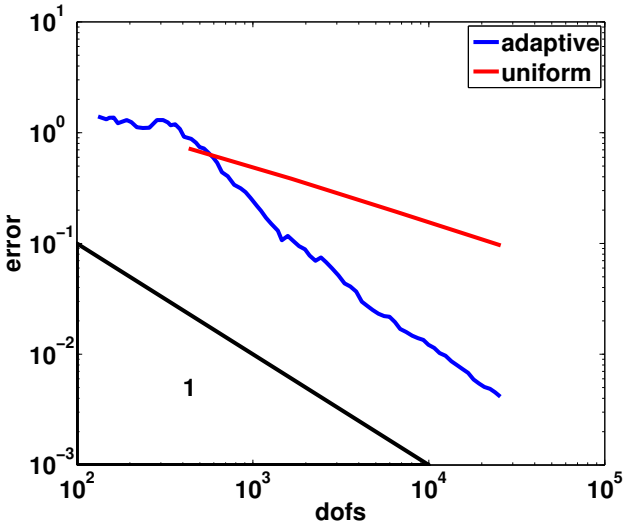
2D Laplace

1D mixed
Laplace

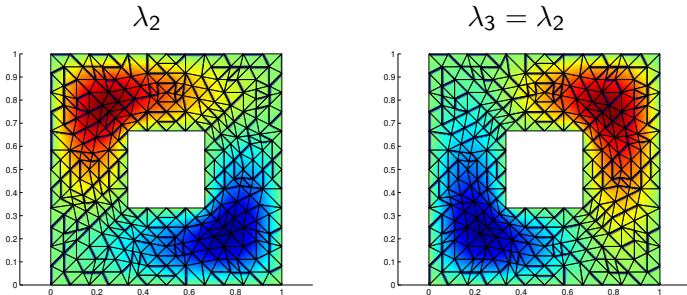
2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis



Multiple eigenvalues



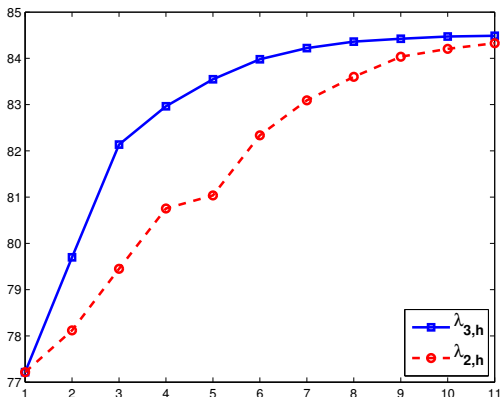
Question

What is the best adaptive strategy for the approximation of the multiple eigenvalue?

- ① Indicator based on $(\lambda_{h,2}, u_{h,2})$
- ② Indicator based on $(\lambda_{h,3}, u_{h,3})$
- ③ Indicator based on both $(\lambda_{h,2}, u_{h,2})$ and $(\lambda_{h,3}, u_{h,3})$

Refinement based on $\lambda_{h,3}$

Remark: here we are using a nonconforming discretization which provides eigenvalue approximation from below



Refinement based on $\lambda_{h,3}$ (eigenfunction $u_{h,3}$)

1D Laplace

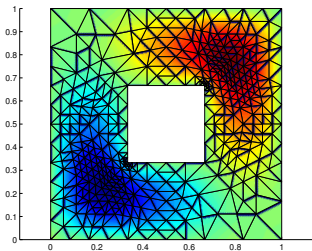
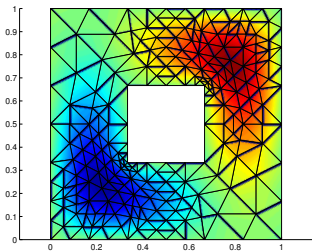
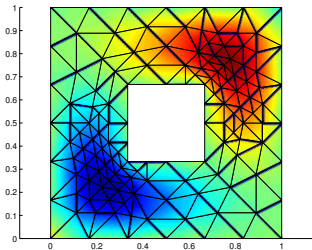
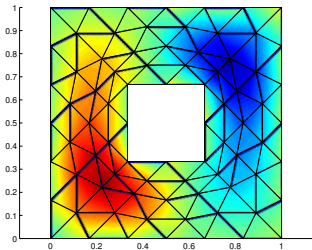
2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis



Refinement based on $\lambda_{h,2}$

1D Laplace

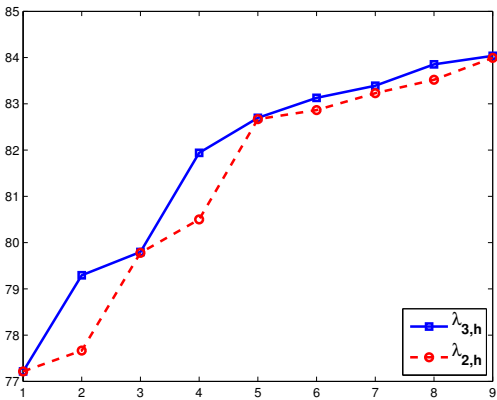
2D Laplace

1D mixed
Laplace

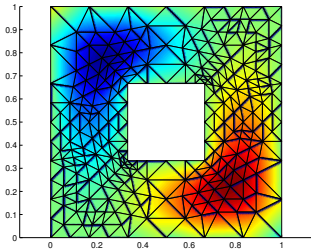
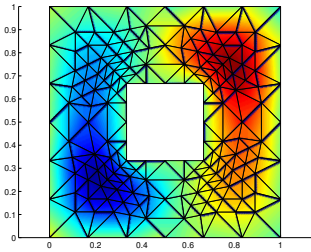
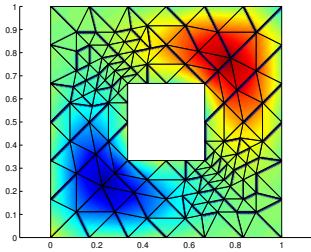
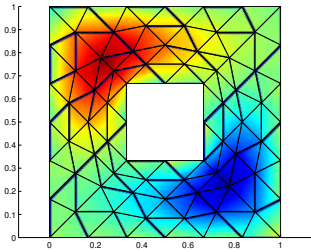
2D mixed
Laplace

Multiple
eigenvalues

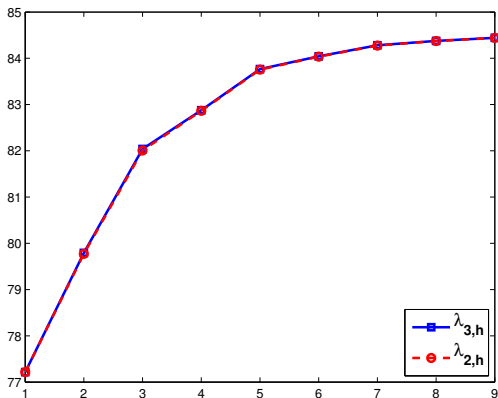
A posteriori
analysis



Refinement based on $\lambda_{h,2}$ (eigenfunction $u_{h,2}$)



Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenvalues)



Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenfunction $u_{h,2}$)

1D Laplace

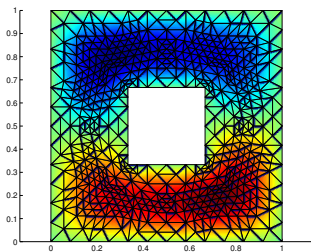
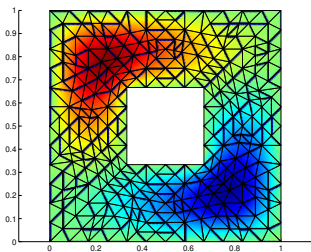
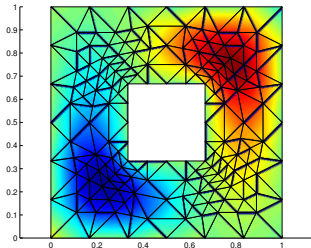
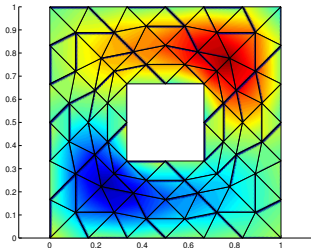
2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis



Cluster of eigenvalues

1D Laplace

2D Laplace

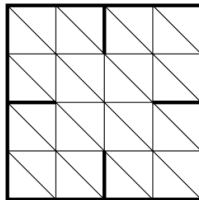
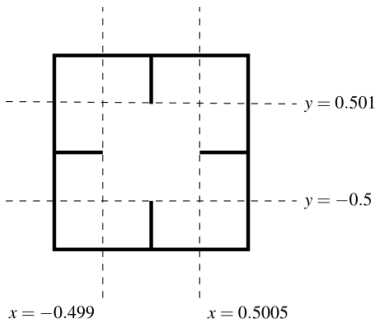
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

A slightly non-symmetric domain



Now $\lambda_2 < \lambda_3$ but they are very close to each other

Non-symmetric slit domain

1D Laplace

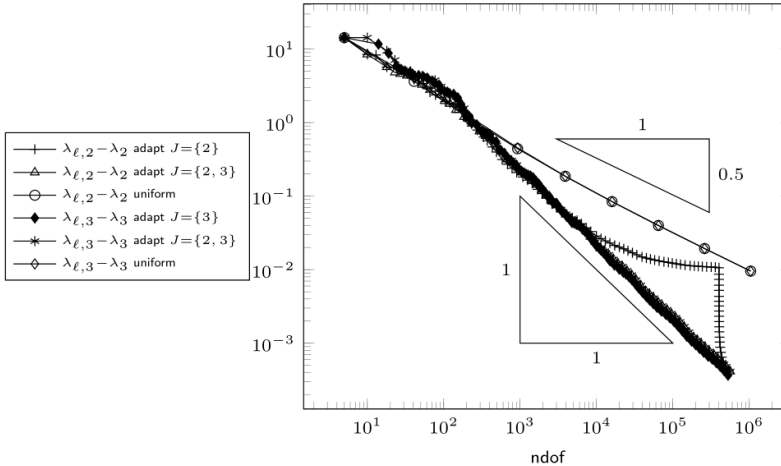
2D Laplace

1D mixed
Laplace

2D mixed
Laplace

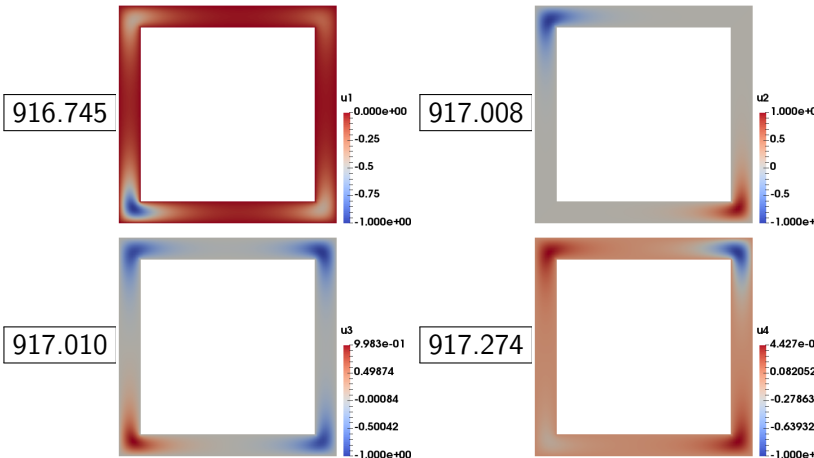
Multiple
eigenvalues

A posteriori
analysis



A slightly non-symmetric domain

A square ring for which the first four modes are the following ones (computed on an adapted mesh with 4,122,416 dof's)



The computational framework

1D Laplace

2D Laplace

1D mixed
Laplace

2D mixed
Laplace

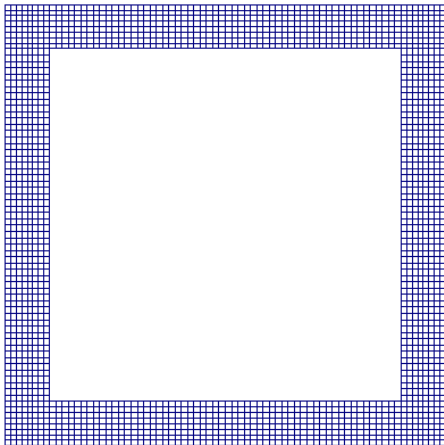
Multiple
eigenvalues

A posteriori
analysis

All computations performed on a **desktop computer** (2 Intel Xeon CPU's @ 3.60GHz, 2 cores each, 16Gb RAM) and on a **laptop computer** (Intel i7 CPU @ 1.90GHz, 2 cores, 8Gb RAM)
AFEM algorithm implemented within the **deal.II** library
Tensor product mesh
Finite element space: continuous bilinear space Q_1
Solution of algebraic eigenvalue problem by **SLEPc**
<http://slepc.upv.es> using generalized Davidson iterations
SLEPc is build on top of PETSc for the parallelization

Approximation of first frequency

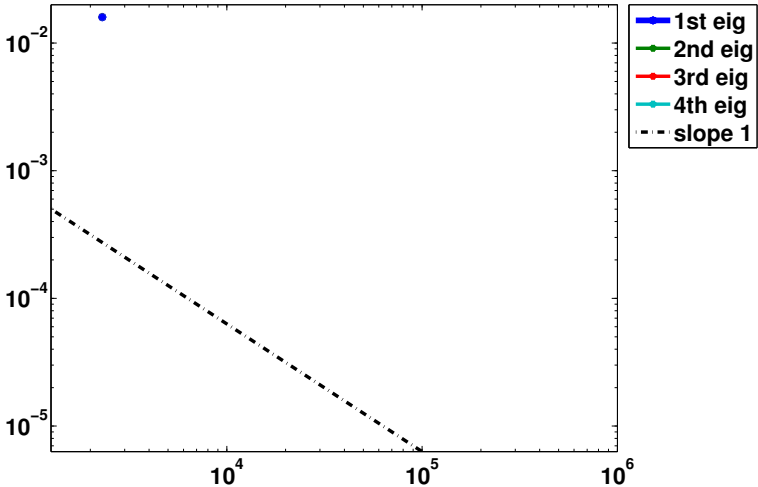
Initial mesh



At each refinement level we **compute** the first **four** eigenmodes and drive the adaptive strategy according to the **error indicator** related to the **first** eigenmode

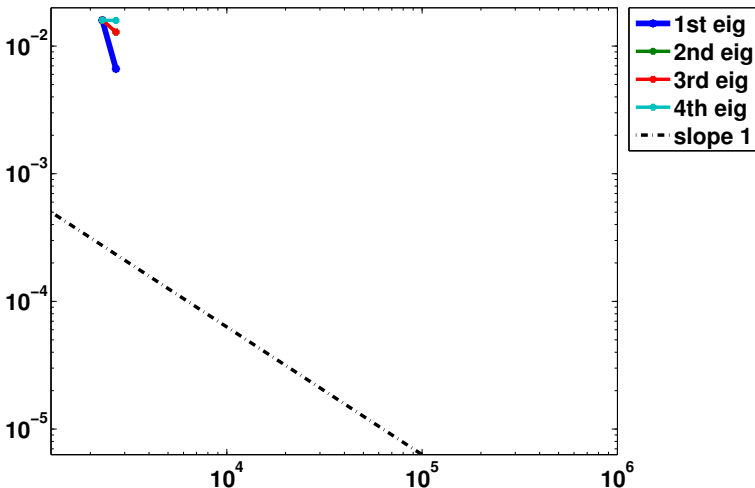
Approximation of first frequency

Bulk parameter=0.3, Refinement level=0 (initial mesh)



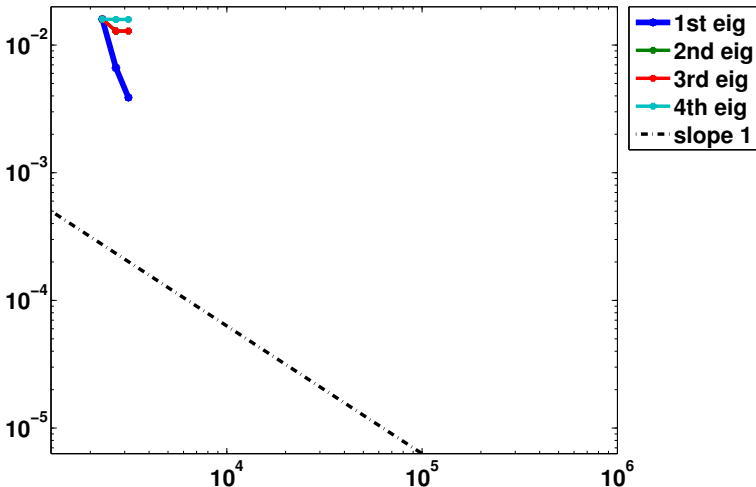
Approximation of first frequency

Bulk parameter=0.3, Refinement level=1



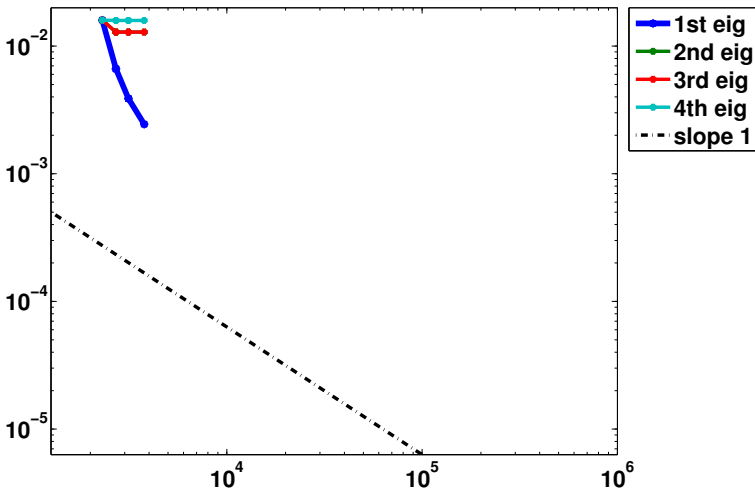
Approximation of first frequency

Bulk parameter=0.3, Refinement level=2



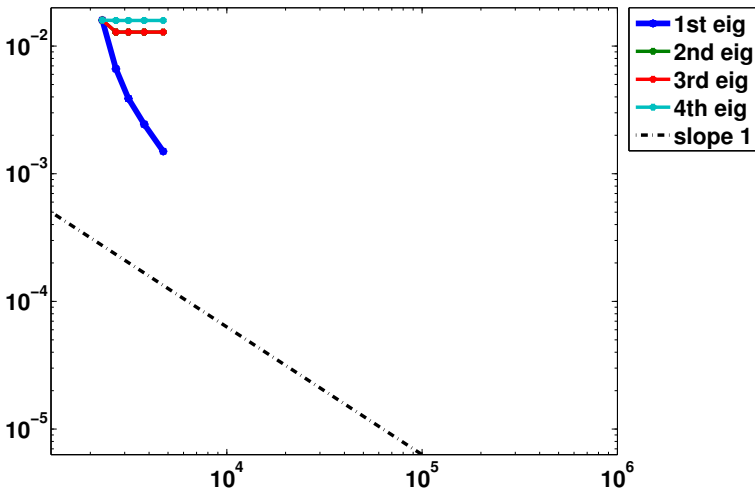
Approximation of first frequency

Bulk parameter=0.3, Refinement level=3



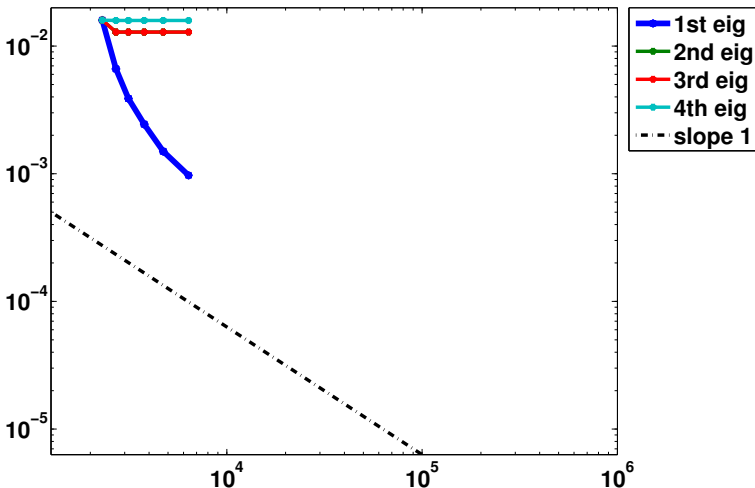
Approximation of first frequency

Bulk parameter=0.3, Refinement level=4



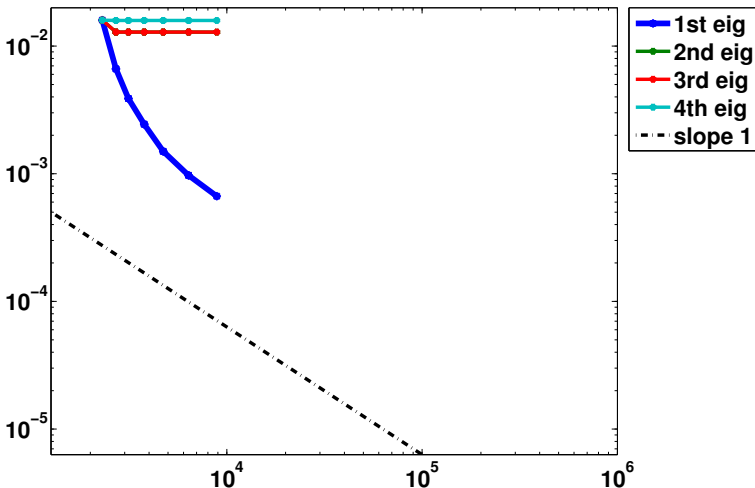
Approximation of first frequency

Bulk parameter=0.3, Refinement level=5



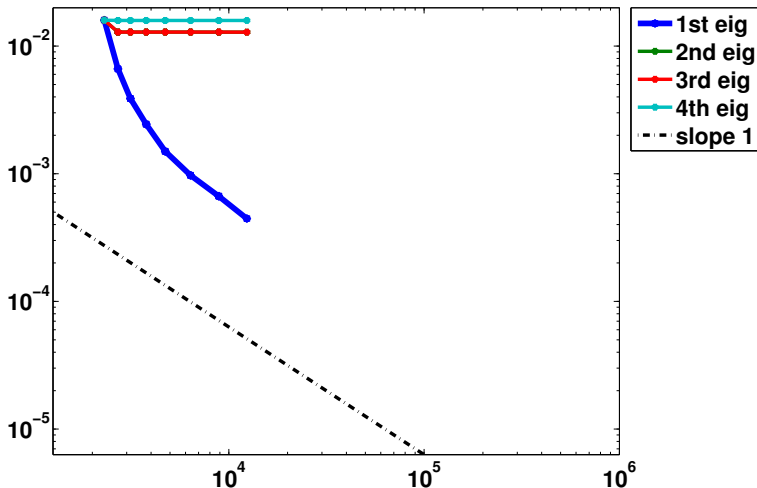
Approximation of first frequency

Bulk parameter=0.3, Refinement level=6



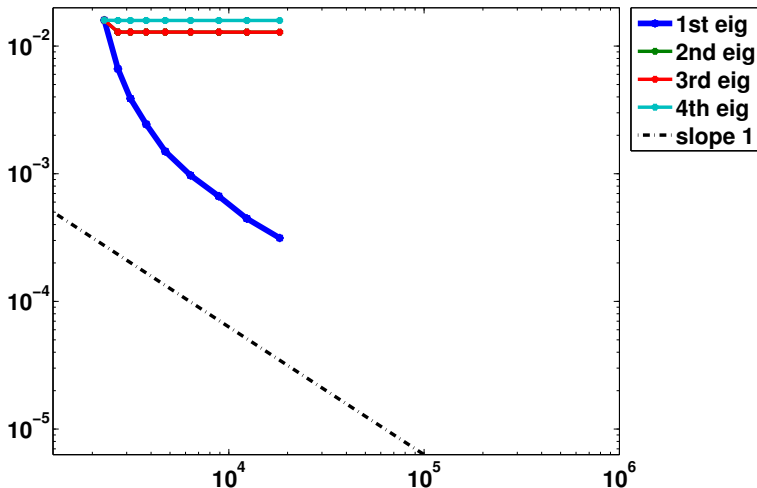
Approximation of first frequency

Bulk parameter=0.3, Refinement level=7



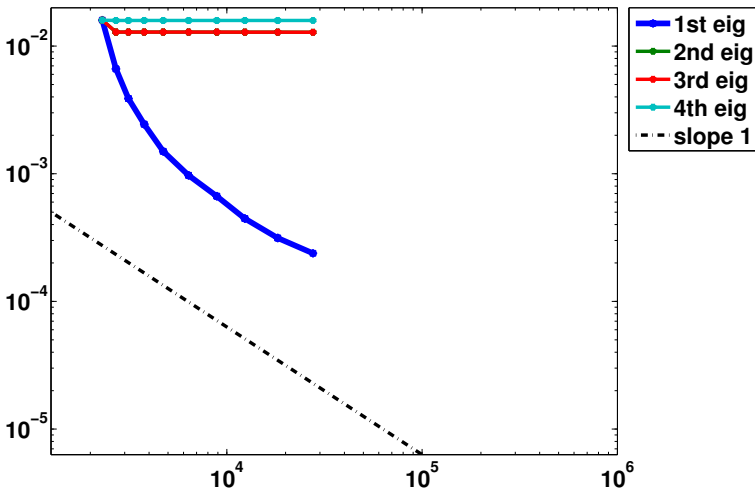
Approximation of first frequency

Bulk parameter=0.3, Refinement level=8



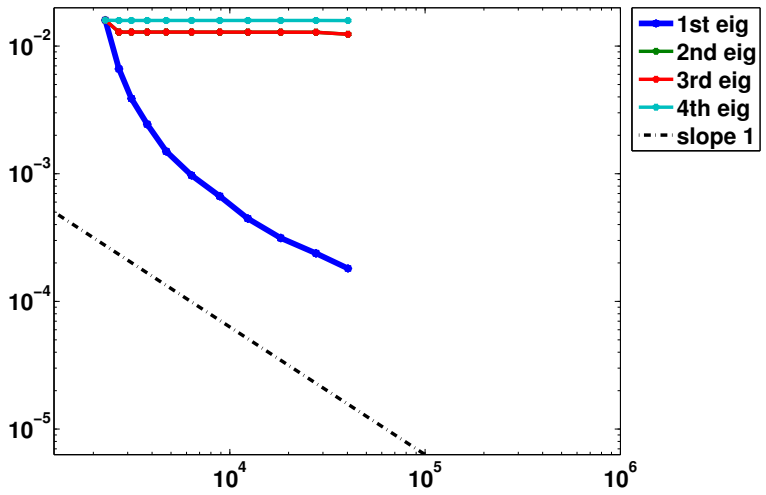
Approximation of first frequency

Bulk parameter=0.3, Refinement level=9



Approximation of first frequency

Bulk parameter=0.3, Refinement level=10



Approximation of first frequency

1D Laplace

2D Laplace

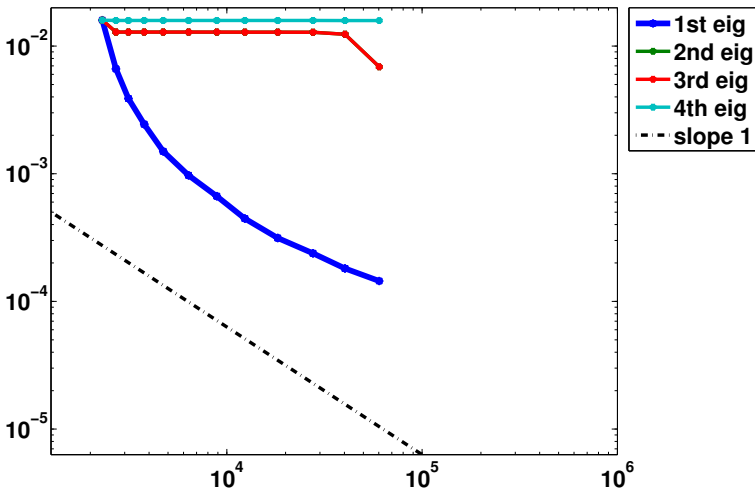
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=11



Approximation of first frequency

1D Laplace

2D Laplace

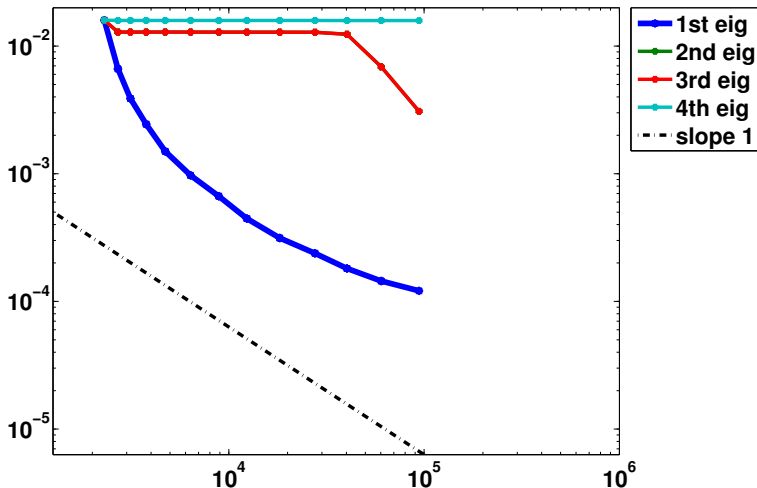
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=12



Approximation of first frequency

1D Laplace

2D Laplace

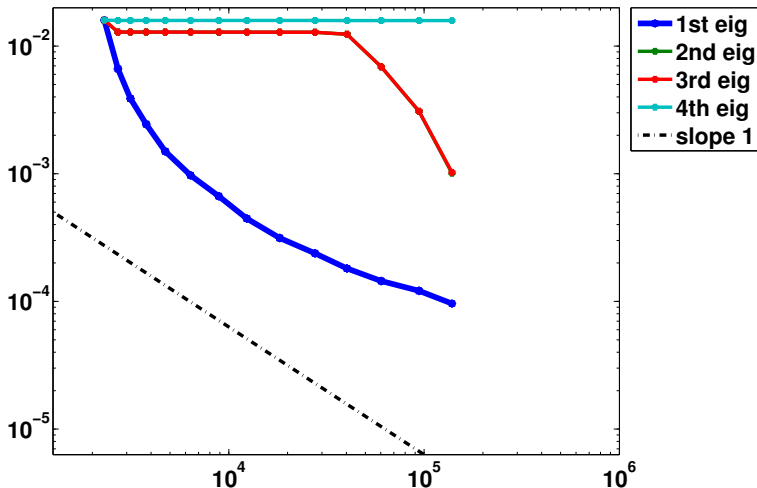
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=13



Approximation of first frequency

1D Laplace

2D Laplace

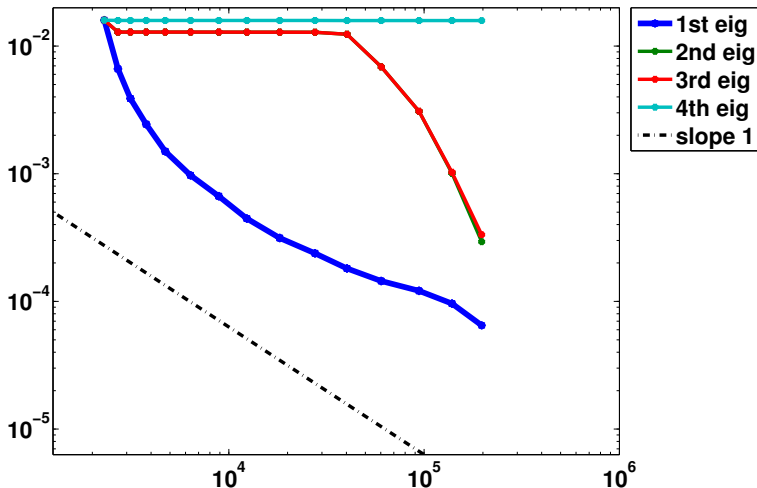
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=14



Approximation of first frequency

1D Laplace

2D Laplace

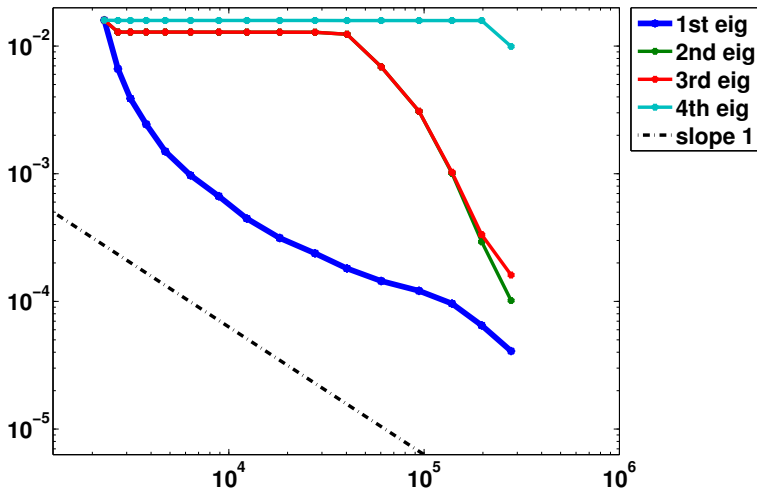
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=15



Approximation of first frequency

1D Laplace

2D Laplace

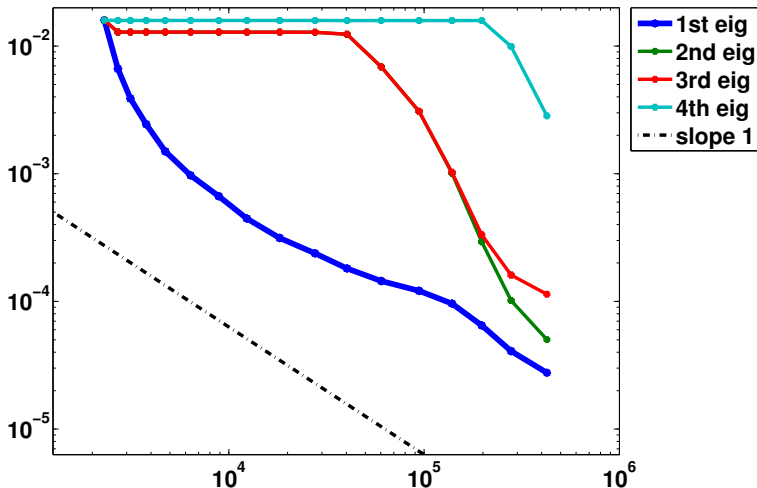
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=16



Approximation of first frequency

1D Laplace

2D Laplace

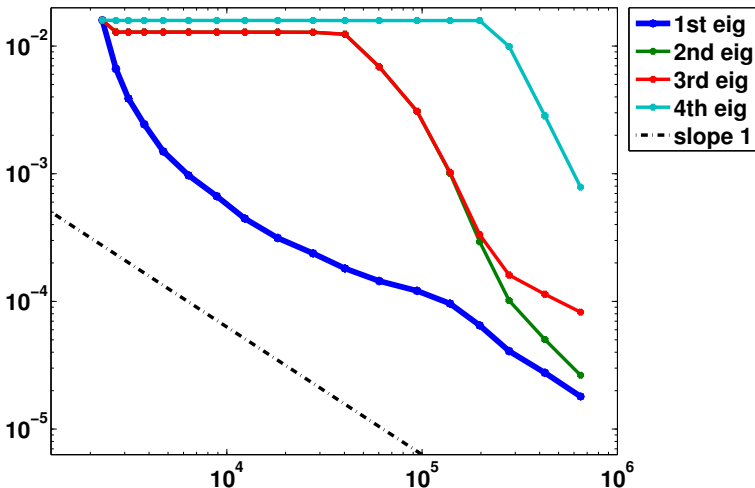
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=17



Approximation of first frequency

1D Laplace

2D Laplace

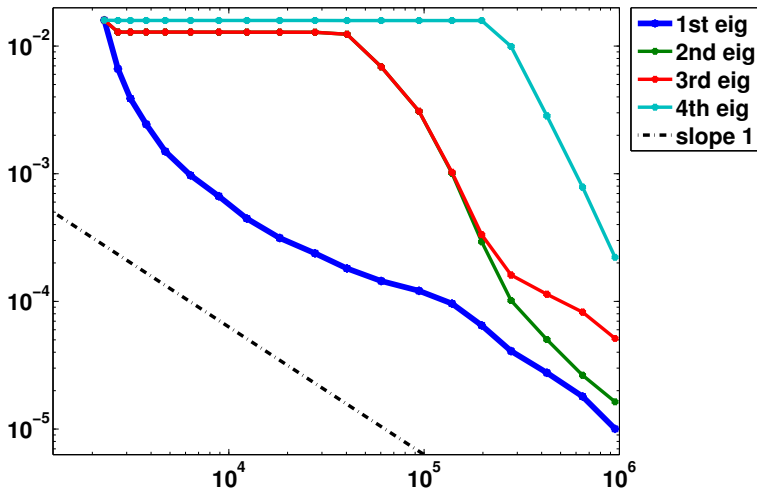
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

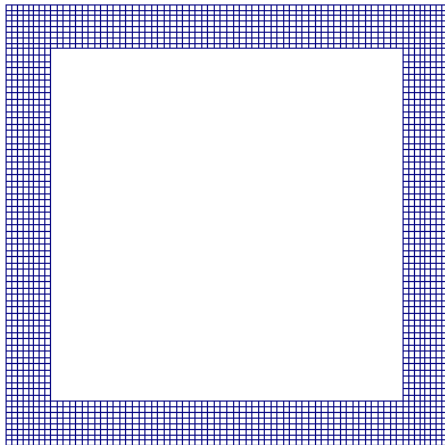
A posteriori
analysis

Bulk parameter=0.3, Refinement level=18



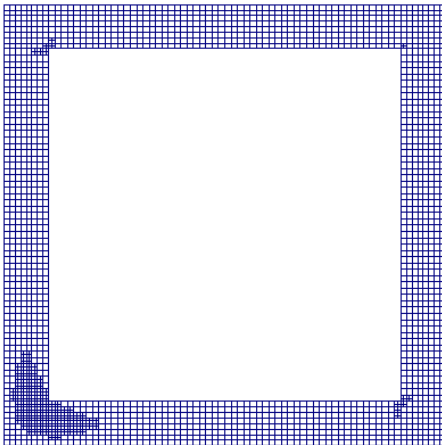
Approximation of first frequency: underlying mesh

Initial mesh



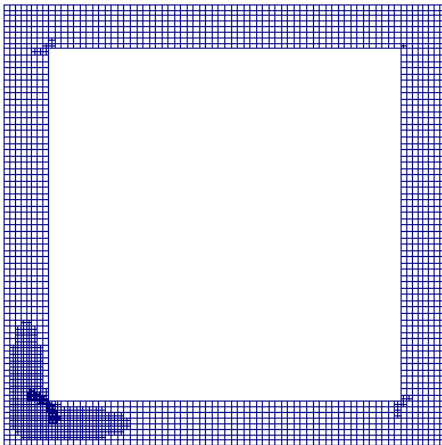
Approximation of first frequency: underlying mesh

Refinement level=1



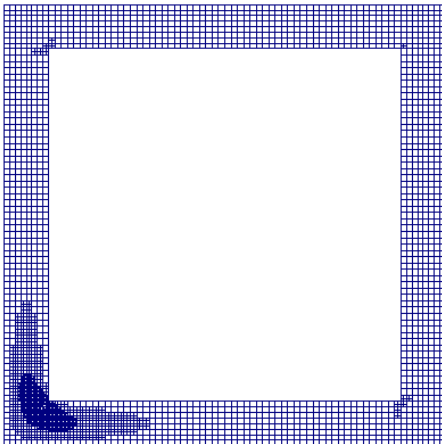
Approximation of first frequency: underlying mesh

Refinement level=2



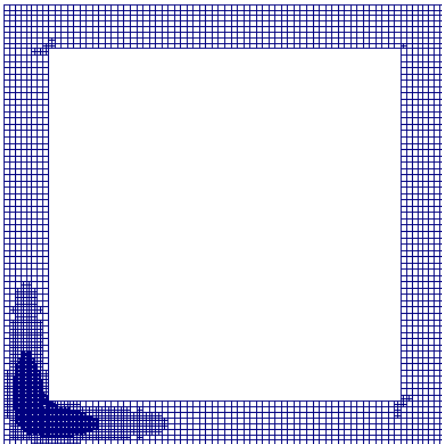
Approximation of first frequency: underlying mesh

Refinement level=3



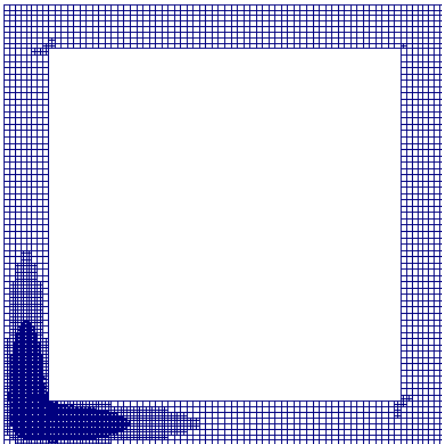
Approximation of first frequency: underlying mesh

Refinement level=4



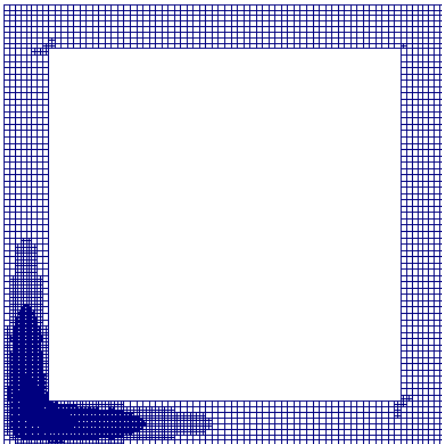
Approximation of first frequency: underlying mesh

Refinement level=5



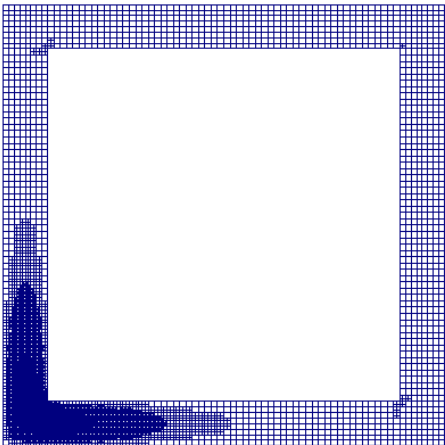
Approximation of first frequency: underlying mesh

Refinement level=6



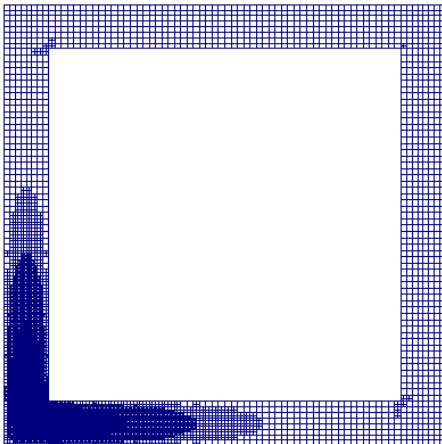
Approximation of first frequency: underlying mesh

Refinement level=7



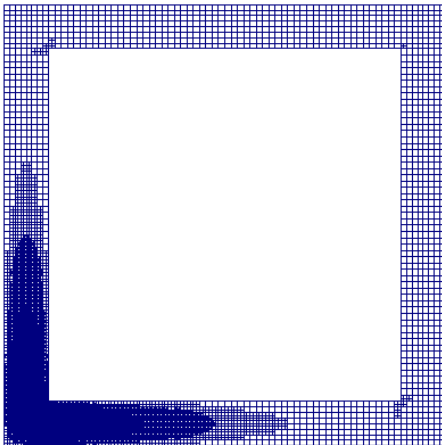
Approximation of first frequency: underlying mesh

Refinement level=8



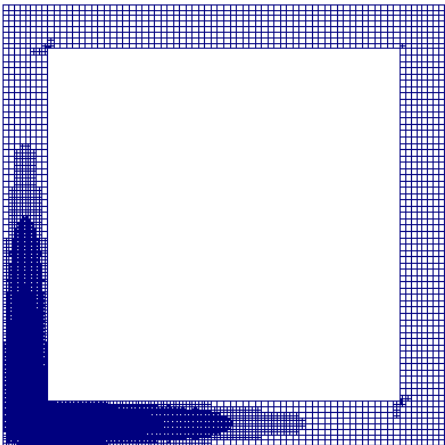
Approximation of first frequency: underlying mesh

Refinement level=9



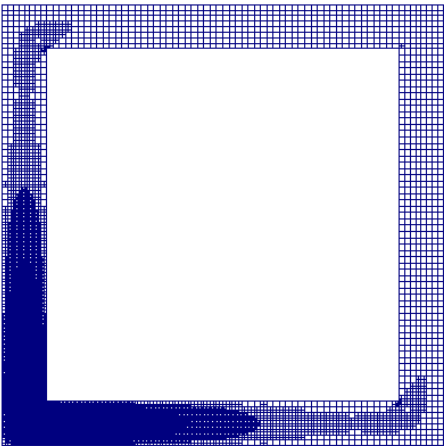
Approximation of first frequency: underlying mesh

Refinement level=10



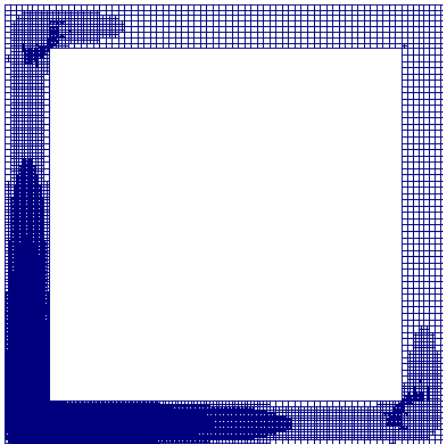
Approximation of first frequency: underlying mesh

Refinement level=11



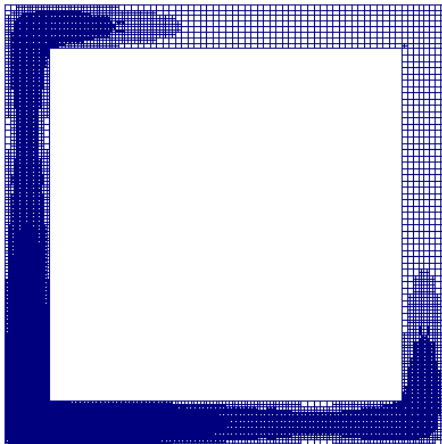
Approximation of first frequency: underlying mesh

Refinement level=12



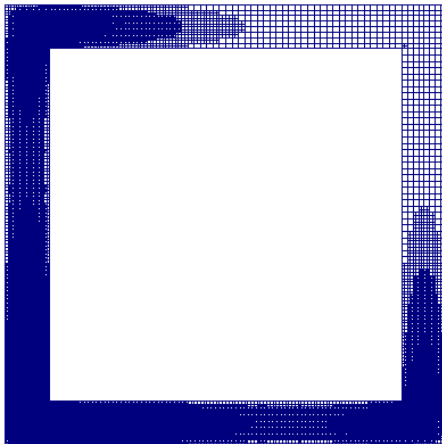
Approximation of first frequency: underlying mesh

Refinement level=13



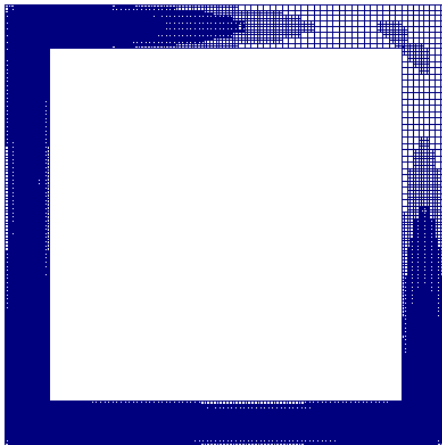
Approximation of first frequency: underlying mesh

Refinement level=14



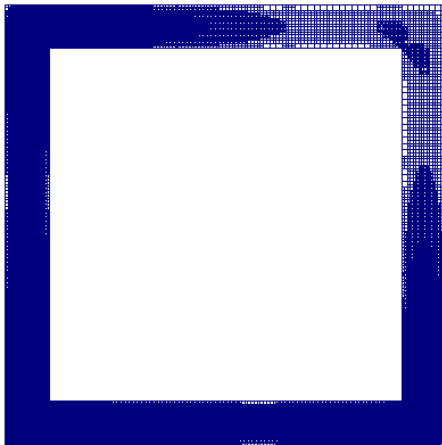
Approximation of first frequency: underlying mesh

Refinement level=15



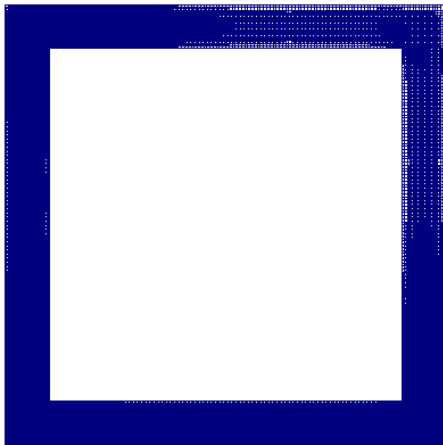
Approximation of first frequency: underlying mesh

Refinement level=16



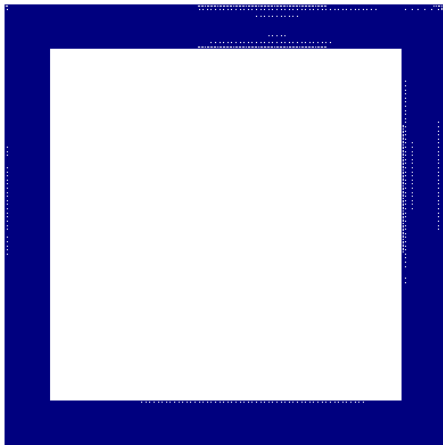
Approximation of first frequency: underlying mesh

Refinement level=17



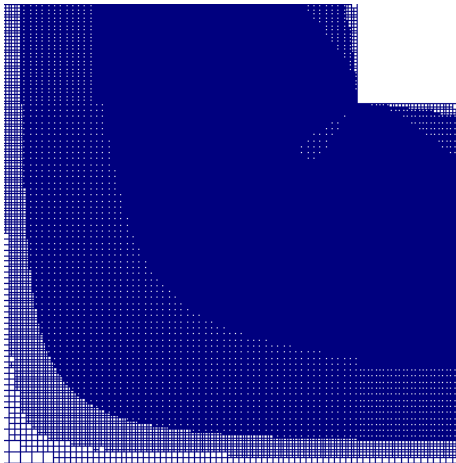
Approximation of first frequency: underlying mesh

Refinement level=18

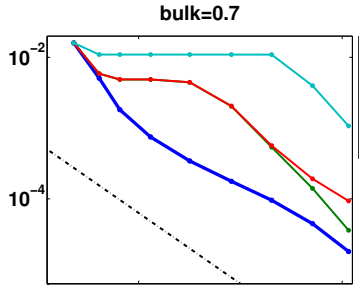
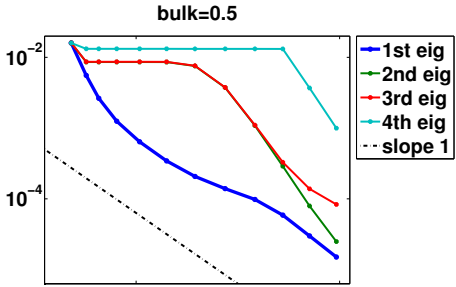
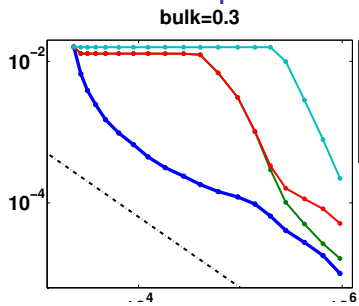
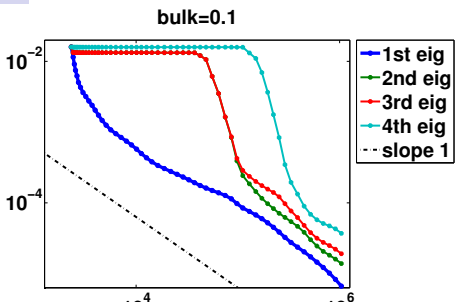


Approximation of first frequency: underlying mesh

Detail of the last mesh



Changing the bulk parameter doesn't help



Approximation of the second frequency

1D Laplace

2D Laplace

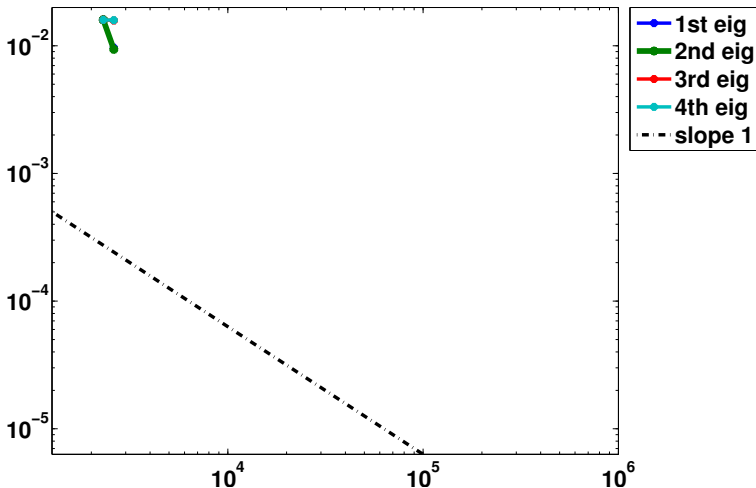
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=1



Approximation of the second frequency

1D Laplace

2D Laplace

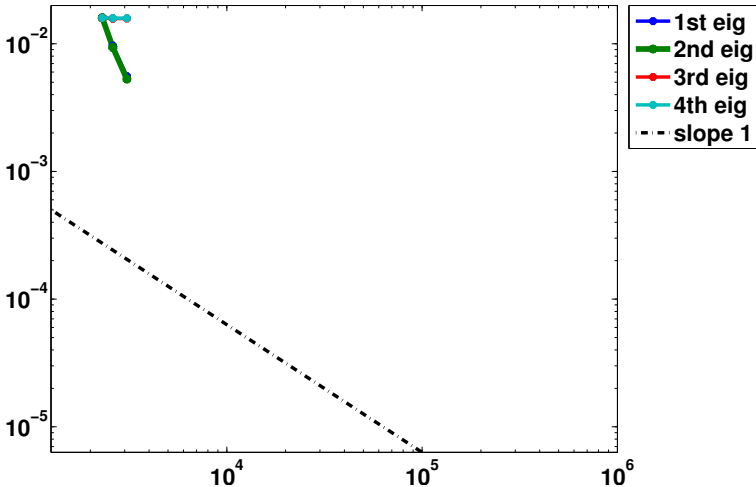
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=2



Approximation of the second frequency

1D Laplace

2D Laplace

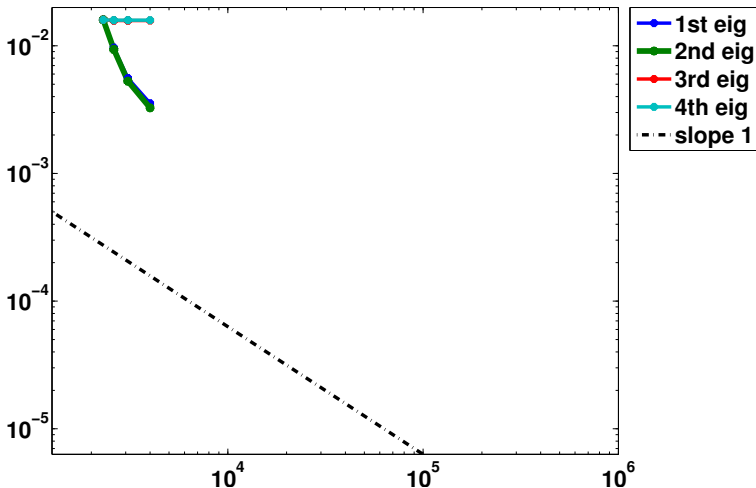
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

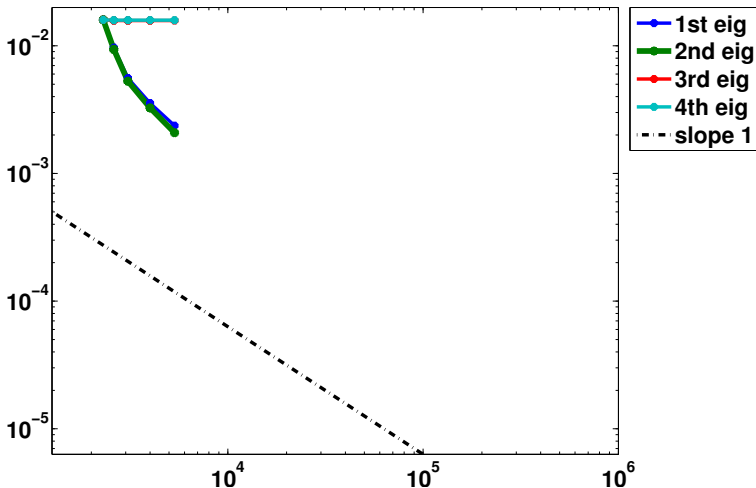
A posteriori
analysis

Bulk parameter=0.3, Refinement level=3



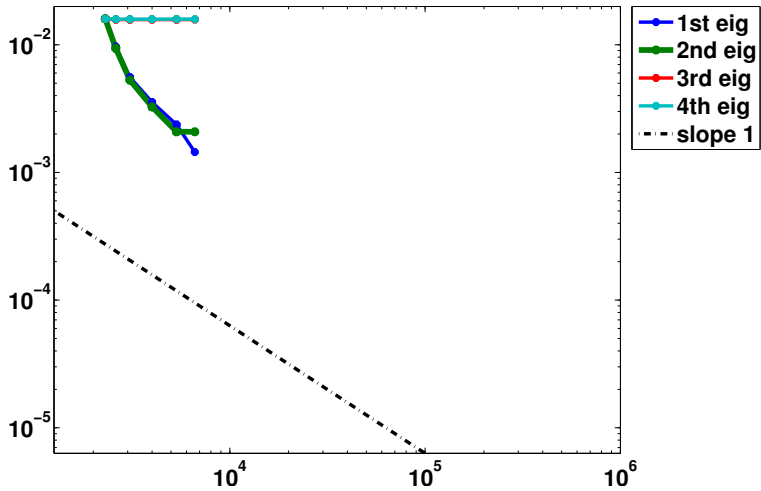
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=4



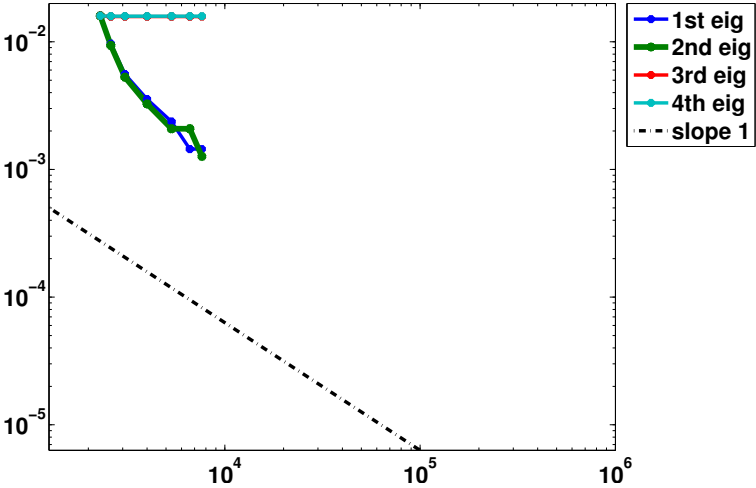
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=5



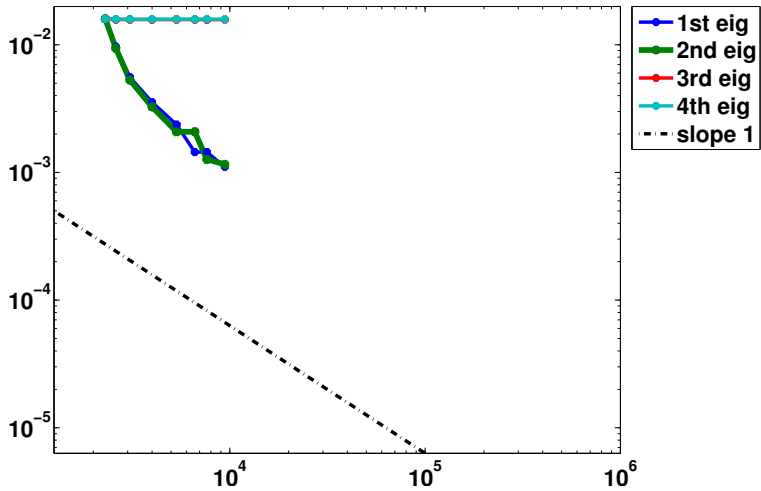
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=6



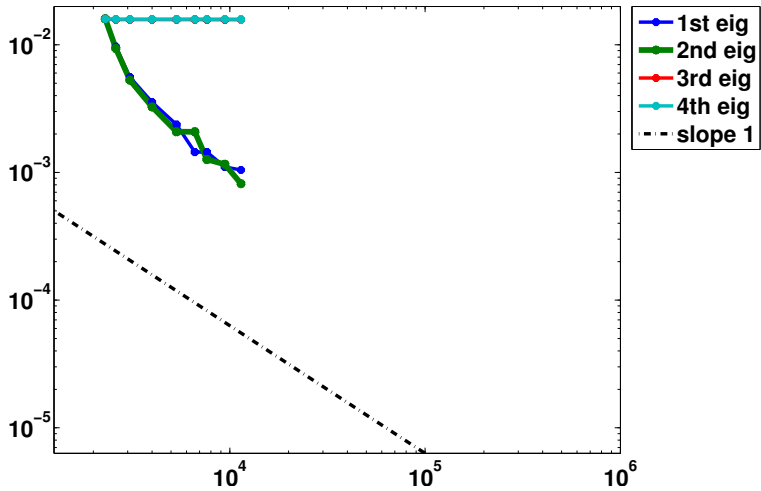
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=7



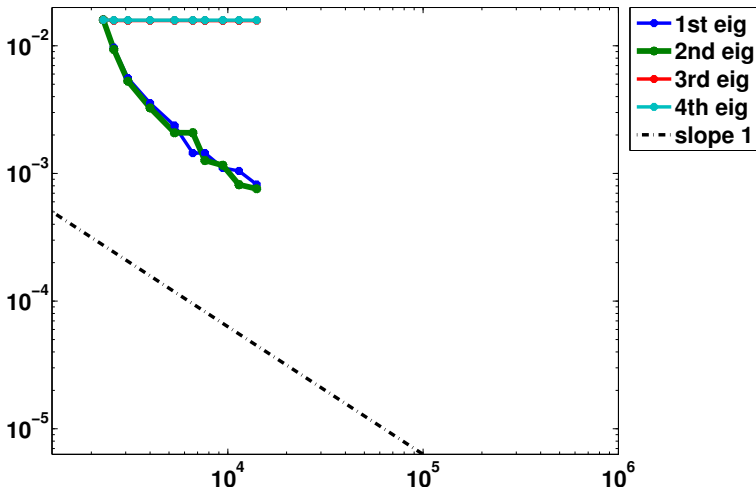
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=8



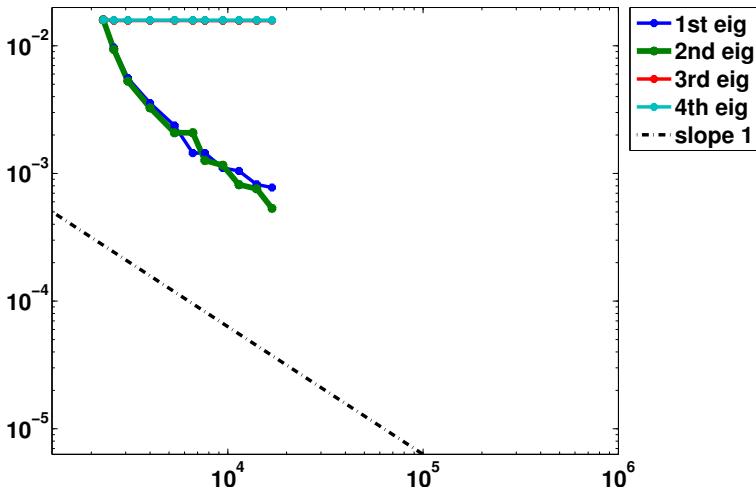
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=9



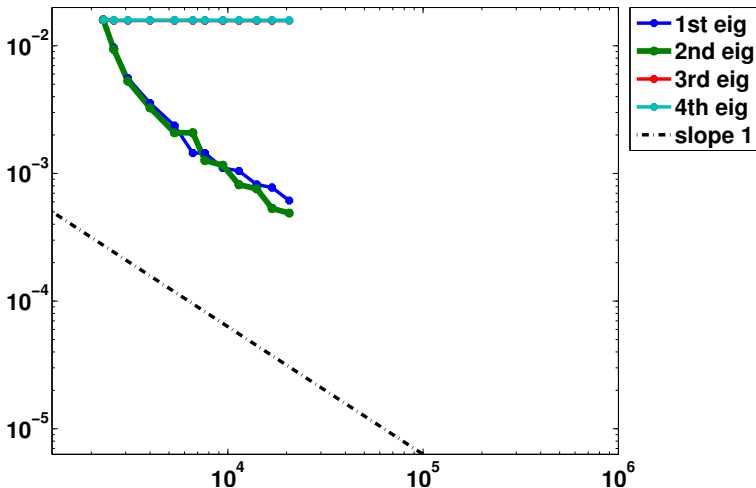
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=10



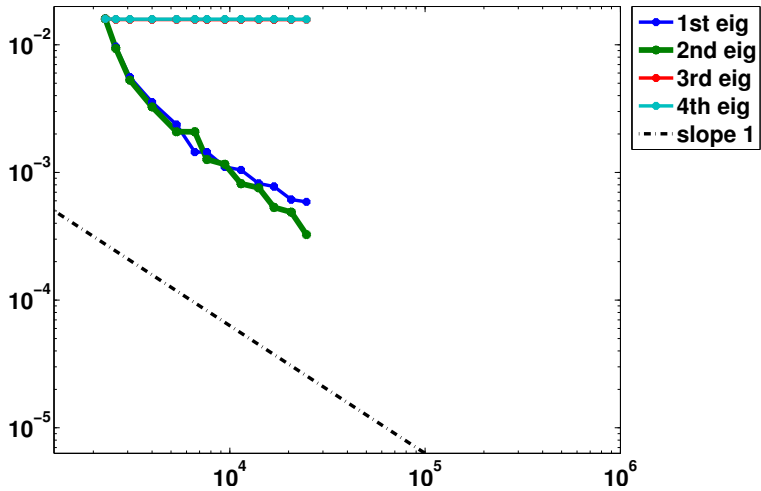
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=11



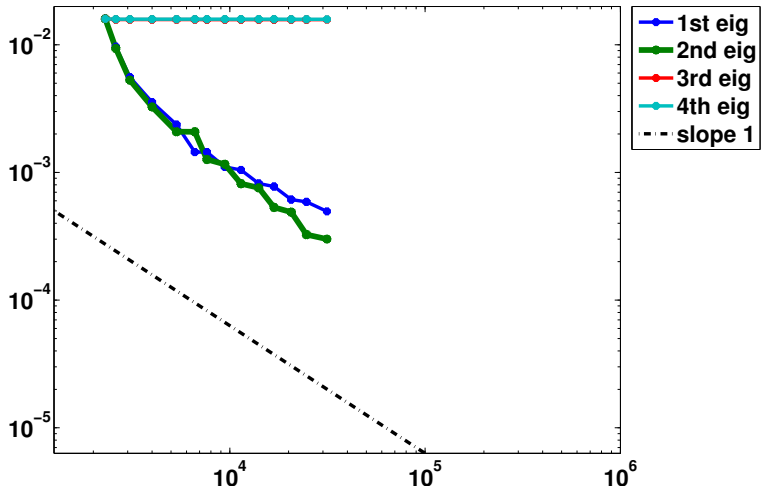
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=12



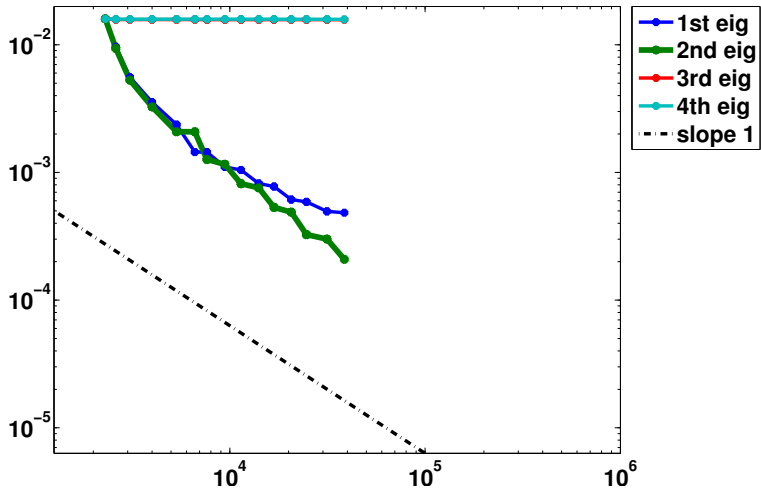
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=13



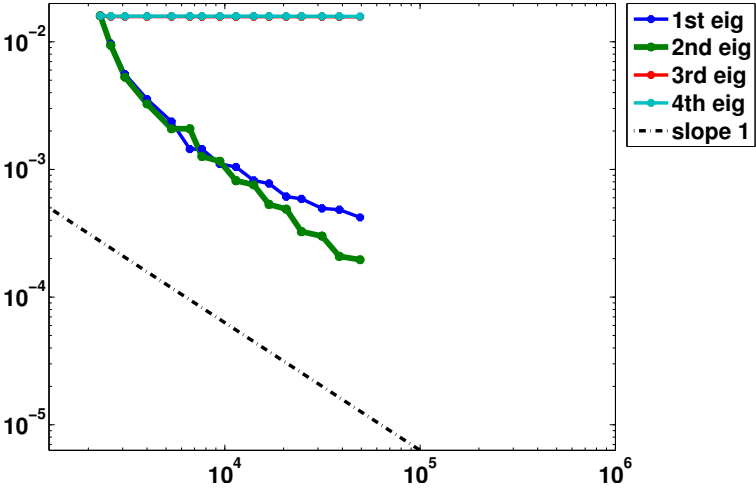
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=14



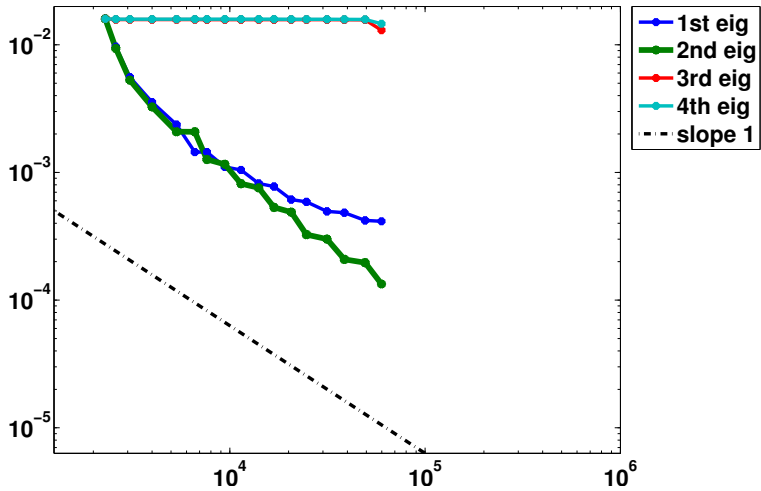
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=15



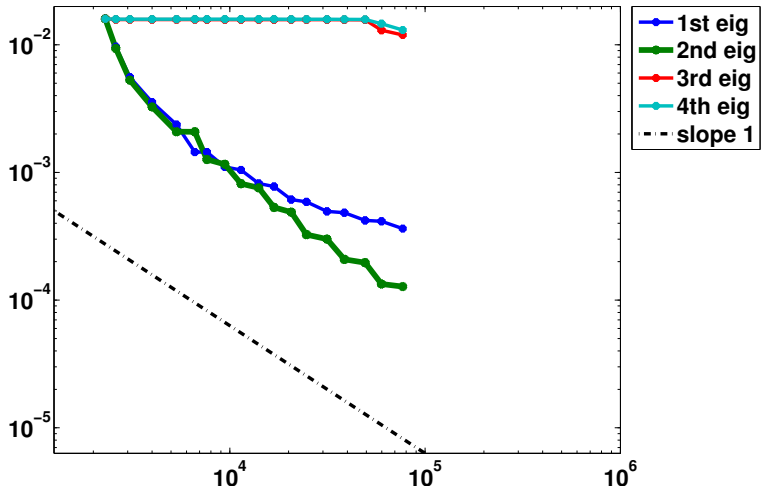
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=16



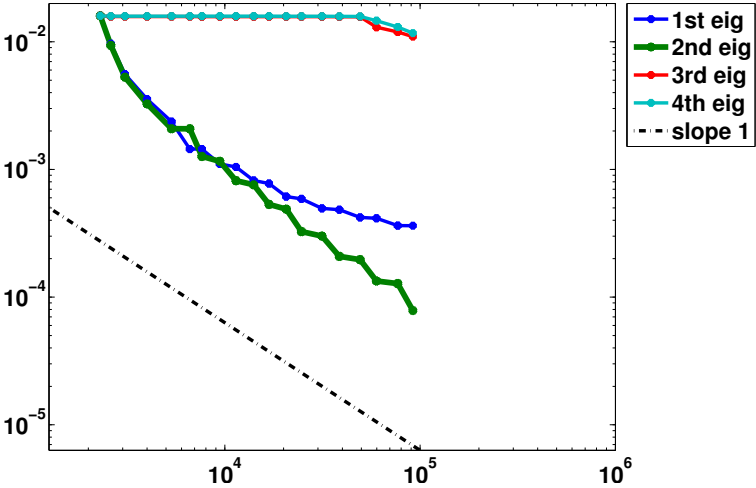
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=17



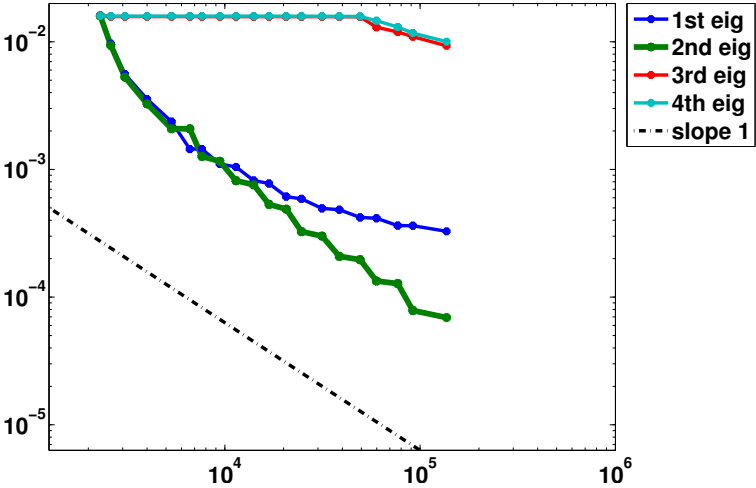
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=18



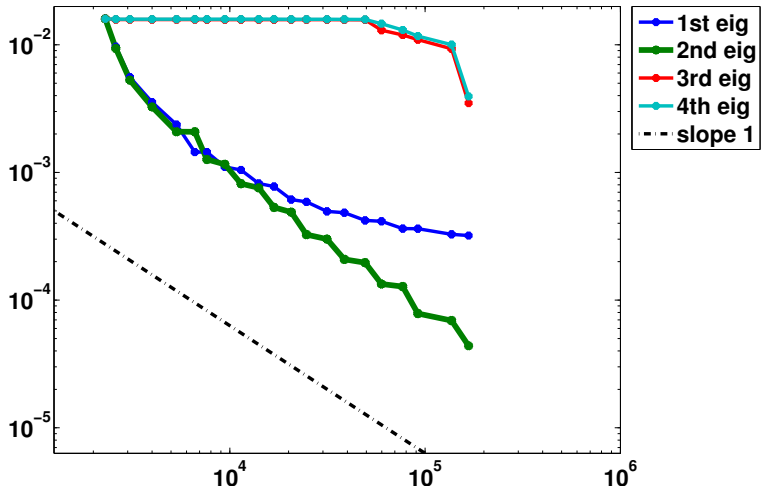
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=19



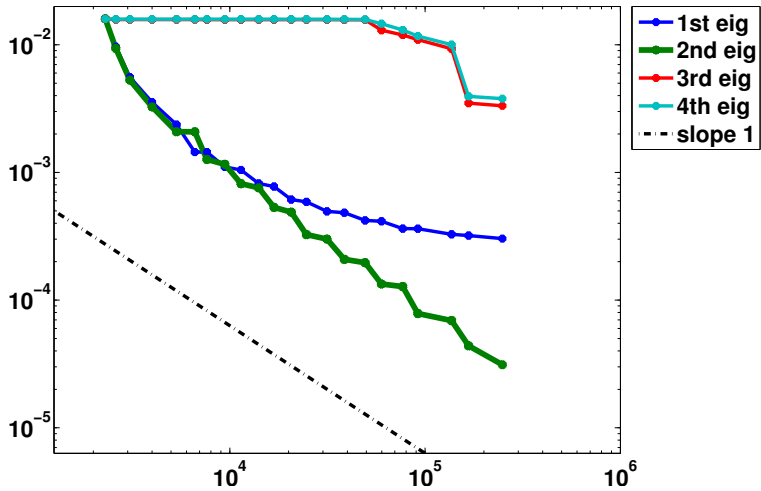
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=20



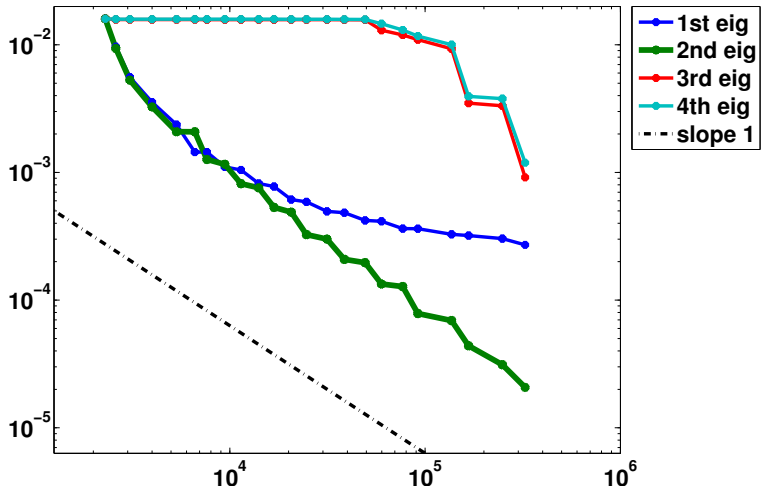
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=21



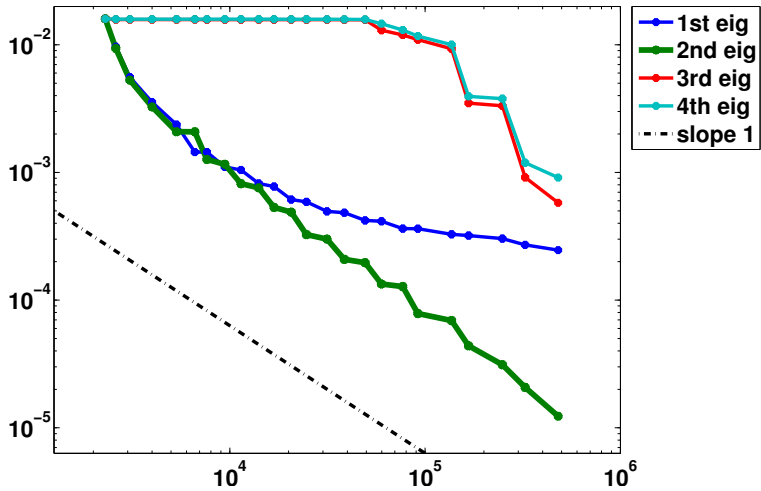
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=22



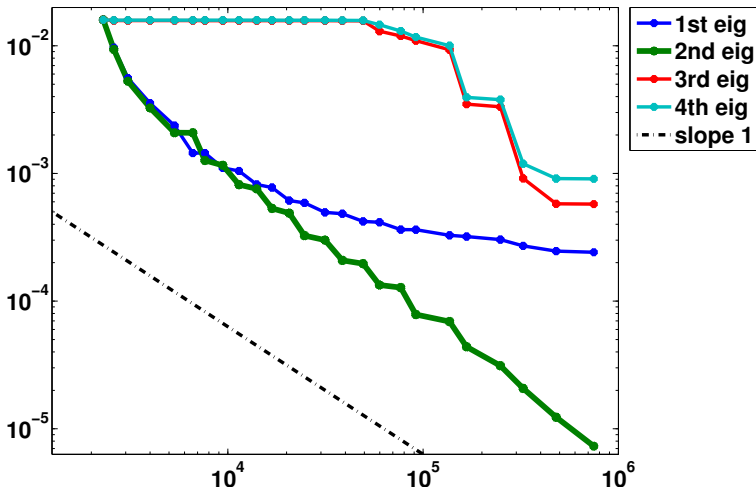
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=23



Approximation of the second frequency

Bulk parameter=0.3, Refinement level=24



Approximation of the third frequency

1D Laplace

2D Laplace

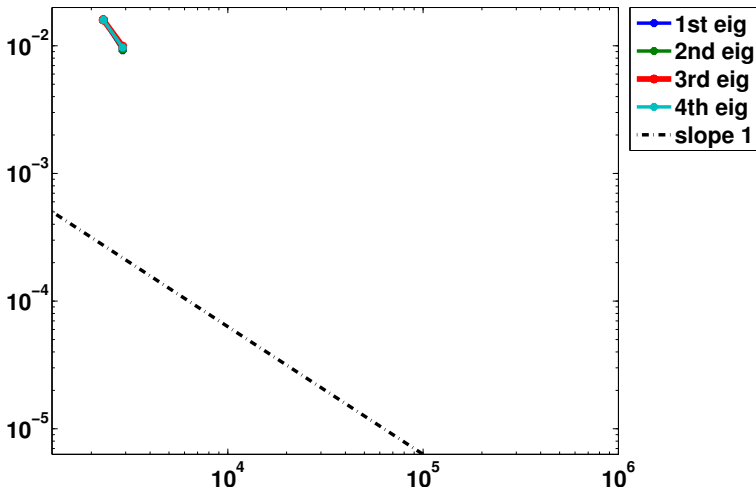
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=1



Approximation of the third frequency

1D Laplace

2D Laplace

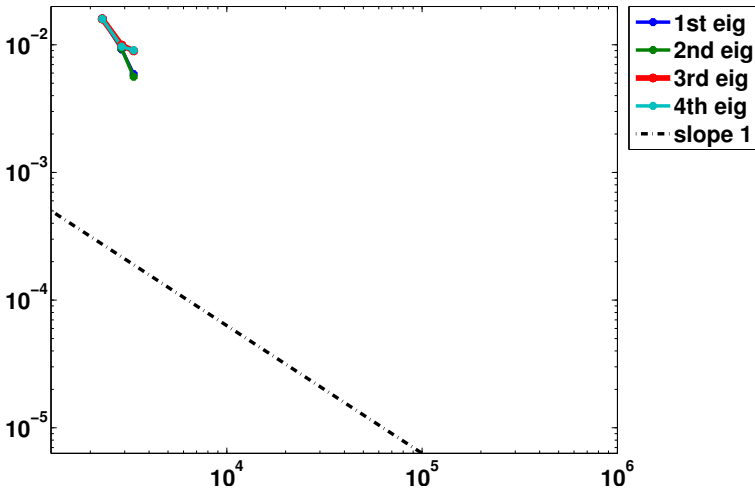
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=2



Approximation of the third frequency

1D Laplace

2D Laplace

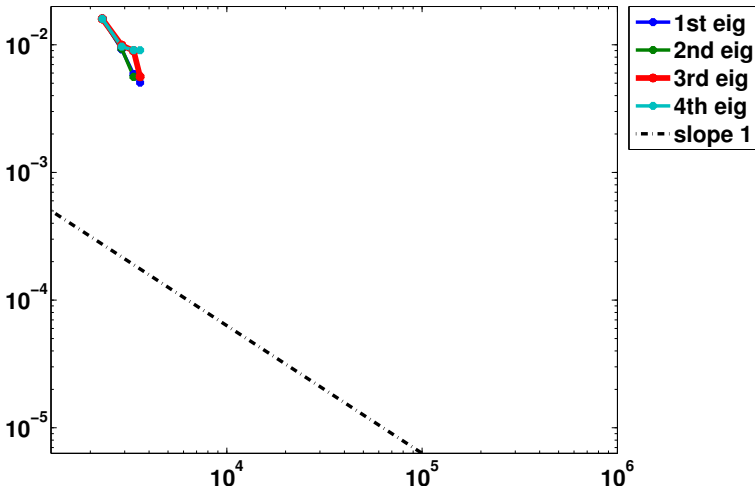
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

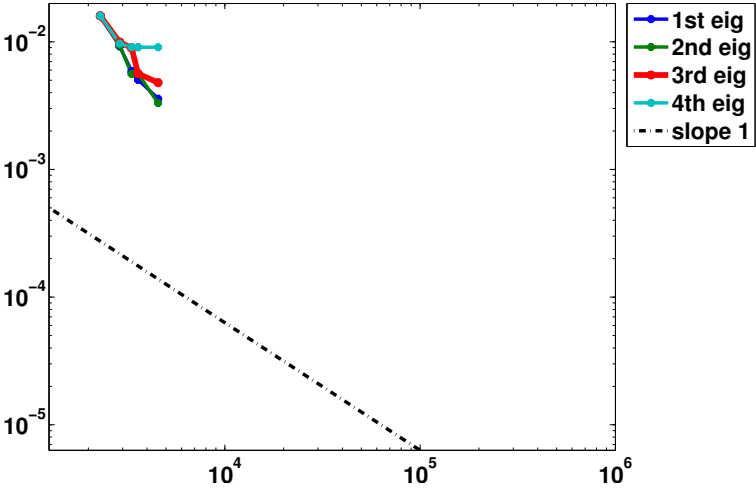
A posteriori
analysis

Bulk parameter=0.3, Refinement level=3



Approximation of the third frequency

Bulk parameter=0.3, Refinement level=4



Approximation of the third frequency

1D Laplace

2D Laplace

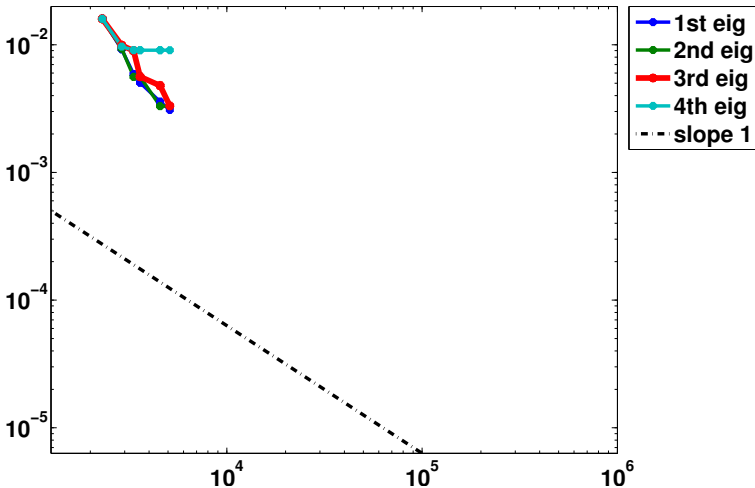
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

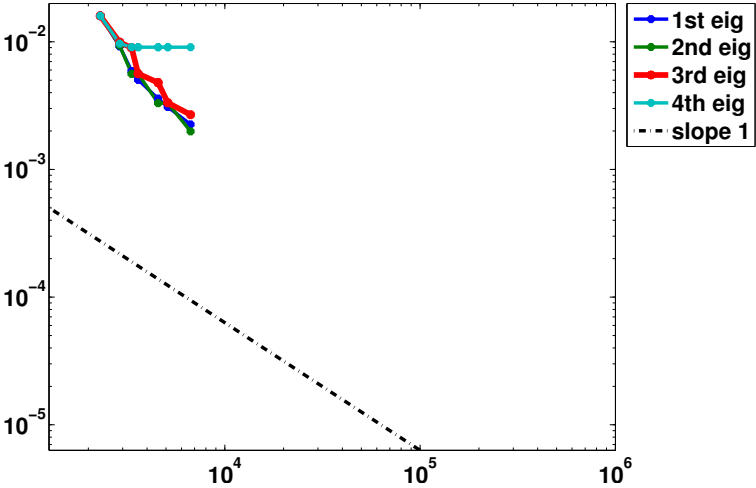
A posteriori
analysis

Bulk parameter=0.3, Refinement level=5



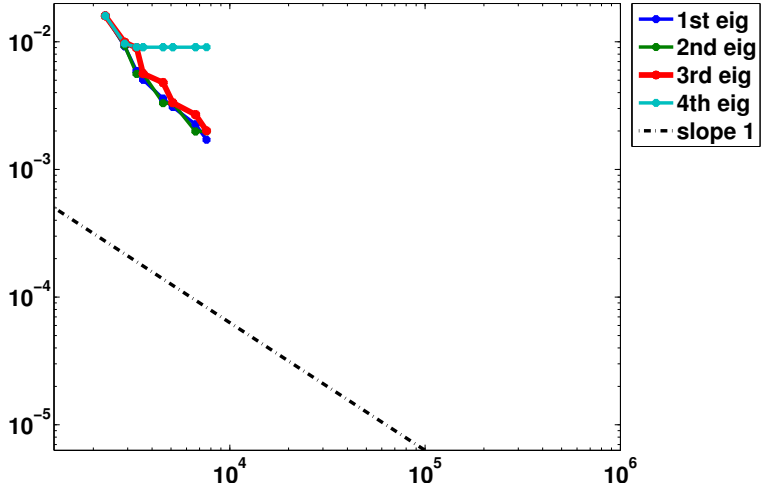
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=6



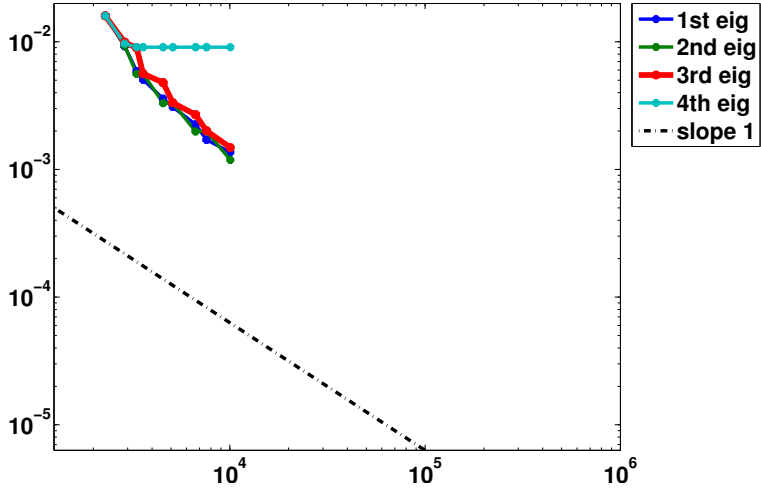
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=7



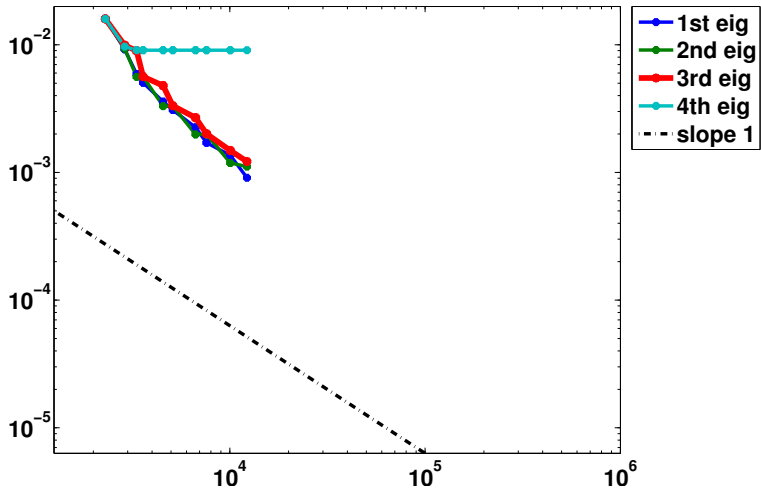
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=8



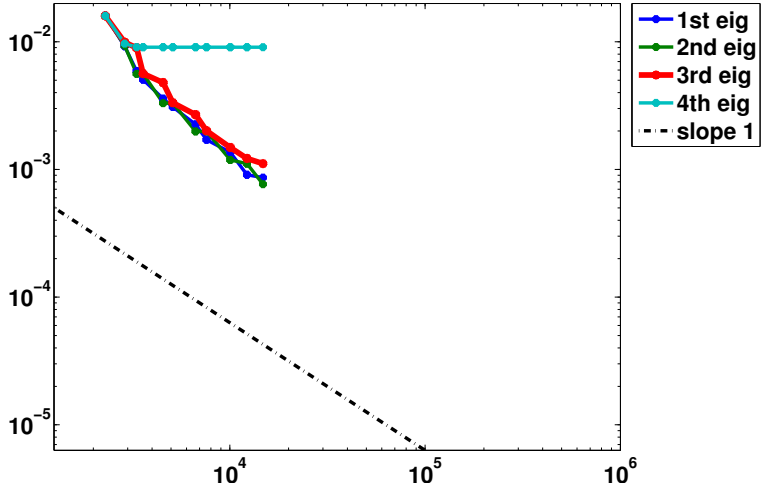
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=9



Approximation of the third frequency

Bulk parameter=0.3, Refinement level=10



Approximation of the third frequency

1D Laplace

2D Laplace

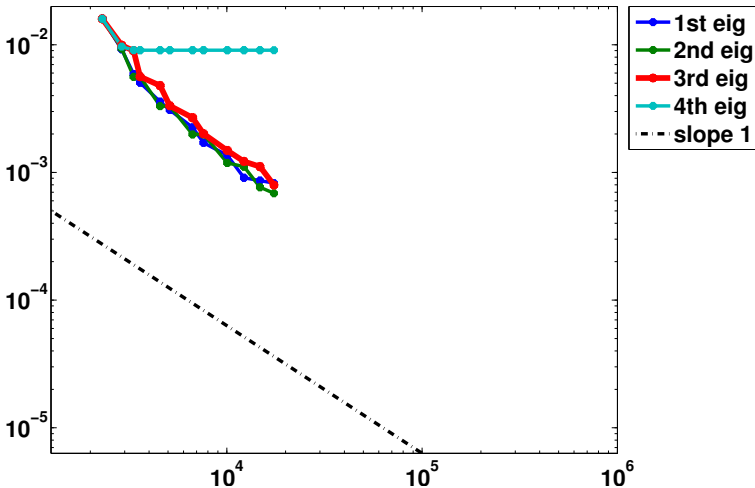
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=11



Approximation of the third frequency

1D Laplace

2D Laplace

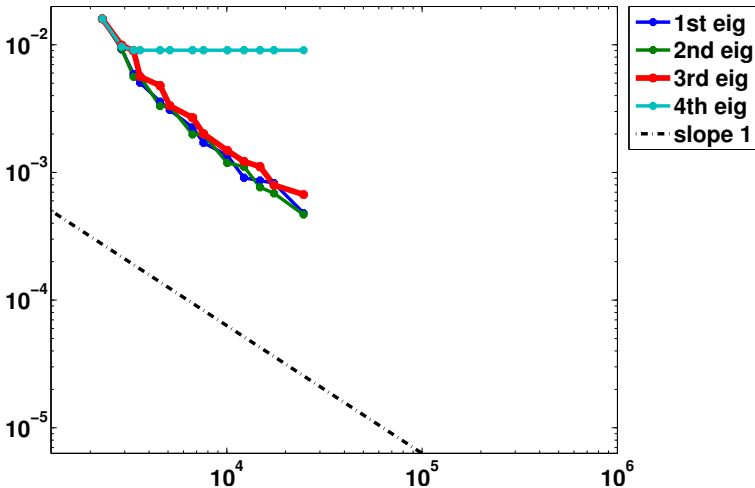
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

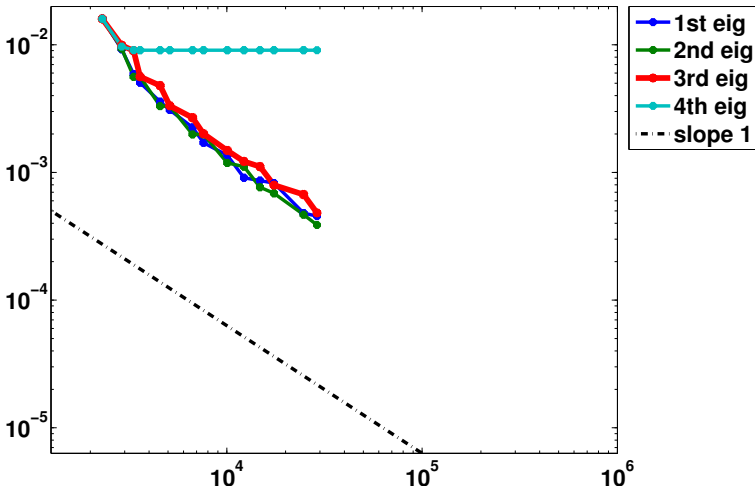
A posteriori
analysis

Bulk parameter=0.3, Refinement level=12



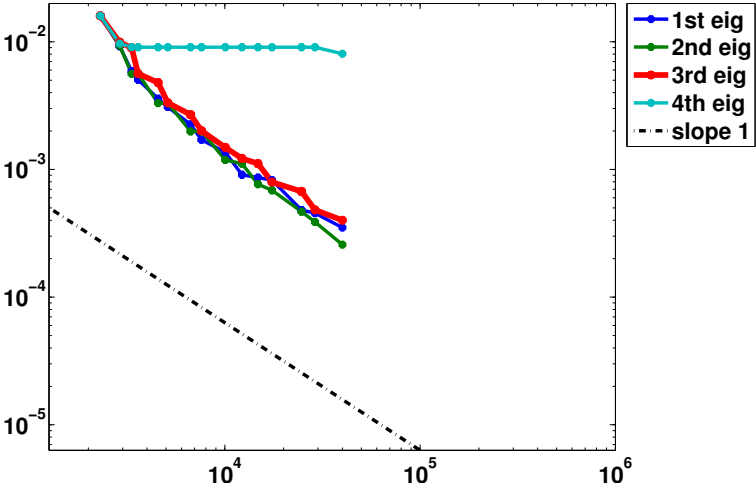
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=13



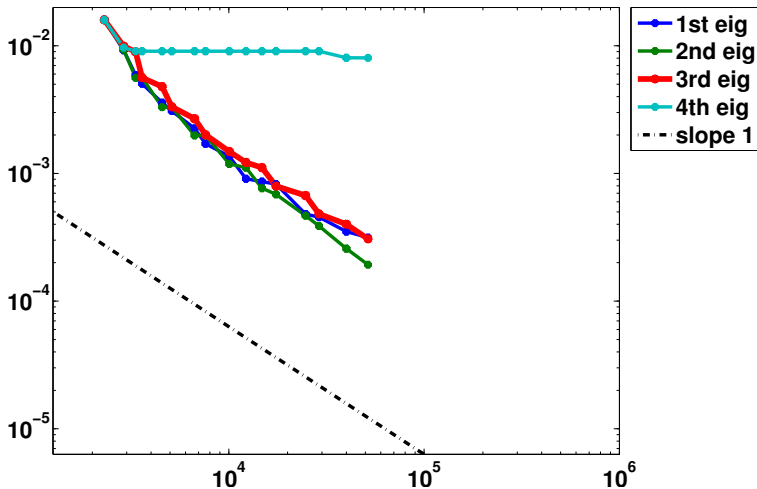
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=14



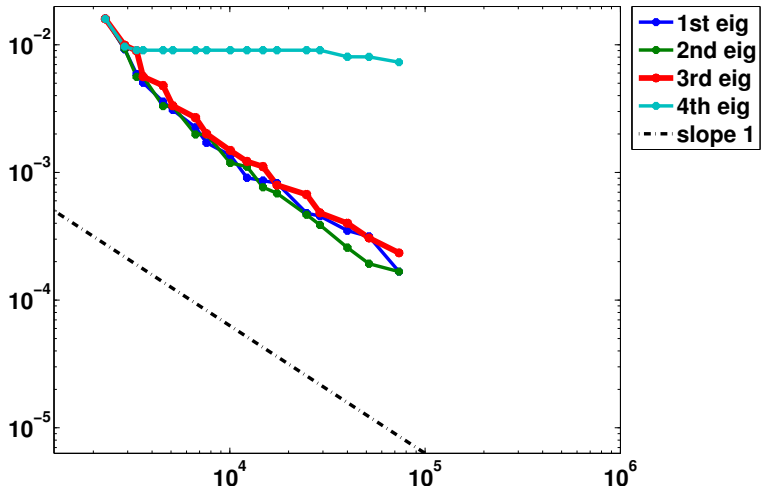
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=15



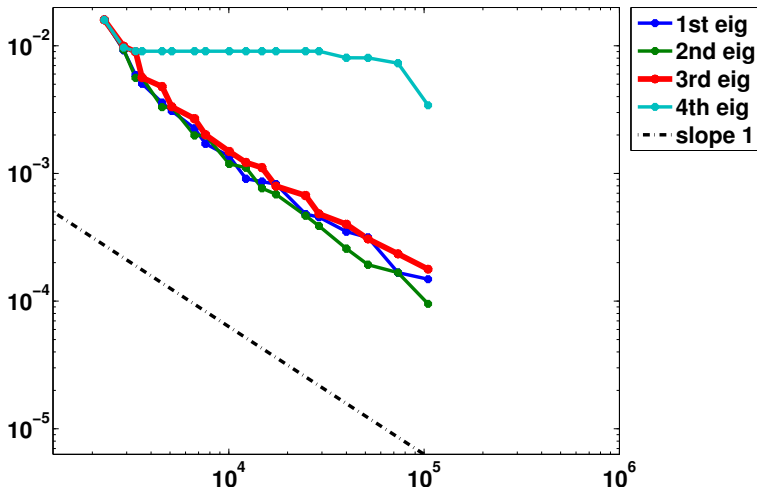
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=16



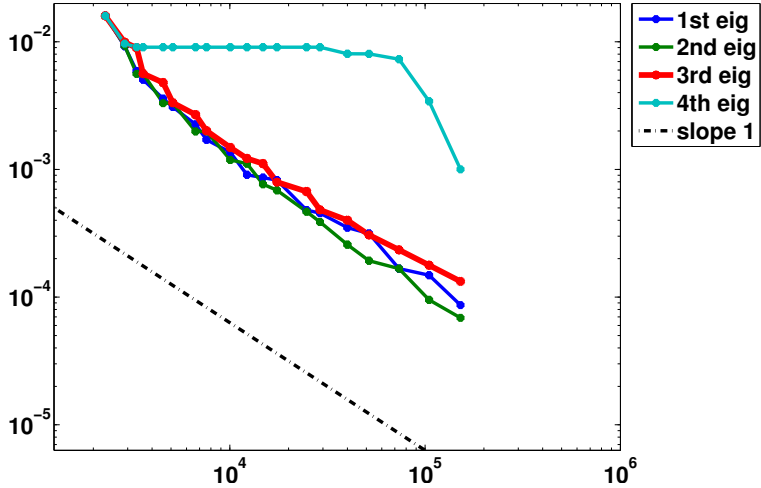
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=17



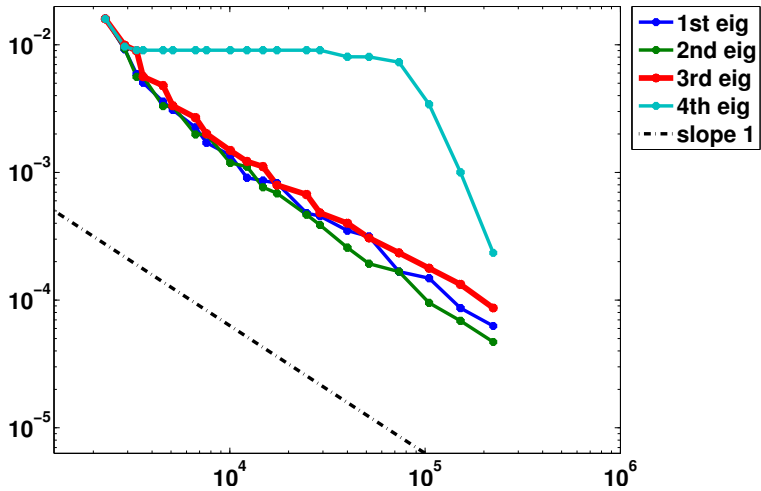
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=18



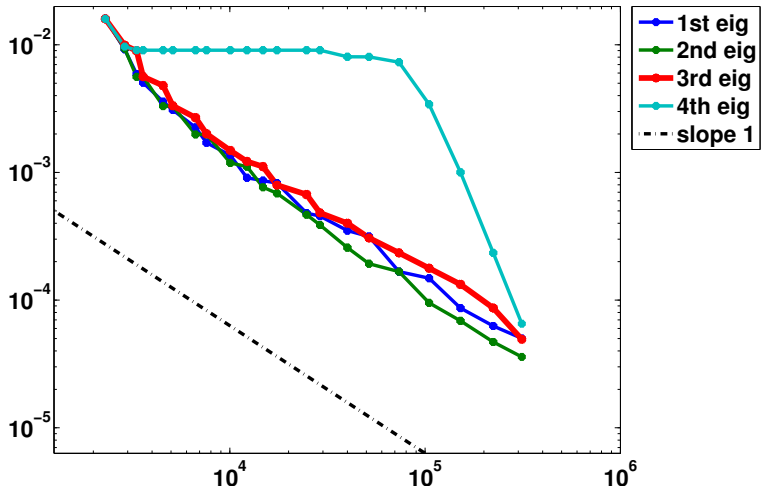
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=19



Approximation of the third frequency

Bulk parameter=0.3, Refinement level=20



Approximation of the third frequency

1D Laplace

2D Laplace

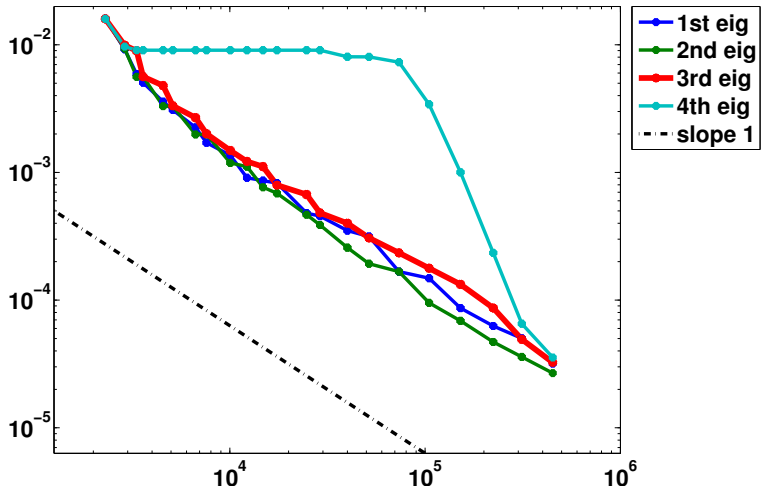
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=21



Approximation of the third frequency

1D Laplace

2D Laplace

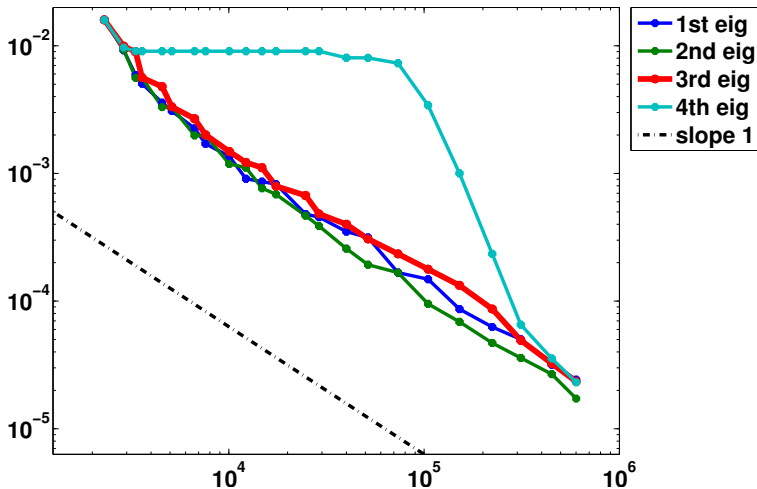
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=22



Approximation of the third frequency

1D Laplace

2D Laplace

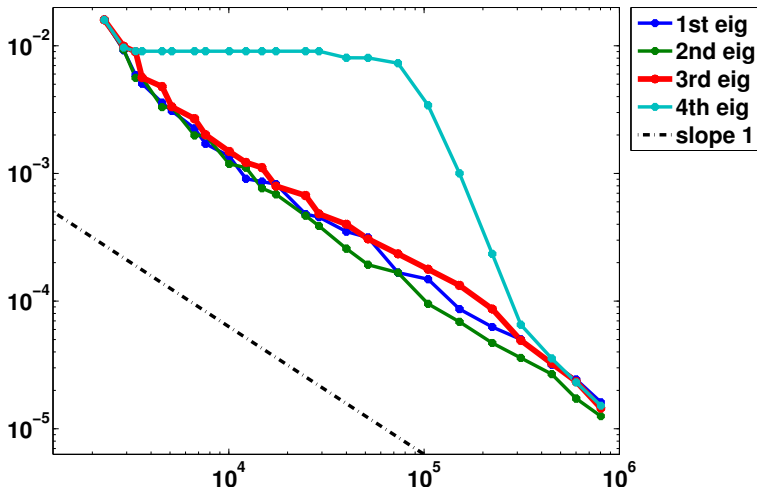
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=23



Approximation of the fourth frequency

1D Laplace

2D Laplace

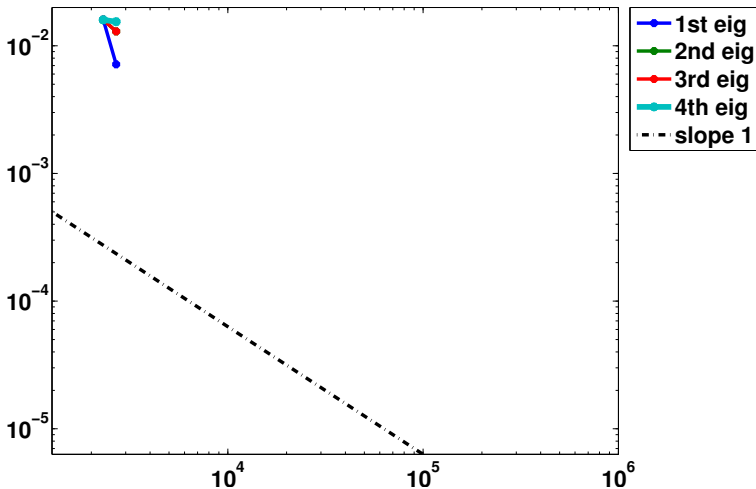
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

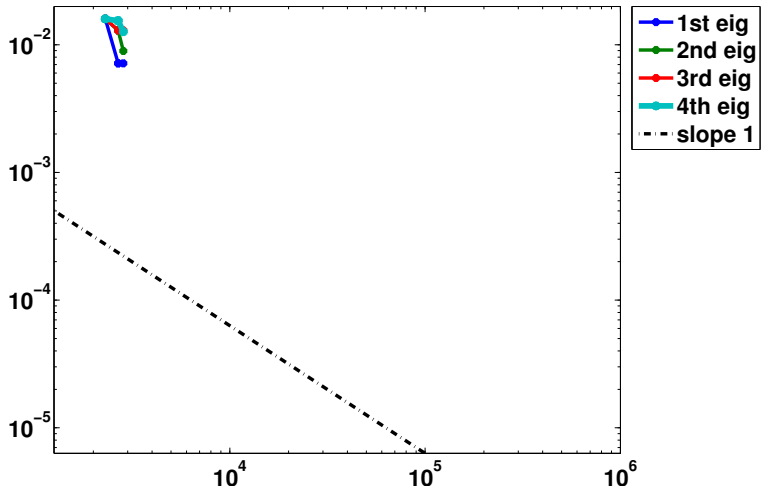
A posteriori
analysis

Bulk parameter=0.3, Refinement level=1



Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=2



Approximation of the fourth frequency

1D Laplace

2D Laplace

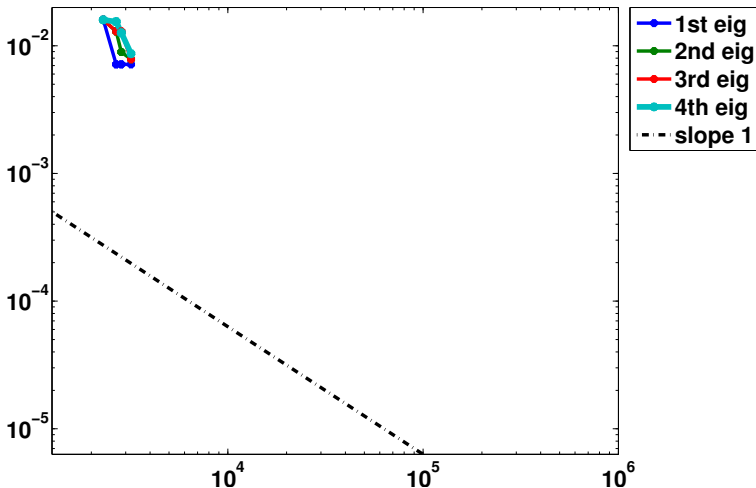
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=3



Approximation of the fourth frequency

1D Laplace

2D Laplace

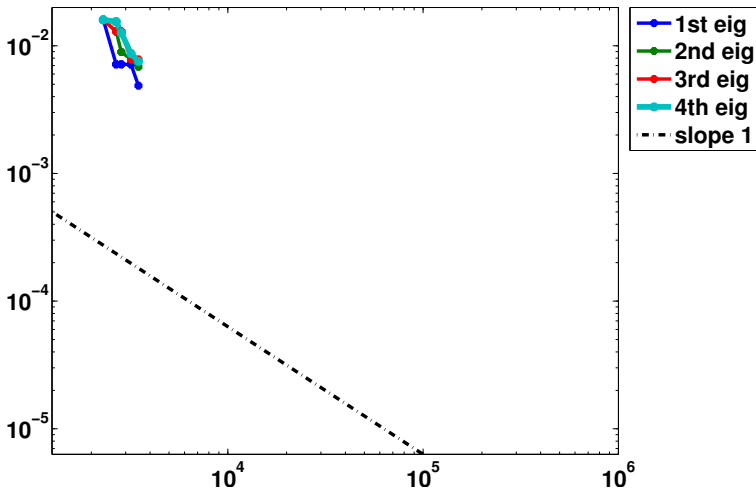
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

A posteriori
analysis

Bulk parameter=0.3, Refinement level=4



Approximation of the fourth frequency

1D Laplace

2D Laplace

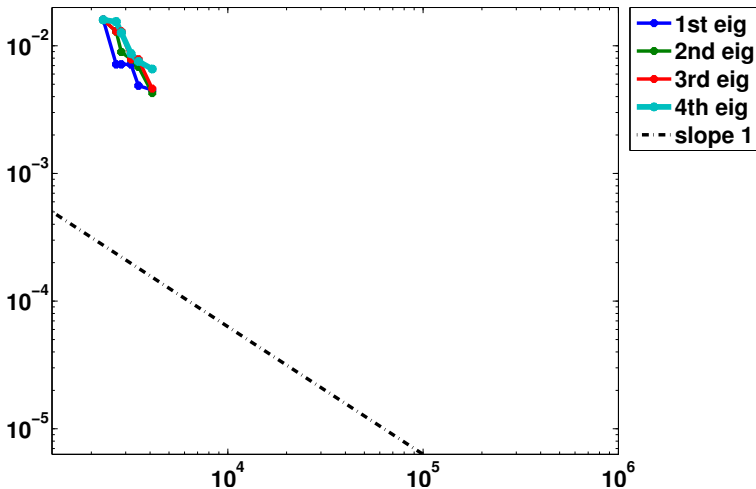
1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalues

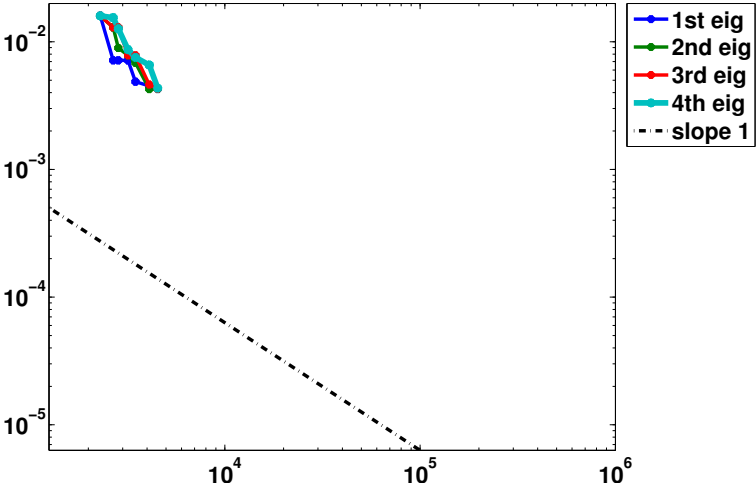
A posteriori
analysis

Bulk parameter=0.3, Refinement level=5



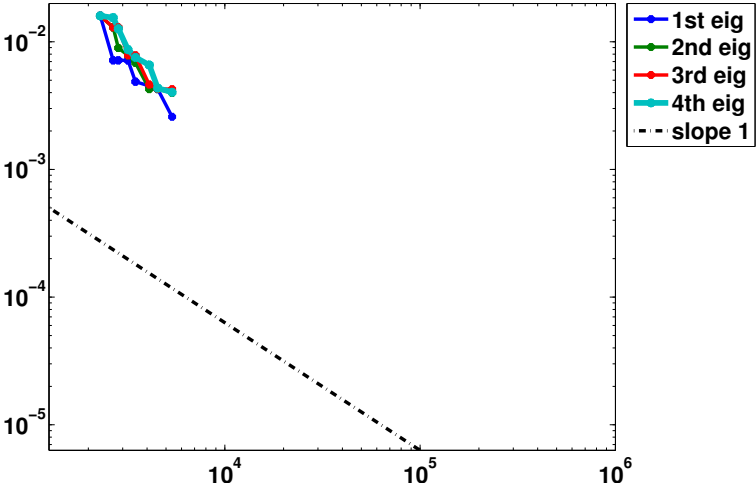
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=6



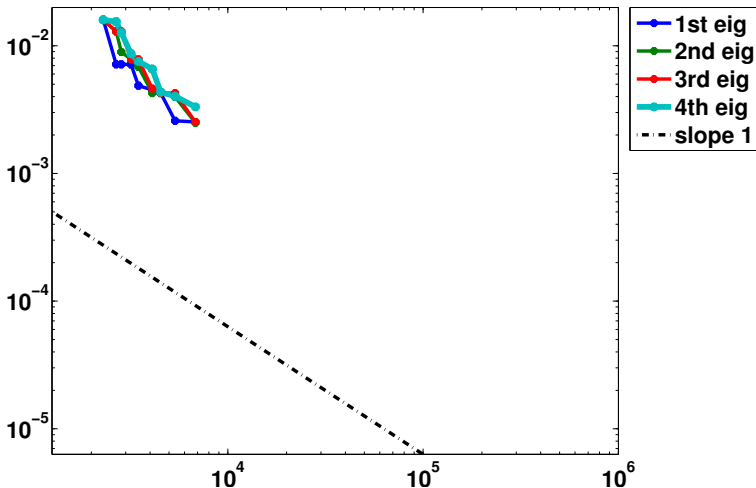
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=7



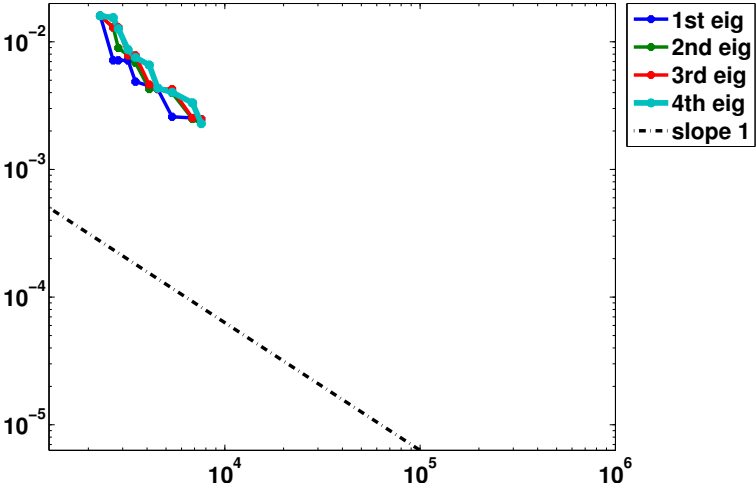
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=8



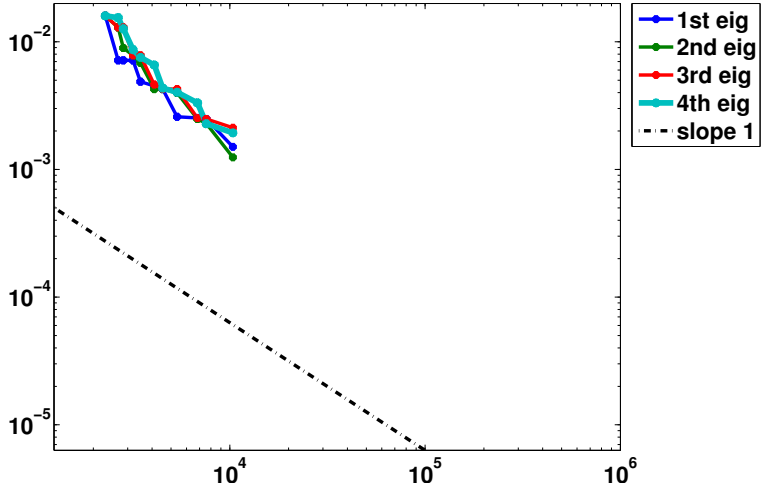
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=9



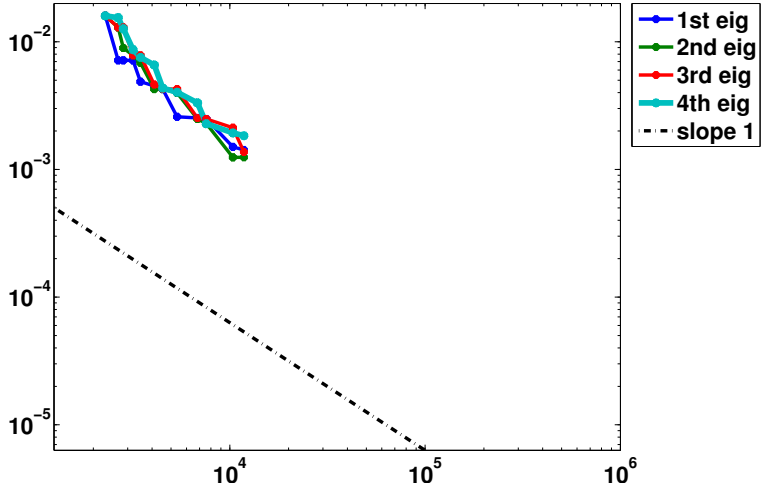
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=10



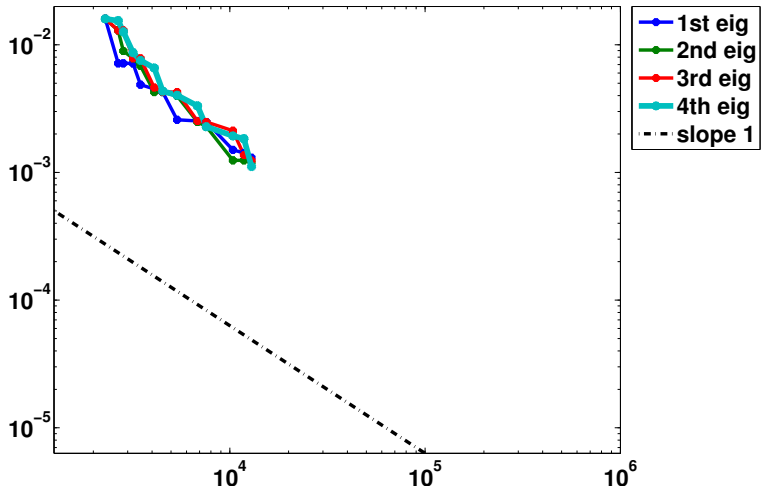
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=11



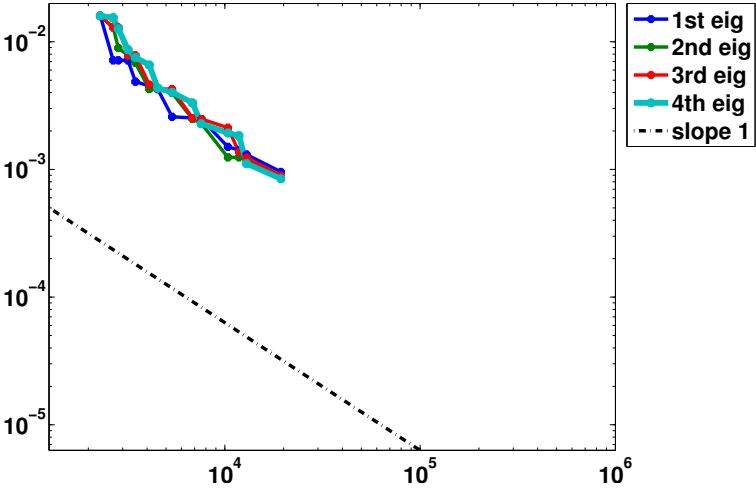
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=12



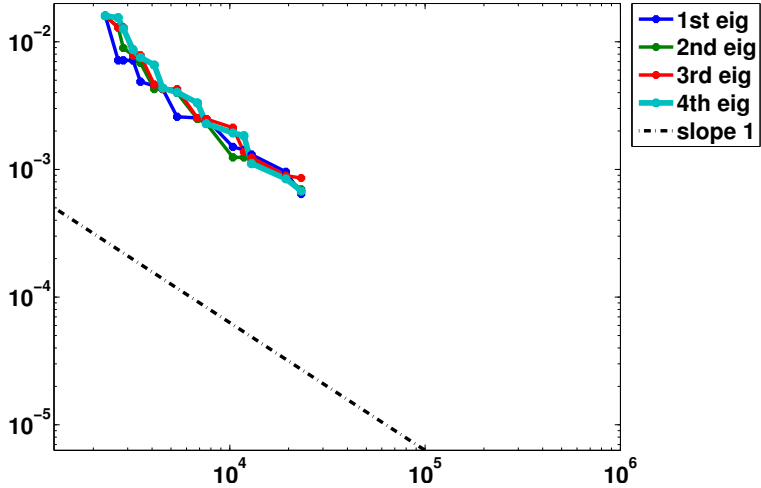
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=13



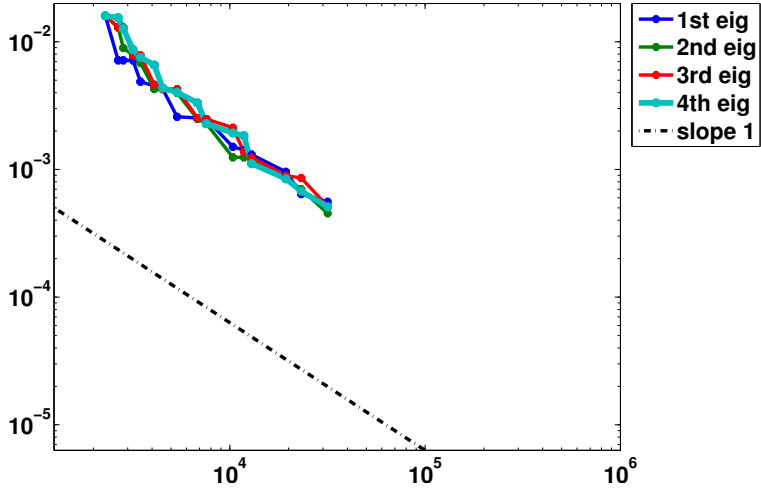
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=14



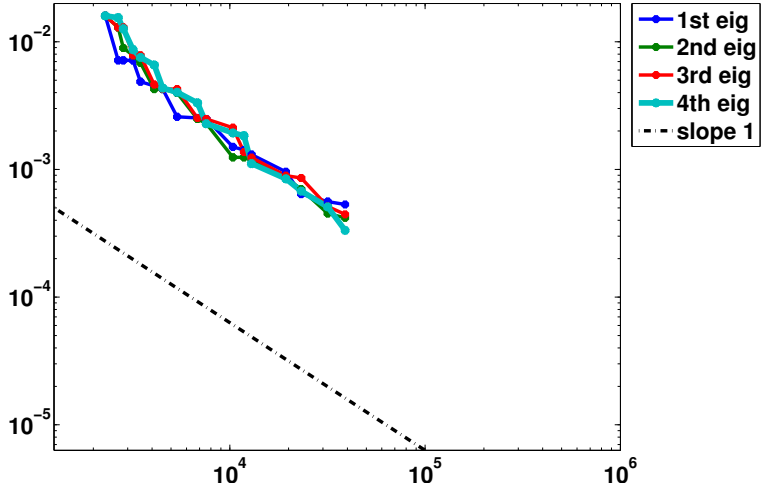
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=15



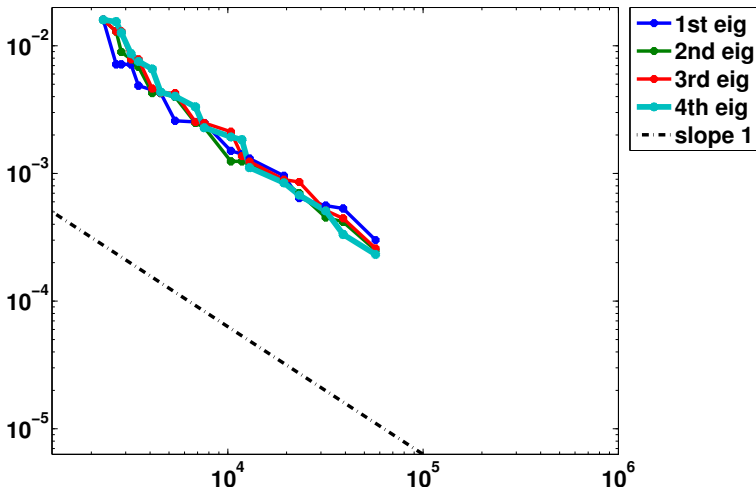
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=16



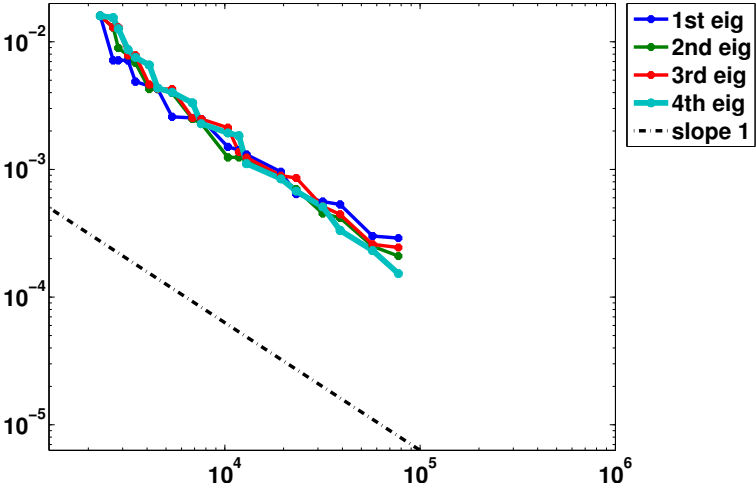
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=17



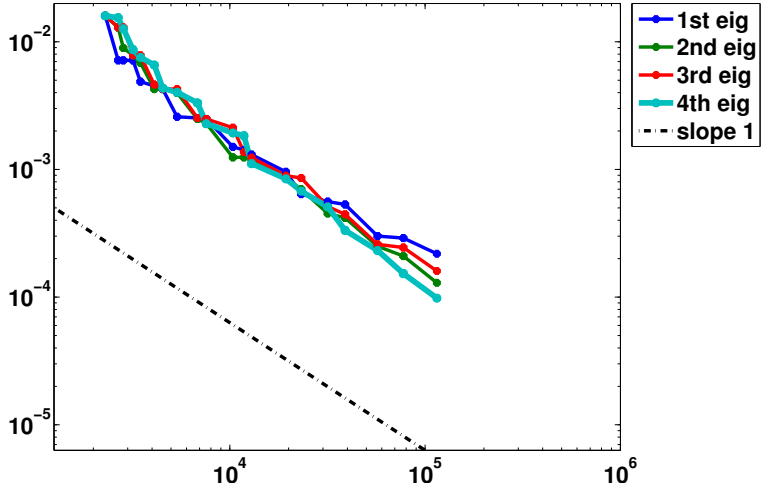
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=18



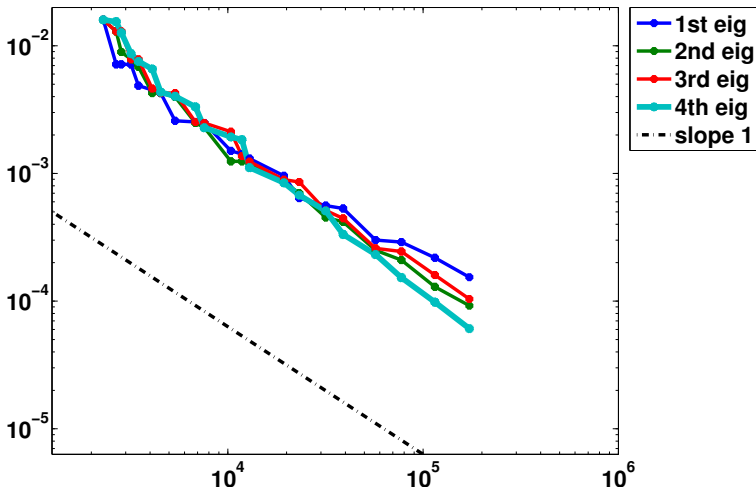
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=19



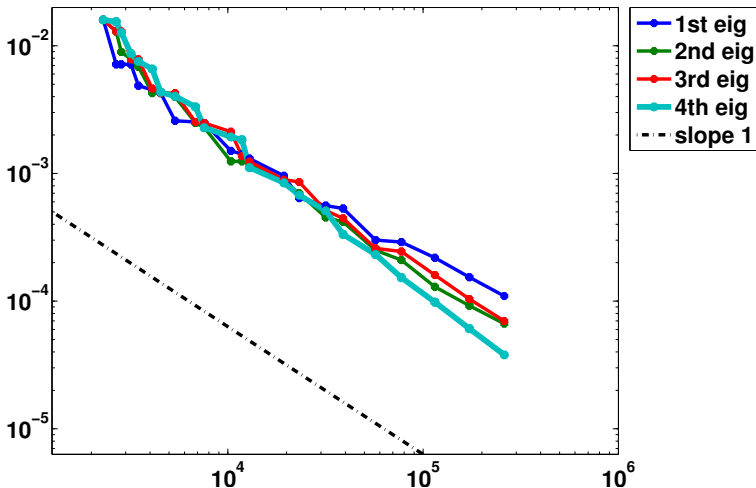
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=20



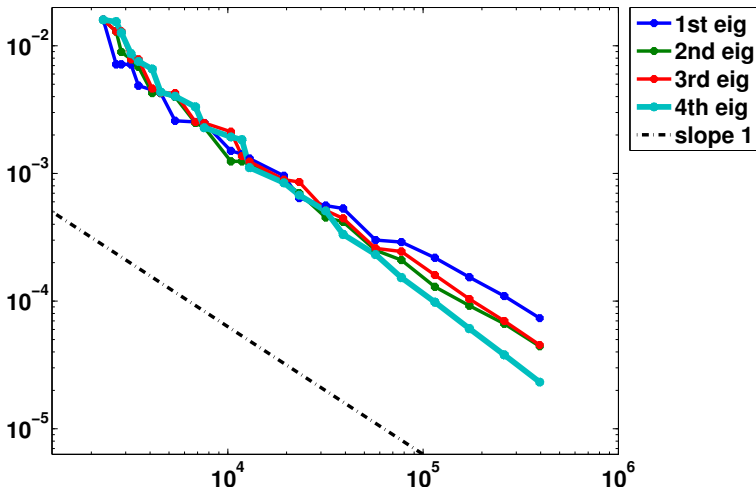
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=21



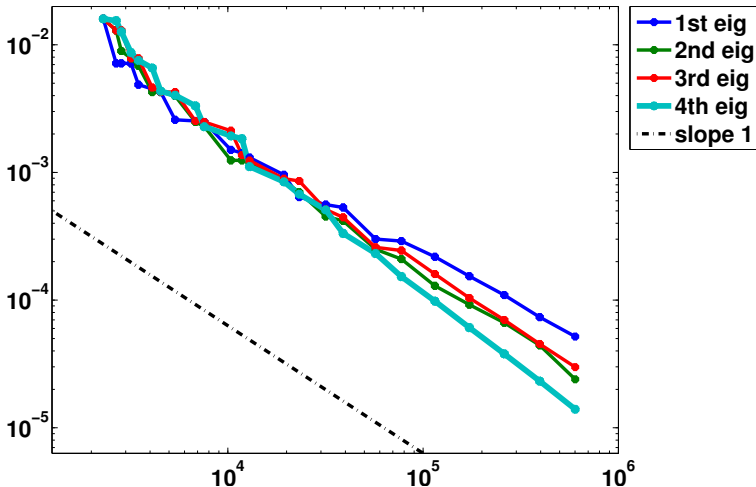
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=22



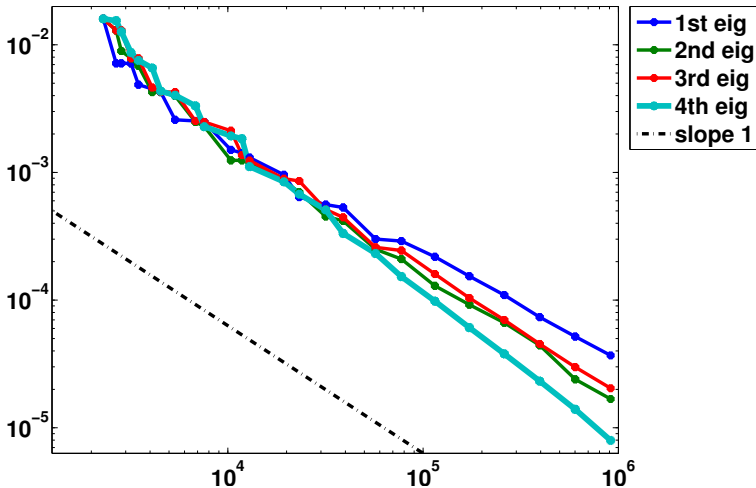
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=23



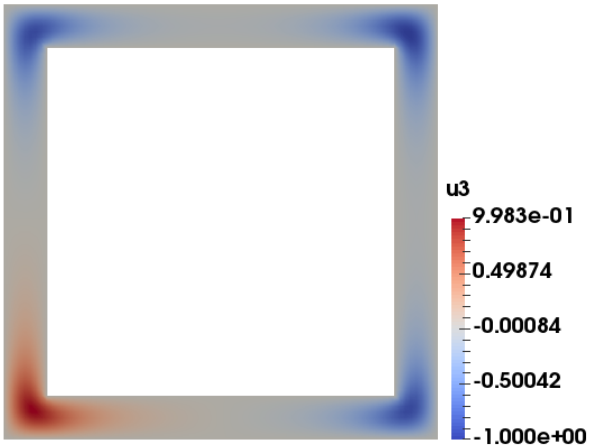
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=24



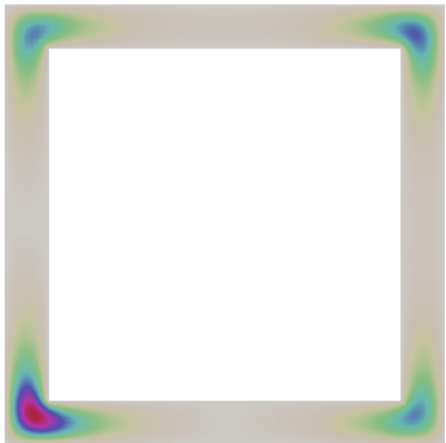
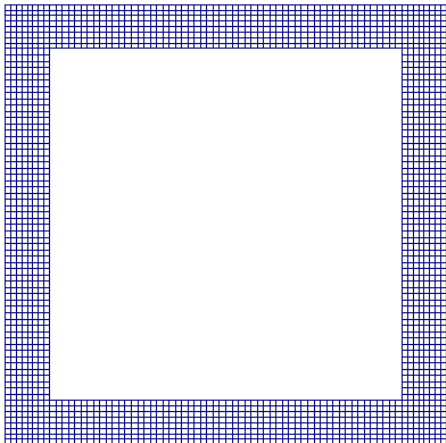
What is going on?

Let's see the mesh sequence and the computed eigenfunctions, for instance, in the case of the third frequency



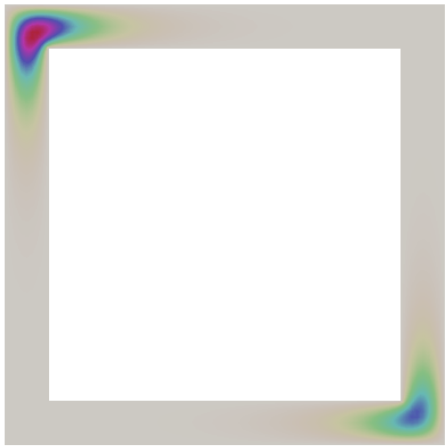
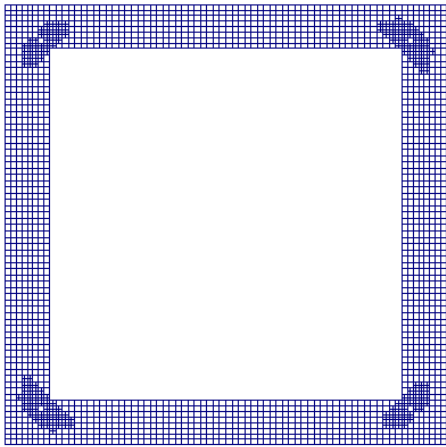
What is going on?

Initial mesh



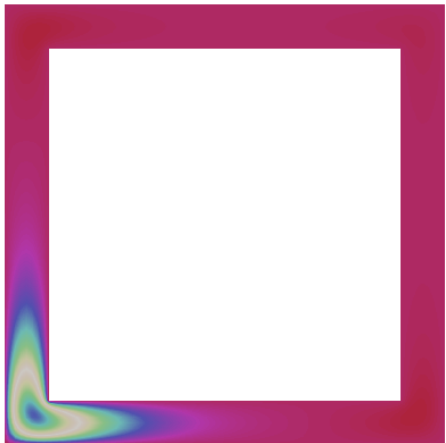
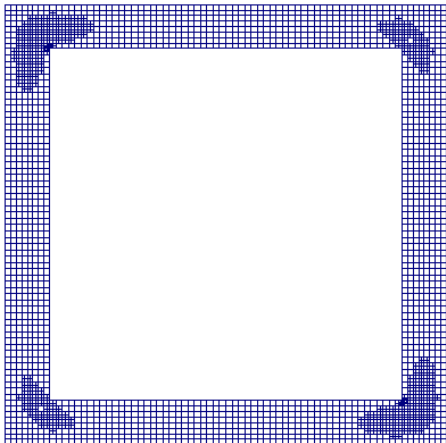
What is going on?

Refinement level=1



What is going on?

Refinement level=2



What is going on?

1D Laplace

2D Laplace

1D m

Laplace

2D m

Laplace

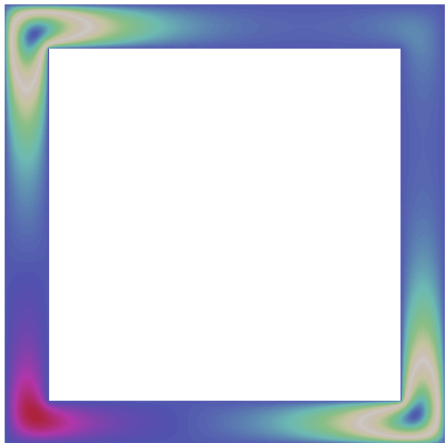
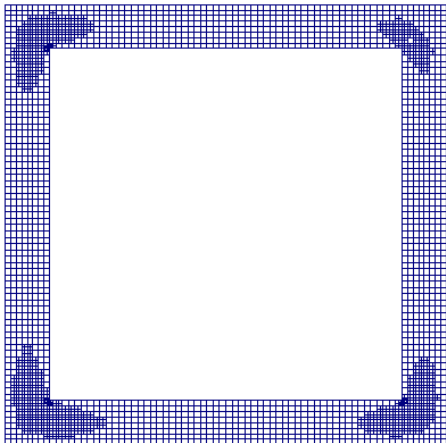
Multi

eigen

A po

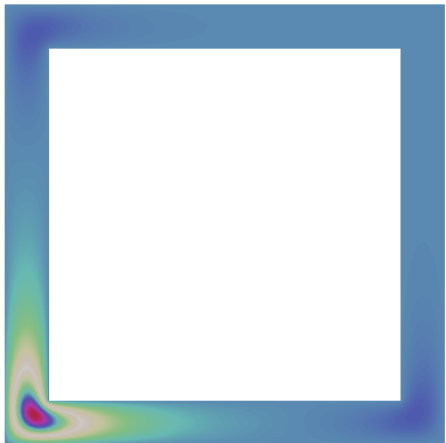
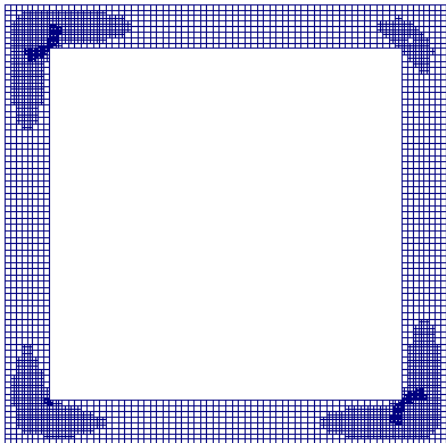
analy

Refinement level=3



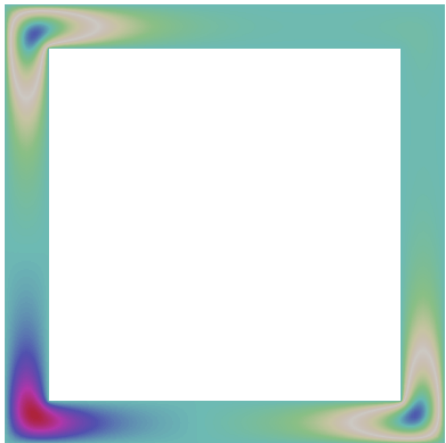
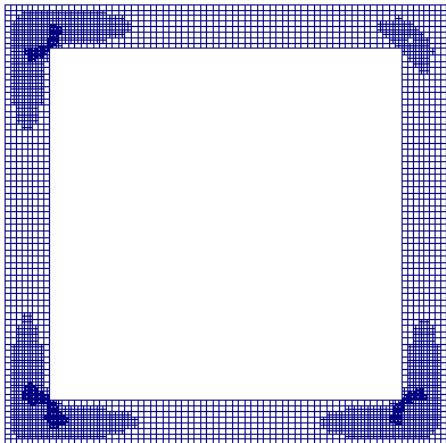
What is going on?

Refinement level=4



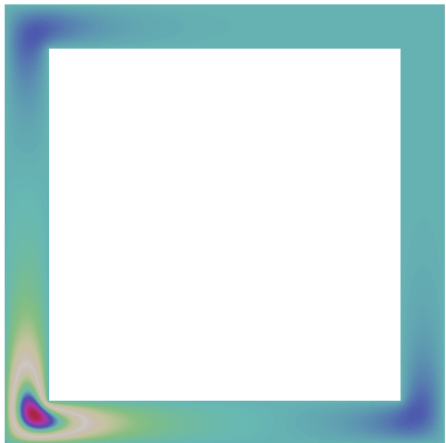
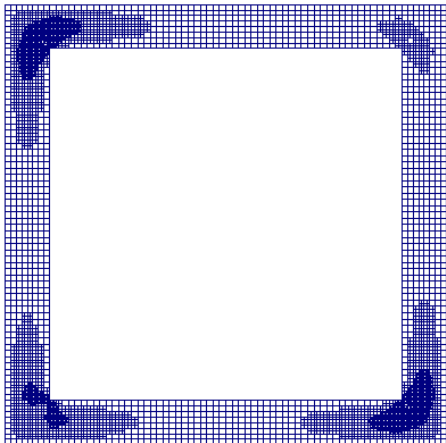
What is going on?

Refinement level=5



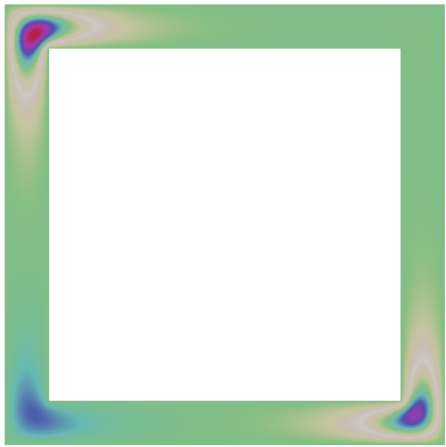
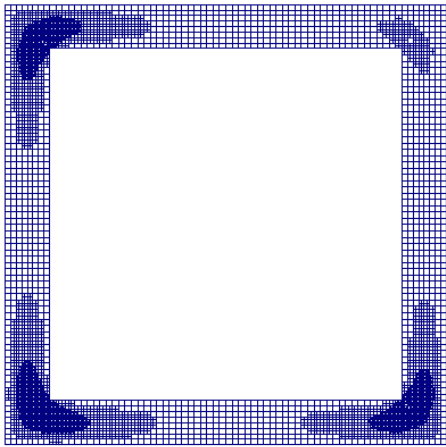
What is going on?

Refinement level=6



What is going on?

Refinement level=7



What is going on?

1D Laplace

2D Laplace

1D m

Laplace

2D m

Laplace

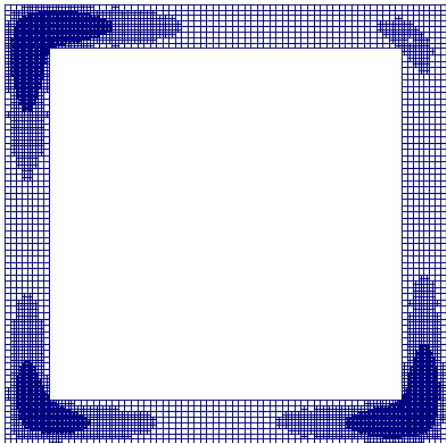
Multi

eigen

A po

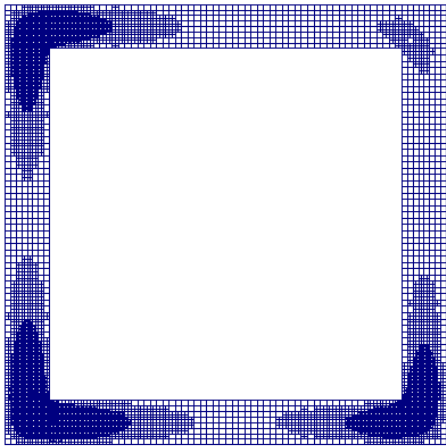
analy

Refinement level=8



What is going on?

Refinement level=9



What is going on?

1D Laplace

2D Laplace

1D m

Laplace

2D m

Laplace

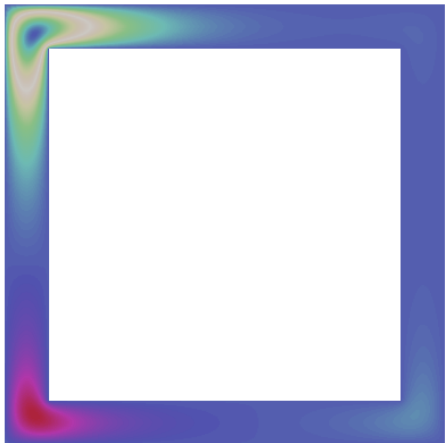
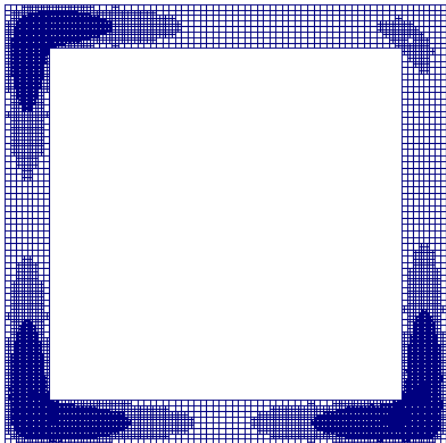
Multi

eigen

A po

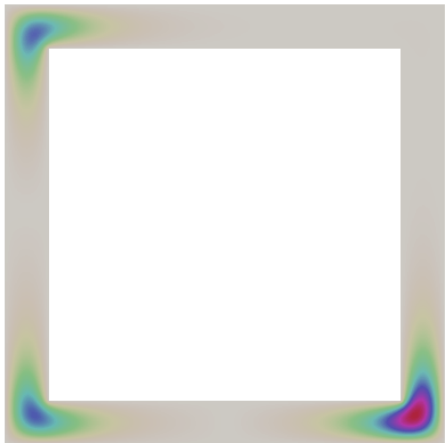
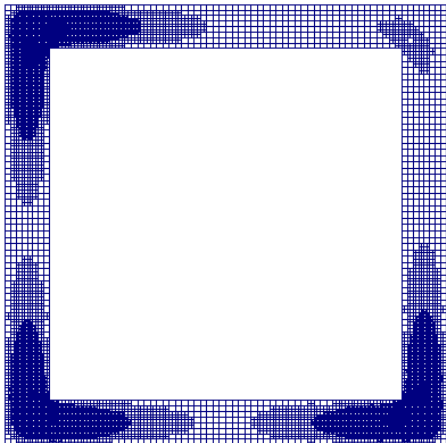
analy

Refinement level=10



What is going on?

Refinement level=11



What is going on?

1D Laplace

2D Laplace

1D m

Laplace

2D m

Laplace

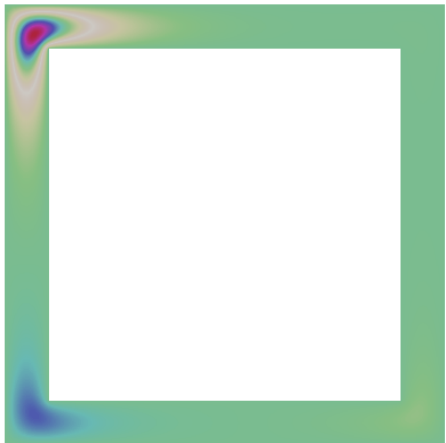
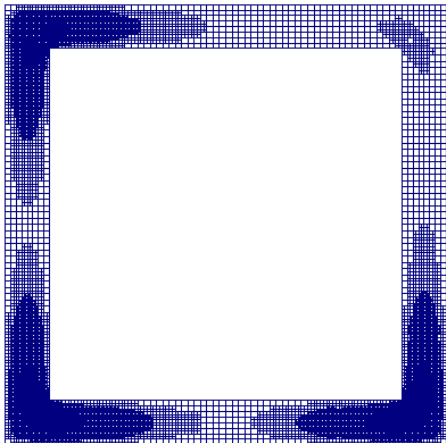
Multi

eigen

A po

analy

Refinement level=12



What is going on?

1D Laplace

2D Laplace

1D m

Laplace

2D m

Laplace

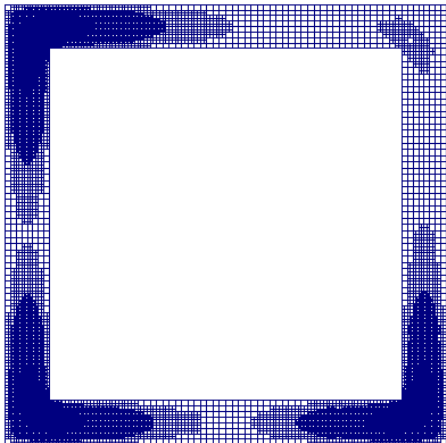
Multi

eigen

A po

analy

Refinement level=13



What is going on?

1D Laplace

2D Laplace

1D m

Laplace

2D m

Laplace

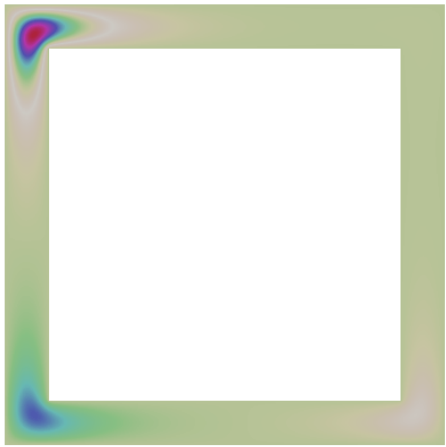
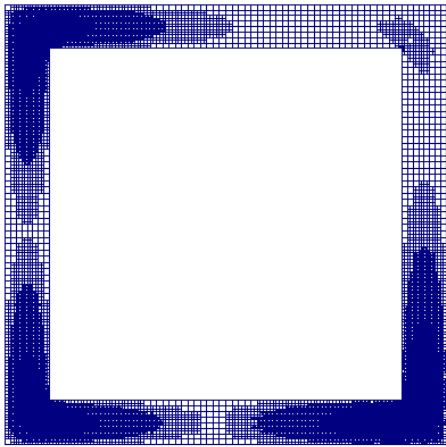
Multi

eigen

A po

analy

Refinement level=14



What is going on?

1D Laplace

2D Laplace

1D m

Laplace

2D m

Laplace

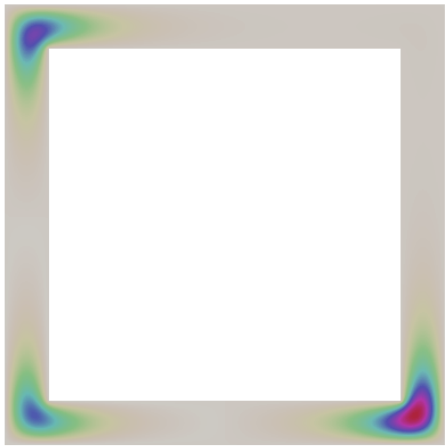
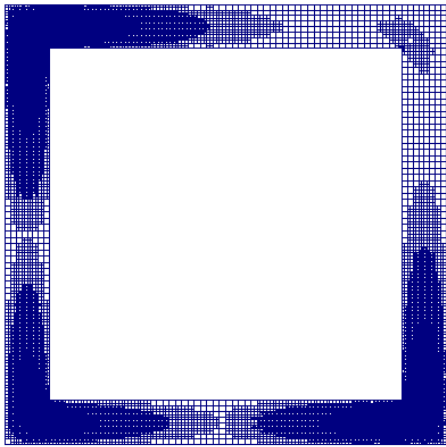
Multi

eigen

A po

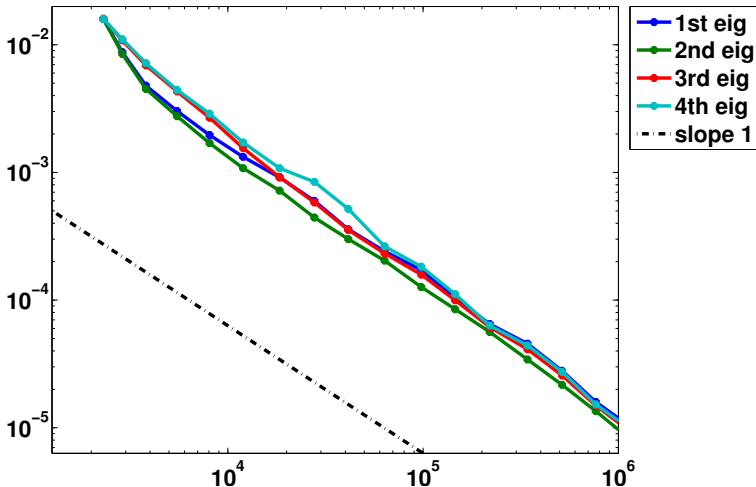
analy

Refinement level=15



Approximation of the first four frequencies altogether

Bulk parameter=0.3, Refinement level=16



Convergence rate vs. computational cost

1D Laplace

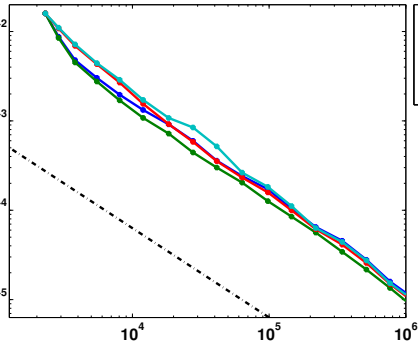
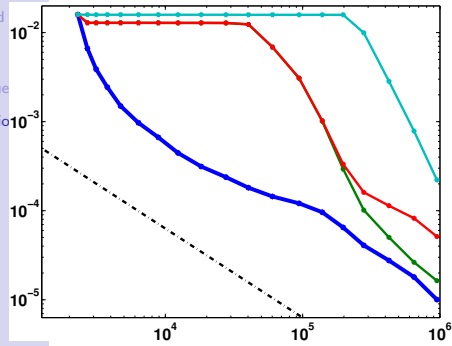
2D Laplace

1D mixed
Laplace

2D mixed
Laplace

Multiple
eigenvalue

A posteriori
analysis



Influence of the bulk parameter

1D Lapl
2D Lapl
1D mixe
Laplace
2D mixe
Laplace
Multiple
eigenval
A poster
analysis

