Adaptive finite element approximation of mixed eigenvalue problems

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Based on a joint work with
D. Gallistl, F. Gardini, and L. Gastaldi

Laplace eigenvalue problem in mixed form

Find $\lambda \in \mathbb{R}$ and $u \in L^2(\Omega)$ with $u \neq 0$ such that for $\boldsymbol{\sigma} \in H(\mathrm{div};\Omega)$

$$\begin{cases} \displaystyle\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, d\mathbf{x} + \int_\Omega u \operatorname{div} \boldsymbol{\tau}\, d\mathbf{x} = 0 & \forall \boldsymbol{\tau} \in H(\mathrm{div};\Omega) \\ \displaystyle\int_\Omega v \operatorname{div} \boldsymbol{\sigma}\, d\mathbf{x} = -\lambda \int_\Omega uv\, d\mathbf{x} & \forall v \in L^2(\Omega) \end{cases}$$

$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\tau}\, d\mathbf{x}$
$b(\boldsymbol{\tau}, v) = \int_\Omega v \operatorname{div} \boldsymbol{\tau}\, d\mathbf{x}$

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Finite element approximation

$\Sigma_h \subset H(\mathrm{div};\Omega)$ and $M_h \subset L^2(\Omega)$
Find $\lambda_h \in \mathbb{R}$ and $u_h \in M_h$ with $u_h \neq 0$ such that for $\boldsymbol{\sigma}_h \in \Sigma_h$

$$\begin{cases} \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau}\, d\mathbf{x} + \int_\Omega u_h \,\mathrm{div}\, \boldsymbol{\tau}\, d\mathbf{x} = 0 & \forall \boldsymbol{\tau} \in \Sigma_h \\ \int_\Omega v \,\mathrm{div}\, \boldsymbol{\sigma}_h\, d\mathbf{x} = -\lambda_h \int_\Omega u_h v\, d\mathbf{x} & \forall v \in M_h \end{cases}$$

Structure of this talk

Adaptive finite element method for Laplace eigenvalue problem in mixed form

Convergence of the adaptive scheme with optimal rate

Auxiliary results

Proofs of some auxiliary results

Clusters of eigenvalues

When multiple eigenvalues are present, a posteriori error indicators should be based simultaneously on all eigenfunctions belonging to the corresponding eigenspace

⟨Solin–Giani 2012⟩
⟨B.–Durán–Gardini–Gastaldi 2015⟩

It is now recognized that an adaptive scheme for the approximation of eigenvalue problems should be designed taking into account error indicators based on all eigenmodes belonging to clusters of eigenvalues

⟨Gallistl 2014⟩

A posteriori analysis for the mixed Laplacian

Let $\lambda_{h,j} \in \mathbb{R}$, $\sigma_{h,j} \in \Sigma_h$, $u_{h,j} \in M_h$ denote an eigensolution.

We consider the following error indicator

$$\eta_{h,j}(T)^2 = \|h_T(\sigma_{h,j} - \nabla u_{h,j})\|_T^2 \qquad \text{1}^{\text{st}} \text{ equation } \text{“}\sigma = \nabla u\text{”}$$

$$\left.\begin{aligned} &+ \|h_T \operatorname{curl} \sigma_{h,j}\|_T^2 \\ &+ \sum_{E \in \mathcal{E}(T)} h_E \|[\sigma_{h,j}]_E \cdot t_E\|_E^2 \end{aligned}\right\} \text{2}^{\text{nd}} \text{ residual term}$$

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2^{nd} residual term

Take $\tau = \operatorname{curl} \varphi$

$$(\sigma - \sigma_h, \tau) = -(\sigma_h, \operatorname{curl} \varphi) = -\sum_T \left\{ (\operatorname{curl} \sigma_h, \varphi)_T + \int_{\partial T} (\sigma_h \cdot t) \varphi \, ds \right\}$$

Some references for mixed Laplacian

Source problem

⟨Alonso 1996⟩: 2^{nd} residual term

⟨Carstensen 1997⟩: adds term for 1st equation

⟨Chen–Holst–Xu 2008⟩: 2^{nd} residual term, convergence of the adaptive scheme

⟨Braess–Verfürth 1996, Wohlmuth–Hoppe 1999, Gatica–Maischak 2005, Lovadina–Stenberg 2006, Larson–Målqvist 2008, Huang–Xu 2012⟩

Eigenvalue problem

⟨Durán–Gastaldi–Padra 1999⟩: 2^{nd} residual term, equivalence with non-conforming scheme

⟨Gardini 2009⟩: 2^{nd} residual term, superconvergence arising from equivalence with non-conforming scheme

Adaptive FEM

$\boxed{\text{Input}}$

Parameter $\theta \in (0,1]$ and initial triangulation $\mathcal{T}_0$

$\boxed{\text{Output}}$

Sequence of meshes $\{\mathcal{T}_\ell\}$, sol.'s $\{(\lambda_\ell, \sigma_\ell, u_\ell)\}$, indicators $\{\eta_\ell(\mathcal{T}_\ell)\}$

Adaptive FEM

Input

Parameter $\theta \in (0,1]$ and initial triangulation $\mathcal{T}_0$

SOLVE, ESTIMATE, MARK, REFINE

Solve:	Compute discrete solution $(\lambda_\ell, \sigma_\ell, u_\ell)$ on $\mathcal{T}_\ell$
Estimate:	Compute local contributions of the error estimator $\{\eta_\ell^2(T)\}_{T\in\mathcal{T}_\ell}$
Mark:	Choose minimal subset $\mathcal{M}_\ell \subset \mathcal{T}_\ell$ such that $\theta\eta_\ell^2(\mathcal{T}_\ell) \le \eta_\ell^2(\mathcal{M}_\ell) \quad (0 < \theta \le 1)$
Refine:	Generate new triangulation as the smallest refinement of $\mathcal{T}_\ell$ satisfying $\mathcal{M}_\ell \cap \mathcal{T}_{\ell+1} = \emptyset$

Output

Sequence of meshes $\{\mathcal{T}_\ell\}$, sol.'s $\{(\lambda_\ell, \sigma_\ell, u_\ell)\}$, indicators $\{\eta_\ell(\mathcal{T}_\ell)\}$

AFEM for clusters of eigenvalues

Cluster of length N

$\lambda_{n+1}, \ldots, \lambda_{n+\mathsf{N}}$

$J = \{n+1, \ldots, n+\mathsf{N}\}$

Corresponding combination of eigenspaces

$W = \operatorname{span}\{u_j \mid j \in J\}$

$W_{\mathcal{T}_h} = W_h = \operatorname{span}\{u_{h,j} \mid j \in J\}$

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How to implement the AFEM scheme

Consider contribution of all elements in W_ℓ simultaneously

$$\theta \sum_{j\in J} \eta_{\ell,j}(\mathcal{T}_\ell)^2 \leq \sum_{j\in J} \eta_{\ell,j}(\mathcal{M}_\ell)^2$$

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Remark: notation $\mathcal{T}_\ell$ or $\mathcal{T}_h$, $\mathcal{T}_H$

Error quantity

Let us introduce the gradient $\mathbf{G}$ and the discrete gradient $\mathbf{G}_h$

$\mathbf{G}(w) \in H(\mathrm{div};\Omega)$ is the solution to

$$a(\mathbf{G}(w),\tau) + b(\tau, w) = 0 \quad \text{for all } \tau \in H(\mathrm{div};\Omega)$$

$\mathbf{G}_h(w_h) \in \Sigma_h$ is the solution to

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Error quantity

$$d(v, w) = \sqrt{\|v - w\|^2 + \|\mathbf{G}(v) - \mathbf{G}(w)\|^2}$$

N.B: when v (resp. w) belongs to M_h, then $\mathbf{G}_h(v)$ (resp. $\mathbf{G}_h(w)$) should be used

$$\boxed{\delta(W, W_h) = \sup_{\substack{u \in W \\ \|u\|=1}} \inf_{v_h \in W_h} d(u, v_h)}$$

Theoretical error indicator

Seminorm

$$\begin{aligned}|g_h|^2_{\eta,T} = \|h_T(\mathbf{G}_h(g_h) - \nabla g_h)\|^2_T \\ + \|h_T \operatorname{curl} \mathbf{G}_h(g_h)\|^2_T \\ + \sum_{E\in\mathcal{E}(T)} h_E \|[\mathbf{G}_h(g_h)]_E \cdot t_E\|^2_E,\end{aligned}$$

so that

$$\eta_{h,j}(T) = |u_{h,j}|_{\eta,T}.$$

Let (λ, σ, u) be an eigensolution to the continuous problem, then

$$\boxed{\mu_h(u; T) = |\Lambda_h u|_{\eta,T}}$$

where $\Lambda_h = P_h^W \circ T_h^\lambda$ (see next page)

Useful operators

P_h^W is the L^2-projection onto W_h

$T_h^\lambda : L^2 \to M_h$ is defined by

$$\begin{cases} a(\mathbf{G}_h(T_h^\lambda g), \tau_h) + b(\tau_h, T_h^\lambda g) = 0 & \forall \tau_h \in \Sigma_h \\ b(\mathbf{G}_h(T_h^\lambda g), v_h) = -(\lambda g, v_h) & \forall v_h \in M_h \end{cases}$$

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Then, we have

$$\Lambda_h = P_h^W \circ T_h^\lambda = T_h^\lambda \circ P_h^W$$

that is,

$$\begin{cases} a(\mathbf{G}_h(\Lambda_h u), \tau_h) + b(\tau_h, \Lambda_h u) = 0 & \forall \tau_h \in \Sigma_h \\ b(\mathbf{G}_h(\Lambda_h u), v_h) = -(\lambda P_h^W u, v_h) & \forall v_h \in M_h \end{cases}$$

Main theorem (convergence and optimal rate)

Nonlinear approximation classes ⟨Binev–Dahmen–DeVore 2004⟩ ⟨Cascon–Kreuzer–Nochetto–Siebert 2008⟩

Best convergence rate $s \in (0, +\infty)$ characterized in terms of

$$|W|_{\mathcal{A}_s} = \sup_{m \in \mathbb{N}} m^s \inf_{\mathcal{T} \in \mathbb{T}(m)} \delta(W, W_{\mathcal{T}}).$$

In particular, $|W|_{\mathcal{A}_s} < \infty$ if $\delta(W, W_{\mathcal{T}}) = O(m^{-s})$ for the optimal triangulations in $\mathbb{T}(m)$, that is, with $\mathrm{card}(\mathcal{T}) - \mathrm{card}(\mathcal{T}_0) \leq m$

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Theorem

Provided the initial mesh-size and the bulk parameter θ are small enough, if for the eigenvalue cluster W it holds $|W|_{\mathcal{A}_s} < \infty$, then the sequence of discrete clusters W_ℓ computed on the mesh $\mathcal{T}_\ell$ satisfies the optimal estimate

$$\delta(W, W_\ell)(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0))^s \le C|W|_{\mathcal{A}_s}$$

Convergence of the eigenvalues

The previous theorem implies that the eigenfunctions in the cluster are optimally approximated. The next theorem shows that the eigenvalues are well approximated as well

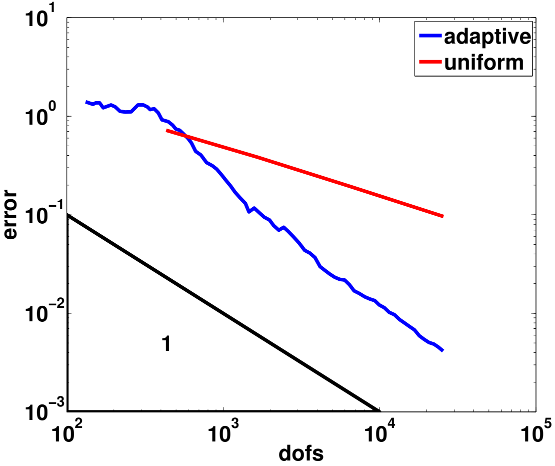
Theorem

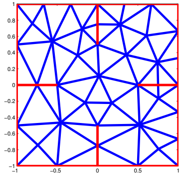
Let J denote the set of indices corresponding to the eigenvalues in the cluster W. Then

$$\sup_{i \in J} \inf_{j \in J} |\lambda_i - \lambda_{\ell,j}| \leq C\delta(W, W_\ell)^2$$

Numerical results (non-symmetric slit domain)

Convergence plot for the second eigenfunction (indicator based on both eigenvalues in the cluster)





Superconvergence

Let Π_h denote the orthogonal projection onto M_h

Proposition (Superconvergence for the eigenvalue problem)

Any eigensolution (λ, σ, u) *in the cluster satisfies*

$$\|\Pi_h u - \Lambda_h u\| \leq \rho(h)\|\sigma - \mathbf{G}_h(T_h^\lambda u)\|_{\mathrm{div}}$$

with $\rho(h)$ *tending to zero as* h *goes to zero*

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Lemma (Bound for the $\boldsymbol{H}(\mathrm{div})$ norm)

Any eigensolution $(\lambda, \sigma, u) \in \mathbb{R} \times \Sigma \times M$ *satisfies*

$$\|\sigma - \mathbf{G}_h(\Lambda_h u)\|_{\mathrm{div}} \lesssim \|\sigma - \mathbf{G}_h(\Lambda_h u)\| + (1 + \lambda)\|u - \Lambda_h u\|.$$

Efficiency and reliability

Proposition (Efficiency)

$$\mu_h(u; \mathcal{T}_h) \le C_{\mathrm{eff}}\, d(u, \Lambda_h u)$$

Proposition (Discrete reliability)

Provided the mesh-size of $\mathcal{T}_H$ is sufficiently small, we have

$$\begin{aligned}\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\| + \|\Lambda_h u - \Lambda_H u\| \\ \le C_{\mathsf{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))\end{aligned}$$

Corollary (Reliability)

Provided the initial mesh-size is sufficiently fine, we have

$$\sum_{j\in J} d(u_j, \Lambda_h u_j)^2 \le C_{\mathsf{rel}}^2 \sum_{j\in J} \mu_h(u_j, \mathcal{T}_h)^2$$

Quasi-orthogonality

Proposition (Quasi-orthogonality)

There exists a constant C_{qo} *such that*

$$\begin{aligned} d(\Lambda_h u, \Lambda_H u)^2 \leq{}& d(u, \Lambda_H u)^2 - d(u, \Lambda_h u)^2 \\ &+ C_{\mathrm{qo}}\, \rho(h)\big(d(u, \Lambda_h u)^2 + d(u, \Lambda_H u)^2\big) \end{aligned}$$

Equivalence of estimators and contraction property

N eigenvalues contained in $[A, B]$

Lemma (Local comparison of the error estimators)

Provided the initial mesh-size is small enough, for any $T \in \mathcal{T}_h$

$$\mathrm{N}^{-1} \sum_{j \in J} \mu_h(u_j; T)^2 \leq \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^2 \sum_{j \in J} \eta_{h,j}(T)^2 \leq \left(\frac{\mathrm{B}}{\mathrm{A}}\right)^2 (2\mathrm{N}+4\mathrm{N}^2) \sum_{j \in J} \mu_h(u_j; T)^2$$

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Proposition (Contraction property)

Set $\xi_\ell^2 = \sum_{j \in J} \mu_\ell(u_j, \mathcal{T}_\ell)^2 + \beta \sum_{j \in J} d(u_j, \Lambda_\ell u_j)^2$

Provided the initial mesh-size is sufficiently small, there exist $\rho_2 \in (0,1)$ *and* $\beta \in (0, +\infty)$ *such that*

$$\xi_{\ell+1}^2 \leq \rho_2 \xi_\ell^2 \quad \textit{for all } \ell \in \mathbb{N}$$

Outline

List of proofs

- Superconvergence
- Discrete reliability
- Quasi-orthogonality

- Convergence of the eigenvalues

Last chance to skip proofs

Superconvergence

Lemma (Superconvergence for the source problem)

There exist $\rho(h)$ tending to zero as h goes to zero such that

$$\|\Pi_h u - T_h^\lambda u\| \leq \rho(h)\|\sigma - \mathbf{G}_h(T_h^\lambda u)\|_{\mathrm{div}}$$

Known result

Superconvergence

Lemma (Superconvergence for the source problem)

There exist $\rho(h)$ tending to zero as h goes to zero such that

$\|\Pi_h u - T_h^\lambda u\| \leq \rho(h)\|\sigma - \mathbf{G}_h(T_h^\lambda u)\|_{\mathrm{div}}$ Known result

Proposition (Superconvergence for the eigenvalue problem)

Any eigensolution $(\lambda, \sigma, u) \in \mathbb{R} \times \Sigma \times W$ in the cluster satisfies

$$\|\Pi_h u - \Lambda_h u\| \leq \rho(h)\|\sigma - \mathbf{G}_h(T_h^\lambda u)\|_{\mathrm{div}}$$

Superconvergence

Lemma (Superconvergence for the source problem)

There exist $\rho(h)$ *tending to zero as* h *goes to zero such that*

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Proof.

$$\|\Pi_h u - \Lambda_h u\| \le \|\Pi_h u - T_h^\lambda u\| + \|T_h^\lambda u - \Lambda_h u\|$$

Superconvergence (cont'ed)

Lemma

Any eigensolution $(\lambda, \sigma, u) \in \mathbb{R} \times \Sigma \times W$ *in the cluster satisfies*

$$\|T_h^\lambda u - \Lambda_h u\| \leq K \|\Pi_h u - T_h^\lambda u\|$$

Superconvergence (cont'ed)

Lemma

Any eigensolution $(\lambda, \sigma, u) \in \mathbb{R} \times \Sigma \times W$ *in the cluster satisfies*

$$\|T_h^\lambda u - \Lambda_h u\| \le K \|\Pi_h u - T_h^\lambda u\|$$

Proof.

$$T_h^\lambda u - \Lambda_h u = \sum_{j \in J^C} \alpha_j u_{h,j}, \quad \sum_{j \in J^C} \alpha_j^2 = \|T_h^\lambda u - \Lambda_h u\|^2$$

$$\|T_h^\lambda u - \Lambda_h u\|^2 = \sum_{j \in J^C} \alpha_j (T_h^\lambda u, u_{h,j}) = \sum_{j \in J^C} \alpha_j \frac{\lambda}{\lambda - \lambda_{h,j}} (T_h^\lambda u - \Pi_h u, u_{h,j})$$

$$\le K \left(\sum_{j \in J^C} \alpha_j^2 \right)^{1/2} \|T_h^\lambda u - \Pi_h u\|$$

□

Discrete reliability

Proposition (Discrete reliability)

Provided the mesh-size of $\mathcal{T}_H$ is sufficiently small, we have

$$\begin{aligned}
&\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\| + \|\Lambda_h u - \Lambda_H u\| \\
&\qquad \leq C_{\mathrm{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))
\end{aligned}$$

Discrete reliability

$$\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\| \; + \; \|\Lambda_h u - \Lambda_H u\|$$
$$\leq C_{\mathrm{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

Discrete reliability

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$$\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u) = \mathbf{G}_h(\alpha_h) + \mathbf{curl}\, \beta_h$$

Discrete reliability

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Discrete reliability

$$\boxed{\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|} + \|\Lambda_h u - \Lambda_H u\| \\ \leq C_{\mathrm{drel}}\mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

$$\begin{aligned} \|\operatorname{\mathbf{curl}} \beta_h\|^2 &= a(\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u), \operatorname{\mathbf{curl}} \beta_h) \\ &= -a(\mathbf{G}_H(\Lambda_H u), \operatorname{\mathbf{curl}}(\beta_h - \overline{\beta}_H)) \\ &= \sum_{T \in \mathcal{T}_H \setminus \mathcal{T}_h} \Big(\int_T (\beta_h - \overline{\beta}_H) \operatorname{curl} \mathbf{G}_H(\Lambda_H u)\, dx \\ &\qquad - \int_{\partial T} (\beta_h - \overline{\beta}_H) \mathbf{G}_H(\Lambda_H u) \cdot t\, ds \Big) \end{aligned}$$

Here $\overline{\beta}_H$ is the Scott-Zhang interpolant of β_h on $\mathcal{T}_H$

$$\|\operatorname{\mathbf{curl}} \beta_h\| \leq C\mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h).$$

Discrete reliability

$$\boxed{\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|} + \|\Lambda_h u - \Lambda_H u\|$$
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$$\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u) = \mathbf{G}_h(\alpha_h) + \mathbf{curl}\, \beta_h$$

Discrete reliability

$$\boxed{\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|} + \|\Lambda_h u - \Lambda_H u\| \\ \leq C_{\mathrm{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

$$\begin{aligned} \|\mathbf{G}_h(\alpha_h)\|^2 &= a(\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u), \mathbf{G}_h(\alpha_h)) \\ &= \lambda(P_h^W u - P_H^W u, \alpha_h) \\ &= \lambda\big((P_h^W u - \Pi_h u, \alpha_h) \\ &\quad + (\Pi_h u - \Pi_H u, \alpha_h - \Pi_H \alpha_h) \\ &\quad + (\Pi_H u - P_H^W u, \alpha_h)\big) \end{aligned}$$

Discrete reliability

$$\boxed{\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|} + \|\Lambda_h u - \Lambda_H u\|$$
$$\leq C_{\mathsf{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

$$\boxed{(P_h^W u - \Pi_h u, \alpha_h)} + (\Pi_h u - \Pi_H u, \alpha_h - \Pi_H \alpha_h) + \boxed{(\Pi_H u - P_H^W u, \alpha_h)}$$

$$\begin{aligned}(P_h^W u - \Pi_h u, \alpha_h) + (\Pi_H u - P_H^W u, \alpha_h) \\ &\leq C\big(\|P_h^W u - \Pi_h u\| + \|\Pi_H u - P_H^W u\|\big)\|\mathbf{G}_h(\alpha_h)\| \\ &\leq \rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))\|\mathbf{G}_h(\alpha_h)\|\end{aligned}$$

Discrete reliability

$$\boxed{\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|} + \|\Lambda_h u - \Lambda_H u\|$$
$$\leq C_{\mathrm{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

$$(P_h^W u - \Pi_h u, \alpha_h) + \boxed{(\Pi_h u - \Pi_H u, \alpha_h - \Pi_H \alpha_h)} + (\Pi_H u - P_H^W u, \alpha_h)$$

$$\|\xi\| = \|\alpha_h - \Pi_H \alpha_h\| \leq CH\|\mathbf{G}_h(\alpha_h)\|$$ ⟨Huang–Xu 2012⟩

$$\begin{aligned}(\Pi_h u - \Pi_H u, \xi) &= (\Pi_h u - \Lambda_H u, \xi) \\ &= (\Pi_h u - \Lambda_h u, \xi) + (\Lambda_h u - \Lambda_H u, \xi) \\ &\leq CH(\rho(H) d(u, \Lambda_h u) + \|\Lambda_h u - \Lambda_H u\|)\|\mathbf{G}_h(\alpha_h)\|\end{aligned}$$

Discrete reliability

$$\boxed{\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|} + \|\Lambda_h u - \Lambda_H u\|$$
$$\leq C_{\mathrm{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

$$\begin{aligned} &\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\| \\ &\quad \leq C \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) \\ &\qquad + \rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u)) + H\|\Lambda_h u - \Lambda_H u\| \end{aligned}$$

Discrete reliability

$$\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\| + \boxed{\|\Lambda_h u - \Lambda_H u\|}$$
$$\leq C_{\mathrm{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

Let $\hat{z}$ be the gradient of the solution $\hat{\phi}$ of the problem

$$\begin{cases} \Delta\hat{\phi} = \Lambda_h u - \Lambda_H u & \text{in } \Omega \\ \hat{\phi} = 0 & \text{on } \partial\Omega \end{cases}$$

$z \in H^1(\Omega)$ such that ⟨Pasciak–Zhao, 2002⟩ ⟨Gatica–Oyarzua–Sayas, 2011⟩

$$\hat{z} = z + \mathbf{curl}\,\psi$$

so that

$$\operatorname{div} z = \boxed{\Lambda_h u - \Lambda_H u}$$
$$\|z\| + \|\nabla z\| \leq C \left\| \boxed{\Lambda_h u - \Lambda_H u} \right\|$$

Discrete reliability

$$\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\| + \boxed{\|\Lambda_h u - \Lambda_H u\|}$$
$$\leq C_{\mathrm{drel}} \mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

$$\begin{aligned}
\|\Lambda_h u - \Lambda_H u\|^2 &= b(z, \Lambda_h u - \Lambda_H u) \\
&= b(\Pi_h^F z, \Lambda_h u) - b(\Pi_H^F z, \Lambda_H u) \\
&= -a(\mathbf{G}_h(\Lambda_h u), \Pi_h^F z) + a(\mathbf{G}_H(\Lambda_H u), \Pi_H^F z) \\
&= a(\mathbf{G}_H(\Lambda_H u) - \mathbf{G}_h(\Lambda_h u), \Pi_h^F z) \\
&\quad + a(\mathbf{G}_H(\Lambda_H u), (\Pi_H^F - \Pi_h^F) z) \\
&\leq \|\mathbf{G}_H(\Lambda_H u) - \mathbf{G}_h(\Lambda_h u)\| \|\Pi_h^F z\| \\
&\quad + a(\mathbf{G}_H(\Lambda_H u) - \nabla_H(\Lambda_H u), (\Pi_H^F - \Pi_h^F) z)
\end{aligned}$$

N.B.: $a(\nabla_H(\Lambda_H u), (\Pi_H^F - \Pi_h^F) z) = 0$

Discrete reliability

$$\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\| + \boxed{\|\Lambda_h u - \Lambda_H u\|}$$
$$\leq C_{\mathrm{drel}}\mu_H(u; \mathcal{T}_H \setminus \mathcal{T}_h) + C\rho(H)(d(u, \Lambda_h u) + d(u, \Lambda_H u))$$

$$\begin{aligned}\|\Lambda_h u - \Lambda_H u\|^2 \leq{}& \|\mathbf{G}_H(\Lambda_H u) - \mathbf{G}_h(\Lambda_h u)\| \|\Pi_h^F z\| \\ &+ a(\mathbf{G}_H(\Lambda_H u) - \nabla_H(\Lambda_H u), (\Pi_H^F - \Pi_h^F) z)\end{aligned}$$

$$\begin{aligned}&\leq \|\mathbf{G}_H(\Lambda_H u) - \mathbf{G}_h(\Lambda_h u)\| \|\Pi_h^F z\| \\ &\quad + \|H(\mathbf{G}_H(\Lambda_H u) - \nabla_H(\Lambda_H u))\|_{\mathcal{T}_H \setminus \mathcal{T}_h} \|H^{-1}(\Pi_h^F z - \Pi_H^F z)\| \\ &\leq C\|\Lambda_h u - \Lambda_H u\| \\ &\quad (\|\mathbf{G}_H(\Lambda_H u) - \mathbf{G}_h(\Lambda_h u)\| + \mu_H(\Lambda_H u; \mathcal{T}_H \setminus \mathcal{T}_h))\end{aligned}$$

Recall: $\|z\| + \|\nabla z\| \leq C\|\Lambda_h u - \Lambda_H u\|$ and $\Pi_h^F z - \Pi_H^F z$ vanishes on unrefined elements $\mathcal{T}_H \cap \mathcal{T}_h$ $\square$

Quasi-orthogonality

Proposition (Quasi-orthogonality)

There exists a constant C_{qo} *such that*

$$\begin{aligned} d(\Lambda_h u, \Lambda_H u)^2 \leq\ & d(u, \Lambda_H u)^2 - d(u, \Lambda_h u)^2 \\ & + C_{\mathrm{qo}}\, \rho(h)(d(u, \Lambda_h u)^2 + d(u, \Lambda_H u)^2) \end{aligned}$$

Quasi-orthogonality

$$
\begin{aligned}
d(\Lambda_h u, \Lambda_H u)^2 \leq\ & d(u, \Lambda_H u)^2 - d(u, \Lambda_h u)^2 \\
& +\ C_{\mathrm{qo}} \rho(h) (d(u, \Lambda_h u)^2 + d(u, \Lambda_H u)^2)
\end{aligned}
$$

Quasi-orthogonality

$$d(\Lambda_h u, \Lambda_H u)^2 \leq \boxed{d(u, \Lambda_H u)^2 - d(u, \Lambda_h u)^2}$$
$$+ \; C_{\mathrm{qo}} \rho(h) (d(u, \Lambda_h u)^2 + d(u, \Lambda_H u)^2)$$

$$\boxed{d(\Lambda_h u, \Lambda_H u)^2 = \|\Lambda_h u - \Lambda_H u\|^2 + \|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|^2}$$

$$\|\Lambda_h u - \Lambda_H u\|^2 = \boxed{\|u - \Lambda_H u\|^2 - \|u - \Lambda_h u\|^2}$$
$$-2(\Pi_h u - \Lambda_h u, \Lambda_h u - \Lambda_H u)$$

$$\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|^2 = \boxed{\|\sigma - \mathbf{G}_H(\Lambda_H u)\|^2 - \|\sigma - \mathbf{G}_h(\Lambda_h u)\|^2}$$
$$-2a(\sigma - \mathbf{G}_h(\Lambda_h u), \mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u))$$

Quasi-orthogonality

$$d(\Lambda_h u, \Lambda_H u)^2 \leq \; d(u, \Lambda_H u)^2 - d(u, \Lambda_h u)^2 + \boxed{C_{\mathrm{qo}} \rho(h) (d(u, \Lambda_h u)^2 + d(u, \Lambda_H u)^2)}$$

$$d(\Lambda_h u, \Lambda_H u)^2 = \|\Lambda_h u - \Lambda_H u\|^2 + \|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|^2$$

$$\|\Lambda_h u - \Lambda_H u\|^2 = \; \|u - \Lambda_H u\|^2 - \|u - \Lambda_h u\|^2 - 2(\Pi_h u - \Lambda_h u, \Lambda_h u - \Lambda_H u)$$

$$\|\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)\|^2 = \; \|\sigma - \mathbf{G}_H(\Lambda_H u)\|^2 - \|\sigma - \mathbf{G}_h(\Lambda_h u)\|^2 - 2a(\sigma - \mathbf{G}_h(\Lambda_h u), \mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u))$$

Quasi-orthogonality

$$d(\Lambda_h u, \Lambda_H u)^2 \leq d(u, \Lambda_H u)^2 - d(u, \Lambda_h u)^2 + \boxed{C_{\mathrm{qo}} \rho(h) (d(u, \Lambda_h u)^2 + d(u, \Lambda_H u)^2)}$$

$$\begin{aligned} a(\sigma - \mathbf{G}_h(\Lambda_h u), \mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u)) \\ &= -b(\mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u), u - \Lambda_h u) \\ &= \lambda(P_h^W u - P_H^W u, \Pi_h u - \Lambda_h u) \end{aligned}$$

$$\begin{aligned} |a(\sigma - \mathbf{G}_h(\Lambda_h u), \mathbf{G}_h(\Lambda_h u) - \mathbf{G}_H(\Lambda_H u))| + |(\Pi_h u - \Lambda_h u, \Lambda_h u - \Lambda_H u)| \\ &\leq \|\Pi_h u - \Lambda_h u\| \left(\|\Lambda_h u - \Lambda_H u\| + \lambda \|P_h^W u - P_H^W u\|\right) \\ &\leq \boxed{C \rho(h) (d(u, \Lambda_h u)^2 + d(u, \Lambda_H u)^2)} \end{aligned}$$

Convergence of the eigenvalues — sketch

T and T_ℓ solution operators

$E : L^2 \to L^2$ orthogonal projection onto W
$E_\ell : L^2 \to L^2$ orthogonal projection onto W_ℓ

$F_\ell = E_\ell|_W$

Convergence of the eigenvalues — sketch

T and T_ℓ solution operators

$E : L^2 \to L^2$ orthogonal projection onto W
$E_\ell : L^2 \to L^2$ orthogonal projection onto W_ℓ

$F_\ell = E_\ell|_W$

Proposition

For ℓ large enough F_ℓ is a bijection from W to W_ℓ

Convergence of the eigenvalues — sketch

T and T_ℓ solution operators

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$F_\ell = E_\ell|_W$

Proposition

For ℓ large enough F_ℓ is a bijection from W to W_ℓ

Some useful estimates

$$\|(T - T_\ell)x\| \le C\delta(W, W_\ell)$$
$$\|(A - A_\ell)x\|_{\mathrm{div}} \le C\delta(W, W_\ell)$$

Convergence of the eigenvalues (cont'ed)

Define the following operators

$$\hat{T} = T|_W, \quad \hat{T}_\ell = F_\ell^{-1} T_\ell F_\ell$$

so that the eigenvalues of $\hat{T}$ ($\hat{T}_\ell$, resp.) are equal to $\mu_j = 1/\lambda_j$ ($\mu_{\ell,j} = 1/\lambda_{\ell,j}$ resp.), $j \in J$

Convergence of the eigenvalues (cont'ed)

Define the following operators

$$\hat{T} = T|_W, \quad \hat{T}_\ell = F_\ell^{-1} T_\ell F_\ell$$

so that the eigenvalues of $\hat{T}$ ($\hat{T}_\ell$, resp.) are equal to $\mu_j = 1/\lambda_j$ ($\mu_{\ell,j} = 1/\lambda_{\ell,j}$ resp.), $j \in J$

Proposition

The following estimate holds true

$$\|\hat{T} - \hat{T}_\ell\|_{\mathcal{L}(W)} \leq C\delta(W, W_\ell)^2$$

Convergence of the eigenvalues (cont'ed)

Define the following operators

$$\hat{T} = T|_W, \quad \hat{T}_\ell = F_\ell^{-1} T_\ell F_\ell$$

so that the eigenvalues of $\hat{T}$ ($\hat{T}_\ell$, resp.) are equal to $\mu_j = 1/\lambda_j$ ($\mu_{\ell,j} = 1/\lambda_{\ell,j}$ resp.), $j \in J$

Proposition

The following estimate holds true

$$\|\hat{T} - \hat{T}_\ell\|_{\mathcal{L}(W)} \leq C\delta(W, W_\ell)^2$$

The operators $\hat{T}$ and $\hat{T}_\ell$ can be represented by symmetric positive definite matrices of dimension $\mathrm{N} \times \mathrm{N}$ (N being the dimension of W) Standard matrix perturbation theory gives the final result

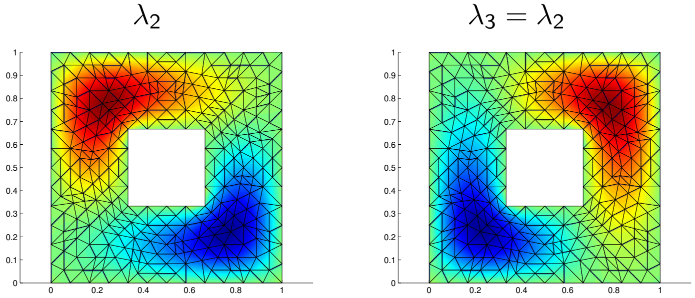
Convergence of the eigenvalues (cont'ed)

Theorem

Let J denote the set of indices corresponding to the eigenvalues in the cluster W. Then

$$\sup_{i\in J}\inf_{j\in J}|\lambda_i-\lambda_{\ell,j}|\leq C\delta(W,W_\ell)^2$$

Multiple eigenvalues: the square ring



Question

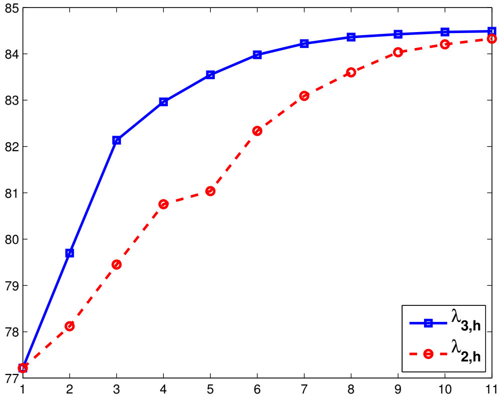
What is the best adaptive strategy for the approximation of the multiple eigenvalue?

1. Indicator based on $(\lambda_{h,2}, u_{h,2})$
2. Indicator based on $(\lambda_{h,3}, u_{h,3})$
3. Indicator based on both $(\lambda_{h,2}, u_{h,2})$ and $(\lambda_{h,3}, u_{h,3})$

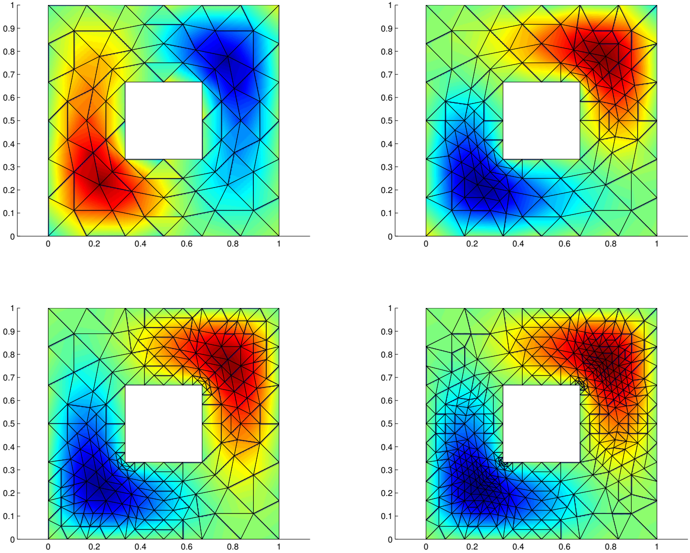
Refinement based on $\lambda_{h,3}$

⟨B.–Durán–Gardini–Gastaldi 2015⟩

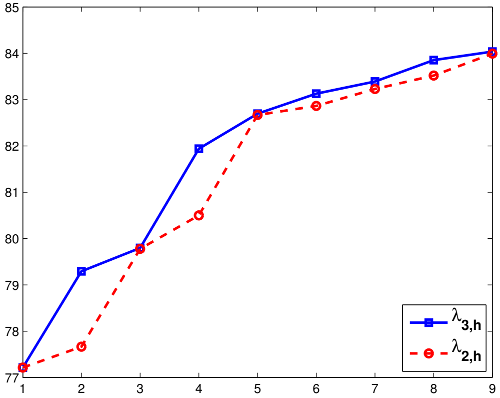
Remark: here we are using a nonconforming discretization which provides eigenvalue approximation from below



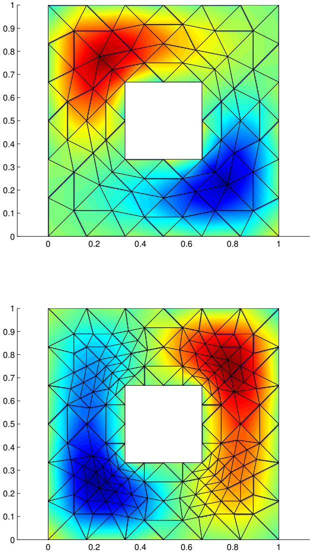
Refinement based on $\lambda_{h,3}$ (eigenfunction $u_{h,3}$)



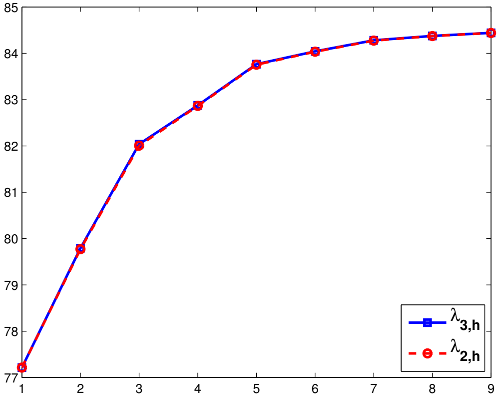
Refinement based on $\lambda_{h,2}$



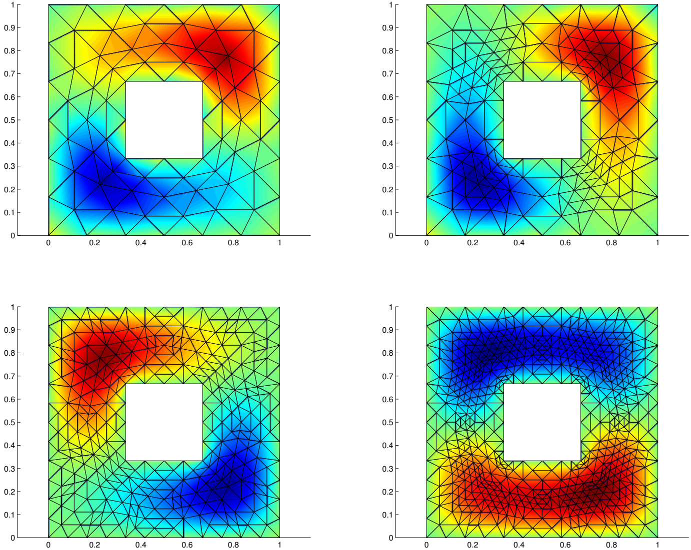
Refinement based on $\lambda_{h,2}$ (eigenfunction $u_{h,2}$)



Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenvalues)



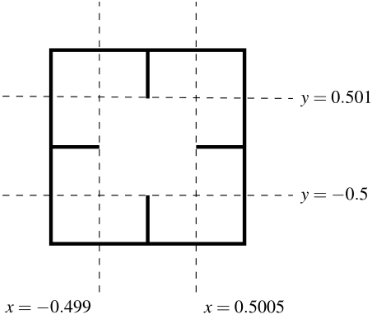
Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenfunction $u_{h,2}$)

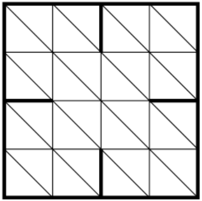


Cluster of eigenvalues

⟨Gallistl '14⟩

A slightly non-symmetric domain





Now $\lambda_2 < \lambda_3$ but they are very close to each other

Non-symmetric slit domain