Finite element quasi-interpolation and best approximation

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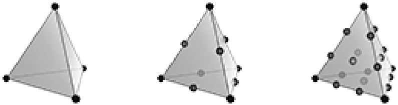
Setting

- A polyhedral Lipschitz domain $D \subset \mathbb{R}^d$
- A mesh sequence $(\mathcal{T}_h)_{h>0}$ (affine and shape-regular)
- A finite element space $P^{\mathrm{x}}(\mathcal{T}_h; \mathbb{R}^q)$ with some conformity property, composed of scalar- or vector-valued functions
  - $(\mathrm{x} = \mathrm{g})$ $P^{\mathrm{g}}(\mathcal{T}_h)$ is H^1-conforming ($q = 1$, integrable gradient)
  - $(\mathrm{x} = \mathrm{c})$ $\boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h)$ is $\boldsymbol{H}(\mathrm{curl})$-conforming ($q = d = 3$, integrable curl)
  - $(\mathrm{x} = \mathrm{d})$ $\boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h)$ is $\boldsymbol{H}(\mathrm{div})$-conforming ($q = d$, integrable divergence)

Examples

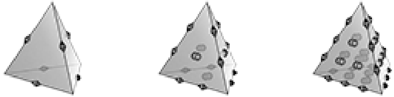
Periodic table of finite elements [Arnold & Logg 14]

$x = g$



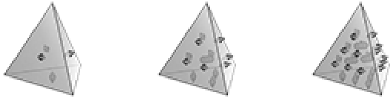
point evaluation	$(k \geq 1)$
edge integral	$(k \geq 2)$
face integral	$(k \geq 3)$
cell integral	$(k \geq 4)$

$x = c$



edge integral	$(k \geq 0)$
face integral	$(k \geq 1)$
cell integral	$(k \geq 2)$

$x = d$



face integral	$(k \geq 0)$
cell integral	$(k \geq 1)$

Best approximation

- Let $v \in W^{s,p}(D;\mathbb{R}^q)$, $p \in [1,\infty]$, and $s > 0$ **possibly very small**
- What is the decay of the **best-approximation** error in L^p

$$\inf_{p \in P^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)} \|v - p\|_{L^p(D;\mathbb{R}^q)}$$

- More generally, has $P^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$ the same **local approximation power** in $W^{m,p}$ as the local finite element space?
- This is a key question in FEM error analysis
  - H^1-conforming FEM for diffusive problem; Céa's Lemma yields

$$|u - u_h|_{W^{1,2}(D;\mathbb{R})} \leq c \inf_{p \in P^{\mathrm{g}}(\mathcal{T}_h)} |u - p|_{W^{1,2}(D;\mathbb{R})}$$

  - to conclude, we need a (localized) decay estimate of the RHS

Quasi-interpolation

- The canonical interpolation operator is stable only in $W^{s,p}(D;\mathbb{R}^q)$

  $$s > \frac{d}{p} \text{ if } \mathrm{x}=\mathrm{g}, \quad s > \frac{d-1}{p} = \frac{2}{p} \text{ if } \mathrm{x}=\mathrm{c}, \quad s > \frac{1}{p} \text{ if } \mathrm{x}=\mathrm{d}$$

  (equalities are possible for $p=1$)

- We want to build quasi-interpolation operators

  $$\mathcal{I}_h^{\mathrm{x}} : L^1(D;\mathbb{R}^q) \to P^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$$

  such that
  - $\mathcal{I}_h^{\mathrm{x}}$ is L^p-stable for all $p \in [1,\infty]$
  - leaves $P^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$ pointwise invariant ($\mathcal{I}_h^{\mathrm{x}} \circ \mathcal{I}_h^{\mathrm{x}} = \mathcal{I}_h^{\mathrm{x}}$)
  - has optimal local approximation properties in any $W^{s,p}$, $s>0$

- We devise a **unified framework** for design and analysis

- It is also possible to prescribe **homogeneous boundary conditions**

Literature review

- [Clement 75] 2D Lagrange; sketch of proof is believable ... but not a projection and BC's not preserved
- [Bernardi & Girault 98] real proof in 2D for Lagrange elements
- [Scott & Zhang 90] Lagrange elements, L^1-stability without BC's
  - [Girault & Lions 01] L^1-stability with BC's
  - [Ciarlet Jr. 13] analysis in $W^{s,p}(D)$, $s > \frac{1}{p}$
- **Literature on Nédélec or Raviart–Thomas elements ??**
- [Arnold, Falk & Winther, Acta Numerica 06] invokes Clement interpolation to prove best approximation in FEEC ... **is it trivial?**

Outline

- Finite element spaces
- Main result without BC's
- Main result with boundary prescription
- Mollification and commuting projections

Finite element spaces

- Affine and shape-regular mesh sequence $(\mathcal{T}_h)_{h>0}$
- Reference finite element $(\widehat{K}, \widehat{P}^{\mathrm{x}}, \widehat{\Sigma}^{\mathrm{x}})$ with $\widehat{P}^{\mathrm{x}} \subset W^{1,\infty}(\widehat{K}; \mathbb{R}^q)$
- Local interpolation operator $\mathcal{I}_{\widehat{K}}^{\mathrm{x}} : V^{\mathrm{x}}(\widehat{K}) \to \widehat{P}^{\mathrm{x}} \subset V^{\mathrm{x}}(\widehat{K})$
  - $(\mathrm{x} = \mathrm{g})$ $V^{\mathrm{g}}(\widehat{K}) = W^{s,p}(\widehat{K})$, $s > \frac{d}{p}$
  - $(\mathrm{x} = \mathrm{c})$ $V^{\mathrm{c}}(\widehat{K}) = W^{s,p}(\widehat{K})$, $s > \frac{d-1}{p} = \frac{2}{p}$
  - $(\mathrm{x} = \mathrm{d})$ $V^{\mathrm{d}}(\widehat{K}) = W^{s,p}(\widehat{K})$, $s > \frac{1}{p}$

Finite element generation

- Mesh cell generated by affine geometric map $\boldsymbol{T}_K : \widehat{K} \to K$ with constant Jacobian $\mathbb{J}_K \in \mathbb{R}^{d\times d}$
- $(K, P_K^{\mathrm{x}}, \Sigma_K^{\mathrm{x}})$ generated using functional map $\psi_K^{\mathrm{x}} : V^{\mathrm{x}}(K) \to V^{\mathrm{x}}(\widehat{K})$ so that $P_K^{\mathrm{x}} = (\psi_K^{\mathrm{x}})^{-1}(\widehat{P}^{\mathrm{x}})$ and $\Sigma_K^{\mathrm{x}} = \widehat{\Sigma}^{\mathrm{x}} \circ \psi_K^{\mathrm{x}}$
- Key example: $\psi_K^{\mathrm{x}}(v) = \mathbb{A}_K^{\mathrm{x}}(v \circ \boldsymbol{T}_K)$ for all $v \in V^{\mathrm{x}}(K)$
  - $(\mathrm{x} = \mathrm{g})$ $\mathbb{A}_K^{\mathrm{g}} = \mathbb{I}$ (pullback by $\boldsymbol{T}_K$)
  - $(\mathrm{x} = \mathrm{c})$ $\mathbb{A}_K^{\mathrm{c}} = \mathbb{J}_K^{\mathrm{T}}$ (covariant Piola transf.)
  - $(\mathrm{x} = \mathrm{d})$ $\mathbb{A}_K^{\mathrm{d}} = \det(\mathbb{J}_K)\mathbb{J}_K^{-1}$ (contravariant Piola transf.)
- **Main property**: $\|\mathbb{A}_K^{\mathrm{x}}\|_{\ell^2}\|(\mathbb{A}_K^{\mathrm{x}})^{-1}\|_{\ell^2} \leq c\|\mathbb{J}_K\|_{\ell^2}\|(\mathbb{J}_K)^{-1}\|_{\ell^2}$ $(c = 1)$
- The broken FE space $P^{\mathrm{x,b}}(\mathcal{T}_h; \mathbb{R}^q) \subset W^{1,\infty}(\mathcal{T}_h; \mathbb{R}^q)$ is defined as

$$P^{\mathrm{x,b}}(\mathcal{T}_h; \mathbb{R}^q) := \{v_h \in L^\infty(\mathcal{T}_h; \mathbb{R}^q) \mid v_{h|K} \in P_K^{\mathrm{x}}, \forall K \in \mathcal{T}_h\}$$

Conforming finite element subspaces

- Jump operator $[\![\cdot]\!]_F^{\mathrm{x}}$ across interfaces $F = \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ and corresponding global functional space $V^{\mathrm{x}}(D)$

$$
\begin{aligned}
&[\![v]\!]_F^{\mathrm{g}} := (v_{|K_l} - v_{|K_r})_{|F} && V^{\mathrm{g}}(D) := \{v \in L^1(D) \mid \nabla v \in \boldsymbol{L}^1(D)\}\\
&[\![\boldsymbol{v}]\!]_F^{\mathrm{c}} := (\boldsymbol{v}_{|K_l} - \boldsymbol{v}_{|K_r})_{|F} \times \boldsymbol{n}_F && \boldsymbol{V}^{\mathrm{c}}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^1(D) \mid \nabla \times \boldsymbol{v} \in \boldsymbol{L}^1(D)\}\\
&[\![\boldsymbol{v}]\!]_F^{\mathrm{d}} := (\boldsymbol{v}_{|K_l} - \boldsymbol{v}_{|K_r})_{|F} \cdot \boldsymbol{n}_F && \boldsymbol{V}^{\mathrm{d}}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^1(D) \mid \nabla \cdot \boldsymbol{v} \in L^1(D)\}
\end{aligned}
$$

- The jump operator $[\![\cdot]\!]_F^{\mathrm{x}}$ is well-defined on $W^{1,1}(\mathcal{T}_h;\mathbb{R}^q)$ and $\|[\![v]\!]_F^{\mathrm{x}}\|_{\ell^2} \le c\|[\![v]\!]_F\|_{\ell^2}$ a.e. in F $(c=1)$

- Conforming finite element subspaces

$$
\begin{aligned}
P^{\mathrm{g}}(\mathcal{T}_h) &= P^{\mathrm{g,b}}(\mathcal{T}_h) \cap V^{\mathrm{g}}(D) = \{v_h \in P^{\mathrm{g,b}}(\mathcal{T}_h) \mid [\![v_h]\!]_F^{\mathrm{g}} = 0,\ \forall F \in \mathcal{F}_h^\circ\}\\
\boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) &= \boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h) \cap \boldsymbol{V}^{\mathrm{c}}(D) = \{\boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{c,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}} = \boldsymbol{0},\ \forall F \in \mathcal{F}_h^\circ\}\\
\boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) &= \boldsymbol{P}^{\mathrm{d,b}}(\mathcal{T}_h) \cap \boldsymbol{V}^{\mathrm{d}}(D) = \{\boldsymbol{v}_h \in \boldsymbol{P}^{\mathrm{d,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}} = 0,\ \forall F \in \mathcal{F}_h^\circ\}
\end{aligned}
$$

Connectivity array and classes

- Local shape functions $\theta_{K,i}$, $(K,i) \in \mathcal{T}_h \times \mathcal{N}$
- Standard construction of global shape functions $(\varphi_a)_{a\in\mathcal{A}_h}$
- **Connectivity array** $\mathsf{a} : \mathcal{T}_h \times \mathcal{N} \to \mathcal{A}_h$ s.t. $\varphi_{\mathsf{a}(K,i)|K} = \theta_{K,i}$
- Connectivity set $\mathcal{C}_a$ for all $a \in \mathcal{A}_h$ s.t.

$$\mathcal{C}_a := \mathsf{a}^{-1}(a) = \{(K,i) \in \mathcal{T}_h{\times}\mathcal{N} \mid a = \mathsf{a}(K,i)\}$$

- For all $K \in \mathcal{T}_h$, $\mathcal{T}_K$ is the **set of cells sharing global shape functions** with K, and D_K^{x} collects the points in these cells
  - D_K^{g} results from cells sharing at least a vertex with K
  - D_K^{c} results from cells sharing at least an edge with K
  - D_K^{d} results from cells sharing at least a face with K

Main result: assumptions

- Affine shape-regular mesh sequence
- $\psi_K^{\mathrm{x}}(v) = \mathbb{A}_K^{\mathrm{x}}(v \circ \boldsymbol{T}_K)$ and $\|\mathbb{A}_K^{\mathrm{x}}\|_{\ell^2}\|(\mathbb{A}_{K'}^{\mathrm{x}})^{-1}\|_{\ell^2} \leq c$ for all $K' \in D_K^{\mathrm{x}}$
- $\|[\![v]\!]_F^{\mathrm{x}}\|_{\ell^2} \leq c\|[\![v]\!]_F\|_{\ell^2}$ for all $v \in W^{1,1}(\mathcal{T}_h;\mathbb{R}^q)$
- Control of DoFs across interfaces: For all $v_h \in P^{\mathrm{x,b}}(\mathcal{T}_h;\mathbb{R}^q)$,

  $$\left|\sigma_{K,i}^{\mathrm{x}}(v_h) - \sigma_{K',i'}^{\mathrm{x}}(v_h)\right| \leq c \min(\|\mathbb{A}_K^{\mathrm{x}}\|_{\ell^2}, \|\mathbb{A}_{K'}^{\mathrm{x}}\|_{\ell^2})\|[\![v_h]\!]_F^{\mathrm{x}}\|_{L^\infty(F;\mathbb{R}^t)}$$

  for all pairs $(K,i),(K',i') \in \mathcal{C}_a$ such that $F = K \cap K'$
- All assumptions satisfied by all known FE elements (all degree, all type, all kind)
- Let k be the largest integer s.t. $\mathbb{P}_{k,d}(\widehat{K};\mathbb{R}^q) \subset \widehat{P}^{\mathrm{x}}$

Main result: statement

Theorem (EG15)

There is a quasi-interpolation operator $\mathcal{I}_h^{\mathrm{x}} : L^1(D;\mathbb{R}^q) \to P^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$ s.t.

- $\mathcal{I}_h^{\mathrm{x}}$ leaves $P^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$ pointwise invariant ($\mathcal{I}_h^{\mathrm{x}} \circ \mathcal{I}_h^{\mathrm{x}} = \mathcal{I}_h^{\mathrm{x}}$)
- $\mathcal{I}_h^{\mathrm{x}}$ is L^p-stable for all $p \in [1,\infty]$
- Optimal local approximation (integer-order)

$$|v - \mathcal{I}_h^{\mathrm{x}}(v)|_{W^{m,p}(K;\mathbb{R}^q)} \leq c\, h_K^{l-m} |v|_{W^{l,p}(D_K^{\mathrm{x}};\mathbb{R}^q)}$$

for all $l \in \{0{:}\,k+1\}$, all $m \in \{0{:}\,l\}$, and all $p \in [1,\infty]$

- Optimal local approximation (fractional-order)

$$|v - \mathcal{I}_h^{\mathrm{x}}(v)|_{W^{m,p}(K;\mathbb{R}^q)} \leq c\, h_K^{s-m} |v|_{W^{s,p}(D_K^{\mathrm{x}};\mathbb{R}^q)}$$

for all $s \in [0,k+1]$, all $m \in \{0{:}\,\lfloor s \rfloor\}$, and all $p \in [1,\infty)$

Idea of proof (1)

- $\mathcal{I}_h^{\mathrm{x}} = \mathcal{I}_h^{\mathrm{x,av}} \circ \mathcal{I}_h^{\mathrm{x},\sharp}$
  - construct an approximation operator on the broken space first
  - stitch the result by averaging
  - $\Rightarrow$ avoids working with continuous functions on patches

- $\mathcal{I}_h^{\mathrm{x},\sharp} : L^1(D) \to P^{\mathrm{x,b}}(\mathcal{T}_h)$ built locally using the dual basis technique of [Scott & Zhang 90] with functions $\widehat{\rho}_i^{\mathrm{x}} \in \widehat{P}^{\mathrm{x}}$ s.t.

$$\frac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{\rho}_i^{\mathrm{x}} \widehat{p} \,\mathrm{d}\widehat{x} = \widehat{\sigma}_i^{\mathrm{x}}(\widehat{p}) \qquad \forall \widehat{p} \in \widehat{P},\ \forall i \in \mathcal{N}$$

- Bramble–Hilbert/Deny–Lions Lemma

$$|v - \mathcal{I}_h^{\mathrm{x},\sharp}(v)|_{W^{m,p}(K;\mathbb{R}^q)} \leq c\, h_K^{l-m} |v|_{W^{l,p}(K;\mathbb{R}^q)}$$

for all $l \in \{0\!:k+1\}$, all $m \in \{0\!:l\}$, and all $p \in [1,\infty]$

Idea of proof (2)

- $\mathcal{I}_h^{\mathrm{x,av}} : P^{\mathrm{x,b}}(\mathcal{T}_h) \to P^{\mathrm{x}}(\mathcal{T}_h)$ defined by averaging

$$\mathcal{I}_h^{\mathrm{x,av}}(v_h)(\boldsymbol{y}) = \sum_{a\in\mathcal{A}_h} \frac{1}{\mathrm{card}(\mathcal{C}_a)} \sum_{(K,i)\in\mathcal{C}_a} \sigma_{K,i}(v_{h|K})\varphi_a(\boldsymbol{y})$$

  - see [Oswald 93] for scalar case
  - a posteriori error analysis [Achdou, Bernardi & Coquel 03; Karakashian & Pascal 03; AE, Stephansen & Vohralik 07], preconditioning [Schöberl & Lehrenfeld 13], stabilization [Burman & AE 07], compatible dG [Cockburn, Kanschat & Schötzau 07; Campos Pinto & Sonnendrücker 15]

- Approximation by averaging

$$|v_h - \mathcal{I}_h^{\mathrm{x,av}}(v_h)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{d\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{r}-m} \sum_{F\in\mathcal{F}_K^\circ} \|[\![v_h]\!]_F^{\mathrm{x}}\|_{L^r(F;\mathbb{R}^t)}$$

for all $l \in \{0{:}\,k+1\}$, all $p, r \in [1,\infty]$, and all $v_h \in P^{\mathrm{x,b}}(\mathcal{T}_h;\mathbb{R}^q)$

Fractional-order estimates

- Sobolev–Slobodeckij norm ($r = m + s$, $m \in \mathbb{N}$, $s \in (0,1)$)

$$\|v\|^p_{W^{r,p}(D)} := \|v\|^p_{W^{m,p}(D)} + \sum_{|\alpha|=m} \int_D \int_D \frac{\|\partial_\alpha v(\boldsymbol{x}) - \partial_\alpha v(\boldsymbol{y})\|^p_{\ell^2}}{\|\boldsymbol{x} - \boldsymbol{y}\|^{sp+d}_{\ell^2}} \,\mathrm{d}x\,\mathrm{d}y$$

- We observe that

$$\begin{aligned}
\|v - \mathcal{I}_h^{\mathrm{x,av}}(v)\|_{L^p(K)} &= \|v - \overline{v}_{D_K} - \mathcal{I}_h^{\mathrm{x,av}}(v - \overline{v}_{D_K})\|_{L^p(K)} \\
&\le \|v - \overline{v}_{D_K}\|_{L^p(K)} + \|\mathcal{I}_h^{\mathrm{x,av}}(v - \overline{v}_{D_K})\|_{L^p(K)} \\
&\le c\,\|v - \overline{v}_{D_K}\|_{L^p(D_K)}
\end{aligned}$$

- **Poincaré inequality in fractional Sobolev norms** (see also [Dupont & Scott 80; Heuer 14])

$$\|v - \overline{v}_U\|_{L^p(U)} \le h_U^s \left(\frac{h_U^d}{|U|}\right)^{\frac{1}{p}} |v|_{W^{s,p}(U)}$$

- For $r \ge 1$, we use Morrey's polynomial s.t. $\int_{D_K} \partial^\alpha(v - \pi(v))\,\mathrm{d}x = 0$

Boundary conditions

- There is a **trace map** $\gamma^{\mathrm{x}} : W^{1,1}(D;\mathbb{R}^q) \to L^1(\partial D;\mathbb{R}^t)$
  - $\gamma^{\mathrm{g}}(v) = v_{|\partial D}$, $\gamma^{\mathrm{c}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\times\boldsymbol{n}$, $\gamma^{\mathrm{d}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\cdot\boldsymbol{n}$
- Extension into bounded linear operator $\gamma^{\mathrm{x}} : V^{\mathrm{x}}(D) \to V^{\mathrm{x}}(\partial D)$ (exact structure of trace space not important here)
  - $V^{\mathrm{g}}(D) = \{v \in L^1(D) \mid \nabla v \in \boldsymbol{L}^1(D)\}$, etc.
- Define $V_0^{\mathrm{x}}(D) := \ker(\gamma^{\mathrm{x}}) = \{v \in V^{\mathrm{x}}(D) \mid \gamma^{\mathrm{x}}(v) = 0\}$
- Define $P_0^{\mathrm{x}}(\mathcal{T}_h) = P^{\mathrm{x}}(\mathcal{T}_h) \cap V_0^{\mathrm{x}}(D)$
- Control of DoFs at boundary: For all $v_h \in P^{\mathrm{x,b}}(\mathcal{T}_h;\mathbb{R}^q)$,

$$\left|\sigma^{\mathrm{x}}_{K,i}(v_h)\right| \le c\,\|\mathbb{A}^{\mathrm{x}}_K\|_{\ell^2}\|\gamma^{\mathrm{x}}(v_h)\|_{L^\infty(F;\mathbb{R}^t)}$$

- Internal degrees of freedom: $a \in \mathcal{A}_h^\circ$ means that $\gamma^{\mathrm{x}}(\varphi_a) = 0$

Main result

Theorem (EG15)

There is a quasi-interpolation operator $\mathcal{I}_{h0}^{\mathrm{x}} : L^1(D;\mathbb{R}^q) \to P_0^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$ s.t.

- $\mathcal{I}_{h0}^{\mathrm{x}}$ leaves $P_0^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$ pointwise invariant ($\mathcal{I}_{h0}^{\mathrm{x}} \circ \mathcal{I}_{h0}^{\mathrm{x}} = \mathcal{I}_{h0}^{\mathrm{x}}$)
- $\mathcal{I}_{h0}^{\mathrm{x}}$ is L^p-stable for all $p \in [1,\infty]$
- Optimal local approximation

$$|v - \mathcal{I}_{h0}^{\mathrm{x}}(v)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{s-m} |v|_{W^{s,p}(D_K^{\mathrm{x}};\mathbb{R}^q)}$$

  - for all $s \in [0, k+1]$, $m \in \{0{:}\lfloor s \rfloor\}$, $p \in [1,\infty)$ ($p \in [1,\infty]$ if $s \in \mathbb{N}$)
  - all $v \in W^{s,p}(D_K^{\mathrm{x}};\mathbb{R}^q)$
  - all $K \in \mathcal{T}_h^\circ := \{K \in \mathcal{T}_h \mid \forall i \in \mathcal{N},\ \mathrm{a}(K,i) \in \mathcal{A}_h^\circ\}$, i.e., if K "does not touch the boundary"

Idea of proof

- $\mathcal{I}_{h0}^{\mathrm{x}} = \mathcal{I}_{h0}^{\mathrm{x,av}} \circ \mathcal{I}_{h}^{\mathrm{x},\sharp}$
- Boundary values are prescribed to zero for averaging operator $\mathcal{I}_{h0}^{\mathrm{x,av}}$

$$\mathcal{I}_{h0}^{\mathrm{x,av}}(v_h)(\boldsymbol{y}) = \sum_{a\in\mathcal{A}_h^{\circ}} \frac{1}{\mathrm{card}(\mathcal{C}_a)} \sum_{(K,i)\in\mathcal{C}_a} \sigma_{K,i}(v_{h|K})\varphi_a(\boldsymbol{y})$$

Analysis close to the boundary

- $\mathcal{T}_h^\partial := \mathcal{T}_h \setminus \mathcal{T}_h^\circ$ collects the cells having at least one **boundary degree of freedom**; let $D^\partial := \mathrm{int}(\cup_{K\in\mathcal{T}_h^\partial} K)$
- If $K \in \mathcal{T}_h^\partial$, $sp > 1$ and $\gamma^{\mathrm{x}}(v) = 0$, the following local bound holds (with c depending on $|sp-1|$)

$$\|v - \mathcal{I}_{h0}^{\mathrm{x}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c\, h_K^s |v|_{W^{s,p}(D_K^{\mathrm{x}};\mathbb{R}^q)}$$

- If $sp < 1$, the following holds:

$$\|v - \mathcal{I}_{h0}^{\mathrm{x}}(v)\|_{L^p(D^\partial;\mathbb{R}^q)} \le c\, h^s \|v\|_{W^{s,p}(D;\mathbb{R}^q)}$$

i.e., **localization is lost** (as expected)

Technical result and corollary

- Trace inequality in fractional Sobolev spaces ($sp > 1$)

$$\|v\|_{L^p(F)} \leq c\,\big(h_K^{-\frac{1}{p}}\|v\|_{L^p(K)} + h_K^{s-\frac{1}{p}}|v|_{W^{s,p}(K)}\big)$$

- **Corollary**: Best approximation in fractional spaces with BC's
  - If $sp > 1$ and $v \in \{w \in W^{s,p}(D;\mathbb{R}^q) \mid \gamma^{\mathrm{x}}(w) = 0\}$

$$\inf_{w_h \in P_0^{\mathrm{x}}(\mathcal{T}_h)} \|v - w_h\|_{L^p(D;\mathbb{R}^q)} \leq c\, h^s |v|_{W^{s,p}(D;\mathbb{R}^q)}$$

  - If $sp < 1$ and $v \in W^{s,p}(D;\mathbb{R}^q)$

$$\inf_{w_h \in P_0^{\mathrm{x}}(\mathcal{T}_h)} \|v - w_h\|_{L^p(D;\mathbb{R}^q)} \leq c\, h^s \|v\|_{W^{s,p}(D;\mathbb{R}^q)}$$

L^1-stable commuting projections (1)

We build operators $\mathcal{J}_h^{\mathrm{x}} : L^1(D;\mathbb{R}^q) \to P^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$, $\mathrm{x} \in \{\mathrm{g},\mathrm{c},\mathrm{d},\mathrm{b}\}$, s.t.

- $\mathcal{J}_h^{\mathrm{x}}$ leaves $P^{\mathrm{x}}(\mathcal{T}_h;\mathbb{R}^q)$ pointwise invariant ($\mathcal{J}_h^{\mathrm{x}} \circ \mathcal{J}_h^{\mathrm{x}} = \mathcal{J}_h^{\mathrm{x}}$)
- $\mathcal{J}_h^{\mathrm{x}}$ is L^p-stable for all $p \in [1,\infty]$
- $\mathcal{J}_h^{\mathrm{x}}$ **commutes with the standard differential operators**

$$
\begin{array}{ccccccc}
V^{\mathrm{g}}(D) & \xrightarrow{\nabla} & \boldsymbol{V}^{\mathrm{c}}(D) & \xrightarrow{\nabla\times} & \boldsymbol{V}^{\mathrm{d}}(D) & \xrightarrow{\nabla\cdot} & L^1(D) \\
\downarrow \mathcal{J}_h^{\mathrm{g}} & & \downarrow \mathcal{J}_h^{\mathrm{c}} & & \downarrow \mathcal{J}_h^{\mathrm{d}} & & \downarrow \mathcal{J}_h^{\mathrm{b}} \\
P^{\mathrm{g}}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\nabla\times} & \boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) & \xrightarrow{\nabla\cdot} & P^{\mathrm{b}}(\mathcal{T}_h)
\end{array}
$$

A similar construction is possible with **boundary prescription**

L^1-stable commuting projections (2)

- The operators $\mathcal{J}_h^{\mathrm{x}}$ are important in many situations
  - discrete Poincaré inequalities for curl and div operators
  - analysis of compatible approximation of PDEs
  - bounded cochain projections in FEEC [Arnold, Falk & Winther 06]

- **Stability and polynomial invariance imply approximation**

  For all $v \in L^p(D)$,

$$\begin{aligned} \|v - \mathcal{J}_h^{\mathrm{x}}(v)\|_{L^p(D;\mathbb{R}^q)} &= \inf_{v_h \in P^{\mathrm{x}}(\mathcal{T}_h)} \|v - v_h - \mathcal{J}_h^{\mathrm{x}}(v - v_h)\|_{L^p(D;\mathbb{R}^q)} \\ &\leq \inf_{v_h \in P^{\mathrm{x}}(\mathcal{T}_h)} (1 + \|\mathcal{J}_h^{\mathrm{x}}\|_{\mathcal{L}(L^p;L^p)})\|v - v_h\|_{L^p(D;\mathbb{R}^q)} \\ &\leq c \inf_{v_h \in P^{\mathrm{x}}(\mathcal{T}_h)} \|v - v_h\|_{L^p(D;\mathbb{R}^q)} \end{aligned}$$

  Decay estimates of best-approximation error with $v_h = \mathcal{I}_h^{\mathrm{x}}(v)$

Discrete Poincaré inequality for curl (1)

- **Continuous Poincaré inequality**
  - assume ∂D to be connected
  - let ϵ be piecewise smooth and uniformly bounded away from zero
  - let $\boldsymbol{H}_{\times\boldsymbol{n}} := \{\boldsymbol{z} \in \boldsymbol{H}(\mathrm{curl}) \mid \nabla\cdot(\epsilon\boldsymbol{z}) = 0,\ \boldsymbol{z}\times\boldsymbol{n}_{|\partial D} = \boldsymbol{0}\}$
  - there is $c_{\mathrm{P,c}}$ s.t. $\|\boldsymbol{z}\|_{\boldsymbol{L}^2(D)} \le c_{\mathrm{P,c}}\|\nabla\times\boldsymbol{z}\|_{\boldsymbol{L}^2(D)}$ for all $\boldsymbol{z} \in \boldsymbol{H}_{\times\boldsymbol{n}}$
- **Discrete Poincaré inequality**
  - let $\boldsymbol{H}_{h,\times\boldsymbol{n}} := \{\boldsymbol{z}_h \in \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) \mid (\epsilon\boldsymbol{z}_h, \nabla q_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall q_h \in P_0^{\mathrm{g}}(\mathcal{T}_h)\}$
  - there is $\hat{c}_{\mathrm{P,c}}$ s.t. $\|\boldsymbol{z}_h\|_{\boldsymbol{L}^2(D)} \le \hat{c}_{\mathrm{P,c}}\|\nabla\times\boldsymbol{z}_h\|_{\boldsymbol{L}^2(D)}$ for all $\boldsymbol{z}_h \in \boldsymbol{H}_{h,\times\boldsymbol{n}}$
- Classical routes for proving discrete Poincaré inequality
  - if ϵ is smooth, (subtle) regularity estimates for vector potentials [Amrouche et al. 98] can be invoked [Hiptmair 02]
  - non-constructive proof based on discrete compactness argument [Kikuchi 89; Caorsi, Fernandes & Raffetto 00; Monk & Demkowicz 01]

Discrete Poincaré inequality for curl (2)

- Simple proof based on stable commuting projection
  - inspired from [Arnold, Falk & Winther 10]
  - cf. [Bonelle & AE 15] for lowest-order schemes on polyhedral meshes

- Sketch of proof: Let $\boldsymbol{z}_h \in \boldsymbol{H}_{h,\times\boldsymbol{n}}$
  - Let $\phi \in H^1_0(D)$ solve $\nabla\cdot(\epsilon\nabla\phi) = \nabla\cdot(\epsilon\boldsymbol{z}_h)$ and $\phi_{|\partial D} = 0$
  - Then $\boldsymbol{z} = \boldsymbol{z}_h - \nabla\phi \in \boldsymbol{H}_{\times\boldsymbol{n}}$ and $\nabla\times(\boldsymbol{z} - \boldsymbol{z}_h) = \boldsymbol{0}$

$$\begin{aligned}\|\epsilon^{\frac{1}{2}}\boldsymbol{z}_h\|^2_{\boldsymbol{L}^2} &= (\epsilon\boldsymbol{z}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{z}_h))_{\boldsymbol{L}^2} = (\epsilon\boldsymbol{z}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\nabla\phi))_{\boldsymbol{L}^2} + (\epsilon\boldsymbol{z}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{z}))_{\boldsymbol{L}^2}\\ &= (\epsilon\boldsymbol{z}_h, \nabla(\mathcal{J}^{\mathrm{g}}_{h0}\phi))_{\boldsymbol{L}^2} + (\epsilon\boldsymbol{z}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{z}))_{\boldsymbol{L}^2} = (\epsilon\boldsymbol{z}_h, \mathcal{J}^{\mathrm{c}}_{h0}(\boldsymbol{z}))_{\boldsymbol{L}^2}\end{aligned}$$

  - CS + continuous Poincaré inequality lead to

$$\|\epsilon^{\frac{1}{2}}\boldsymbol{z}_h\|_{\boldsymbol{L}^2} \le \epsilon_\sharp^{\frac{1}{2}}\|\mathcal{J}^{\mathrm{c}}_{h0}\|_{\mathcal{L}(\boldsymbol{L}^2)}\|\boldsymbol{z}\|_{\boldsymbol{L}^2} \le \epsilon_\sharp^{\frac{1}{2}}\|\mathcal{J}^{\mathrm{c}}_{h0}\|_{\mathcal{L}(\boldsymbol{L}^2)} c_{\mathrm{P,c}}\|\nabla\times\boldsymbol{z}_h\|_{\boldsymbol{L}^2}$$

- $\hat{c}_{\mathrm{P,c}} = (\epsilon_\sharp/\epsilon_\flat)^{\frac{1}{2}}\|\mathcal{J}^{\mathrm{c}}_{h0}\|_{\mathcal{L}(\boldsymbol{L}^2)} c_{\mathrm{P,c}}$

Constructing L^1-stable commuting projections

- **Canonical interpolation** operator $\widehat{\mathcal{I}}_h^{\mathrm{x}}$ commutes with differential operators, ... but is not stable in L^1
- **Mollification** operator $\mathcal{K}_\delta^{\mathrm{x}} : L^1(D;\mathbb{R}^q) \to C^\infty(\overline{D};\mathbb{R}^q)$, $\delta > 0$
- $\widehat{\mathcal{J}}_h^{\mathrm{x}} := \widehat{\mathcal{I}}_h^{\mathrm{x}} \circ \mathcal{K}_\delta^{\mathrm{x}}$ achieves stability and commutation [Schöberl 01; Christiansen 07], ... but is not a projection
- $\widehat{\mathcal{J}}_h^{\mathrm{x}}$ is invertible on $P^{\mathrm{x}}(\mathcal{T}_h)$ if $\delta \le ch$, c small enough [Schöberl 05]
  - on shape-regular meshes, δ is a (smooth) space-dependent function [Christiansen & Winther 08]
- $\mathcal{J}_h^{\mathrm{x}} := (\widehat{\mathcal{J}}_h^{\mathrm{x}}|_{P^{\mathrm{x}}(\mathcal{T}_h)})^{-1} \circ \widehat{\mathcal{I}}_h^{\mathrm{x}} \circ \mathcal{K}_\delta^{\mathrm{x}}$ satisfies **all the required properties**
- Boundary conditions can be prescribed
  - same mollification, just change the canonical interpolation operator

Mollification in strongly Lipschitz domains (1)

- Strongly Lipschitz domain $D \subset \mathbb{R}^d$ ($\Rightarrow$ uniform cone property)
- There is $\boldsymbol{j} \in \boldsymbol{C}^\infty(\mathbb{R}^d)$ whose restriction to ∂D is **globally transversal** and with unit norm [Hofmann, Mitrea & Taylor 07]
- We introduce the map $\varphi_\delta : \mathbb{R}^d \ni \boldsymbol{x} \mapsto \boldsymbol{x} - \delta \boldsymbol{j}(\boldsymbol{x}) \in \mathbb{R}^d$
- This map defines a **shrinking** of the domain D: There is $r > 0$ s.t.

$$\varphi_\delta(D) + B(\mathbf{0}, \delta r) \subset D, \qquad \forall \delta \in [0,1]$$

- The shrinking technique avoids **nontrivial extensions** outside D
- Let $\mathbb{J}_\delta(\boldsymbol{x})$ be the Jacobian matrix of φ at $\boldsymbol{x} \in D$
  - $\mathbb{J}_\delta$ converges uniformly to $\mathbb{I}$ as $\delta \to 0$

Mollification in strongly Lipschitz domains (2)

- Let us define [Schöberl 01]

$$(\mathcal{K}^{\mathrm{g}}_\delta f)(\boldsymbol{x}) := \int_{B(\boldsymbol{0},1)} \rho(\boldsymbol{y}) f(\varphi_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y$$

$$(\mathcal{K}^{\mathrm{c}}_\delta \boldsymbol{g})(\boldsymbol{x}) := \int_{B(\boldsymbol{0},1)} \rho(\boldsymbol{y}) \mathbb{J}^{\mathsf{T}}_\delta(\boldsymbol{x}) \boldsymbol{g}(\varphi_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y$$

$$(\mathcal{K}^{\mathrm{d}}_\delta \boldsymbol{g})(\boldsymbol{x}) := \int_{B(\boldsymbol{0},1)} \rho(\boldsymbol{y}) \det(\mathbb{J}_\delta(\boldsymbol{x})) \mathbb{J}^{-1}_\delta(\boldsymbol{x}) \boldsymbol{g}(\varphi_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y$$

$$(\mathcal{K}^{\mathrm{b}}_\delta f)(\boldsymbol{x}) := \int_{B(\boldsymbol{0},1)} \rho(\boldsymbol{y}) \det(\mathbb{J}_\delta(\boldsymbol{x})) f(\varphi_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y$$

  with smooth kernel ρ supported in $B(\boldsymbol{0},1)$

- **Commuting with differential operators**

$$\begin{array}{ccccccc}
V^{\mathrm{g}}(D) & \xrightarrow{\nabla} & \boldsymbol{V}^{\mathrm{c}}(D) & \xrightarrow{\nabla\times} & \boldsymbol{V}^{\mathrm{d}}(D) & \xrightarrow{\nabla\cdot} & L^1(D) \\
\downarrow \mathcal{K}^{\mathrm{g}}_\delta & & \downarrow \mathcal{K}^{\mathrm{c}}_\delta & & \downarrow \mathcal{K}^{\mathrm{d}}_\delta & & \downarrow \mathcal{K}^{\mathrm{b}}_\delta \\
C^\infty(D) & \xrightarrow{\nabla} & \boldsymbol{C}^\infty(D) & \xrightarrow{\nabla\times} & \boldsymbol{C}^\infty(D) & \xrightarrow{\nabla\cdot} & C^\infty(D)
\end{array}$$

Expansion-based mollification

- We introduce the map $\boldsymbol{\vartheta}_\delta : \mathbb{R}^d \ni \boldsymbol{x} \longmapsto \boldsymbol{x} + \delta \boldsymbol{k}(\boldsymbol{x}) \in \mathbb{R}^d$
  - $\boldsymbol{k}$ is a globally transversal field for $\mathcal{O} := B(\boldsymbol{x}_D, r_D)\backslash\overline{D}$ where $D \subset B(\boldsymbol{x}_D, r_D)$
  - there is $\zeta > 0$ s.t. $\boldsymbol{\vartheta}_\delta(\overline{\mathcal{O}}) + B(\boldsymbol{0}, 2\delta\zeta) \subset \mathcal{O}$ for all $\delta \in (0,1]$
- Let us define (see also [Bonito, Guermond & Luddens 15])

$$(\mathcal{K}^{\mathrm{g}}_{\delta,0} f)(\boldsymbol{x}) := \int_{B(\boldsymbol{0},1)} \rho(\boldsymbol{y}) \widetilde{f}(\boldsymbol{\vartheta}_\delta(\boldsymbol{x}) + (\delta\zeta)\boldsymbol{y})\,\mathrm{d}y \quad \text{etc.}$$

  where $\widetilde{f}$ is the extension by zero of $f \in L^1(D)$ over $\mathbb{R}^d$
- $\mathcal{K}^{\mathrm{x}}_{\delta,0} f \in C^\infty_0(D;\mathbb{R}^q)$ for all $f \in L^1(D;\mathbb{R}^q)$ and all $\delta \in (0,1]$

Application: traces of vector fields

- Let $p \in (1, \infty)$ and let us set

$$\boldsymbol{Z}^{\mathrm{c},p}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^p(D) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^p(D)\}$$
$$\boldsymbol{Z}^{\mathrm{d},p}(D) := \{\boldsymbol{v} \in \boldsymbol{L}^p(D) \mid \nabla\cdot\boldsymbol{v} \in L^p(D)\}$$

- **Tangential trace map** $\gamma_{\times\boldsymbol{n}} : \boldsymbol{Z}^{\mathrm{c},p}(D) \to \boldsymbol{W}^{-\frac{1}{p},p}(\partial D)$ s.t.

$$\langle\gamma_{\times\boldsymbol{n}}(\boldsymbol{v}), \boldsymbol{l}\rangle_{\partial D} := \int_D \boldsymbol{v}\cdot\nabla\times\boldsymbol{w}(\boldsymbol{l})\,\mathrm{d}x - \int_D \boldsymbol{w}(\boldsymbol{l})\cdot\nabla\times\boldsymbol{v}\,\mathrm{d}x \qquad (\gamma_0(\boldsymbol{w}(\boldsymbol{l})) = \boldsymbol{l} \in \boldsymbol{W}^{\frac{1}{p},p'})$$

- **Normal trace map** $\gamma_{\cdot\boldsymbol{n}} : \boldsymbol{Z}^{\mathrm{d},p}(D) \to W^{-\frac{1}{p},p}(\partial D)$ s.t.

$$\langle\gamma_{\cdot\boldsymbol{n}}(\boldsymbol{v}), l\rangle_{\partial D} := \int_D \boldsymbol{v}\cdot\nabla q(l)\,\mathrm{d}x + \int_D q(l)\nabla\cdot\boldsymbol{v}\,\mathrm{d}x \qquad (\gamma_0(q(l)) = l \in W^{\frac{1}{p},p'})$$

- **Characterization of kernel of traces**

$$\overline{\boldsymbol{C}_0^\infty(D)}^{\boldsymbol{Z}^{\mathrm{c},p}(D)} = \ker(\gamma_{\times\boldsymbol{n}}), \qquad \overline{\boldsymbol{C}_0^\infty(D)}^{\boldsymbol{Z}^{\mathrm{d},p}(D)} = \ker(\gamma_{\cdot\boldsymbol{n}})$$

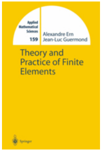
Converse inclusions proved using expansion-based mollification

Summary

- **Unified design and analysis** of quasi-interpolation operators for H^1-, $\boldsymbol{H}(\mathrm{curl})$-, and $\boldsymbol{H}(\mathrm{div})$-finite elements
- Decay estimates of best approximation errors in usual conforming FE subspaces for **arbitrary Sobolev norms**
- Two mollification techniques (**shrinking or expansion-based**), leading to stable, commuting projections onto conforming FE subspaces
- With or without boundary prescription

- References for this presentation
  - Quasi-interpolation and best approximation, arXiv 1505.06931 (to appear in ESAIM M2AN)
  - Mollification and commuting projections, Comput. Methods Appl. Math. 16:51–75, 2016

- The material (and much more) can be found in new book (2016?)
  - 10 chapters of 50 pages $\rightarrow$ 60 chapters of 10 pages + exercices



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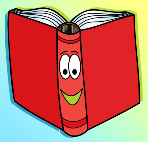

Thank you for your attention