Université Pierre et Marie Curie

École Doctorale de Sciences Mathématiques de Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Clément Dupont

Périodes des arrangements d'hyperplans et coproduit motivique

---

dirigée par Francis Brown

Rapporteurs :

Alexandru Dimca	Université Nice Sophia Antipolis
Marc Levine	Universität Duisburg-Essen

Soutenue le 26 septembre 2014 devant le jury composé de :

Yves André	ÉNS
Francis Brown	IHÉS
Pierre Cartier	IHÉS
Loïc Foissy	Université du Littoral Côte d'Opale
Marc Levine	Universität Duisburg-Essen
Ivan Marin	Université de Picardie Jules Verne
Joseph Oesterlé	Université Pierre et Marie Curie

Institut de Mathématiques de Jussieu -
Paris Rive Gauche
(UMR 7586)
4 place Jussieu
75005 Paris

UPMC
Ecole Doctorale de Sciences
Mathématiques de Paris Centre
4 place Jussieu
75252 Paris Cedex 05
Boite courrier 290

Remerciements

Je voudrais en premier lieu exprimer mon immense reconnaissance à mon directeur de thèse, Francis Brown, pour sa générosité et sa constante confiance en mon travail. Francis, merci d'avoir partagé avec moi ton goût et ta vision des mathématiques, de m'avoir initié au métier de chercheur et à un domaine de recherche passionnant, d'avoir su me conseiller, m'encourager, me guider, m'écouter, me supporter, me soutenir. C'était tout simplement un plaisir de travailler avec toi.

Je tiens à remercier Alexandru Dimca et Marc Levine qui ont accepté de rapporter cette thèse et de me transmettre leur précieux commentaires. Merci aussi à Pierre Cartier pour m'avoir toujours prêté une oreille attentive et pour sa présence à la soutenance. Yves André, Loïc Foissy, Ivan Marin et Joseph Oesterlé me font l'honneur d'être membres du jury de ma thèse et je les en remercie vivement.

Ma gratitude s'adresse également à Pierre Lochak qui m'a fait faire mes premiers pas dans la recherche mathématique. A la même époque, j'ai eu la chance de rencontrer Bruno Vallette (et sa grande famille opéradique) que je remercie pour ses encouragements et son amitié. Merci aussi à Spencer Bloch pour nos discussions mathématiques toujours enrichissantes et son soutien. J'ai énormément profité de mes interactions avec nombre de physiciens de tout poil et de toutes origines - notamment le groupe de recherche de Dirk Kreimer à Berlin - que je tiens à remercier collectivement. Merci aussi à ma jeune famille mathématique pour tous les moments passés ensemble: mes frangins Claire et David, à qui je souhaite tout le bonheur mathématique qu'ils méritent, et le cousin Samuel qui a eu la patience de relire un premier jet de cette thèse.

J'ai bénéficié de conditions de travail idéales à l'IMJ-PRG. Que tous ses membres, chercheurs et personnel administratif, soient ici remerciés. Merci surtout aux doctorants de Chevaleret et à ceux du couloir 15-16 de toutes les générations que j'ai eu le plaisir de côtoyer. Votre bonne humeur, votre folie douce, votre alcoolisme, votre enthousiasme, votre écoute, ont fait briller un beau soleil sur chaque jour de ces longues années de travail. A ceux pour qui le voyage de la thèse n'est pas encore terminé, je vous souhaite le meilleur.

Plus généralement, merci à tou(te)s mes ami(e)s, de Paris et d'ailleurs. Je mesure ma chance de vous avoir à mes côtés.

Mes derniers remerciements vont évidemment à ma famille: Camou, Lénou, Mamou, vous êtes géniales.

Résumé

Résumé

Dans cette thèse, on s'intéresse à des questions relatives aux arrangements d'hyperplans du point de vue des périodes motiviques. Suivant un programme initié par Beilinson *et al.*, on étudie une famille de périodes appelée polylogarithmes d'Aomoto et leurs variantes motiviques, vues comme éléments de l'algèbre de Hopf fondamentale de la catégorie des structures de Hodge-Tate mixtes, ou de la catégorie des motifs de Tate mixtes sur un corps de nombres.

On commence par calculer le coproduit motivique d'une famille de telles périodes, appelées polylogarithmes de dissection génériques, en montrant qu'il est régi par une formule combinatoire. Ce résultat généralise un théorème de Goncharov sur les intégrales itérées.

Puis, on introduit les bi-arrangements d'hyperplans, objets géométriques et combinatoires qui généralisent les arrangements d'hyperplans classiques. Le calcul de groupes de cohomologie relative associés aux bi-arrangements d'hyperplans est une étape cruciale dans la compréhension du coproduit motivique des polylogarithmes d'Aomoto. On définit des outils cohomologiques et combinatoires pour calculer ces groupes de cohomologie, qui éclairent dans un cadre global des objets classiques tels que l'algèbre d'Orlik-Solomon.

Mots-clefs

Périodes, polylogarithmes, arrangements d'hyperplans, motifs de Tate mixtes, structures de Hodge mixtes, algèbres de Hopf combinatoires.

Periods of hyperplane arrangements and motivic coproduct

Abstract

In this thesis, we deal with some questions about hyperplane arrangements from the viewpoint of motivic periods. Following a program initiated by Beilinson *et al.*, we study a family of periods called Aomoto polylogarithms and their motivic variants, viewed as elements of the fundamental Hopf algebra of the category of mixed Hodge-Tate structures, or the category of mixed Tate motives over a number field.

We start by computing the motivic coproduct of a family of such periods, called generic dissection polylogarithms, showing that it is governed by a combinatorial formula. This result generalizes a theorem of Goncharov on iterated integrals.

Then, we introduce bi-arrangements of hyperplanes, which are geometric and combinatorial objects which generalize classical hyperplane arrangements. The computation of relative cohomology groups associated to bi-arrangements of hyperplanes is a crucial step in the understanding of the motivic coproduct of Aomoto polylogarithms. We define cohomological and combinatorial tools to compute these cohomology groups, which recast classical objects such as the Orlik-Solomon algebra in a global setting.

Keywords

Periods, polylogarithms, hyperplane arrangements, mixed Tate motives, mixed Hodge structures, combinatorial Hopf algebras.

Contents

Chapter 1

Introduction

1.1 The philosophy of periods

Following [KZ01], we introduce periods and some very general ideas related to them.

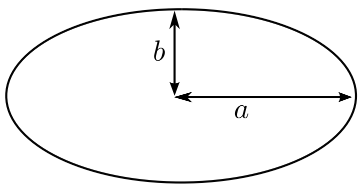
1.1.1 The algebra of periods

In their modern definition, periods are the values of the integrals that "come from algebraic geometry", even though it is fair to say that an important part of modern algebraic geometry and number theory actually finds its source in the study of these integrals.

For instance, let us consider the classical problem of finding a formula for the arc length of an ellipse, studied by Fagnano and Euler in the first half of the 18^{th} century. If we consider an ellipse with major radius a and minor radius b, then its arc length is given by the value of the integral

$$\ell = 4a \int_0^1 \frac{1 - \varepsilon^2 x^2}{\sqrt{(1-x^2)(1-\varepsilon^2 x^2)}} dx \tag{1.1}$$

where $\varepsilon^2 = 1 - \frac{a^2}{b^2}$ denotes the square of the eccentricity of the ellipse.



In terms of algebraic geometry, this leads to the introduction of the *elliptic curve* given by the equation

$$y^2 = (1-x^2)(1-\varepsilon^2 x^2).$$

We may indeed rewrite the above integral as

$$\ell = 4a \int_{\substack{0 \leqslant x \leqslant 1\,;\, y \geqslant 0\,; \\ y^2 = (1-x^2)(1-\varepsilon^2 x^2)}} \frac{(1-\varepsilon^2 x^2)}{y} dx.$$

If ε^2 is a rational number, then the dimensionless quantity $\frac{\ell}{a}$ is an effective period in the sense of the following definition.

Definition 1.1.1. An *effective period* is a complex number whose real and imaginary part can be written as an absolutely convergent integral

$$\int_\sigma f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n$$

where

- $f(x_1, \ldots, x_n)$ is a rational function with rational coefficients, and
- σ is a subset of $\mathbb{R}^n$ defined by a finite union and intersection of subsets of the form $\{g(x_1, \ldots, x_n) \geqslant 0\}$ with $g(x_1, \ldots, x_n)$ a rational function with rational coefficients.

Effective periods form a subring $\mathcal{P}^{\text{eff}}$ of the complex numbers. It contains the field $\overline{\mathbb{Q}}$ of algebraic numbers; for instance, one can realize $\sqrt{2}$ as an effective period by writing

$$\sqrt{2} = \int_{x^2 \leqslant 2\,;\, x \geqslant 0} dx.$$

More generally, we may use rational functions with algebraic coefficients (and even algebraic functions) to define periods, as in (1.1). Most examples of interesting periods are transcendental numbers, such as

$$\log(2) = \int_1^2 \frac{dx}{x} \qquad \text{and} \qquad \pi = \iint_{x^2+y^2 \leqslant 1} dx\, dy.$$

It is conjectured that $\frac{1}{\pi}$ is not an effective period. For reasons that will soon be made clearer (see Example 1.3.3), it is convenient to add it to the ring of effective periods and make the following definition.

Definition 1.1.2. A *period* is a complex number p such that for some non-negative integer N, the product $\pi^N p$ is an effective period.

The periods form an algebra

$$\mathcal{P} = \mathcal{P}^{\text{eff}} \left[\frac{1}{\pi}\right].$$

Although by definition $\mathcal{P}$ is countable, there is no known example of a complex number that is not a period (apart from "artificial" examples using cardinal-theoretic arguments). It is conjectured that the basis e of the natural logarithm, or the Euler-Mascheroni constant γ, are not periods.

1.1.2 Relations between periods

A very important problem is to understand the algebraic relations between periods. Since $\mathcal{P}$ is an algebra, it can be reduced to the study of linear relations. In the case of one-dimensional integrals, we have the following identities.

1. Bilinearity:
   - $\int_a^b (f(x) + \lambda g(x))dx = \int_a^b f(x)dx + \lambda \int_a^b g(x)dx$;
   - $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$.
2. Change of variables: $\int_a^b f(\varphi(u))\varphi'(u)du = \int_{\varphi(a)}^{\varphi(b)} f(x)dx$.
3. Stokes' formula: $\int_a^b f'(t)dt = f(b) - f(a)$.

For higher-dimensional integrals, these relations have to be replaced by their well-known generalizations.

Conjecture 1.1.3 ([KZ01], Conjecture 1)**.** *The above relations generate all linear relations over $\mathbb{Q}$ between periods.*

For instance, it is an easy exercise to prove the relation $\log(ab) = \log(a) + \log(b)$, $a, b \in \mathbb{Q}_{>0}$, using only these relations.

1.1.3 Periods and cohomology

It is convenient to give a more abstract definition of a period in terms of algebraic geometry. Let X be a smooth algebraic variety defined over $\mathbb{Q}$, let Y be a closed subvariety of X also defined over $\mathbb{Q}$, and let n be any integer. Let $\alpha \in \Omega^n(X)$ be a closed algebraic differential form on X (with rational coefficients) whose restriction to Y is zero. Let σ be a singular n-chain on $X(\mathbb{C})$ whose boundary lies on $Y(\mathbb{C})$. Then the value of the integral

$$\int_\sigma \alpha \tag{1.2}$$

is an effective period, and all effective periods arise in this way.

A period can then be viewed as a coefficient of the comparison isomorphism between two cohomology theories, as follows.

– Let $H^\bullet_{\mathrm{dR}}(X, Y)$ be the algebraic de Rham cohomology groups of X relative to Y [Gro66]; they are finite-dimensional vector spaces over $\mathbb{Q}$.

– Let $H^\bullet_{\mathrm{B}}(X, Y)$ be the singular [1] cohomology groups of $X(\mathbb{C})$ relative to $Y(\mathbb{C})$; they are finite-dimensional vector spaces over $\mathbb{Q}$.

– Let
$$\mathrm{comp}_{\mathrm{B,dR}} : H^\bullet_{\mathrm{dR}}(X, Y) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\cong} H^\bullet_{\mathrm{B}}(X, Y) \otimes_{\mathbb{Q}} \mathbb{C} \tag{1.3}$$
be the comparison isomorphism between de Rham and Betti cohomology, given by the integration of algebraic differential forms on singular chains. A matrix representing this isomorphism in $\mathbb{Q}$-bases of $H^\bullet_{\mathrm{dR}}(X, Y)$ and $H^\bullet_{\mathrm{B}}(X, Y)$ is called a *period matrix* of the pair (X, Y).

By definition, α defines a class in the de Rham cohomology group $H^n_{\mathrm{dR}}(X, Y)$, and σ defines a class in the Betti homology group $H^B_n(X, Y) = H^n_{\mathrm{B}}(X, Y)^\vee$. The period (1.2) is thus the corresponding coefficient $\langle \sigma, \mathrm{comp}_{\mathrm{B,dR}}(\alpha) \rangle$ of the period matrix.

Example 1.1.4. Let us look at $X = \mathbb{A}^1 \setminus \{0\}$ the punctured affine line, and $Y = \varnothing$. The de Rham cohomology group $H^1_{\mathrm{dR}}(\mathbb{A}^1 \setminus \{0\})$ is one-dimensional with basis the class of $\frac{dx}{x}$; the Betti homology group $H^1_{\mathrm{B}}(\mathbb{A}^1 \setminus \{0\})^\vee$ is one-dimensional with basis the class of a circle δ winding counterclockwise around 0 in $\mathbb{C}^*$. The corresponding period is $2i\pi = \int_\delta \frac{dz}{z}$.

Example 1.1.5. The integral $\log(2) = \int_1^2 \frac{dx}{x}$ corresponds to $X = \mathbb{A}^1 \setminus \{0\}$, $Y = \{1, 2\}$, $\alpha = \frac{dx}{x}$ and σ the straight path from 1 to 2. A basis of $H^1_{\mathrm{B}}(\mathbb{A}^1 \setminus \{0\}, \{1, 2\})^\vee$ is given by the classes of σ and δ, while a basis of $H^1_{\mathrm{dR}}(\mathbb{A}^1 \setminus \{0\}, \{1, 2\})$ is given by the classes of the forms dx and $\frac{dx}{x}$. The corresponding period matrix is then

$$\begin{pmatrix} 1 & \log(2) \\ 0 & 2i\pi \end{pmatrix}. \tag{1.4}$$

1. The letter B stands for Betti (co)homology, which is a standard term for singular (co)homology in algebraic geometry.

1.2 Aomoto polylogarithms

Among all periods, we will focus on certain subspaces of periods that are "defined by linear data". In other words, the algebraic varieties that we are going to consider will be defined by (products of) linear equations. We start by formalizing this idea, following [BVGS90].

1.2.1 Definition

Let us fix an integer $n \geqslant 1$. A *n-simplex* is an ordered family $L = (L_0, L_1, \ldots, L_n)$ of $n+1$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$. We use the same letter to denote the union $L = L_0 \cup L_1 \cup \cdots \cup L_n$ of the hyperplanes. A *stratum* of this simplex is a non-empty intersection $L_{i_1} \cap \cdots \cap L_{i_r}$ of hyperplanes from the family; the whole space $\mathbb{P}^n(\mathbb{C})$ is a stratum of L, and the other strata are called *strict*. We say that L is *degenerate* if the hyperplanes $L_0, L_1, \ldots, L_n$ are linearly dependent, i.e. if $L_0 \cap L_1 \cap \cdots \cap L_n$ is non-empty.

An *admissible pair of n-simplices* is a pair $(L; M) = (L_0, L_1, \ldots, L_n; M_0, M_1, \ldots, M_n)$ of simplices such that L and M do not have any strict stratum in common.

We fix an admissible pair of n-simplices $(L; M)$; let us choose for each L_i a defining linear form f_i (it is unique up to a non-zero multiplicative constant). We then have a canonical differential form

$$\omega_L = \sum_{i=0}^{n} (-1)^i \frac{df_0}{f_0} \wedge \cdots \wedge \widehat{\frac{df_i}{f_i}} \wedge \cdots \wedge \frac{df_n}{f_n}$$

on $\mathbb{P}^n(\mathbb{C}) \setminus L$. If L is degenerate, then a standard computation shows that ω_L is zero. Else we may choose projective coordinates $(x_0, x_1, \ldots, x_n)$ such that $L_i = \{x_i = 0\}$. In this system of coordinates and in the principal affine chart $\{x_0 \neq 0\}$, ω_L is nothing but the logarithmic n-form

$$\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$$

on the torus $(\mathbb{C}^*)^n$.

Now let $\Delta^n = \{0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant 1\}$ be the standard topological n-simplex and

$$\Delta_M : \Delta^n \to \mathbb{P}^n(\mathbb{C})$$

be an embedding that maps the interior of Δ^n to $\mathbb{P}^n(\mathbb{C}) \setminus L$, and the j-th face $\partial_j \Delta^n$ to the hyperplane M_j, for all $j = 0, \ldots, n$. Such an embedding exists if M is not degenerate, otherwise we may choose Δ_M to be constant.

Definition 1.2.1. The *Aomoto polylogarithm* associated to the admissible pair of simplices $(L; M)$ and the choice of Δ_M is the value of the absolutely convergent integral

$$I(L; M) = \int_{\Delta_M} \omega_L.$$

If $(L; M)$ is defined over a number field $F \hookrightarrow \mathbb{C}$, then $I(L; M)$ is a period in the sense of Definition 1.1.2 and we call it an *Aomoto period*.

The terminology is a reference to the work of Aomoto on the analytic properties of families of integrals indexed by sets of hyperplanes [Aom77, Aom82], that are close to those considered here.

Remark 1.2.2. Although it is not clear from the notation, $I(L; M)$ depends on a choice of integration simplex Δ_M, see Example 1.2.3.

1.2.2 Examples

Example 1.2.3 (Logarithms). Let $n = 1$. Then an admissible pair of 1-simplices is a quadruple $(a, b; c, d)$ of points of $\mathbb{P}^1(\mathbb{C})$ such that $\{a, b\} \cap \{c, d\} = \varnothing$. It is non-degenerate if and only if $a \neq b$ and $c \neq d$; in this case, we may choose a coordinate such that $(a, b; c, d) = (\infty, 0; r, 1)$, where by definition $r = \frac{(a-d)(b-c)}{(a-c)(b-d)} \in \mathbb{C}^*$ is the cross-ratio of (a, b, c, d).

The form ω_L is $\frac{dx}{x}$, and $\Delta_M : [0, 1] \to \mathbb{C}^*$ is a path that starts at 1 and ends at r. We picture this geometric situation in Figure 1.1, with $\mathbb{P}^1(\mathbb{C})$ being depicted as the Riemann sphere. Then $I(\infty, 0; r, 1) = \log(r)$ is the value of the logarithm of r computed via the path Δ_M.[2] Changing Δ_M by winding around 0 amounts to adding an integral multiple of $2i\pi$ to the value of $I(\infty, 0; r, 1)$. If r belongs to a number field $F \hookrightarrow \mathbb{C}$, then $\log(r)$ is an Aomoto period.

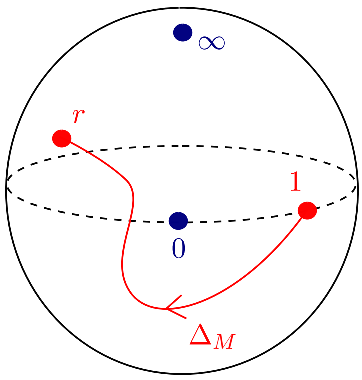


Figure 1.1: The example of $\log(r)$

Example 1.2.4 (Classical polylogarithms). Let $n \geqslant 1$ be an integer. The n-th *polylogarithm* is a function of a complex variable defined by the series

$$\mathrm{Li}_n(z) = \sum_{k \geqslant 1} \frac{z^k}{k^n}$$

for $|z| < 1$. For instance, $\mathrm{Li}_1(z) = -\log(1 - z)$. Using the differential system

$$\mathrm{Li}_n'(z) = \mathrm{Li}_{n-1}(z) \frac{dz}{z},$$

it is possible [Hai94] to analytically continue the polylogarithms as multivalued functions

$$\mathrm{Li}_n : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C},$$

i.e. holomorphic functions on a universal covering space of $\mathbb{C} \setminus \{0, 1\}$. It is easy to see that all the values of these functions are naturally Aomoto polylogarithms, and that their values at algebraic numbers are Aomoto periods. We explain how this works for $\mathrm{Li}_2(t) = \sum_{k \geqslant 0} \frac{t^k}{k^2}$ with a real argument $0 < t < 1$ to keep things simple. By integrating the differential equation above (or expanding $\frac{1}{1-x}$ as a geometric series and integrating), one gets the integral representation

$$\mathrm{Li}_2(t) = \iint_{0<x<y<t} \frac{dx\, dy}{(1-x)y}.$$

2. Our convention is the classical one according to which for a path $\gamma : [0, 1] \to \mathbb{C}^*$, we have $\partial_0 \gamma = \gamma(1)$ and $\partial_1 \gamma = \gamma(0)$. In [BVGS90] the authors use a different convention and get $I(\infty, 0; r, 1) = -\log(r)$.

We work in the projective plane $\mathbb{P}^2(\mathbb{C})$ with affine coordinates (x, y), whose real points are pictured in Figure 1.2. Let $L = (L_0, L_1, L_2)$ be the 2-simplex with L_0 the line at infinity, $L_1 = \{x = 1\}$ and $L_2 = \{y = 0\}$, pictured in dashed lines in Figure 1.2. Then

$$\omega_L = \frac{dx}{x-1} \wedge \frac{dy}{y}.$$

Now let $M = (M_0, M_1, M_2)$ be the 2-simplex with $M_0 = \{x = 0\}$, $M_1 = \{x = y\}$ and $M_2 = \{y = t\}$, pictured in full lines in Figure 1.2. If we choose for Δ_M the truncated standard triangle $0 < x < y < t$, shaded in Figure 1.2, then we have $\mathrm{Li}_2(t) = -I(L; M)$.

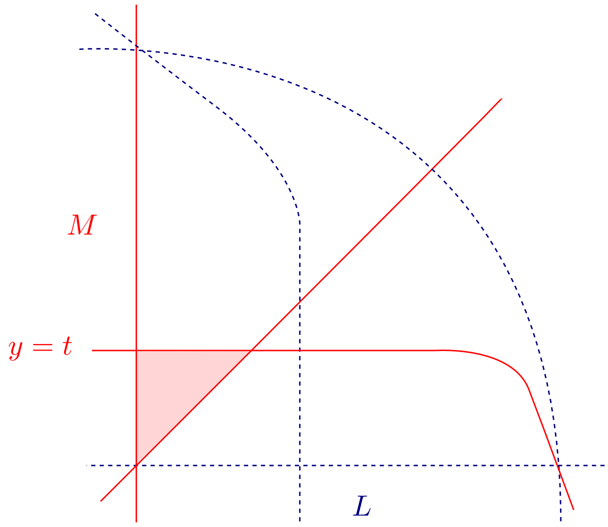


Figure 1.2: The example of $\mathrm{Li}_2(t)$

Example 1.2.5 (Multiple zeta values). For $n \geqslant 2$, the value at 1 of the n-th polylogarithm is the special value of the Riemann zeta function

$$\zeta(n) = \sum_{k \geqslant 1} \frac{1}{k^n}.$$

It has an integral representation:

$$\zeta(n) = \int_{0<t_1<\cdots<t_n<1} \frac{dt_1\, dt_2 \cdots dt_n}{(1-t_1)t_2 \cdots t_n},$$

which shows that it is an Aomoto period.

More generally, we define the *multiple zeta values* (originally introduced by Euler for $r \leqslant 2$)

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leqslant k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}} \tag{1.5}$$

for integer indices $n_1, \ldots, n_{r-1} \geqslant 1$, $n_r \geqslant 2$. It has been observed by Kontsevich that these numbers all have an integral representation

$$\zeta(n_1, \ldots, n_r) = (-1)^r \int_{0<t_1<\cdots<t_n<1} \frac{dt_1 \cdots dt_n}{(t_1 - \varepsilon_1) \cdots (t_n - \varepsilon_n)}$$

where $n = n_1 + \cdots + n_r$ and

$$(\varepsilon_1, \ldots, \varepsilon_n) = (\underbrace{1, 0, \ldots, 0}_{n_1}, \underbrace{1, 0, \ldots, 0}_{n_2} \ldots, \underbrace{1, 0, \ldots, 0}_{n_r})$$

This shows that multiple zeta values are Aomoto periods.

Example 1.2.6 (Iterated integrals on the punctured affine line). Let us fix pairwise distinct complex numbers $a_0, a_1, \ldots, a_n, a_{n+1}$ and a path γ from a_0 to a_{n+1} in $\mathbb{C} \setminus \{a_1, \ldots, a_n\}$. Let $\Delta^n(\gamma) \subset \mathbb{C}^n$ be the image of the standard simplex $0 < t_1 < \cdots < t_n < 1$ under $(t_1, \ldots, t_n) \mapsto (\gamma(t_1), \ldots, \gamma(t_n))$. Then the integral

$$\mathbb{I}(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{\Delta^n(\gamma)} \frac{dt_1 \cdots dt_n}{(t_1 - a_1) \cdots (t_n - a_n)} \tag{1.6}$$

is an Aomoto polylogarithm. The above integral may even converge (or be regularized) without any assumption on the complex numbers a_i. They form an important family of Aomoto polylogarithms called *iterated integrals* (on the punctured affine line), introduced and studied in full generality by Goncharov [Gon05]. For pairwise distinct indices, we will call them *generic iterated integrals*. For indices

$$(a_0; a_1, \ldots, a_n; a_{n+1}) = (0; \underbrace{1, 0, \ldots, 0}_{n}; t),$$

we recover the value $Li_n(t)$ of the n-th polylogarithm (up to a minus sign). For indices

$$(a_0; a_1, \ldots, a_n; a_{n+1}) = (0; \underbrace{1, 0, \ldots, 0}_{n_1}, \ldots, \underbrace{1, 0, \ldots, 0}_{n_r}; 1),$$

we recover the multiple zeta value $\zeta(n_1, \ldots, n_r)$ (up to the sign $(-1)^r$).

Example 1.2.7. Apart from iterated integrals, there are many interesting families of Aomoto polylogarithms. In this thesis (see § 1.8 in this Introduction), we will introduce and study the family of (generic) *dissection polylogarithms*, which are integrals of the form

$$\int_{\Delta} \frac{dt_1 \cdots dt_n}{f_1 \cdots f_n}$$

where f_i has the form $t_i - a_i$ with $a_i \in \mathbb{C}$ or $t_i - t_j - a_i$ for $a_i \in \mathbb{C}$ and $j \neq i$, and Δ is a singular simplex bounded by the hyperplanes $t_1 = a_0, t_1 = t_2, t_2 = t_3, \ldots, t_{n-1} = t_n, t_n = a_{n+1}$.

1.2.3 Scissors congruence relations

Some linear relations between Aomoto polylogarithms are easily written in terms of the corresponding pairs of simplices. In the following relations, whose proofs are straightforward, one has to choose the different integration domains in a consistent way. The name "scissors congruence relations" comes from the analogy with Hilbert's third problem [DS82].

– Degeneracy: if L or M is degenerate, then

$$I(L; M) = 0.$$

– Projective invariance: for $g \in \mathrm{PGL}_{n+1}(\mathbb{C})$ a projective transformation, we have

$$I(g.L; g.M) = I(L; M).$$

– Anti-symmetry in L: for σ a permutation of $\{0, \ldots, n\}$, we have

$$I(\sigma.L; M) = \mathrm{sgn}(\sigma)\, I(L; M).$$

where $\sigma.L = (L_{\sigma^{-1}(0)}, \ldots, L_{\sigma^{-1}(n)})$ and $\mathrm{sgn}(\sigma)$ is the signature of σ.

– Anti-symmetry in M: for σ a permutation of $\{0,\dots,n\}$, we have

$$I(L;\sigma.M) = \mathrm{sgn}(\sigma)\, I(L;M).$$

where $\sigma.M = (M_{\sigma^{-1}(0)},\dots,M_{\sigma^{-1}(n)})$ and $\mathrm{sgn}(\sigma)$ is the signature of σ.

– Additivity in L: If $L_0,\dots,L_{n+1}$ are such that for each i, $(L^{(i)};M) = (L_0,\dots,\widehat{L_i},\dots,L_{n+1};M)$ is an admissible pair of n-simplices, we have

$$\sum_{i=0}^{n+1}(-1)^i\, I(L^{(i)};M) = 0.$$

– Additivity in M: If $M_0,\dots,M_{n+1}$ are such that for each i, $(L;M^{(i)}) = (L;M_0,\dots,\widehat{M_i},\dots,M_{n+1})$ is an admissible pair of n-simplices, we have

$$\sum_{i=0}^{n+1}(-1)^i\, I(L;M^{(i)}) = 0.$$

The following conjecture, in the spirit of Conjecture 1.1.3, is implicit in [BVGS90].

Conjecture 1.2.8. *The above relations generate all linear relations over $\mathbb{Q}$ between Aomoto periods.*

1.3 Motivic periods and motivic Galois theory

The (still conjectural) theory of motives, initiated by Grothendieck, provides an abstract framework for working with periods. We review this formalism, following [And04, And09, Bro13].

1.3.1 Categories of mixed motives, and motivic periods

Let us assume that we have a category M of mixed motives over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$. It is an abelian $\mathbb{Q}$-linear category, and an algebraic variety X over $\mathbb{Q}$ defines objects $H^n(X)$ in M that are contravariant in X and should be thought of as "universal cohomology groups" of X. In addition, there should exist two functors

$$\omega_{\mathrm{dR}} : \mathsf{M} \to \mathsf{Vect}_{\mathbb{Q}}\ ,\ M \mapsto M_{\mathrm{dR}}$$

$$\omega_{\mathrm{B}} : \mathsf{M} \to \mathsf{Vect}_{\mathbb{Q}}\ ,\ M \mapsto M_{\mathrm{B}}$$

respectively called the de Rham and Betti realization functors, which map $H^n(X)$ to $H^n_{\mathrm{dR}}(X)$ and $H^n_{\mathrm{B}}(X)$ respectively. The same should be true for relative cohomology groups, i.e. a pair (X,Y) should define objects $H^n(X,Y)$ in M that are mapped to the usual de Rham and Betti relative cohomology groups by the corresponding realization functors. The comparison isomorphisms (1.3) should give an isomorphism of functors

$$\mathrm{comp}_{\mathrm{B,dR}} : \omega_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\cong} \omega_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Remark 1.3.1. At the present time, we do not have such an abelian category of mixed motives, but only triangulated categories of mixed motives [Lev98, VSF00]. Nevertheless, we will see in §1.4.2 that we have categories of mixed Tate motives over number fields (which should be thought of as subcategories of a still-to-be-defined category of mixed motives). Only certain pairs (X,Y) will define objects in these categories of mixed Tate motives, which will be enough for all our purposes.

Let us further assume that the category M is Tannakian [3], with ω_{dR} and ω_B two fiber functors. In particular, this means that M is endowed with a tensor product $\otimes$ such that ω_{dR} and ω_{B} are tensor functors; we assume that the comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ is compatible with the tensor product. Let

$$T = \underline{\mathrm{Isom}}^{\otimes}_{\mathsf{M}}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}})$$

be the torsor of tensor-preserving isomorphisms from the de Rham to the Betti realization functors. It is an affine $\mathbb{Q}$-scheme. More precisely, for F a field of characteristic zero, an F-point of T is a set of isomorphisms

$$M_{\mathrm{dR}} \otimes_{\mathbb{Q}} F \xrightarrow{\cong} M_{\mathrm{B}} \otimes_{\mathbb{Q}} F$$

which are functorial in $M \in \mathsf{M}$ and compatible with the tensor product. Thus, the comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ is a $\mathbb{C}$-point of T.

Let us denote by

$$\mathcal{P}^{\mathsf{M}} = \mathcal{O}(T)$$

the $\mathbb{Q}$-algebra of functions on T. By definition, it is spanned by triples (M, α, σ) with $M \in \mathsf{M}$, $\alpha \in M_{\mathrm{dR}}$ and $\sigma \in M_{\mathrm{B}}^{\vee}$. We call $\mathcal{P}^{\mathsf{M}}$ the *algebra of motivic periods.* The comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ gives a morphism of algebras

$$\mathrm{per} : \mathcal{P}^{\mathsf{M}} \to \mathbb{C}$$

called the *period morphism.* The algebra of periods $\mathcal{P}$ contains the image of per.

Conjecture 1.3.2 (Grothendieck's period conjecture for M)**.** *The period morphism is injective.*

If we start with a period $p = \int_\sigma \alpha$ as in §1.1.3, with $\alpha \in H^n_{\mathrm{dR}}(X,Y)$ and $\sigma \in H^n_{\mathrm{B}}(X,Y)^{\vee}$, then we define a triple

$$p^{\mathsf{M}} = (H^n(X,Y), \alpha, \sigma) \in \mathcal{P}^{\mathsf{M}}$$

which we call the *motivic period* corresponding to p. It maps to p via the period map:

$$\mathrm{per}(p^{\mathsf{M}}) = p.$$

If we start with another representation $p = \int_{\sigma'} \alpha'$ of p as a period, with $\alpha' \in H^n_{\mathrm{dR}}(X',Y')$ and $\sigma' \in H^n_{\mathrm{B}}(X',Y')^{\vee}$, then we get another definition of the motivic period corresponding to p. If Grothendieck's period conjecture for M holds, then we actually get the same element of $\mathcal{P}^{\mathsf{M}}$. Thus, in principle, the motivic period p^{M} is canonically attached to p.

Example 1.3.3. Going back to Example 1.1.4, the motive $H^1(\mathbb{A}^1 \setminus \{0\})$ is called the Lefschetz motive, denoted by $\mathbb{Q}(-1)$. Its period is $2i\pi$. If we want the category M to have a duality, there should exist a dual $\mathbb{Q}(1)$ to the Lefschetz motive, called the Tate motive. Its period is then necessarily $\frac{1}{2i\pi}$. This explains the inversion of π in Definition 1.1.2. More generally, we should have motives $\mathbb{Q}(-k)$ for $k \in \mathbb{Z}$, such that $\mathbb{Q}(-k)^{\vee} = \mathbb{Q}(k)$ and $\mathbb{Q}(-k) \otimes \mathbb{Q}(-l) \cong \mathbb{Q}(-k-l)$. The period of $\mathbb{Q}(-k)$ is $(2i\pi)^k$.

Example 1.3.4. Coming back to Example 1.1.5, the period $\log(2) = \int_1^2 \frac{dx}{x}$ corresponds to the motive $H = H^1(\mathbb{A}^1 \setminus \{0\}, \{1,2\})$. It is called a Kummer motive. In any reasonable category of mixed motives we should have a short exact sequence (implied in particular by the long exact sequence in relative cohomology)

$$0 \to \mathbb{Q}(0) \to H \to \mathbb{Q}(-1) \to 0$$

which is parallel to the period matrix (1.4).

3. Throughout this text, "Tannakian category" means "neutral Tannakian category over $\mathbb{Q}$".

1.3.2 Motivic Galois theory of periods

Let us denote by
$$G^{\mathsf{M}} = \underline{\mathrm{Aut}}^{\otimes}_{\mathsf{M}}(\omega_{\mathrm{dR}})$$
the Tannakian group of M relative to the fiber functor ω_{dR}. It is an affine group scheme over $\mathbb{Q}$, and ω_{dR} induces an equivalence of categories
$$\mathsf{M} \cong \mathsf{Rep}(G^{\mathsf{M}})$$
between M and the category of finite-dimensional representations of G^{M}.

By definition $T = \underline{\mathrm{Isom}}^{\otimes}_{\mathsf{M}}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}})$ is a G^{M}-torsor. Thus, there is a G^{M}-action on the algebra $\mathcal{P}^{\mathsf{M}}$ of functions on T. This action
$$G^{\mathsf{M}} \times \mathcal{P}^{\mathsf{M}} \to \mathcal{P}^{\mathsf{M}} \tag{1.7}$$
is called the *motivic Galois theory of periods*. If Grothendieck's period conjecture holds, then this gives an action of G^{M} on (a subalgebra of) the algebra of periods $\mathcal{P}$.

The motivic Galois theory of periods generalizes the classical Galois theory of algebraic numbers in the following sense: if $a \in \overline{\mathbb{Q}}$ is an algebraic number and F the number field generated by a and its conjugates, then G^{M} acts on $F \subset \mathcal{P}$ by the classical Galois theory through a quotient $G^{\mathsf{M}} \twoheadrightarrow \mathrm{Gal}(F/\mathbb{Q})$.

More generally, if p is a period, there should be a well-defined notion of the Galois group of p, which should be an algebraic group over $\mathbb{Q}$. For instance, the Galois group of π should be the multiplicative group $\mathbb{G}_m$. See [And09] for a more detailed treatment of this notion.

1.4 The Tannakian formalism in the mixed Tate setting

1.4.1 Mixed Hodge-Tate structures

The category of mixed Hodge structures is a good prototype for a category of mixed motives. We describe here the Tannakian formalism for mixed Hodge-Tate structures.

The fundamental Hopf algebra

Definition 1.4.1. A *mixed Hodge-Tate structure* is a mixed Hodge structure (H, W, F) such that for every integer k, $\mathrm{gr}^W_{2k+1} H = 0$ and $\mathrm{gr}^W_{2k} H$ is a sum of Tate structures $\mathbb{Q}(-k)$.

Equivalently, a mixed Hodge-Tate structure is the data of

- a finite-dimensional $\mathbb{Q}$-vector space H;
- an increasing filtration (the *weight filtration*) $W_{2\bullet} H$ of H indexed by even integers;
- a decreasing filtration (the *Hodge filtration*) $F^{\bullet} H_{\mathbb{C}}$ of the complexification $H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C}$ indexed by integers,

such that for every integer k we have a direct sum decomposition
$$W_{2k} H_{\mathbb{C}} = W_{2(k-1)} H_{\mathbb{C}} \oplus (W_{2k} H_{\mathbb{C}} \cap F^k H_{\mathbb{C}}). \tag{1.8}$$

Note that for a mixed Hodge-Tate structure (H, W, F), the direct sum decomposition (1.8) provides a canonical splitting of the weight filtration over $\mathbb{C}$:
$$\bigoplus_k \mathrm{gr}^W_{2k} H_{\mathbb{C}} \xrightarrow{\cong} H_{\mathbb{C}}. \tag{1.9}$$

We denote by MHTS the category of mixed Hodge-Tate structures. It is a Tannakian category, with two distinguished fiber functors:

$$\begin{aligned} \omega_{\mathrm{dR}} &: \mathsf{MHTS} \to \mathsf{Vect}_{\mathbb{Q}} \qquad (H, W, F) \mapsto \bigoplus_k \mathrm{Hom}_{\mathsf{MHTS}}(\mathbb{Q}(-k), \mathrm{gr}^W_{2k} H) \\ \text{and} \qquad \omega_{\mathrm{B}} &: \mathsf{MHTS} \to \mathsf{Vect}_{\mathbb{Q}} \qquad (H, W, F) \mapsto H. \end{aligned}$$

The canonical splitting (1.9) gives a comparison isomorphism

$$\mathrm{comp}_{\mathrm{B,dR}} : \omega_{\mathrm{dR}} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\cong} \omega_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}.$$

Let G^{MHTS} be the Tannakian group of MHTS for the de Rham fiber functor ω_{dR}. It is an affine group scheme over $\mathbb{Q}$, and we have an equivalence of categories

$$\mathsf{MHTS} \cong \mathsf{Rep}(G^{\mathsf{MHTS}})$$

between MHTS and the category of finite-dimensional representations of G^{MHTS}. Using in particular the fact that ω_{dR} is naturally graded, one derives a semi-direct product decomposition

$$G^{\mathsf{MHTS}} = \mathbb{G}_m \ltimes U^{\mathsf{MHTS}}$$

where $\mathbb{G}_m$ is the multiplicative group and U^{MHTS} is a pro-unipotent affine group scheme over $\mathbb{Q}$.

We let $\mathcal{H}^{\mathsf{MHTS}}$ be the Hopf algebra of functions on U^{MHTS}. Since $\mathbb{G}_m$ acts on U^{MHTS}, $\mathcal{H}^{\mathsf{MHTS}}$ is graded. The category MHTS is then equivalent to the category of finite-dimensional graded comodules over $\mathcal{H}^{\mathsf{MHTS}}$:

$$\mathsf{MHTS} \cong \mathsf{grComod}(\mathcal{H}^{\mathsf{MHTS}}).$$

We call $\mathcal{H}^{\mathsf{MHTS}}$ the *fundamental Hopf algebra* of the category MHTS.

The fact that the extension groups $\mathrm{Ext}^1_{\mathsf{MHTS}}(\mathbb{Q}(-n), \mathbb{Q}(0))$ vanish for $n \leqslant 0$ implies that $\mathcal{H}$ is non-negatively graded and connected:

$$\mathcal{H}^{\mathsf{MHTS}} = \bigoplus_{n \geqslant 0} \mathcal{H}^{\mathsf{MHTS}}_n \quad , \quad \mathcal{H}^{\mathsf{MHTS}}_0 = \mathbb{Q}.$$

An element of $\mathcal{H}^{\mathsf{MHTS}}_n$ is an equivalence class of triples (H, v, φ) where

- H is a mixed Hodge-Tate structure,
- $v \in \mathrm{Hom}_{\mathsf{MHTS}}(\mathbb{Q}(-n), \mathrm{gr}^W_{2n} H)$,
- $\varphi \in \mathrm{Hom}_{\mathsf{MHTS}}(\mathrm{gr}^W_0 H, \mathbb{Q}(0))$.

The equivalence relation is generated by the fact that $(H, v, \varphi) \equiv (H', v', \varphi')$ if there exists a morphism of mixed Hodge-Tate structures $f : H \to H'$ such that $\mathrm{gr}^W_{2n} f \circ v = v'$ and $\varphi' \circ \mathrm{gr}^W_0 f = \varphi$.

A triple (H, v, φ) as above is called an *n-framed mixed Hodge-Tate structure*, where v and φ are called the *framings*. The expression (H, v, φ) is bilinear in v and φ.

The product in $\mathcal{H}^{\mathsf{MHTS}}$ is defined via the tensor product:

$$(H, v, \varphi)(H', v', \varphi') = (H \otimes H', v \otimes v', \varphi \otimes \varphi').$$

The coproduct $\Delta_{n-k,k} : \mathcal{H}^{\mathsf{MHTS}}_n \to \mathcal{H}^{\mathsf{MHTS}}_{n-k} \otimes \mathcal{H}^{\mathsf{MHTS}}_k$ is abstractly defined by the formula

$$\Delta_{n-k,k}(H, v, \varphi) = \sum_i (H(k), v, b_i^\vee) \otimes (H, b_i, \varphi) \tag{1.10}$$

where (b_i) is any basis of $\mathrm{Hom}_{\mathsf{MHTS}}(\mathbb{Q}(-k), \mathrm{gr}^W_{2k} H)$ and $(b_i^\vee)$ the dual basis, with the identification $\mathrm{Hom}_{\mathsf{MHTS}}(\mathrm{gr}^W_0 H(k), \mathbb{Q}(0)) \cong \mathrm{Hom}_{\mathsf{MHTS}}(\mathbb{Q}(-k), \mathrm{gr}^W_{2k} H)^\vee$.

The algebra of mixed Hodge-Tate periods

Following the construction of §1.3.1 , we let $\mathcal{P}^{\mathsf{MHTS}}$ be the algebra of functions on the torsor of tensor-preserving isomorphisms between ω_{dR} and ω_{B}. We call it the *algebra of mixed Hodge-Tate periods*; it is graded:

$$\mathcal{P}^{\mathsf{MHTS}} = \bigoplus_n \mathcal{P}_n^{\mathsf{MHTS}}.$$

An element of $\mathcal{P}_n^{\mathsf{MHTS}}$ is an equivalence class of triples (H, v, δ) with H a mixed Hodge-Tate structure, $v \in \mathrm{Hom}_{\mathsf{MHTS}}(\mathbb{Q}(-n), \mathrm{gr}^W_{2n} H)$ and $\delta \in H^\vee$.

The product is defined in the same way as in $\mathcal{H}^{\mathsf{MHTS}}$. The action of G^{MHTS} on $\mathcal{P}^{\mathsf{MHTS}}$ translates as a comodule structure

$$\rho_{n-k,k} : \mathcal{P}_n^{\mathsf{MHTS}} \to \mathcal{H}_{n-k}^{\mathsf{MHTS}} \otimes \mathcal{P}_k^{\mathsf{MHTS}}$$

given by the formula

$$\rho_{n-k,k}(H, v, \delta) = \sum_i (H(k), v, b_i^\vee) \otimes (H, b_i, \delta) \tag{1.11}$$

where (b_i) is any basis of $\mathrm{Hom}_{\mathsf{MHTS}}(\mathbb{Q}(-k), \mathrm{gr}^W_{2k} H)$ and $(b_i^\vee)$ the dual basis.

Let $\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$ denote the subalgebra of $\mathcal{P}^{\mathsf{MHTS}}$ spanned by the triples (H, v, δ) where H has non-negative weights: $W_{-1}H = 0$. It is non-negatively graded, and is stable under the coaction by $\mathcal{H}^{\mathsf{MHTS}}$. We call it the elements of $\mathcal{P}^{\mathsf{MHTS}}$ *effective mixed Hodge-Tate periods.* There is a surjective morphism of graded algebras

$$\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}} \twoheadrightarrow \mathcal{H}^{\mathsf{MHTS}} \tag{1.12}$$

which sends (H, v, δ) to (H, v, φ) where φ is the image of δ via the map $H^\vee \twoheadrightarrow \left(\mathrm{gr}_0^W H\right)^\vee$. This is well-defined since by assumption $W_{-1}H = 0$. The surjection (1.12) is compatible with the structures of graded comodules over $\mathcal{H}^{\mathsf{MHTS}}$.

The comparison isomorphism $\mathrm{comp}_{\mathrm{B,dR}}$ gives a period morphism $\mathrm{per} : \mathcal{P}^{\mathsf{MHTS}} \to \mathbb{C}$. Note that contrary to what is expected in a motivic setting, this period morphism is not injective, and its image is $\mathbb{C}$. Actually, we will see in §1.6.2 that $\mathcal{P}_1^{\mathsf{MHTS},\mathrm{eff}}$ contains elements $\log^{\mathcal{P}}(r)$ for $r \in \mathbb{C}^*$, whose image by per is $\log(r)$, hence its image by per is already all of $\mathbb{C}$.

1.4.2 Mixed Tate motives

Let F be a number field and let $\mathsf{MTM}(F)$ be the category of mixed Tate motives over F with coefficients in $\mathbb{Q}$, defined in [Lev93]. It should be thought of as a subcategory of a still-to-be-defined category of mixed motives over $\mathbb{Q}$.

For simplicity, let us fix $\sigma : F \hookrightarrow \mathbb{C}$ a complex embedding of F; according to [Hub00, Hub04], we have a Hodge realization functor

$$\mathrm{real}_\sigma : \mathsf{MTM}(F) \to \mathsf{MHTS}. \tag{1.13}$$

This allows us to define fiber functors ω_{dR} and ω_{B} on $\mathsf{MTM}(F)$. It so happens that ω_{dR} is independent of σ and is given by the formula

$$\omega_{\mathrm{dR}}(H) = \bigoplus_k \mathrm{Hom}_{\mathsf{MTM}(F)}(\mathbb{Q}(-k), \mathrm{gr}^W_{2k} H)$$

where $W_{2\bullet}H$ is the canonical weight filtration on a mixed Tate motive H.

Remark 1.4.2. The "true" de Rham realization is defined over F; in this case it may be checked [DG05, Proposition 2.10] that it is nothing but $\omega_{\mathrm{dR}} \otimes_{\mathbb{Q}} F$, hence the (slightly abusive) notation.

As in the case of mixed Hodge-Tate structures, one may use the fiber functor ω_{dR} to define a fundamental Hopf algebra $\mathcal{H}^{\mathsf{MTM}(F)}$, which is non-negatively graded and connected, and such that MHTS is equivalent to the category of graded comodules over $\mathcal{H}^{\mathsf{MHTS}}$. The realization functor (1.13) induces a morphism of graded Hopf algebras

$$\mathrm{real}_\sigma : \mathcal{H}^{\mathsf{MTM}(F)} \to \mathcal{H}^{\mathsf{MHTS}}. \tag{1.14}$$

One may also define algebras $\mathcal{P}^{\mathsf{MTM}(F),\mathrm{eff}} \subset \mathcal{P}^{\mathsf{MTM}(F)}$ of (effective) mixed Tate periods over F, which depend on the choice of σ; we have a natural surjection

$$\mathcal{P}^{\mathsf{MTM}(F),\mathrm{eff}} \twoheadrightarrow \mathcal{H}^{\mathsf{MTM}(F)}.$$

The relation between mixed Tate motives and higher algebraic K-theory resides in the isomorphisms [DG05, 1.6]

$$\mathrm{Ext}^1_{\mathsf{MTM}(F)}(\mathbb{Q}(-n), \mathbb{Q}(0)) \cong K_{2n-1}(F) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{1.15}$$

which rely on the work of Borel, Beilinson, Bloch, Levine ...

1.5 Motivic Aomoto polylogarithms

We want to define a motivic version of a given Aomoto polylogarithm $I(L; M)$. As a first step, we have to find a cohomology group which has $I(L; M)$ as a period. We first discuss this step for the case of $\zeta(2)$, following the discussion in [GM04].

1.5.1 The example of $\zeta(2)$

We work with the Aomoto period

$$\zeta(2) = \iint_{0<x<y<1} \frac{dx}{1-x} \frac{dy}{y}.$$

Let us consider the geometric situation pictured in the left-hand side of Figure 1.3 below. In the projective plane $\mathbb{P}^2$ with affine coordinates (x, y), let L (the dashed lines) be the divisor of poles of the form $\omega = \frac{dx}{1-x} \frac{dy}{y}$. It is the union of the line at infinity and the lines $\{x = 1\}$, $\{y = 0\}$. Let now M (the full lines) be the Zariski closure of the boundary of the domain of integration $\Delta = \{0 < x < y < 1\}$ (the shaded triangle). It is the union of the lines $\{x = 0\}$, $\{x = y\}$, $\{y = 1\}$.

The divisor $L \cup M$ is not normal crossing in $\mathbb{P}^2$. We let $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the blow-up along the points P_1, P_2, Q_1, Q_2, and let E_1, E_2, F_1, F_2 be the corresponding exceptional divisors. We let $\widetilde{L}$ be the union of E_1, E_2, and the strict transforms of the three lines from L; we let $\widetilde{M}$ be the union of F_1, F_2, and the strict transforms of the three lines from M. Now $\widetilde{L} \cup \widetilde{M}$ is a normal crossing divisor in $\widetilde{\mathbb{P}^2}$, pictured in the right-hand side of Figure 1.3.

Let us introduce the relative cohomology group (with coefficients in $\mathbb{Q}$)

$$H = H^2(\widetilde{\mathbb{P}^2} \setminus \widetilde{L}, \widetilde{M} \setminus \widetilde{M} \cap \widetilde{L}). \tag{1.16}$$

The differential form $\pi^*(\omega)$ is closed and has poles along $\widetilde{L}$, hence defines a class in the de Rham cohomology group H_{dR}. The domain $\pi^{-1}(\Delta)$ (the shaded pentagon in Figure 1.3) has its boundary on $\widetilde{M}$, hence defines a class in the Betti homology group $H_{\mathrm{B}}^\vee$. Hence,

$$\zeta(2) = \int_\Delta \omega = \int_{\pi^{-1}(\Delta)} \pi^*(\omega)$$

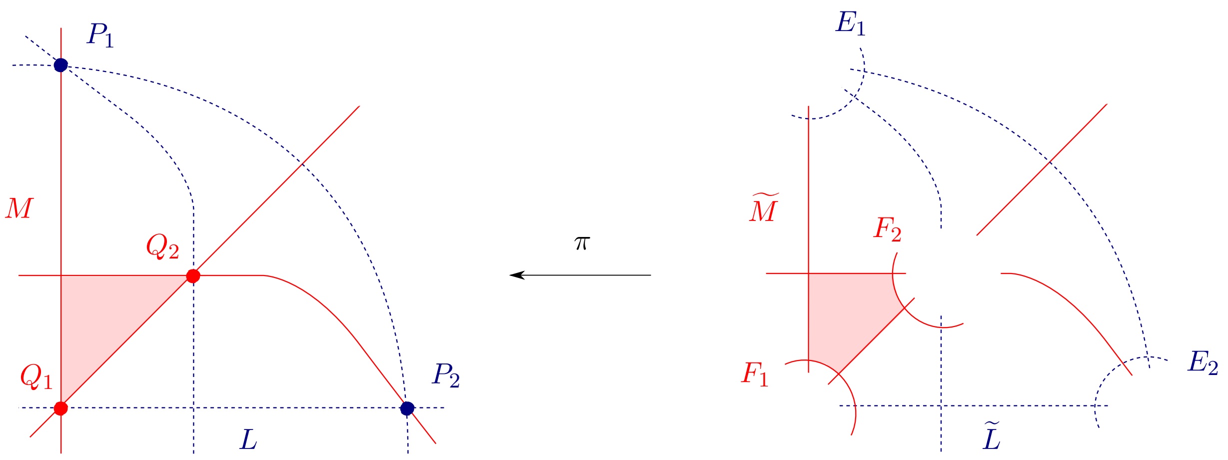


Figure 1.3: The example of $\zeta(2)$

is a period of H.

Why work in the blow-up? One could want to replace H with the (simpler) relative cohomology group $H' = H^2(\mathbb{P}^2 \setminus L, M \setminus M \cap L)$. This is not the right thing to do because the boundary of Δ intersects L, hence Δ does not define a homology class in H'_{B}. This is why we have to work in the blown-up situation.

One can note that such a blow-up procedure is also standard in the algebro-geometric study of Feynman integrals [BEK06], in the process of finding a motive which has a given Feynman integral as a period.

1.5.2 The general case

Let $(L; M)$ be an admissible pair of n-simplices, that we assume to be non-degenerate. We fix a choice of integration simplex Δ_M for the corresponding Aomoto polylogarithm $I(L; M)$. We are going to define

- an element $I^{\mathcal{P}}(L; M) \in \mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$;
- an element $I^{\mathcal{H}}(L; M) \in \mathcal{H}^{\mathsf{MHTS}}$.

They will both be called the *motivic Aomoto polylogarithm* corresponding to $(L; M)$. They satisfy the following properties.

- $\mathrm{per}(I^{\mathcal{P}}(L; M)) = I(L; M)$;
- $I^{\mathcal{H}}(L; M)$ is the image of $I^{\mathcal{P}}(L; M)$ under the surjection $\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}} \twoheadrightarrow \mathcal{H}^{\mathsf{MHTS}}$;
- $I^{\mathcal{H}}(L; M)$ is independent of the choice of Δ_M.

This last point is the main motivation for the notation $I(L; M)$ where Δ_M does not appear.

As in the case of $\zeta(2)$, we first want to resolve the singularities of $L \cup M$. A *stratum* of $(L; M)$ is a non-empty intersection $L_I \cap M_J$. An *L-stratum* is a stratum of type L_I with $|I| > 0$, and an *M-stratum* is a stratum of type M_J with $|J| > 0$; the other strata are called *mixed strata*. By the admissibility condition, an L-stratum and an M-stratum are never equal.

Let

$$\pi : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$$

be the iterated blow-up defined by the following procedure.

– Blow-up the strata of $(L; M)$ of dimension 0 (the points);

– Blow-up the strict transforms of the strata of $(L; M)$ of dimension 1;

– Blow-up the strict transforms of the strata of $(L; M)$ of dimension 2;

– ...

– Blow-up the strict transforms of the strata of $(L; M)$ of dimension $(n-1)$;

One notes that at each step, the center of the blow-up is a disjoint union of smooth varieties.

The total transform $\pi^{-1}(L \cup M)$ is a normal crossing divisor whose irreducible components are in bijection with the strict strata of $(L; M)$; they are thus of three types:

– the L-components, which are the exceptional divisors corresponding to L-strata;

– the M-components, which are the exceptional divisors corresponding to M-strata;

– the mixed components, which are the exceptional divisors corresponding to mixed strata.

The following Lemma is easily proved using local coordinates.

Lemma 1.5.1. *1. The divisor of the poles of the form $\pi^*(\omega_L)$ is the union of the L-components.*

2. The Zariski closure of the boundary of $\pi^{-1}(\Delta_M)$ is the union of the M-components.[4]

Let us write $\widetilde{L}$ for the union of the L-components and $\widetilde{M}$ for the union of the M-components. We then look at the relative cohomology group (with coefficients in $\mathbb{Q}$)

$$H(L; M) = H^n(\widetilde{\mathbb{P}^n} \setminus \widetilde{L}, \widetilde{M} \setminus \widetilde{M} \cap \widetilde{L}).$$

It is endowed with a mixed Hodge structure which is easily seen to be a mixed Hodge-Tate structure, hence $H(L; M)$ is an object of the category MHTS. Its weights are between 0 and $2n$, and one easily proves the following facts.

– The weight $2n$ part

$$\mathrm{gr}^W_{2n} H(L; M) \cong \mathrm{gr}^W_{2n} H^n(\widetilde{\mathbb{P}^n} \setminus \widetilde{L}) \cong H^n(\mathbb{P}^n \setminus L) \cong \mathbb{Q}(-n)$$

is 1-dimensional with a canonical generator α (fixed by the ordering of L) corresponding to $\pi^*(\omega_L)$ under the morphism

$$H(L; M) \twoheadrightarrow \mathrm{gr}^W_{2n} H(L; M).$$

– The dual of the weight 0 part

$$\left(\mathrm{gr}^W_0 H(L; M)\right)^\vee \cong \left(\mathrm{gr}^W_0 H^n(\widetilde{\mathbb{P}^n}, \widetilde{M})\right)^\vee \cong H_n(\mathbb{P}^n, M) \cong \mathbb{Q}(0)$$

is 1-dimensional with canonical generator φ (fixed by the ordering of M) corresponding to $\pi^{-1}(\Delta_M)$ under the morphism

$$H(L; M)^\vee \twoheadrightarrow \left(\mathrm{gr}^W_0 H(L; M)\right)^\vee.$$

4. Here, as in the example of §1.5.1, Δ_M denotes the *open* simplex. By its boundary, we mean the image of the boundary of the standard simplex Δ^n.

We define

$$I^{\mathcal{P}}(L;M) = (H(L;M), \alpha, \pi^{-1}(\Delta_M)) \quad \in \mathcal{P}_n^{\mathsf{MHTS},\mathrm{eff}}$$

and

$$I^{\mathcal{H}}(L;M) = (H(L;M), \alpha, \varphi) \quad \in \mathcal{H}_n^{\mathsf{MHTS}},$$

which is independent of the choice of Δ_M. By definition, the period of $I^{\mathcal{P}}(L;M)$ is

$$\mathrm{per}(I^{\mathcal{P}}(L;M)) = \int_{\pi^{-1}(\Delta_M)} \pi^*(\omega_L) = \int_{\Delta_M} \omega_L = I(L;M)$$

and its image under the surjection $\mathcal{P}_n^{\mathsf{MHTS},\mathrm{eff}} \twoheadrightarrow \mathcal{H}_n^{\mathsf{MHTS}}$ is $I^{\mathcal{H}}(L;M)$.

1.5.3 Variant: the need for a coloring

In the construction of the previous paragraph, we did not take into account the mixed components. In practice, in order to compute the cohomology groups $H^n(\widetilde{\mathbb{P}^n} \setminus \widetilde{L}, \widetilde{M} \setminus \widetilde{M} \cap \widetilde{L})$, it is convenient to be in a situation where $\widetilde{L} \cup \widetilde{M}$ is all of $\pi^{-1}(L \cup M)$. We may then define $\widetilde{L}$ to be the union of the L-components and the mixed components, and $\widetilde{M}$ to be the union of the M-components; dually, we may define $\widetilde{L}$ to be the union of the L-components and $\widetilde{M}$ to be the union of the M-components and the mixed components. These are the two "extreme" choices that Goncharov looks at in [Gon02]. In between these two possibilities, one may simply choose to partition the mixed components into two sets, one that will form part in $\widetilde{L}$ and the other one in $\widetilde{M}$.

In order to keep track of these choices, one introduces a *coloring function* χ which associates to every strict stratum of $(L;M)$ a *color* $\chi(S) \in \{\lambda, \mu\}$. The L-strata should have the color λ, and the M-strata should have the color μ. This coloring induces a coloring on the irreducible components of $\pi^{-1}(L\cup M)$, which by definition are in bijection with the strict strata. Then we may define $\widetilde{L}$ to be the union of the components with color λ, and $\widetilde{M}$ to be the union of the components with color μ.

We then have relative cohomology groups $H(L;M;\chi) = H^n(\widetilde{\mathbb{P}^n}\setminus\widetilde{L}, \widetilde{M}\setminus\widetilde{M}\cap\widetilde{L})$ which depend on the coloring function χ. One easily sees that the corresponding framed objects $I^{\mathcal{P}}(L;M)$ and $I^{\mathcal{H}}(L;M)$ that we define out of these cohomology groups are *independent of* χ.

1.5.4 The setting of mixed Tate motives

If we start with an Aomoto period $I(L;M)$ with $(L;M)$ defined over a number field $F \hookrightarrow \mathbb{C}$, then $H^n(\widetilde{\mathbb{P}^n}\setminus\widetilde{L}, \widetilde{M}\setminus\widetilde{M}\cap\widetilde{L})$ defines an object of the category $\mathsf{MTM}(F)$ whose Hodge realization is the mixed Hodge-Tate structure defined above (see [Gon02], Proposition 3.6). We then have motivic Aomoto polylogarithms $I^{\mathcal{P}}(L;M)$ and $I^{\mathcal{H}}(L;M)$ in $\mathcal{P}_n^{\mathsf{MTM}(F),\mathrm{eff}}$ and $\mathcal{H}_n^{\mathsf{MTM}(F)}$ respectively.

The following conjecture is implicit in [BVGS90].

Conjecture 1.5.2. *The elements $I^{\mathcal{P}}(L;M)$ span $\mathcal{P}^{\mathsf{MTM}(F),\mathrm{eff}}$, hence the elements $I^{\mathcal{H}}(L;M)$ span $\mathcal{H}^{\mathsf{MTM}(F)}$.*

The initial motivation of [BVGS90] for introducing motivic Aomoto polylogarithms comes from higher algebraic K-theory. Indeed, Conjecture 1.5.2 and (the motivic version of) Conjecture 1.2.8 give a set of explicit generators and relations for $\mathcal{H}^{\mathsf{MTM}(F)}$. If we knew how to compute the motivic coproduct of these generators, we would be able to produce explicit "combinatorial" complexes that compute the rational K-groups of F, in view of (1.15). One hope is that these complexes would be natural enough to compute the rational K-groups of *any* field F, in the spirit of the Bloch group and its variants (see [Gon95] for a survey on this topic).

An alternative approach for finding generators of categories of mixed Tate motives is Deligne and Goncharov's theory of motivic fundamental groups [Del89, Gon05, DG05, Del10]. Brown's theorem [Bro12] shows that the motivic fundamental group of $\mathbb{P}^1 \setminus \{\infty, 0, 1\}$ generates the Tannakian category $\mathsf{MTM}(\mathbb{Z}) \subset \mathsf{MTM}(\mathbb{Q})$ of mixed Tate motives over the integers. Similar results for rings of integers in cyclotomic number fields were previously obtained by Deligne [Del10]. More precisely, let F_N denotes the N-th cyclotomic field, $\mathcal{O}_N$ denotes its ring of integers, and μ_N denote the group of N-th roots of unity. It is proved that for $N \in \{2, 3, 4, 8\}$ the motivic fundamental group of $\mathbb{P}^1 \setminus \{\infty, 0, \mu_N\}$ generates the Tannakian category $\mathsf{MTM}(\mathcal{O}_N[1/N]) \subset \mathsf{MTM}(F_N)$. The limits of such an approach were discovered by Goncharov [Gon01], who showed that it is no longer the case for N prime $\geqslant 5$. Finding the "missing generators" is one of the motivation for introducing all motivic Aomoto polylogarithms.

1.6 The motivic coproduct of Aomoto polylogarithms

1.6.1 The general question

Summing up what we have done so far, we have associated to each admissible pair of simplices $(L; M)$ an element $I^{\mathcal{P}}(L; M)$ in the algebra of effective mixed Hodge-Tate periods $\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$, and its image $I^{\mathcal{H}}(L; M)$ in the fundamental Hopf algebra $\mathcal{H}^{\mathsf{MHTS}}$ of the category of mixed Hodge-Tate structures.

Both $\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$ and $\mathcal{H}^{\mathsf{MHTS}}$ are acted upon by the motivic Galois group G^{MHTS}. We want to understand how the latter acts on the motivic Aomoto polylogarithms. This amounts to understanding the motivic Galois theory for Aomoto polylogarithms.

In practice, it is easier to work with the Hopf algebra $\mathcal{H}^{\mathsf{MHTS}}$ and compute the coaction

$$\rho : \mathcal{P}^{\mathsf{MHTS},\mathrm{eff}} \to \mathcal{H}^{\mathsf{MHTS}} \otimes \mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$$

and the coproduct

$$\Delta : \mathcal{H}^{\mathsf{MHTS}} \to \mathcal{H}^{\mathsf{MHTS}} \otimes \mathcal{H}^{\mathsf{MHTS}}.$$

We then pose the following general problem.

Problem A: For a given admissible pair of simplices $(L; M)$, compute the motivic coaction ρ on $I^{\mathcal{P}}(L; M)$ and the motivic coproduct Δ on $I^{\mathcal{H}}(L; M)$.

Of course, the answer for the motivic coaction gives an answer for the motivic coproduct by applying the surjection $\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}} \twoheadrightarrow \mathcal{H}^{\mathsf{MHTS}}$. This being said, we will focus on the motivic coproduct because it allows us to state the results in a more symmetric manner, and without taking care of choosing integration simplices for Aomoto polylogarithms. In practice, though, these two questions are equally difficult.

If $(L;M)$ is defined over a number field $F \hookrightarrow \mathbb{C}$, then we may ask the same question in the setting of mixed Tate motives described in §1.5.4. Although we will not mention it any further, all the formulas that follow are valid in this setting.

Remark 1.6.1. In view of Conjecture 1.5.2, one should be able to express the motivic coproduct $\Delta(I^{\mathcal{H}}(L;M))$ in terms of motivic Aomoto polylogarithms $I^{\mathcal{H}}(L',M')$.

1.6.2 The case of dimension 1: logarithms

Let $r \in \mathbb{C}^*$ be a fixed complex number, and let $\log^{\mathcal{P}}(r) = I^{\mathcal{P}}(\infty,0;r,1) \in \mathcal{P}_1^{\mathsf{MHTS},\mathrm{eff}}$ and $\log^{\mathcal{H}}(r) = I^{\mathcal{H}}(\infty,0;r,1) \in \mathcal{H}_1^{\mathsf{MHTS}}$ be the motivic periods corresponding to $\log(r)$. For $r=1$ we have the motivic version $(2i\pi)^{\mathcal{P}}$ of $2i\pi$. Note that $(2i\pi)^{\mathcal{H}} = 0$. The motivic coaction is given by

$$\rho(\log^{\mathcal{P}}(r)) = 1 \otimes \log^{\mathcal{P}}(r) + \log^{\mathcal{H}}(r) \otimes 1. \tag{1.17}$$

More symmetrically, the motivic coproduct is given by

$$\Delta(\log^{\mathcal{H}}(r)) = 1 \otimes \log^{\mathcal{H}}(r) + \log^{\mathcal{H}}(r) \otimes 1.$$

1.6.3 The case of dimension 2

The authors of [BVGS90] give formulas for $\Delta(I^{\mathcal{H}}(L;M))$ for $(L;M) = (L_0,L_1,L_2;M_0,M_1,M_2)$ an admissible pair of 2-simplices. Although these formulas are not proved, one can check that they indeed give the right answer (up to a sign, see Remark 1.6.3 below).

For instance, we have a motivic dilogarithm $Li_2^{\mathcal{H}}(t)$ for $t \in \mathbb{C}\setminus\{0,1\}$ whose motivic coproduct is given by

$$\Delta_{1,1}(Li_2^{\mathcal{H}}(t)) = -\log^{\mathcal{H}}(t) \otimes \log^{\mathcal{H}}(1-t).$$

Remark 1.6.2. In order to compare with the formulas of [BVGS90], one has to interchange the left-hand side and the right-hand side of the coproduct. This is just a matter of convention.

Remark 1.6.3. There is a sign error in the formulas of [BVGS90]: the component $\Delta_{1,1}$ should be multiplied by the sign -1. It so happens that the formula (2.14) *op.cit.* for $\Delta_{1,1}(Li_2^{\mathcal{H}}(t))$ is correct, but is actually derived from $-Li_2^{\mathcal{H}}(t)$ (L_1 and L_2 should be interchanged in Figure 1.4 *op. cit.*).

1.6.4 The case of dimension 3

Zhao [Zha04] gives formulas to compute the motivic coproduct $\Delta(I^{\mathcal{H}}(L;M))$ for $(L;M) = (L_0,L_1,L_2,L_3;M_0,M_1,M_2,M_3)$ an admissible pair of 3-simplices. Although these formulas are not proved, one can check that they indeed give the right answer (up to a sign).

1.6.5 The generic case

A pair of n-simplices $(L;M) = (L_0,L_1,\ldots,L_n;M_0,M_1,\ldots,M_n)$ is *generic* if its $(2n+2)$ hyperplanes are in general position in $\mathbb{P}^n$. It is then automatically admissible. For such a pair, $L \cup M$ is a normal-crossing divisor and the corresponding mixed Hodge-Tate structure is simply $H^n(\mathbb{P}^n \setminus L, M \setminus L \cap M)$. In this case, it is easy to compute the motivic coproduct. First we introduce a little bit of notation.

For a subset $I \subset \{1,\ldots,n\}$, we denote by $\overline{I}$ its complement in $\{1,\ldots,n\}$. We write L_I for the intersection of the hyperplanes L_i, $i \in I$, and $L(I)$ for the tuple formed by the L_i, $i \in I$. For $A \subset \mathbb{P}^n$ a projective subspace of dimension k, and $L_i, M_j \subset \mathbb{P}^n$ hyperplanes, we use the notation $(A|L_0,\ldots,L_k;M_0,\ldots,M_k)$ for a pair of k-simplices on $A \cong \mathbb{P}^k$ formed by the intersections $A \cap L_i$ and $A \cap M_j$.

One then proves the formula

$$\Delta(I^{\mathcal{H}}(L;M)) = \sum_{\substack{I,J\subset\{1,\dots,n\}\\|I|+|J|=n}} \varepsilon_{I,J}\, I^{\mathcal{H}}\left(L_I|L_0,L(\overline{I});M_0,M(J)\right)\otimes I^{\mathcal{H}}\left(M_J|L_0,L(I);M_0,M(\overline{J})\right) \tag{1.18}$$

where $\varepsilon_{I,J} = (-1)^{|I||J|}\mathrm{sgn}(I,\overline{I})\mathrm{sgn}(J,\overline{J})$. Here $(L_I|L_0,L(\overline{I});M_0,M(J))$ is a generic pair of simplices in $L_I \cong \mathbb{P}^{n-|I|}$.

The formula for the motivic coaction is exactly the same, replacing $I^{\mathcal{H}}(M_J|L_0,L(I);M_0,M(\overline{J}))$ on the right-hand side by $I^{\mathcal{P}}(M_J|L_0,L(I);M_0,M(\overline{J}))$; this has to be understood as being computed with the integration simplex $\partial_J\Delta_M \subset M_J$.

Remark 1.6.4. As in Remark 1.6.3, there is a sign error in most references where formula (1.18) appears [BMS87, BVGS90, Zha00], where the sign $(-1)^{|I||J|}$ is forgotten. Since

$$(-1)^{k(n-k)} = (-1)^{\frac{n(n+1)}{2}}(-1)^{\frac{k(k+1)}{2}}(-1)^{\frac{(n-k)(n-k+1)}{2}},$$

one recovers the right formulas by multiplying the component $\mathcal{H}_n^{\mathsf{MHTS}}$ by the sign $(-1)^{\frac{n(n+1)}{2}}$.

However, in references such as [Gon00, GZ01, Gon13], the formula for the component $\Delta_{1,n-1}$ of the generic motivic coproduct is actually correct, the sign being, for $i,j\in\{1,\dots,n\}$:

$$\varepsilon_{\{i\},\{1,\dots,n\}\setminus\{j\}} = (-1)^{n-1}(-1)^{i-1}(-1)^{n-j} = (-1)^{i+j}.$$

1.6.6 Iterated integrals

In [Gon05], Theorem 1.2, Goncharov computes the coproduct for motivic iterated integrals. More precisely, he shows that they generate a Hopf subalgebra of $\mathcal{H}^{\mathsf{MHTS}}$ and that the coproduct $\Delta(\mathbb{I}^{\mathcal{H}}(a_0;a_1,\dots,a_n;a_{n+1}))$ is given by the formula

$$\sum_{\substack{0\leqslant k\leqslant n\\ 0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1}} \left(\prod_{s=0}^{k}\mathbb{I}^{\mathcal{H}}(a_{i_s};a_{i_s+1},\dots,a_{i_{s+1}-1};a_{i_{s+1}})\right)\otimes \mathbb{I}^{\mathcal{H}}(a_0;a_{i_1},\dots,a_{i_k};a_{n+1}). \tag{1.19}$$

When applied to indices $a_i\in\{0,1\}$, one sees that the motivic multiple zeta values are closed under the coproduct. This special feature and the corresponding formula for the motivic coaction have been used by Brown [Bro12] to prove the Deligne-Ihara conjecture and the motivic Hoffman basis conjecture.

1.7 Arrangements and cohomology

1.7.1 The general question

As a preliminary step towards computing the motivic coproduct of Aomoto polylogarithms, we want to understand the underlying relative cohomology groups. Let us forget for a moment about the periods and look at the following general situation. In the projective space $\mathbb{P}^n$, let $L=\{L_1,\dots,L_l\}$ and $M=\{M_1,\dots,M_m\}$ be two disjoint sets of hyperplanes. In view of the discussion of §1.5.3, let us assign to each strict stratum S of the pair $(L;M)$ a *color* $\chi(S)\in\{\lambda,\mu\}$ such that the strata L_I ($I\subset\{1,\dots,l\}$, $|I|>0$) are colored λ and the strata M_J ($J\subset\{1,\dots,m\}$, $|J|>0$) are colored μ. For now, let us call such a triple $(L;M;\chi)$ a (projective) *bi-arrangement of hyperplanes* (the definition is actually a little bit more technical, see Definition 4.1.1, but this one will do for now).

Let $\pi : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$ be the iterated blow-up described in §1.5.2. Then $\pi^{-1}(L \cup M)$ is a normal crossing divisor in $\widetilde{\mathbb{P}^n}$ and its irreducible components are in bijection with the strict strata of $(L; M)$. We let $\widetilde{L}$ be the union of those irreducible components corresponding to strata colored λ, and $\widetilde{M}$ be the union of those irreducible components corresponding to strata colored μ. We then let

$$H^\bullet(L; M; \chi) = H^\bullet(\widetilde{\mathbb{P}^n} \setminus \widetilde{L}, \widetilde{M} \setminus \widetilde{M} \cap \widetilde{L})$$

be the collection of the corresponding relative cohomology groups. We call it the *motive* of the bi-arrangement of hypersurfaces $(L; M; \chi)$. A general problem is thus the following.

Problem B: For a given bi-arrangement of hyperplanes $(L; M; \chi)$ in the projective space $\mathbb{P}^n$, set up tools to compute the motive $H^\bullet(L; M; \chi)$.

We will be more precise about this problem in §1.10. For now let us just mention a "global" version of **Problem B**. We take X a complex variety, $L = \{L_1, \ldots, L_l\}$ and $M = \{M_1, \ldots, M_m\}$ two disjoint sets of hypersurfaces in X such that around every point of X we can find local coordinates in which all hypersurfaces L_i and M_j are defined by linear equations. If we add a coloring χ of all strata of $(L; M)$, then we get a triple $(L; M; \chi)$ that we call a *bi-arrangement of hypersurfaces* in X. By exactly the same procedure as in the setting of bi-arrangements of hyperplanes, one may attach to such a triple the *motive*

$$H^\bullet(L; M; \chi) = H^\bullet(\widetilde{X} \setminus \widetilde{L}, \widetilde{M} \setminus \widetilde{M} \cap \widetilde{L})$$

where $\widetilde{X} \to X$ is an iterated blow-up.

Remark 1.7.1. Here, the word "motive" is used in a non-technical sense, as a synonym for the expression "relative cohomology group". Nonetheless, if X is a smooth algebraic variety over a field, then we may give a more technical sense to it by realizing $H^\bullet(L; M; \chi)$ in Voevodsky's triangulated category of motives. In the case of bi-arrangements of hyperplanes in $\mathbb{P}^n$ defined over a number field F, we have seen in §1.5.4 that $H^\bullet(L; M; \chi)$ defines an object of the category of mixed Tate motives over F.

We then pose the following general problem, which includes **Problem B** as a special case.

Problem B': For a given bi-arrangement of hypersurfaces $(L; M; \chi)$ in a complex manifold X, set up tools to compute the motive $H^\bullet(L; M; \chi)$.

1.7.2 The classical Orlik-Solomon algebra

As a very special case of **Problem B**, let us look at the case $M = \varnothing$. The coloring is thus constant of value λ, and we are left with a set of hyperplanes $L = \{L_1, \ldots, L_l\}$. This is simply called an *arrangement of hyperplanes* in $\mathbb{P}^n$. The corresponding cohomology groups are

$$H^\bullet(L; \varnothing; \lambda) = H^\bullet(\mathbb{P}^n \setminus L).$$

Arrangements of hyperplanes have been much studied from multiple points of view (algebraic topology, combinatorics, algebraic geometry, etc.) since the pioneering work [Arn69] of Arnol'd.

To recall this classical setting, we now work in $\mathbb{C}^n$, looking at a set $L = \{L_1, \ldots, L_l\}$ of hyperplanes of $\mathbb{C}^n$ that pass through the origin. Arnol'd posed the following problem: compute the cohomology ring $H^\bullet(\mathbb{C}^n \setminus L)$ of the complement of $L = L_1 \cup \cdots \cup L_l$ in $\mathbb{C}^n$. This problem was settled in two steps by Brieskorn [Bri73] and Orlik and Solomon [OS80] and led to the introduction of the Orlik-Solomon algebra which we now discuss.

We set $\Lambda_\bullet(L) = \Lambda^\bullet(e_1, \ldots, e_l)$, the exterior algebra over $\mathbb{Q}$ with a generator e_i in degree 1 for each L_i. Let $d : \Lambda_\bullet(L) \to \Lambda_{\bullet-1}(L)$ be the unique derivation of $\Lambda_\bullet(L)$ such that $d(e_i) = 1$ for $i = 1, \ldots, l$.

A subset $I \subset \{1, \ldots, l\}$ is said to be **dependent** if the hyperplanes L_i, for $i \in I$, are linearly dependent. Let $J_\bullet(L)$ be the homogeneous ideal of $\Lambda_\bullet(L)$ generated by the elements $d(e_I)$ for $I \subset \{1, \ldots, l\}$ dependent. The quotient

$$A_\bullet(L) = \Lambda_\bullet(L)/J_\bullet(L)$$

is a graded $\mathbb{Q}$-algebra called the **Orlik-Solomon algebra** of L.

An important feature of the Orlik-Solomon algebra is the direct sum decomposition with respect to the set of strata. If $\mathscr{S}_r(L)$ denotes the set of strata of L of codimension r, then we have

$$A_r(L) = \bigoplus_{S \in \mathscr{S}_r(L)} A_S(L)$$

where $A_S(L)$ is spanned by the classes of the monomials e_I for $I \subset \{1, \ldots, l\}$ such that $L_I = S$. Combining this with the fact that the Orlik-Solomon algebra is exact as a complex, we get a more geometric but less explicit way of defining the components $A_\Sigma(L)$ by induction on the codimension of the strata, as follows. Suppose that we have already defined the components $A_S(L)$ for $S \in \mathscr{S}_{k-1}(L)$, the components $A_T(L)$ for $T \in \mathscr{S}_{k-2}(L)$, and the differential $d : A_{k-1}(L) \to A_{k-2}(L)$. Then for $\Sigma \in \mathscr{S}_k(L)$, we define $A_\Sigma(L)$ as the kernel of the differential

$$\bigoplus_{\substack{S \in \mathscr{S}_{k-1}(L) \\ S \supset \Sigma}} A_S(L) \xrightarrow{d} \bigoplus_{\substack{T \in \mathscr{S}_{k-2}(L) \\ T \supset \Sigma}} A_T(L)$$

which means that we impose an exact sequence

$$0 \to A_\Sigma(L) \xrightarrow{d} \bigoplus_{\substack{S \in \mathscr{S}_{k-1}(L) \\ S \supset \Sigma}} A_S(L) \xrightarrow{d} \bigoplus_{\substack{T \in \mathscr{S}_{k-2}(L) \\ T \supset \Sigma}} A_T(L).$$

This point of view is adopted in [Loo93, Lemma 2.2].

The Brieskorn-Orlik-Solomon theorem asserts that there is an isomorphism of graded algebras

$$H^\bullet(\mathbb{C}^n \setminus L) \cong A_\bullet(L)$$

where a generator e_i corresponds to the class of the logarithmic differential form $\frac{1}{2i\pi}\frac{df_i}{f_i}$, with f_i a linear equation for L_i. In particular, this theorem implies that the cohomology ring $H^\bullet(\mathbb{C}^n \setminus L)$ only depends on the combinatorics of L, i.e. on the poset of its strata.

In terms of mixed Hodge structures, the Brieskorn-Orlik-Solomon theorem implies that the cohomology group $H^k(\mathbb{C}^n \setminus L)$ is a sum of Tate structures $\mathbb{Q}(-k)$, and thus a pure Hodge structure of weight $2k$. The point of introducing the motives $H^\bullet(L; M; \chi)$ is that they contain non-trivial extensions between Tate structures $\mathbb{Q}(-k)$.

1.8 The results of Chapter 2

In Chapter 2, which is a slightly rewritten version of the article [Dup14a], we solve **Problem A** for a special family of Aomoto polylogarithms called dissection polylogarithms. This family is indexed by combinatorial objects called dissection diagrams, and includes Goncharov's generic iterated integrals (1.6). We show that their motivic coproduct is related to a combinatorial Hopf algebra on dissection diagrams. This Hopf algebra is part of a growing family of Hopf algebras based on combinatorial objects, whose most famous representative is the Connes-Kreimer Hopf algebra [CK99] (see [LR10] for a tentative definition of the term "combinatorial Hopf algebra" and references).

The combinatorial objects that we consider are called *dissection diagrams*. A dissection diagram of degree n is a set of n non-intersecting chords of a rooted oriented polygon (the polygons will always be drawn as circles) with $(n+1)$ vertices such that the graph formed by the chords is acyclic.

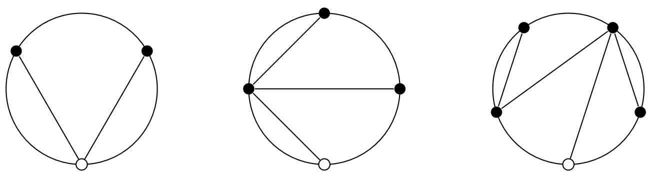


Figure 1.4: Examples of dissection diagrams of respective degrees 2, 3, 4. The polygons are drawn as circles. Every polygon has a distinguished vertex, which is called the root and is depicted as a white dot.

Let $\mathcal{D}$ be the free commutative $\mathbb{Q}$-algebra generated by dissection diagrams, with a grading given by the degrees of the dissection diagrams. We give $\mathcal{D}$ the structure of a graded Hopf algebra (see 2.1 for the precise definition). The coproduct $\Delta : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ is uniquely defined by its value on the dissection diagrams D, and is given by a formula of the form

$$\Delta(D) = \sum_{C \subset \mathscr{C}(D)} \pm q_C(D) \otimes r_C(D). \tag{1.20}$$

For now let us just mention that $\mathscr{C}(D)$ denotes the set of chords of D, $q_C(D)$ is a product of dissection diagrams obtained by taking the quotient of D by the chords in C (contraction of chords), and $r_C(D)$ is a single dissection diagram obtained by keeping only the chords in C (deletion of chords). The form of the coproduct (1.20) is reminiscent of several other combinatorial Hopf algebras, such as the Connes-Kreimer Hopf algebra.

There is a decorated version $\mathcal{D}(\mathbb{C})$ of this Hopf algebra, where we attach complex numbers to each side of the polygon and each chord of the dissection diagram (see Figure 1.5).

We mainly consider the Hopf subalgebra $\mathcal{D}^{\mathrm{gen}}(\mathbb{C}) \subset \mathcal{D}(\mathbb{C})$ generated by the decorated dissection diagrams that satisfy a genericity condition on the decorations. To each such generic decorated dissection diagram D, we associate an absolutely convergent integral

$$I(D) = \int_{\Delta_D} \omega_D$$

called a *dissection polylogarithm*, where ω_D is a meromorphic form on $\mathbb{C}^n$ and Δ_D is a singular simplex in $\mathbb{C}^n$ that does not meet the polar locus of ω_D. It is a special case of the definition of an Aomoto polylogarithm given in §1.2.1.

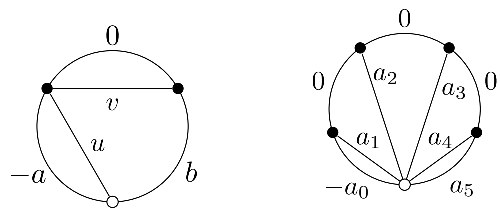


Figure 1.5: Decorated dissection diagrams. On the right, a decorated corolla.

For example, for D the first decorated dissection diagram in Figure 1.5, we get

$$\omega_D = \frac{dx \wedge dy}{(x-u)(y-x-v)}$$

and $\Delta_D : \Delta^2 \to \mathbb{C}^2$ is an embedding of a triangle which is bounded by the lines $\{x = a\}$, $\{x = y\}$ and $\{y = b\}$. In general, the form ω_D is determined by the combinatorial data of the decorated chords and the domain Δ_D is determined by the decorations of the sides of the polygon (as for all Aomoto polylogarithms, the integral $I(D)$ depends on the choice of Δ_D).

The dissection polylogarithms generalize the generic iterated integrals (1.6), which correspond to the special case of corollas (dissection diagrams where the chords are all linked to the root vertex, see Figure 1.5). In this special case, the genericity condition dictates that the decorations a_i are pairwise distinct, hence the terminology is consistent.

As a special case of the construction of §1.5, we introduce motivic counterparts for dissection polylogarithms. Let L be the polar locus of ω_D and let M be the Zariski closure of $\partial\Delta_D$; they are unions of hyperplanes inside $\mathbb{C}^n$. The dissection polylogarithm $I(D)$ is thus a period of the mixed Hodge-Tate structure

$$H(D) = H^n(\mathbb{C}^n \setminus L, M \setminus M \cap L).$$

Because of the genericity assumption on the decorations, the bi-arrangements of hyperplanes $(L; M)$ we are looking at are normal crossing divisors inside $\mathbb{C}^n$: we say that they are *affinely generic*. Nonetheless, they are highly degenerate at infinity when viewed inside $\mathbb{P}^n(\mathbb{C})$ (Remark 2.3.6); they are thus much more degenerate than the generic pairs of simplices from §1.6.5. Furthermore, the affine context enables us to take products of configurations of hyperplanes, an operation which is more involved in the projective setting. Here, because $L \cup M$ is a normal crossing divisor, we do not need a coloring χ.

As in §1.5.2, we use the cohomology groups $H(D)$ to define a motivic version $I^{\mathcal{H}}(D) \in \mathcal{H}^{\mathsf{MHTS}}$ which we call a *motivic dissection polylogarithm*. The main result of Chapter 2 is the computation of the coproduct of the motivic dissection polylogarithms. More precisely, we show that they generate a Hopf subalgebra of $\mathcal{H}^{\mathsf{MHTS}}$ and that their coproduct can be computed combinatorially using formula (1.20).

Theorem 1.8.1 (see Theorem 2.4.9). *Let D be a generic decorated dissection diagram. Then the coproduct of the corresponding motivic dissection polylogarithm in $\mathcal{H}^{\mathsf{MHTS}}$ is given by formula (1.20):*

$$\Delta(I^{\mathcal{H}}(D)) = \sum_{C \subset \mathscr{C}(D)} \pm I^{\mathcal{H}}(q_C(D)) \otimes I^{\mathcal{H}}(r_C(D)).$$

In other words, the morphism

$$\mathcal{D}^{\mathrm{gen}}(\mathbb{C}) \to \mathcal{H}^{\mathsf{MHTS}}\ ,\ D \mapsto I^{\mathcal{H}}(D)$$

is a morphism of graded Hopf algebras.

In the case of corollas, we recover Goncharov's formula (1.19) for the motivic coproduct of (generic) iterated integrals.

In order to prove Theorem 1.8.1, one needs to have a good understanding of the relative cohomology groups $H(D)$. Our main technical tool is the following theorem, which is a solution to **Problem B** in the affinely generic case.

Theorem 1.8.2 (see Theorems 2.2.5 and 2.2.7)**.** *Let $\{L_1, \ldots, L_l, M_1, \ldots, M_m\}$ be a set of hyperplanes of $\mathbb{C}^n$. We write $L = L_1 \cup \cdots \cup L_l$ and $M = M_1 \cup \cdots \cup M_m$ and assume that $L \cup M$ is a normal crossing divisor inside $\mathbb{C}^n$. Then for every k we have a presentation*

$$\mathrm{gr}^W_{2k} H^n(\mathbb{C}^n \setminus L, M \setminus M \cap L) \cong \left(\Lambda^k(e_1, \ldots, e_l) \otimes \Lambda^{n-k}(f_1, \ldots, f_m)\right) / R_k(L; M)$$

where $R_k(L; M)$ is an explicit subspace of relations. This presentation is functorial in $(L; M)$.

The functoriality statement in Theorem 1.8.2, that is made more precise in Theorem 2.2.7, is crucial. Indeed, it allows us to relate the geometric situation coming from D and the geometric situations coming from the terms $q_C(D)$ and $r_C(D)$ in formula (1.20).

1.9 The results of Chapter 3

In Chapter 3, which is a slightly rewritten version of the preprint [Dup13], we study **Problem B'** in the case $M = \varnothing$. Let X be a complex manifold of dimension n. An arrangement of hypersurfaces in X is a union

$$L = L_1 \cup \cdots \cup L_l$$

of smooth hypersurfaces $L_i \subset X$, $i = 1, \ldots, l$, that locally looks like a union of hyperplanes in $\mathbb{C}^n$: around each point of X we can find a system of local coordinates in which each L_i is defined by a linear equation.

This generalizes the notion of a (simple) normal crossing divisor: an arrangement of hypersurfaces is a normal crossing divisor if the local linear equations defining the L_i's are everywhere linearly independent; in other words, if we can always choose local coordinates $(z_1, \ldots, z_n)$ such that L is locally defined by the equation $z_1 \cdots z_r = 0$ for some r.

Besides normal crossing divisors, examples of arrangements of hypersurfaces include unions of hyperplanes in a projective space $\mathbb{P}^n(\mathbb{C})$, or unions of diagonals $\Delta_{i,j} = \{y_i = y_j\} \subset Y^n$ inside the n-fold cartesian product of a Riemann surface Y. The class of hypersurface arrangements is also closed under certain blow-ups.

The aim of Chapter 3 is to define and study a model $M^\bullet(X, L)$ for the cohomology algebra over $\mathbb{Q}$ of the complement $X \setminus L$ of an arrangement of hypersurfaces, when X is a smooth projective variety over $\mathbb{C}$.

Our model, which we call the Orlik-Solomon model, has *combinatorial* inputs coming from the theory of hyperplane arrangements (the local setting) and *geometric* inputs coming from the cohomology of smooth hypersurface complements in a smooth projective variety (the global setting). Roughly speaking, it is the *direct product* of two classical tools related to these two situations, that we first recall.

– *Combinatorics: the Orlik-Solomon algebra.* Let L be an arrangements of hyperplanes in $\mathbb{C}^n$. Its Orlik-Solomon algebra $A_\bullet(L)$, introduced in §1.7.2, has a direct sum decomposition

$$A_\bullet(L) = \bigoplus_{S\in\mathscr{S}_\bullet(L)} A_S(L). \tag{1.21}$$

It has the structure of a graded algebra, via product maps

$$A_S(L)\otimes A_{S'}(L)\to A_{S\cap S'}(L). \tag{1.22}$$

Furthermore, there are differentials

$$A_S(L)\to A_{S'}(L) \tag{1.23}$$

for any inclusion $S\subset S'$ of strata of L such that $\mathrm{codim}(S')=\mathrm{codim}(S)-1$.

One may define an Orlik-Solomon algebra $A_\bullet(L)$ for L any hypersurface arrangement inside a complex manifold X. We still have a direct sum decomposition (1.21), with $\mathscr{S}_\bullet(L)$ the graded poset of the strata of L, as well as product maps (1.22) and natural morphisms (1.23). As in the local case, the Orlik-Solomon algebra $A_\bullet(L)$ only depends on the poset of the strata of L. It is functorial with respect to (X,L) in the sense that any holomorphic map

$$\varphi : X\to X' \ \ , \ \ \varphi^{-1}(L')\subset L \tag{1.24}$$

induces a map of graded algebras $A_\bullet(\varphi):A_\bullet(L')\to A_\bullet(L)$.

– *Geometry: the Gysin long exact sequence.* For a smooth hypersurface V inside a smooth projective variety X over $\mathbb{C}$, the Gysin morphisms of the inclusion $V\subset X$ are the morphisms $H^{k-2}(V)(-1)\to H^k(X)$, where (-1) denotes a Tate twist, obtained as the Poincaré duals of the natural morphisms $H^{2n-k}(X)\to H^{2n-k}(V)$ where $n=\dim_{\mathbb{C}}(X)$. They fit into a long exact sequence, called the Gysin long exact sequence:

$$\cdots\to H^{k-2}(V)(-1)\to H^k(X)\to H^k(X\setminus V)\to H^{k-1}(V)(-1)\to\cdots \tag{1.25}$$

It is worth noting that the connecting homomorphisms $H^k(X\setminus V)\to H^{k-1}(V)(-1)$ are residue morphisms, which are easily described using logarithmic forms.

We can now state the main Theorem of Chapter 3 (see Theorem 3.3.8 for more precise statements).

Theorem 1.9.1. *Let X be a smooth projective variety over $\mathbb{C}$ and L an arrangement of hypersurfaces in X.*

1. *For integers q and n let us consider*

$$M_q^n(X,L)=\bigoplus_{S\in\mathscr{S}_{q-n}(L)} H^{2n-q}(S)(n-q)\otimes A_S(L)$$

where $(n-q)$ is a Tate twist, viewed as a pure Hodge structure of weight q. Then the direct sum

$$M^\bullet(X,L)=\bigoplus_q M_q^\bullet(X,L)$$

has the structure of a differential graded algebra (dga) in the (semi-simple) category of split mixed Hodge structures over $\mathbb{Q}$. The product in $M^\bullet(X,L)$ is obtained by tensoring the product maps (1.22) of the Orlik-Solomon algebra with the cup-product on the cohomology *of the strata. The differential in $M^\bullet(X,L)$ is obtained by tensoring the natural morphisms (1.23) with the Gysin morphisms*

$$H^{2n-q}(S)(n-q) \to H^{2n-q+2}(S')(n+1-q)$$

of the inclusions of strata $S \subset S'$. The dga $M^\bullet(X,L)$ is functorial with respect to (X,L) in the sense explained above.

2. *The dga $M^\bullet(X,L)$ is a model for the cohomology of $X \setminus L$ in the following sense: we have isomorphisms of pure Hodge structures over $\mathbb{Q}$*

$$\mathrm{gr}_q^W H^n(X \setminus L) \cong H^n(M_q^\bullet(X,L))$$

which are compatible with the algebra structures, and functorial with respect to maps (1.24).

The precise definition of the Orlik-Solomon model $M^\bullet(X,L)$ is given in §3.3.4. Theorem 1.9.1 generalizes the case of normal crossing divisors, which is due to P. Deligne [Del71] (see also [Voi02, 8.35]) as a by-product of the definition of the mixed Hodge structure on the cohomology of smooth varieties over $\mathbb{C}$. The Orlik-Solomon model appears as the first page of a spectral sequence, called the Orlik-Solomon spectral sequence.

Before we describe the proof of Theorem 1.9.1 and some of its applications, we mention that it completes a result by E. Looijenga [Loo93] who first considered the Orlik-Solomon spectral sequence. Our approach is totally different, with a prominent use of differential forms. In particular, we introduce a complex of logarithmic differential forms (see §1.9.3 in this Introduction) that should have applications in other situations. The main advantages of the use of differential forms are the following.

1. It allows us to prove the functoriality of the Orlik-Solomon model, whereas Looijenga's spectral sequence cannot be easily proved to be functorial. This is crucial when discussing the behaviour of the Orlik-Solomon model with respect to blow-ups (see §1.9.1 in this Introduction and §3.4). As a consequence, we are able to reconcile Kriz's and Totaro's approaches on models for configuration spaces of points on curves (see §1.9.2 in this Introduction and §3.5.4).

2. It makes the multiplicative structure of the Orlik-Solomon model transparent and closer in spirit to the classical Brieskorn-Orlik-Solomon theorem.

3. Our approach is more down-to-earth in that we prove that the Orlik-Solomon spectral sequence is compatible with Hodge structures using only mixed Hodge theory *à la* Deligne. With Looijenga's formalism, one would have to use Saito's theory of mixed Hodge modules (in this direction, see also [Get99]): indeed, his spectral sequence is defined out of a complex of sheaves built out of the constructible sheaves $i_! i^! \mathbb{Q}$ for i a closed immersion, hence it is not immediate that it is compatible with mixed Hodge theory.

1.9.1 Wonderful compactifications

We should say a word on the usefulness of the generalization from normal crossing divisors to arrangements of hypersurfaces. Indeed, Deligne's approach relies on the fact that any smooth variety over $\mathbb{C}$ can be viewed as the complement of a normal crossing divisor inside a smooth projective variety, using Nagata's compactification theorem and Hironaka's resolution of singularities. Thus the case of normal crossing divisors is (in principle) sufficient to give a model for the cohomology of *any* smooth variety over $\mathbb{C}$.

In the framework of Theorem 1.9.1, we may even produce, following [FM94, DCP95, Hu03, Li09], an explicit sequence of blow-ups (see Theorem 3.4.4)

$$\pi : \widetilde{X} \to X$$

sometimes called a "wonderful compactification", that transforms L into a normal crossing divisor $\widetilde{L} = \pi^{-1}(L)$ inside $\widetilde{X}$ and induces an isomorphism

$$\pi : \widetilde{X} \setminus \widetilde{L} \xrightarrow{\cong} X \setminus L.$$

Thus Deligne's special case of Theorem 1.9.1 applied to $(\widetilde{X}, \widetilde{L})$ gives a model $M^\bullet(\widetilde{X}, \widetilde{L})$ for the cohomology of $X \setminus L$. The functoriality of our construction gives a quasi-isomorphism of differential graded algebras

$$M^\bullet(\pi) : M^\bullet(X, L) \to M^\bullet(\widetilde{X}, \widetilde{L})$$

that we can compute explicitly (see Theorem 3.4.5). Along with the work of Morgan [Mor78], this implies that $M^\bullet(X, L)$ is a model of the space $X \setminus L$ in the sense of rational homotopy theory.

The model $M^\bullet(X, L)$ has three advantages over $M^\bullet(\widetilde{X}, \widetilde{L})$. Firstly, it is in general smaller ($M^\bullet(\pi)$ is always injective). Secondly, its definition only uses geometric and combinatorial information from the pair (X, L) without having to look at the blown-up situation $(\widetilde{X}, \widetilde{L})$. Thirdly, it is functorial with respect to maps (1.24).

1.9.2 Configuration spaces of points on curves

Let Y be a compact Riemann surface and n an integer. For all $1 \leqslant i < j \leqslant n$ we have a diagonal

$$\Delta_{i,j} = \{y_i = y_j\} \subset Y^n$$

inside the n-fold cartesian product of Y. Any union of $\Delta_{i,j}$'s then defines an arrangement of hypersurfaces in Y^n. For example, if we consider the union of all diagonals, the complement is the configuration space of n ordered points in Y:

$$C(Y, n) = \{(y_1, \ldots, y_n) \in Y^n \mid y_i \neq y_j \text{ for } i \neq j\}.$$

Theorem 1.9.1 hence gives an Orlik-Solomon model for the cohomology of $C(Y, n)$. This model is isomorphic to the one independently found by I. Kriz [Kri94] and B. Totaro [Tot96], as we prove in Theorem 3.5.2.

On the one hand, our method is close to Totaro's, since the Orlik-Solomon spectral sequence that we are considering in §3.3.3 is the Leray spectral sequence of the inclusion $j : X \setminus L \hookrightarrow X$. On the other hand, the functoriality of our constructions implies that there exists a quasi-isomorphism $M^\bullet(\pi)$ associated to any wonderful compactification π; in §3.5.4 we prove that this quasi-isomorphism is exactly the one used by Kriz to prove the main result of [Kri94]. Hence, our method reconciles Kriz's and Totaro's approaches in the case of curves.

As a natural generalization, we consider the union of only certain diagonals $\Delta_{i,j}$. Such a generalization has been recently studied by S. Bloch [Blo12], who gives a model in the spirit of Kriz and Totaro's model. We prove that this model is also isomorphic to our Orlik-Solomon model.

1.9.3 Logarithmic forms and mixed Hodge theory

We now discuss the proof of Theorem 1.9.1. Our approach follows Deligne's proof of the case of normal crossing divisors, hence makes extensive use of logarithmic forms and the formalism of mixed Hodge structures.
Let X be a smooth projective variety and $L = L_1 \cup \cdots \cup L_l$ an arrangement of hypersurfaces in X. The first task is to define a complex of sheaves on X, denoted by $\Omega^\bullet_{\langle X,L\rangle}$, of meromorphic forms on X with logarithmic poles along L. In local coordinates where each L_i is defined by a linear equation $f_i = 0$, a section of $\Omega^\bullet_{\langle X,L\rangle}$ is a meromorphic differential form on X which is a linear combination over $\mathbb{C}$ of forms of the type

$$\eta \wedge \frac{df_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{df_{i_s}}{f_{i_s}} \tag{1.26}$$

with η a holomorphic form and $1 \leqslant i_1 < \cdots < i_s \leqslant l$. It has to be noted that the complex $\Omega^\bullet_{\langle X,L\rangle}$ is in general a strict subcomplex of the complex $\Omega^\bullet_X(\log L)$ introduced by Saito [Sai80], even though the two complexes coincide in the case of a normal crossing divisor.

The main point of the complex $\Omega^\bullet_{\langle X,L\rangle}$ is that it computes the cohomology of the complement $X \setminus L$. More precisely, if we denote by $j : X\setminus L \hookrightarrow X$ the open immersion of the complement of L inside X, we prove the following theorem (Theorem 3.2.13).

Theorem 1.9.2. *The inclusion $\Omega^\bullet_{\langle X,L\rangle} \hookrightarrow j_*\Omega^\bullet_{X\setminus L}$ is a quasi-isomorphism, and hence induces isomorphisms*

$$\mathbb{H}^n(\Omega^\bullet_{\langle X,L\rangle}) \cong H^n(X \setminus L, \mathbb{C}). \tag{1.27}$$

It has to be noted (Remark 3.2.10) that according to this theorem, a conjecture of H. Terao [Ter78] is equivalent to the fact that the inclusion $\Omega^\bullet_{\langle X,L\rangle} \subset \Omega^\bullet_X(\log L)$ is a quasi-isomorphism.

The proof of Theorem 1.9.2 is local and relies on the Brieskorn-Orlik-Solomon theorem. Another central technical tool is the weight filtration W on $\Omega^\bullet_{\langle X,L\rangle}$: we define $W_k\Omega^\bullet_{\langle X,L\rangle} \subset \Omega^\bullet_{\langle X,L\rangle}$ to be the subcomplex spanned by the forms (1.26) with $s \leqslant k$. In view of the isomorphism (1.27), we get a filtration on the cohomology of $X \setminus L$ which is proved to be defined over $\mathbb{Q}$. Together with the Hodge filtration $F^p\Omega^\bullet_{\langle X,L\rangle} = \Omega^{\geqslant p}_{\langle X,L\rangle}$, it defines a mixed Hodge structure on $H^\bullet(X \setminus L)$. The functoriality of our construction then implies that this is the same as the mixed Hodge structure defined by Deligne.

According to the general theory of mixed Hodge structures, the hypercohomology spectral sequence associated to the weight filtration degenerates at the E_2-term, hence the E_1-term gives a model for the cohomology of $X\setminus L$. We then prove that this model is indeed the Orlik-Solomon model $M^\bullet(X, L)$. This concludes the proof of Theorem 1.9.1.

It has been pointed out to us by A. Dimca that the sheaves $\Omega^1_{\langle X,L\rangle}$ have been previously defined in [CHKS06] (where they are denoted $\Omega_X(\log L)$) and [Dol07] (where they are denoted $\tilde{\Omega}_X(\log L)$).

1.10 The results of Chapter 4

In Chapter 4, which is a slightly rewritten version of the preprint [Dup14b], we study **Problem B'** by introducing tools to compute the motive of a given bi-arrangement.

– In the local context of hyperplanes in $\mathbb{C}^n$, we define the *Orlik-Solomon bi-complex* of a bi-arrangement of hyperplanes, generalizing the construction of the Orlik-Solomon algebra. This allows us to single out a class of bi-arrangements for which the Orlik-Solomon bi-complex is well-behaved, which we call *exact*, and which includes all classical arrangements of hyperplanes.

– In the global context of hypersurfaces in a complex manifold X, we define the *geometric Orlik-Solomon bi-complex* of a bi-arrangement of hypersurfaces, which incorporates the combinatorial data of the Orlik-Solomon bi-complexes and the cohomological data of the geometric situation.

Our main result can then be vaguely stated as follows: *the motive of an exact bi-arrangement is computed by its Orlik-Solomon bi-complex.* In the special case of arrangements, we recover the classical Brieskorn-Orlik-Solomon theorem and its global counterpart (Theorem 1.9.1).

1.10.1 From the Orlik-Solomon algebra to the Orlik-Solomon bi-complex

Let $(L;M;\chi)$ be a bi-arrangement of hyperplanes in $\mathbb{C}^n$ (all hyperplanes pass through the origin). We recall that a stratum of $(L;M)$ is an intersection $L_I \cap M_J$ of some hyperplanes L_i and M_j. We let $\mathscr{S}_k(L;M)$ denote the set of strata of $(L;M)$ of codimension k, and $\Sigma \overset{c}{\hookrightarrow} \Sigma'$ denote an inclusion of strata of L with $\dim(\Sigma) = \dim(\Sigma') - c$.

The *Orlik-Solomon bi-complex* of $(L;M;\chi)$ is a bi-complex $A_{\bullet,\bullet} = A_{\bullet,\bullet}(L;M;\chi)$ with differentials $d' : A_{\bullet,\bullet} \to A_{\bullet-1,\bullet}$ and $d'' : A_{\bullet,\bullet-1} \to A_{\bullet,\bullet}$. By definition, there is a direct sum decomposition

$$A_{i,j} = \bigoplus_{S\in\mathscr{S}_{i+j}(L;M)} A_{i,j}^S.$$

Following the approach of [Loo93, Lemma 2.2] explained in §1.7.2, we define the components $A_{i,j}^S$ by induction on the codimension $i+j$ of S. According to the color $\chi(\Sigma)$, the component $A_{i,j}^\Sigma$ is defined as a kernel or a cokernel of a previously defined differential, by imposing exact sequences

$$0 \to A_{i,j}^\Sigma \xrightarrow{d'} \bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A_{i-1,j}^S \xrightarrow{d'} \bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A_{i-2,j}^T \quad \text{if } \chi(\Sigma) = \lambda; \tag{1.28}$$

$$0 \leftarrow A_{i,j}^\Sigma \xleftarrow{d''} \bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A_{i,j-1}^S \xleftarrow{d''} \bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A_{i,j-2}^T \quad \text{if } \chi(\Sigma) = \mu. \tag{1.29}$$

Starting with $A_{0,0} = \mathbb{Q}$, this is enough to define the components $A_{i,j}^S$ and the differentials. If $M = \varnothing$ and χ takes only the value λ, we recover the inductive definition of the Orlik-Solomon algebra: $A_{\bullet,0}(L;\varnothing;\lambda) = A_\bullet(L)$. In the world of bi-arrangements there is a duality that exchanges the roles of L and λ on the one hand, and M and μ on the other hand. This duality translates as the linear duality of the Orlik-Solomon bi-complexes.

We are mostly interested in the bi-arrangements $(L;M;\chi)$ such that the exact sequences (1.28) and (1.29) can be extended to exact sequences

$$0 \to A_{i,j}^\Sigma \xrightarrow{d'} \bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A_{i-1,j}^S \xrightarrow{d'} \bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A_{i-2,j}^T \xrightarrow{d'} \cdots \xrightarrow{d'} \bigoplus_{\Sigma \overset{i}{\hookrightarrow} Z} A_{0,j}^Z \to 0 \quad \text{if } \chi(\Sigma) = \lambda;$$

$$0 \leftarrow A_{i,j}^\Sigma \xleftarrow{d''} \bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A_{i,j-1}^S \xleftarrow{d''} \bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A_{i,j-2}^T \xleftarrow{d''} \cdots \xleftarrow{d''} \bigoplus_{\Sigma \overset{j}{\hookrightarrow} Z} A_{i,0}^Z \leftarrow 0 \quad \text{if } \chi(\Sigma) = \mu.$$

These bi-arrangements are called *exact*, and form a very natural class of bi-arrangements that includes the arrangements $(L; \varnothing; \lambda)$ - this is because the Orlik-Solomon algebras are exact as complexes. Inside the class of exact bi-arrangements, we prove the existence of a deletion-restriction short exact sequence that generalizes the classical deletion-restriction exact sequence for arrangements, as follows.

Theorem 1.10.1 (see Theorem 4.2.9 for a precise statement)**.** *If the deletion and the restriction of $(L; M; \chi)$ with respect to some hyperplane $K \in L \cup M$ are exact, then $(L; M; \chi)$ is exact and there is a deletion-restriction short exact sequence which produces the Orlik-Solomon bi-complex of $(L; M; \chi)$ as an extension of that of its restriction and that of its deletion.*

The drawback of the inductive definition of the Orlik-Solomon bi-complexes is that we lack an explicit description as in the case of the Orlik-Solomon algebra. We settle this issue for a subclass of exact bi-arrangements that we call *tame*. The tameness condition (see Definition 4.1.26) is a simple combinatorial condition on the coloring which ensures that the colors λ and μ *do not interfere too much.*

Theorem 1.10.2 (see Theorem 4.1.38)**.** *All tame arrangements are exact. Furthermore, we can describe the Orlik-Solomon algebra of a tame bi-arrangement $(\{L_1, \ldots, L_l\}; \{M_1, \ldots, M_m\}; \chi)$ as an explicit subquotient of the tensor product $\Lambda^\bullet(e_1, \ldots, e_l) \otimes \Lambda^\bullet(f_1^\vee, \ldots, f_m^\vee)$ of two exterior algebras.*

In the affinely generic case, we recover Theorem 1.8.2, with the change of notations $f_j \leftrightarrow f_j^\vee$.

1.10.2 Bi-arrangements of hypersurfaces

We now turn to a global geometric situation. The formalism of the Orlik-Solomon bi-complexes immediately extends from bi-arrangements of hyperplanes to bi-arrangements of hypersurfaces, using the same inductive definition.

The main result of Chapter 4 is the following (see Theorem 4.4.11). It states that for exact bi-arrangements of hypersurfaces, we can compute the corresponding motive via a spectral sequence that involves the cohomology of the strata and the Orlik-Solomon bi-complex of the bi-arrangement. It is important to note that this spectral sequence only uses cohomological and combinatorial information from the geometric situation in X, and not in the blow-up $\widetilde{X}$.

Theorem 1.10.3. *Let $(L; M; \chi)$ be an exact bi-arrangement of hypersurfaces in a complex manifold X, with its Orlik-Solomon bi-complex $A_{\bullet,\bullet}$.*

1. *There is a spectral sequence*[5]

$$E_1^{-p,q} = \bigoplus_{\substack{i-j=p \\ S \in \mathscr{S}_{i+j}(L;M)}} H^{q-2i}(S)(-i) \otimes A^S_{i,j} \implies H^{-p+q}(L; M; \chi). \tag{1.30}$$

2. *If X is a smooth complex variety and all hypersurfaces of $(L; M)$ are smooth divisors in X, then this is a spectral sequence in the category of mixed Hodge structures.*

3. *If X is a smooth and projective complex variety, then this spectral sequence degenerates at the E_2 term and we have*

$$E_\infty^{-p,q} \cong E_2^{-p,q} \cong \mathrm{gr}^W_q H^{-p+q}(L; M; \chi).$$

5. Here, $(-i)$ denotes the Tate twist of weight $2i$. It is important in the algebraic case; otherwise it should be ignored.

The differential of the E_1 page of the above spectral sequence is explicit. It is induced by the differentials of the Orlik-Solomon bi-complex and the Gysin and pullback morphisms corresponding to inclusions of strata.

In the case of an arrangement of hypersurfaces $(L; \varnothing; \lambda)$, this gives a spectral sequence

$$E_1^{-p,q} = \bigoplus_{S \in \mathscr{S}_p(\mathscr{L})} H^{q-2p}(S)(-p) \otimes A_p^S(L) \implies H^{-p+q}(X \setminus L)$$

and we recover the main theorem (Theorem 1.9.1) of Chapter 3. However, the proof of Theorem 1.10.3 is drastically different from the one of Theorem 1.9.1.

We can apply Theorem 1.10.3 to the case of projective bi-arrangements in $X = \mathbb{P}^n(\mathbb{C})$.

Theorem 1.10.4 (see Theorem 4.5.4 for a more precise statement)**.** *Let $(L; M; \chi)$ be an exact bi-arrangement of hyperplanes in $\mathbb{P}^n(\mathbb{C})$. For $k = 0, \ldots, n$, let ${}^{(k)}A_{\bullet,\bullet}$ be the bi-complex obtained by only keeping the rows $0 \leqslant i \leqslant k$ and the columns $0 \leqslant j \leqslant n-k$ of the Orlik-Solomon bi-complex of $(L; M; \chi)$, and let ${}^{(k)}A_\bullet$ be its total complex. Then we have isomorphisms*

$$\mathrm{gr}^W_{2k} H^r(L; M; \chi) \cong H_{2k-r}({}^{(k)}A_\bullet)(-k).$$

We introduce projective bi-arrangements of hyperplanes $\mathscr{Z}(n_1, \ldots, n_r)$ corresponding to the multiple zeta values (1.5) and show that they are tame, hence exact. Theorem 1.10.4 thus provides explicit complexes that compute their motives. The motives $\mathscr{Z}(n_1, \ldots, n_r)$ are alternatives to the approach via the motivic fundamental group of $\mathbb{P}^1 \setminus \{\infty, 0, 1\}$ [Del89, DG05]. One advantage of such an alternative is that it generalizes to a larger family of integrals. More specifically, let us look at the periods of the moduli spaces $\overline{\mathcal{M}}_{0,n}$ considered by Brown in [Bro09]. They are integrals of a rational function over a simplex $0 < t_1 < \cdots < t_n < 1$, such as

$$\iiint_{0<x<y<z<1} \frac{dx\, dy\, dz}{(1-x)y(z-x)}. \tag{1.31}$$

The main result of [Bro09] is that these integrals are all linear combinations (with rational coefficients) of multiple zeta values, although not in an explicit way. It so happens that the projective bi-arrangement of hyperplanes corresponding to the integral (1.31) is also exact, hence the corresponding motive may be computed explicitly via an Orlik-Solomon bi-complex. This will be studied in more detail in a subsequent article.

Theorem 1.10.3 is easy for normal crossing divisors. To prove the general case, we investigate the functoriality of the *geometric Orlik-Solomon bi-complex*

$${}^{(q)}D_{i,j} = \bigoplus_{S \in \mathscr{S}_{i+j}(L;M)} H^{q-2i}(S)(-i) \otimes A^S_{i,j}$$

with respect to blow-ups of some strata. More specifically, we prove that there is a quasi-isomophism between the geometric Orlik-Solomon bi-complex of a bi-arrangement and that of its blow-up. Proceeding by induction, it is enough to treat the case of a single blow-up.

1.11 Open questions and developments

1.11.1 Perspectives on the motivic coproduct of Aomoto polylogarithms

In this thesis, we have answered **Problem A** for a family of Aomoto polylogarithms, called generic dissection polylogarithms. We were able to compute the motivic coproduct of generic dissection polylogarithms because of the genericity assumption, that limited the degeneracy of the geometric situation. Armed with the results of Chapter 4, which solves **Problem B** in the case of tame bi-arrangements, we are now capable of computing an even larger family of motives of bi-arrangements of hyperplanes. Up to the combinatorial problem of finding suitable bases for Orlik-Solomon bi-complexes, this is in principle sufficient for computing the motivic coproduct of a large number of Aomoto polylogarithms, using the abstract formula (1.10).

In order to understand the motivic coproduct of *all* Aomoto polylogarithms, there are at least two (complementary) strategies.

1. The first strategy consists in using the scissors congruence relations (see §1.2.3) to express a given Aomoto polylogarithms in terms of a small family of generators. Thus, we reduce the problem to that of computing the motivic coproduct for a family of *essential* Aomoto polylogarithms. We believe that there is such a family of essential Aomoto polylogarithms for which the corresponding bi-arrangements are tame (for a wise choice of coloring).

2. The second strategy consists in deducing the motivic coproduct of a given Aomoto polylogarithm by a degeneration argument. If $(L_\varepsilon; M_\varepsilon)$ is a "generic" family of admissible pairs of simplices indexed by a parameter ε in a punctured disk and degenerating to an admissible pair of simplices $(L; M)$, then there should be a formalism to deduce the motivic coproduct of $I^{\mathcal{H}}(L; M)$ from the motivic coproduct of the family $I^{\mathcal{H}}(L_\varepsilon; M_\varepsilon)$. Here, the word "generic" can have different meanings: in general position as in §1.6.5, or in a less restrictive way, with "generic arguments", as for dissection polylogarithms. This has been used by Goncharov [Gon05] to deduce the motivic coproduct of all iterated integrals $\mathbb{I}(a_0, a_1, \ldots, a_n; a_{n+1})$ from the case where the indices a_i are pairwise distinct. See also [Gon02, §4].

In view of the second point, one should be able to regularize dissection polylogarithms and give a meaning to the formula for their coproduct without any genericity assumption on the decorations.

1.11.2 Perspective on bi-arrangements

The theory of bi-arrangements that we have developed raises many open questions for future research, such as:

1. give a simple combinatorial characterization of exact bi-arrangements;

2. give an explicit description of the Orlik-Solomon bi-complex of an exact bi-arrangement in the spirit of (and generalizing) the case of tame bi-arrangements.

Another direction of research concerns the homological properties of Orlik-Solomon bi-complexes. The homological properties of Orlik-Solomon algebras have been much studied in their own right (see [Yuz01, Den10] for a survey). In some cases, the Orlik-Solomon bi-complex of a bi-arrangement $(\mathscr{L}, \mathscr{M}, \chi)$ is a module over the Orlik-Solomon algebra of $\mathscr{L}$ or $\mathscr{M}$; it would be interesting to investigate, for instance, the Koszulness properties of these modules in relation to the combinatorial properties of the bi-arrangements, as for the case of arrangements.

1.11.3 Applications

We plan on applying the techniques that we have developed in this thesis in different directions.

As arithmetics are concerned, it is tempting to use motivic Aomoto polylogarithms to investigate the categories of mixed Tate motives over rings of integers of cyclotomic fields. In view of the discussion of §1.5.4, one wants to understand how to generate these categories using explicit mixed Tate motives coming from bi-arrangements, and fill in the gaps of the theory of motivic fundamental groups.

Another direction is the computation of the motivic coproduct of periods of the moduli spaces $\overline{\mathcal{M}}_{0,n}$ studied in [Bro09], such as (1.31). This should help us understand better the combinatorics and arithmetics of multiple zeta values.

Our methods should also apply to the algebro-geometric study of Feynman integrals [BEK06, Dor10, BS12]. Even though graph hypersurfaces are much more complicated than arrangements of hyperplanes, the motives that we consider are “linear prototypes” for the graph motives, and our methods are a first step in the difficult task of computing them.

Notations and terminology

Coefficients: Unless otherwise stated, all vector spaces and algebras are defined over $\mathbb{Q}$, as well as the tensor products of such objects. All (mixed) Hodge structures are defined over $\mathbb{Q}$.

Cohomology: The cohomology groups $H^\bullet(X)$ and relative cohomology groups $H^\bullet(X,Y)$ implicitly denote the singular cohomology groups with coefficients in $\mathbb{Q}$. We will write $H^\bullet(X)\otimes\mathbb{C}$ for the singular cohomology groups with complex coefficients. If X is a manifold, the latter are naturally isomorphic, via the de Rham isomorphism, to the (analytic) de Rham cohomology groups tensored with $\mathbb{C}$, hence we allow ourselves to use smooth differential forms as representatives for cohomology classes.

Homological algebra: Our convention on bi-complexes is not standard since we mix the homological and the cohomological convention. A bi-complex is a collection of vector spaces $C_{i,j}$ with differentials $d' : C_{i,j} \to C_{i-1,j}$ and $d'' : C_{i,j-1} \to C_{i,j}$ such that $d'\circ d' = 0$, $d''\circ d'' = 0$ and $d'\circ d'' = d''\circ d'$. Our convention is to view the total complex $C_n = \bigoplus_{i-j=n} C_{i,j}$ as a homological complex.

Signs: Let I and J be disjoint subsets of a linearly ordered set $\{1,\ldots,n\}$; we then define a sign $\mathrm{sgn}(I,J)\in\{\pm 1\}$ as follows. In the exterior algebra on n independent generators $x_1,\ldots,x_n$, we write $x_I = x_{i_1}\wedge\ldots\wedge x_{i_k}$ for $I=\{i_1<\ldots<i_k\}$. Then $\mathrm{sgn}(I,J)$ is defined by the equation $x_{I\sqcup J} = \mathrm{sgn}(I,J)x_I\wedge x_J$. For example we get $\mathrm{sgn}(\{i_r\}, I\setminus\{i_r\}) = (-1)^{r-1}$.

Consistency: Chapters 2, 3, 4 are slightly rewritten versions of the article [Dup14a] and the preprints [Dup13, Dup14b], and can be read in any order. For the sake of convenience, there are some minor redundancies between chapters. As notations and terminology are concerned, the only sources of inconsistency between chapters are the following:

– In Chapter 2 we use monomials f_j that are denoted by $f_j^\vee$ in Chapter 4. This is because duality is pointless in the affine setting of Chapter 2, and fundamental in the global setting of Chapter 3.

– In Chapter 3, we favored the terminology “hyperplane arrangement” (or “hypersurface arrangement”) over the more cumbersome terminology “arrangement of hyperplanes” (or “arrangement of hypersurfaces”). In Chapter 4, we used the latter terminology because of the appearance of bi-arrangements.

– In Chapter 3, the differential in the Orlik-Solomon algebra is denoted by δ, the letter d being used for the exterior derivative on differential forms.

– In Chapter 4, we have used script letters ($\mathscr{A}$, $\mathscr{L}$, $\mathscr{M}$, $\mathscr{B}$) to denote (bi-)arrangements. This is in accordance with the general convention in the arrangements literature. Italic letters (K, L, M) denote hyperplanes or hypersurfaces from some (bi-)arrangement.

Chapter 2

The combinatorial Hopf algebra of motivic dissection polylogarithms

In §2.1 we introduce dissection diagrams and the Hopf algebra $\mathcal{D}$, as well as its decorated variants; this section is purely combinatorial. In §2.2 we focus on bi-arrangements of hyperplanes and prove Theorem 2.2.2 on the relative cohomology groups for affinely generic bi-arrangements. This section can be read independently from the rest of the chapter. In §2.3 we introduce the dissection polylogarithms $I(D)$ and discuss some of their algebraic relations. In §2.4 we define the motivic dissection polylogarithms $I^{\mathcal{H}}(D)$ and prove the main theorem of this chapter (Theorem 2.4.9), which computes their coproduct. In §§2.5, 2.6 and 2.7, which serve as appendices to this chapter, we prove three technical lemmas.

2.1 A combinatorial Hopf algebra on dissection diagrams

2.1.1 The combinatorics of dissection diagrams

For every integer n we consider a regular oriented $(n+1)$-gon Π_n with a distinguished vertex called the *root*. We draw the polygons as circles so that Π_0 and Π_1 also make sense, hence the *sides* of Π_n are drawn as arcs between two consecutive vertices. A *chord* of Π_n is a line between two distinct vertices.

Definition 2.1.1. A *dissection diagram* of degree n is a set of n non-intersecting chords of Π_n such that the graph formed by the chords is acyclic.

In all the examples the polygons will be drawn with a clockwise orientation. The root will be drawn at the bottom as a white dot, whereas the non-root vertices will be drawn as black dots.
Since there are n chords and $(n+1)$ vertices, the graph formed by the chords is actually a tree that passes through all $(n+1)$ vertices; in other words, it is a spanning tree of the complete graph on the $(n+1)$ vertices of Π_n.
All the dissection diagrams of degree $\leqslant 3$ are pictured in Figure 2.1.

Lemma 2.1.2. *The number of dissection diagrams of degree n is*

$$d_n = \frac{1}{2n+1}\binom{3n}{n}.$$

Proof. We will not use this Lemma in the rest of this chapter, so we just give a sketch of the proof. The sequence $(d_n)_{n\geqslant 0}$ counting the dissection diagrams in each degree satisfies the

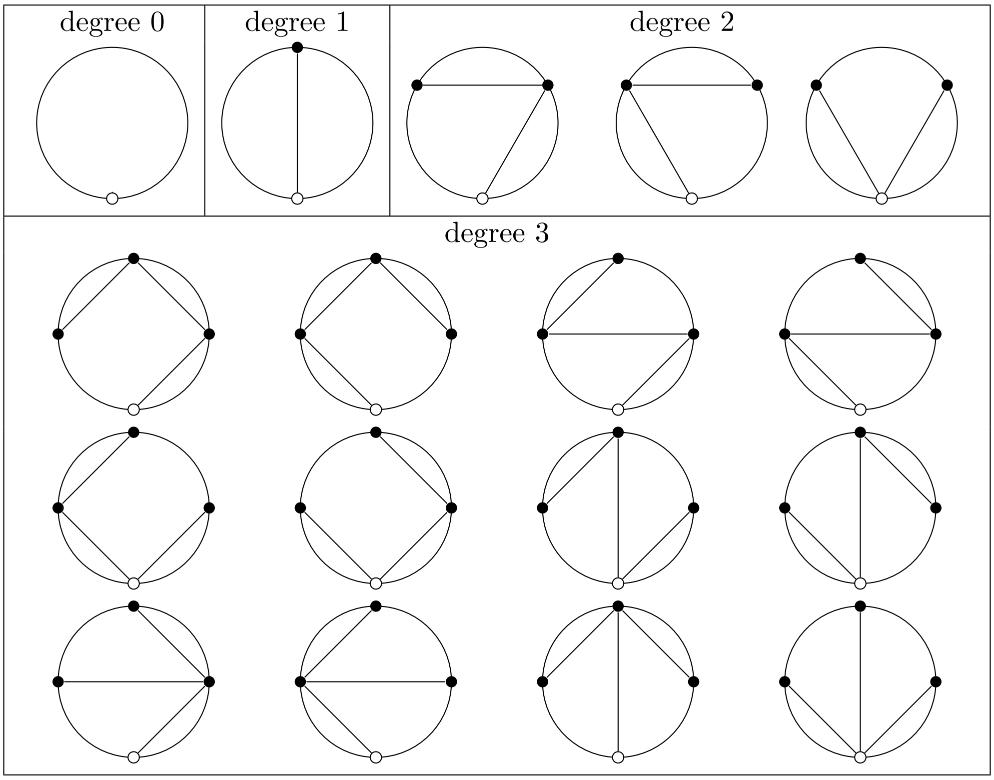
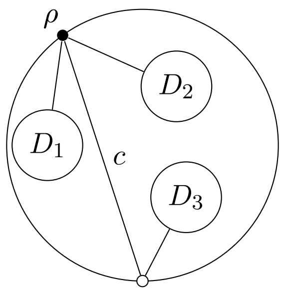


Figure 2.1: The dissection diagrams of degree $\leqslant 3$

recurrence relation, for $n \geqslant 1$:

$$d_n = \sum_{\substack{i_1,i_2,i_3\geqslant 0\\ i_1+i_2+i_3=n-1}} d_{i_1}d_{i_2}d_{i_3}. \tag{2.1}$$

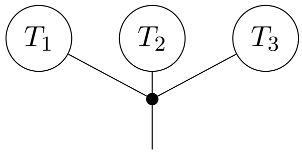
The reason for this recurrence relation is that a dissection diagram D of degree n is uniquely determined by a triple (D_1, D_2, D_3) of dissection diagrams of respective degrees (i_1, i_2, i_3) such that $i_1 + i_2 + i_3 = n - 1$.



In the above picture, ρ is the first (in clockwise order, starting at the root) non-root vertex of Π_n that is attached to the root by a chord of D. Let c be the chord between ρ and the root.
The i_1 chords that are on the left-hand side of c form a rooted tree with i_1 internal vertices, whose root is ρ. Since the internal vertices of this tree are all on the polygon Π_n, we view it as a dissection diagram D_1 of degree i_1.
The $(n-1-i_1)$ chords that are on the right-hand side of c form two connected components: one of cardinality i_2 that is attached to ρ, the other of cardinality i_3 that is attached to the root of Π_n. In the same fashion as above, we get dissection diagrams D_2 and D_3 of respective

degrees i_2 and i_3, with $i_2 + i_3 = n - 1 - i_1$.
Now let $d(x) = \sum_{n \geqslant 0} d_n x^n$ be the ordinary generating series for the enumeration of dissection diagrams. The recurrence relation (2.1), together with $d_0 = 1$, implies the functional equation $d(x) = 1 + xd(x)^3$. Thus the Lagrange inversion formula [Sta99, Theorem 5.4.2] applied to $f(x) = d(x) - 1$ gives the result. $\square$

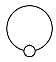
Remark 2.1.3. It is well-known that d_n is also the number of ternary trees (planar rooted trees in which every internal vertex has exactly 3 incoming edges) with n internal vertices. The proof goes along the same lines: a ternary tree T is completely determined by its subtrees T_1, T_2, T_3 attached to the root.

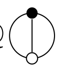
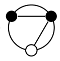
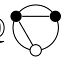
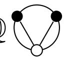
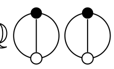


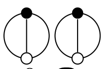
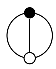
Thus we may recursively build a bijection between ternary trees and dissection diagrams.

We let $\mathcal{D}$ be the free commutative unital algebra (over $\mathbb{Q}$) on the set of dissection diagrams of positive degree. The degrees of the dissection diagrams induce a grading

$$\mathcal{D} = \bigoplus_{n \geqslant 0} \mathcal{D}_n$$

on $\mathcal{D}$. The unit 1 of $\mathcal{D}$ will be identified with the dissection diagram  of degree 0.
In small degree, we have

$\mathcal{D}_0 = \mathbb{Q}$ $\qquad$ $\mathcal{D}_1 = \mathbb{Q}$ $\qquad$ $\mathcal{D}_2 = \mathbb{Q}$ $\oplus \mathbb{Q}$ $\oplus \mathbb{Q}$ $\oplus \mathbb{Q}$

where  represents the square of the only dissection diagram  of degree 1. For every $n \geqslant 0$, $\mathcal{D}_n$ is a finite-dimensional vector space.

Conventions on dissection diagrams We introduce some labeling conventions on dissection diagrams. An example is shown in Figure 2.2.

The non-root vertices of Π_n are labeled $1, \ldots, n$ following the orientation, 1 being just after the root. The sides of Π_n are labeled $0, 1, \ldots, n$ in such a way that the side labeled 0 is between the root and the vertex 1. This side plays a special role in the sequel and is called the *root side*. The other sides are called the *non-root sides*: for $i = 1, \ldots, n-1$, the side labeled i is between the vertices i and $i+1$, and the side labeled n is between the vertex n and the root.

In a dissection diagram of degree n, the n chords form a spanning tree of the complete graph on the $(n+1)$ vertices of Π_n. There is thus a preferred orientation of all the chords, towards the root. We may then label the chords with $1, \ldots, n$ such that the chord labeled i leaves the vertex labeled i.

The sides of Π_n are also implicitly oriented following the orientation of Π_n (clockwise, in all our figures). Thus when we consider the $(n+1)$ sides of Π_n together with the n chords of a dissection diagram D, we get a directed graph with $(n+1)$ vertices and $(2n+1)$ edges that is denoted $\Gamma(D)$ and called the *total directed graph* of D.

Remark 2.1.4. Even though we will not always include them in the pictures, the orientations of the sides and the chords, as well as the labelings of the vertices, sides and chords of a dissection diagram are implicit.

In the sequel it will be more convenient to consider dissection diagrams D where the chords are labeled by some abstract set $\mathscr{C}(D)$ of cardinality n, and the sides of the polygon by some other abstract set $\mathscr{S}(D)$ of cardinality $(n+1)$, which are both linearly ordered.
If we set $\mathscr{S}^+(D) = \mathscr{S}(D) \setminus \{\min(\mathscr{S}(\mathscr{D}))\}$ for the set of non-root sides, the linear orderings give bijections

$$\mathscr{C}(D) \simeq \{1, \dots, n\} \simeq \mathscr{S}^+(D). \tag{2.2}$$

Remark 2.1.5. When the context is clear, we will drop the dissection diagram D from the notations and simply write $\mathscr{C}$, $\mathscr{S}$, $\mathscr{S}^+$, Γ.

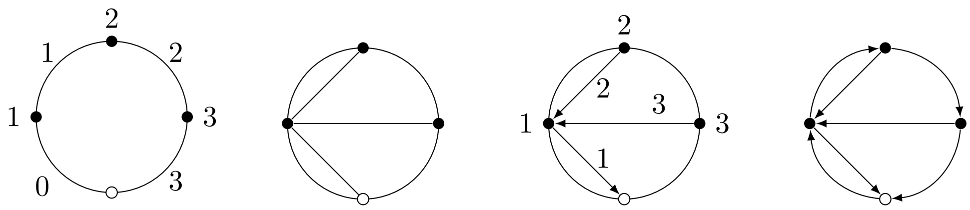


Figure 2.2: Conventions on dissection diagrams. Labeling the vertices and the sides of Π_3; a dissection diagram D of degree 3; the natural orientation and labeling of the chords of D; the total directed graph $\Gamma(D)$ of D.

2.1.2 Operations on dissection diagrams

Let D be a dissection diagram of degree n. We fix a subset $C \subset \mathscr{C}$ of chords of D. We introduce the notations $\mathscr{S}^+_C$, $q_C(D)$, $r_C(D)$, $\mathscr{K}_C(D)$ and $k_C(D)$ that will allow us to make sense of formula (2.4) below for the coproduct in $\mathcal{D}$. The reader may refer to Figure 2.3 for a special case.

The set $\mathscr{S}^+_C \subset \mathscr{S}^+$

We first define a subset $\mathscr{S}^+_C \subset \mathscr{S}^+$ of the non-root sides of D, of cardinality $n - |C|$. It plays a sort of "dual role" to C, see Proposition 2.4.7. In some simple cases (see Example 2.1.14 below), $\mathscr{S}^+_C$ will simply be the complement $\overline{C}$ of C in $\mathscr{S}^+$, using the identification (2.2).
The planar graph $C \cup \mathscr{S}$ has $|C|+1$ faces. Each such face α is the interior of a polygon that we denote $\widetilde{\Pi}(\alpha)$, whose sides are sides of Π_n and chords of D. If we denote by $\mathscr{S}_C(\alpha)$ the set of sides of Π_n that are sides of $\widetilde{\Pi}(\alpha)$, we get a partition

$$\mathscr{S} = \bigsqcup_\alpha \mathscr{S}_C(\alpha). \tag{2.3}$$

Lemma 2.1.6. *Let $J \subset \mathscr{S}$ be a subset of edges of Π_n, with $|C| + |J| = n$. Then the undirected graph $C \cup J$ is acyclic if and only if J has the form*

$$J = \bigsqcup_\alpha J(\alpha) \ \textit{with}\ J(\alpha) = \mathscr{S}_C(\alpha) \setminus \{u_\alpha\}$$

for some choice of $u_\alpha \in \mathscr{S}_C(\alpha)$.

Proof. Let us write $J = \sqcup_\alpha J(\alpha)$ with $J(\alpha) \subset \mathscr{S}_C(\alpha)$. If there exists an α such that $J(\alpha) = \mathscr{S}_C(\alpha)$ then $C \cup J$ contains the whole boundary of the polygon $\widetilde{\Pi}(\alpha)$, hence contains a cycle. Hence if $C \cup J$ is acyclic, then all the inclusions $J(\alpha) \subset \mathscr{S}_C(\alpha)$ are strict. Since $|J| = n - |C|$, we necessarily have $|J(\alpha)| = |\mathscr{S}_C(\alpha)| - 1$ for each α, hence $J(\alpha) = \mathscr{S}_C(\alpha) \setminus \{u_\alpha\}$. We leave it to the reader to show that in that case, $J \cup C$ is indeed acyclic. □

Let us set

$$\mathscr{S}_C^+ = \bigsqcup_\alpha \mathscr{S}_C^+(\alpha) \text{ with } \mathscr{S}_C^+(\alpha) = \mathscr{S}_C(\alpha) \setminus \{\min(\mathscr{S}_C(\alpha))\}.$$

It is a subset of $\mathscr{S}^+$ and has cardinality $n - |C|$.

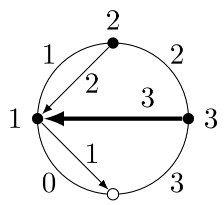
Let $\overline{C} = \mathscr{C} \setminus C$ denote the set of the chords of D which are not in C. Since the chords do not intersect each other, we have a partition

$$\overline{C} = \bigsqcup_\alpha \overline{C}(\alpha)$$

where $\overline{C}(\alpha)$ is the set of chords of D which are inside the polygon $\widetilde{\Pi}(\alpha)$.
It is clear that $\mathscr{S}_C^+(\alpha)$ and $\overline{C}(\alpha)$ have the same cardinality $|\mathscr{S}_C^+(\alpha)| = |\overline{C}(\alpha)| = n(\alpha)$, with $\sum_\alpha n(\alpha) = n - |C|$.

Remark 2.1.7. Despite the notation, $\mathscr{S}_C^+$ does not depend only on C but also on the dissection diagram D.

Example 2.1.8. Let us focus on the dissection diagram D of Figure 2.2 and put $C = \{3\}$ consisting only of the horizontal chord.



Then the partition $\mathscr{S} = \bigsqcup_\alpha \mathscr{S}_C(\alpha)$ is $\{0,1,2,3\} = \{1,2\} \sqcup \{0,3\}$ hence we get $\mathscr{S}_C^+ = \{2\} \sqcup \{3\} = \{2,3\}$. The corresponding partition of $\overline{C} = \{1,2\}$ is $\overline{C} = \{2\} \sqcup \{1\}$.

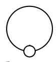
The dissection diagrams $q_C^\alpha(D)$ and their product $q_C(D)$

Starting from the dissection diagram D, let us contract the chords from C. The resulting picture is a “cactus” of dissection diagrams glued together. These dissection diagrams are denoted by $q_C^\alpha(D)$ and we write

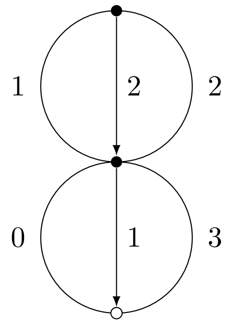
$$q_C(D) = \prod_\alpha q_C^\alpha(D)$$

for their product in $\mathcal{D}$.

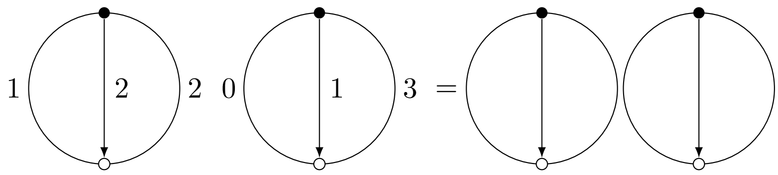
More precisely, let us consider an individual polygon $\widetilde{\Pi}(\alpha)$ and contract all its sides that are chords of C. We get a polygon $\Pi(\alpha)$ that is naturally oriented. The dissection diagram $q_C^\alpha(D)$ naturally lives in $\Pi(\alpha)$. The set of its non-root sides is $\mathscr{S}_C^+(\alpha)$ and the set of its chords is $\overline{C}(\alpha)$. The degree of $q_C^\alpha(D)$ is $n(\alpha)$, hence the degree of $q_C(D)$ is $\sum_\alpha n(\alpha) = n - |C|$.

Let us recall that we identify the dissection diagram  of degree 0 with the unit 1 of $\mathcal{D}$, so that we do not write the dissection diagrams $q_C^\alpha(D)$ of degree $n(\alpha) = 0$ in the product $q_C(D)$.

Example 2.1.9. We come back to the dissection diagram D of degree 3 from Example 2.1.8 with $C = \{3\}$. Contracting the horizontal chord labeled 3 gives the picture



hence $q_C(D) =$



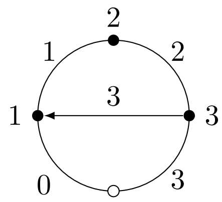
is the square of the dissection diagram of degree 1.

The dissection diagram $r_C(D)$; the set $\mathscr{K}_C(D)$ and its cardinality $k_C(D)$

Going back to the initial dissection diagram D, let us look at the graph obtained by keeping only the chords from C and contracting the sides from $\mathscr{S}_C^+$. By Lemma 2.1.6, this process does not lead to cycles between the chords from C and hence gives a dissection diagram whose set of chords is C and whose set of non-root sides is $\overline{\mathscr{S}_C^+} = \mathscr{S}^+ \setminus \mathscr{S}_C^+$. We call this dissection diagram $r_C(D)$. Its degree is $|C|$.

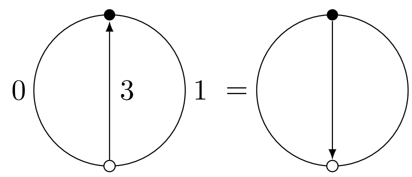
It has to be noted that in general the directions of the chords in $r_C(D)$ may differ from the directions of the chords in D. We let $\mathscr{K}_C(D) \subset C$ be the set of these chords that one has to flip in the process of computing $r_C(D)$, and write $k_C(D) = |\mathscr{K}_C(D)|$ for its cardinality.

Example 2.1.10. We come back to the dissection diagram D of degree 3 from Example 2.1.8 with $C = \{3\}$. Keeping only the horizontal chord labeled 3 gives the picture

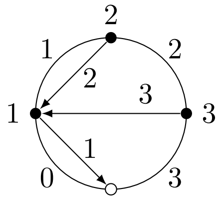


and hence contracting the sides from $\mathscr{S}_C^+ = \{2, 3\}$ gives the picture

$r_C(D) =$



hence $r_C(D)$ is (unsurprisingly) the dissection diagram of degree 1. Since in the above picture we had to flip the chord labeled 3, we get $\mathscr{K}_C(D) = \{3\}$ and $k_C(D) = 1$.



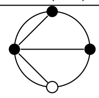
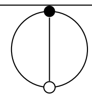
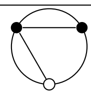
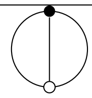
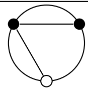
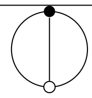
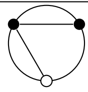
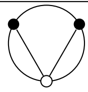
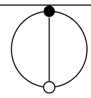
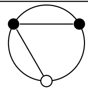
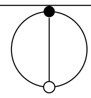
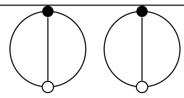
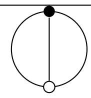
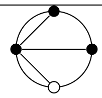
C	$\mathscr{S}_C^+$	$q_C(D)$	$r_C(D)$	$\mathscr{K}_C(D)$	$k_C(D)$
$\{1,2,3\}$	$\varnothing$	1		$\varnothing$	0
$\{1,2\}$	$\{3\}$			$\varnothing$	0
$\{1,3\}$	$\{2\}$			$\varnothing$	0
$\{2,3\}$	$\{3\}$			$\{3\}$	1
$\{1\}$	$\{2,3\}$			$\varnothing$	0
$\{2\}$	$\{2,3\}$			$\{2\}$	1
$\{3\}$	$\{2,3\}$			$\{3\}$	1
$\varnothing$	$\{1,2,3\}$		1	$\varnothing$	0

Figure 2.3: The computations of $\mathscr{S}_C^+$, $q_C(D)$, $r_C(D)$, $\mathscr{K}_C(D)$ and $k_C(D)$ for the dissection diagram D from Example 2.1.8

2.1.3 Definition of the Hopf algebra

We define a map

$$\Delta : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$$

by setting

$$\Delta(D) = \sum_{C \subset \mathscr{C}(D)} (-1)^{k_C(D)} q_C(D) \otimes r_C(D) \tag{2.4}$$

for D a dissection diagram, and extending it to all of $\mathcal{D}$ as a morphism of algebras.
For a dissection diagram D of degree n, $q_C(D)$ has degree $n - |C|$ and $r_C(D)$ has degree $|C|$. Thus the coproduct Δ is compatible with the grading of $\mathcal{D}$, with components

$$\Delta_{n-k,k} : \mathcal{D}_n \to \mathcal{D}_{n-k} \otimes \mathcal{D}_k$$

corresponding to the subsets $C \subset \mathscr{C}(D)$ of cardinality k.

For $C = \mathscr{C}(D)$ we get $\mathscr{S}_C = \varnothing$, $q_C(D) = 1$, $r_C(D) = D$, $\mathscr{K}_C(D) = \varnothing$ and $k_C(D) = 0$, hence the corresponding term in formula (2.4) is $\Delta_{0,n}(D) = 1 \otimes D$. For $C = \varnothing$, we get the term $\Delta_{n,0}(D) = D \otimes 1$.

Proposition 2.1.11. *Formula (2.4) gives $\mathcal{D}$ the structure of a graded connected commutative Hopf algebra.*

Proof. All there is to prove is that Δ is coassociative, since it is well-known that given a graded connected bialgebra there exists a unique antipode that makes it into a Hopf algebra. Let us fix

a dissection diagram D of degree n and prove that $(\mathrm{id}\otimes\Delta)(\Delta(D)) = (\Delta\otimes\mathrm{id})(\Delta(D))$.
On the one hand we have

$$(\mathrm{id}\otimes\Delta)(\Delta(D)) = \sum_{C\subset C'\subset\mathscr{C}(D)} (-1)^{k_{C'}(D)+k_C(r_{C'}(D))} q_{C'}(D)\otimes q_C(r_{C'}(D))\otimes r_C(r_{C'}(D)).$$

On the other hand we have

$$(\Delta\otimes\mathrm{id})(\Delta(D)) = \sum_{C\subset\mathscr{C}(D)} (-1)^{k_C(D)}\Delta(q_C(D))\otimes r_C(D).$$

Let us recall that $q_C(D) = \prod_\alpha q_C^\alpha(D)$. For a given α, the set of chords of $q_C^\alpha(D)$ is $\overline{C}(\alpha)$, hence

$$\Delta(q_C^\alpha(D)) = \sum_{C'_\alpha\subset\overline{C}(\alpha)} (-1)^{k_{C'_\alpha}(q_C^\alpha(D))} q_{C'_\alpha}(q_C^\alpha(D))\otimes r_{C'_\alpha}(q_C^\alpha(D)).$$

Let us perform the change of summation indices $C' = C\sqcup\bigsqcup_\alpha C'_\alpha$. The result then follows from the following Lemma. ☐

Lemma 2.1.12. *1.* $q_{C'}(D) = \prod_\alpha q_{C'_\alpha}(q_C^\alpha(D))$.

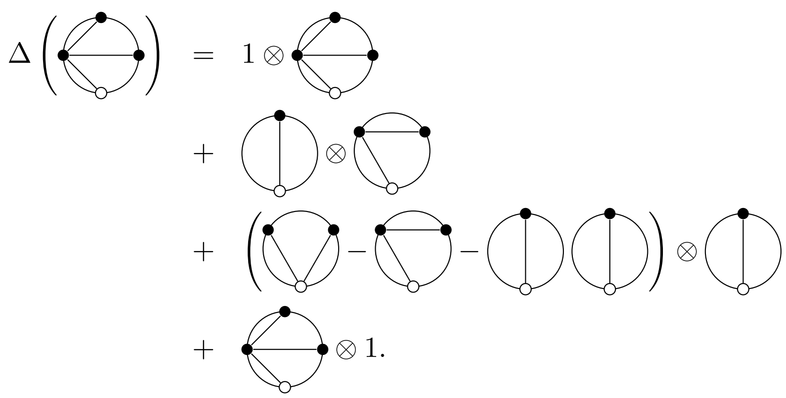
2. $q_C(r_{C'}(D)) = \prod_\alpha r_{C'_\alpha}(q_C^\alpha(D))$.

3. $r_C(r_{C'}(D)) = r_C(D)$.

4. $k_{C'}(D) + k_C(r_{C'}(D)) = k_C(D) + \sum_\alpha k_{C'_\alpha}(q_C^\alpha(D))$.

Proof. See §2.5. ☐

Example 2.1.13. We can use the computations of Figure 2.3 in order to get



Example 2.1.14. 1. For all $n\geqslant 0$ let X_n be the dissection diagram of degree n ("corolla", see Figure 2.4) with all chords pointing towards the root, with the convention $X_0 = 1$.
Then the formula for the coproduct is:

$$\Delta(X_n) = \sum_{k=0}^{n}\left(\sum_{i_0+\cdots+i_k=n-k} X_{i_0}\cdots X_{i_k}\right)\otimes X_k.$$

Indeed, for a subset $C = \{i_0+1, i_0+i_1+2,\ldots,i_0+i_1+\ldots+i_{k-1}+k\}$, we get $\mathscr{S}_C^+ = \overline{C}$, $q_C(X_n) = X_{i_0}\cdots X_{i_k}$, $r_C(X_n) = X_k$, $\mathscr{K}_C(X_n) = \varnothing$ and $k_C(X_n) = 0$.
For instance we get

$$\Delta(X_3) = 1\otimes X_3 + 3X_1\otimes X_2 + (2X_2 + X_1X_1)\otimes X_1 + X_3\otimes 1.$$

2. For all $n \geqslant 0$, let Y_n be the dissection diagram of degree n ("path tree", see Figure 2.4) consisting of the chords betwen 1 and 2, 2 and 3, ..., $(n-1)$ and n, n and the root, with the convention $Y_0 = 1$. Then the formula for the coproduct is:
$$\Delta(Y_n) = \sum_{k=0}^{n} \binom{n}{k} Y_{n-k} \otimes Y_k.$$
Indeed, for any subset $C \subset \{1,\ldots,n\}$ of cardinality k, we get $\mathscr{S}_C^+ = \overline{C}$, $q_C(Y_n) = Y_{n-k}$, $r_C(Y_n) = Y_k$, $\mathscr{K}_C(Y_n) = \varnothing$ and $k_C(Y_n) = 0$.
The above formula is reminiscent of the formula for the coproduct in the Hopf algebra $\mathbb{Q}[t]$ of functions on the additive group $\mathbb{G}_a$.

Figure 2.4: The corolla X_4 and the path tree Y_4.

Remark 2.1.15. The Hopf algebra $\mathcal{D}$ is a right-sided combinatorial Hopf algebra in the sense of [LR10, 5.7]. According to [LR10, Theorem 5.8], there is thus a structure of graded pre-Lie algebra on the free vector space spanned by dissection diagrams of positive degree (more precisely, its graded dual). It would be interesting to know if this pre-Lie structure has a simple presentation.

A family of Hopf algebras Let x be a fixed rational number. If one changes formula (2.4) to
$$\Delta^{(x)}(D) = \sum_{C \subset \mathscr{C}(D)} x^{k_C(D)} q_C(D) \otimes r_C(D) \tag{2.5}$$
then the proof of Proposition 2.1.11 (replace -1 by x) shows that this defines a (graded connected commutative) Hopf algebra $\mathcal{D}^{(x)}$.
Apart from the choice $x = -1$ which gives back $\mathcal{D}^{(-1)} = \mathcal{D}$, there are two other natural choices: for $x = 1$ there is no sign in the formula; for $x = 0$ (with the convention $0^0 = 1$) there is no sign and the sum is restricted to the subsets C with $k_C(D) = 0$. The formulas of Example 2.1.14 are valid for any choice of x since we always have $k_C(D) = 0$.
We may also consider x as a formal parameter and view formula (2.5) as a map of $\mathbb{Q}[x]$-algebras
$$\mathbb{Q}[x] \otimes \mathcal{D} \to \mathbb{Q}[x] \otimes \mathcal{D} \otimes \mathcal{D} \cong (\mathbb{Q}[x] \otimes \mathcal{D}) \otimes_{\mathbb{Q}[x]} (\mathbb{Q}[x] \otimes \mathcal{D}) \tag{2.6}$$
given by
$$D \mapsto \sum_{C \subset \mathscr{C}(D)} x^{k_C(D)} \otimes q_C(D) \otimes r_C(D).$$
In terms of algebraic geometry, we get an algebraic family of affine group schemes parametrized by the affine line
$$\mathrm{Spec}(\mathbb{Q}[x] \otimes \mathcal{D}) = \mathbb{A}^1 \times \mathrm{Spec}(\mathcal{D}) \to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{Q}[x])$$
with constant underlying scheme $\mathrm{Spec}(\mathcal{D})$.

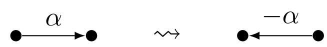
2.1.4 Decorations on dissection diagrams

In this paragraph we fix an abelian group Λ. We define a decorated version $\mathcal{D}(\Lambda)$ of the Hopf algebra $\mathcal{D}$.

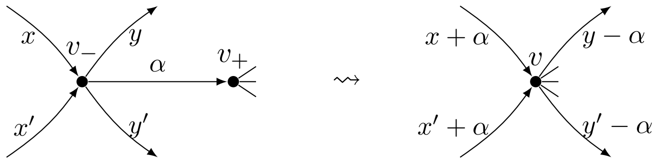
Decorated directed graphs

Let Γ be a directed graph. A Λ-*decoration* on Γ is the data of an element of Λ for each edge of Γ. While performing operations on directed graphs, we will always keep in mind the two following rules for the decorations:

- Let us flip an edge, i.e. change its direction. We then multiply its decoration by -1.



- Let us contract an edge going from a vertex v_- to a vertex v_+ which is decorated by an element $\alpha \in \Lambda$. For any edge of Γ going to v_-, we replace its decoration x by the decoration $x + \alpha$; for any edge of Γ leaving from v_-, we replace its decoration y by the decoration $y - \alpha$. The other decorations (including the decorations of the edges that touch v_+) stay unchanged.



We leave it to the reader to check that if one contracts a set of edges (that does not contain a loop), the resulting decorated graph does not depend on the order in which we perform the contractions.

Decorated dissection diagrams and the Hopf algebra $\mathcal{D}(\Lambda)$

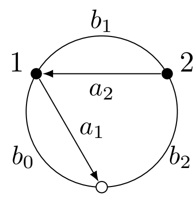


Figure 2.5: A decorated dissection diagram of degree 2

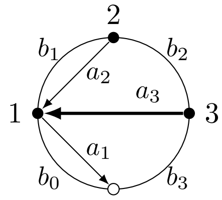
A Λ-*decorated dissection diagram* of degree n is a dissection diagram D of degree n together with a Λ-decoration on the total directed graph $\Gamma(D)$. For $i = 1, \ldots, n$, we denote by $a_i \in \Lambda$ the decoration of the chord i, and for $j = 0, \ldots, n$, we denote by $b_j \in \Lambda$ the decoration of the side j (see Figure 2.5). We use the same letter to denote the decorated dissection diagram and its underlying dissection diagram obtained by forgetting the decorations.

We let $\mathcal{D}(\Lambda)$ be the free commutative unital algebra (over $\mathbb{Q}$) on the set of Λ-decorated dissection diagrams of positive degree. If $\Lambda = 0$ then we recover $\mathcal{D}(0) = \mathcal{D}$. We want to generalize the Hopf algebra structure on $\mathcal{D}$ to all the $\mathcal{D}(\Lambda)$'s.
We define the coproduct

$$\Delta : \mathcal{D}(\Lambda) \to \mathcal{D}(\Lambda) \otimes \mathcal{D}(\Lambda)$$

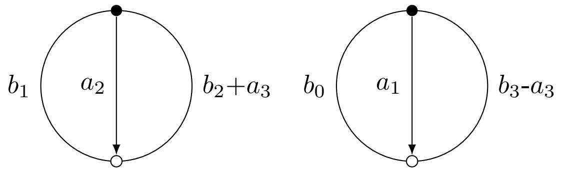
as in $\mathcal{D}$ by formula (2.4). The terms $q_C(D)$ and $r_C(D)$ are understood through the conventions of § 2.1.4.

Example 2.1.16. Let D be a decorated dissection diagram of degree 3 whose underlying dissection diagram is the one from Examples 2.1.8, 2.1.9, 2.1.10:



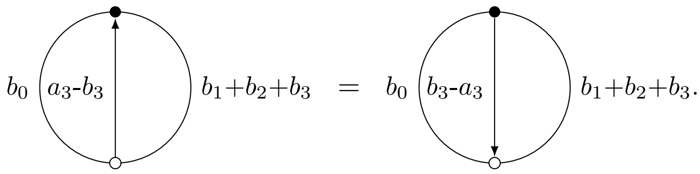
Then for $C = \{3\}$ we get

$q_C(D) =$



and

$r_C(D) =$



The last equality is the application of the convention related to flipping an edge (§2.1.4).

We leave it to the reader to check that properties 1., 2. and 3. from Lemma 2.1.12 remain true, as well as property 4., which is independent of the decorations. Hence the proof of Proposition 2.1.11 can be copied word for word and gives the following extension.

Proposition 2.1.17. *For any abelian group Λ, formula (2.4) gives $\mathcal{D}(\Lambda)$ the structure of a graded connected commutative Hopf algebra. Moreover, for any morphism $\Lambda \to \Lambda'$ of abelian groups, the corresponding morphism $\mathcal{D}(\Lambda) \to \mathcal{D}(\Lambda')$ is a morphism of Hopf algebras. In other words, $\Lambda \rightsquigarrow \mathcal{D}(\Lambda)$ is a functor from the category of abelian groups to the category of Hopf algebras.*

Remark 2.1.18. The variant of formula (2.5) remains valid with decorations.

Generic decorations and the Hopf algebra $\mathcal{D}^{\mathrm{gen}}(\Lambda)$

Let Γ be a directed graph. A *simple cycle* of Γ is an undirected cycle in Γ that does not pass twice through the same vertex. For a given simple cycle in Γ, the *total decoration* of the simple cycle is the signed sum of the decorations in the cycle, the sign being $+1$ if and only if the direction of the edge agrees with the direction of the cycle. We say that a Λ-decoration on Γ is *generic* if for every simple cycle in Γ, the total decoration of the cycle is non-zero.

We say that a Λ-decorated dissection diagram is *generic* if the Λ-decoration on $\Gamma(D)$ is generic. For example, the decorated dissection diagram from Figure 2.5 is generic if and only if the quantities $b_0 + b_1 + b_2$, $b_1 + b_2 - a_1$, $b_0 - a_2 + b_2$, $b_0 + a_1$, $b_1 + a_2$, $b_2 - a_1 - a_2$ are all $\neq 0$.

We leave it to the reader to check that the operations of reversal of arrows and contraction of § 2.1.4 preserve the genericity condition. As a consequence, the generic Λ-decorated dissection diagrams of positive degree generate a Hopf subalgebra

$$\mathcal{D}^{\mathrm{gen}}(\Lambda) \hookrightarrow \mathcal{D}(\Lambda).$$

The functoriality assertion of Proposition 2.1.17 is valid for the Hopf algebras $\mathcal{D}^{\mathrm{gen}}(\Lambda)$ if we restrict to *injective* morphisms $\Lambda \hookrightarrow \Lambda'$.

2.2 Bi-arrangements of hyperplanes and relative cohomology

After recalling some classical results on arrangements of hyperplanes, we introduce and study bi-arrangements of hyperplanes, focusing on the affinely generic case. This section is a solution to **Problem B** in the affinely generic case.

2.2.1 Affinely generic arrangements of hyperplanes

Let $L = \{L_1, \ldots, L_l\}$ be an arrangement of hyperplanes in $\mathbb{C}^n$. The hyperplanes do not necessarily pass through the origin. As the notation suggests, the set L is implicitly linearly ordered. We will use the same letter L to denote the union

$$L = L_1 \cup \ldots \cup L_l$$

of the hyperplanes. For a subset $I \subset \{1, \ldots, l\}$, the stratum of L indexed by I is the affine space

$$L_I = \bigcap_{i \in I} L_i$$

with the convention $L_\varnothing = \mathbb{C}^n$.

We say that L is *affinely generic* if it is a normal crossing divisor inside $\mathbb{C}^n$. It means that for all I, L_I is either empty or has codimension the cardinality $|I|$ of I.

Remark 2.2.1. If the L_i's are in general position in $\mathbb{C}^n$ then L is affinely generic, but the converse is not true. For instance, two parallel lines in $\mathbb{C}^2$ constitute an affinely generic arrangement. In other words, if we work in the projective space $\mathbb{P}^n(\mathbb{C})$ by adding a hyperplane L_0 at infinity, the projective arrangement of hyperplanes $L_0 \cup L_1 \cup \ldots \cup L_n$ is not necessarily normal crossing.

In the sequel, *we will only consider affinely generic hyperplane arrangements*. This class of hyperplane arrangements is stable under the operations of deletion, restriction and product that we now describe.
The *deletion* of L with respect to the last hyperplane L_l is the arrangement $L' = \{L_1, \ldots, L_{l-1}\}$ in $\mathbb{C}^n$. We have a natural morphism $H^\bullet(\mathbb{C}^n \setminus L') \to H^\bullet(\mathbb{C}^n \setminus L)$.
The *restriction* of L with respect to L_l is the arrangement $L'' = \{L_l \cap L_1, \ldots, L_l \cap L_{l-1}\}$ in $L_l \cong \mathbb{C}^{n-1}$ consisting of all the intersections of L_l with the L_i's, $i = 1, \ldots, l-1$. We have a residue morphism $H^\bullet(\mathbb{C}^n \setminus L)(1) \to H^{\bullet-1}(L_l \setminus L'')$, where (1) denotes a Tate twist.
If $L^{(1)} \subset \mathbb{C}^{n_1}$ and $L^{(2)} \subset \mathbb{C}^{n_2}$ are two hyperplane arrangements, then the *product* arrangement $L^{(1)} \times L^{(2)} \subset \mathbb{C}^{n_1+n_2}$ consists of the hyperplanes $L^{(1)}_{i_1} \times \mathbb{C}^{n_2}$ followed by the hyperplanes $\mathbb{C}^{n_1} \times L^{(2)}_{i_2}$. There is a Künneth isomorphism $H^\bullet(\mathbb{C}^{n_1} \setminus L^{(1)}) \otimes H^\bullet(\mathbb{C}^{n_2} \setminus L^{(2)}) \cong H^\bullet(\mathbb{C}^{n_1+n_2} \setminus L^{(1)} \times L^{(2)})$.

Let $\Lambda^\bullet(e_1, \ldots, e_l)$ denote the exterior algebra over $\mathbb{Q}$ with a generator e_i in degree 1 for each hyperplane L_i. For a set $I = \{i_1 < \ldots < i_k\} \subset \{1, \ldots, l\}$ we set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ with the convention $e_\varnothing = 1$.
Let $R_\bullet(L)$ be the ideal of $\Lambda^\bullet(e_1, \ldots, e_l)$ generated by the elements e_I for subsets $I \subset \{1, \ldots, l\}$ such that $L_I = \varnothing$.

The following theorem is a particular case of the Brieskorn-Orlik-Solomon theorem (for a detailed proof of the general case, see [OT92, Theorems 3.126 and 5.89]).

Theorem 2.2.2. *Let L be an affinely generic hyperplane arrangement.*

1. *There is an isomorphism of graded algebras*

$$\Lambda^\bullet(e_1,\ldots,e_l)/R_\bullet(L) \xrightarrow{\cong} H^\bullet(\mathbb{C}^n \setminus L) \tag{2.7}$$

that sends e_i to the class of the form $\omega_i = \dfrac{1}{2i\pi}\dfrac{df_i}{f_i}$, where f_i is any linear form that defines L_i.

2. *This isomorphism is functorial in the following sense.*

   (a) *the deletion morphism $H^\bullet(\mathbb{C}^n \setminus L') \to H^\bullet(\mathbb{C}^n \setminus L)$ is given by $e_I \mapsto e_I$ for $I \subset \{1,\ldots,l-1\}$.*

   (b) *the restriction morphism $H^\bullet(\mathbb{C}^n \setminus L)(1) \to H^{\bullet-1}(L_l \setminus L'')$ is given, for I such that $l \notin I$, by $e_I \mapsto 0$ and $e_I \wedge e_l \mapsto e_I$.*

   (c) *The Künneth isomorphism $H^\bullet(\mathbb{C}^{n_1} \setminus L^{(1)}) \otimes H^\bullet(\mathbb{C}^{n_2} \setminus L^{(2)}) \cong H^\bullet(\mathbb{C}^{n_1+n_2} \setminus L^{(1)} \times L^{(2)})$ is given by $e_{I_1} \otimes e_{I_2} \mapsto e_{I_1 \sqcup I_2}$.*

Remark 2.2.3. The first part of Theorem 2.2.2 implies that the mixed Hodge structure underlying $H^k(\mathbb{C}^n \setminus L)$ is pure of weight $2k$ and of Tate type: it is a direct sum of a certain number of copies of $\mathbb{Q}(-k)$.

Remark 2.2.4. If $I \subset \{1,\ldots,l\}$ has cardinality $> n$ then $L_I = \varnothing$ by definition of an affinely generic hyperplane arrangement. Hence (2.7) implies that $H^k(\mathbb{C}^n \setminus L) = 0$ for $k > n$. This is also a consequence of Artin vanishing since $\mathbb{C}^n \setminus L$ is an affine algebraic variety of dimension n.

2.2.2 Affinely generic bi-arrangements of hyperplanes

A *bi-arrangement of hyperplanes* $(L; M)$ in $\mathbb{C}^n$ is the data of two disjoint sets $L = \{L_1,\ldots,L_l\}$ and $M = \{M_1,\ldots,M_m\}$ of hyperplanes in $\mathbb{C}^n$. Equivalently, it is a 2-partition of the underlying hyperplane arrangement $L \cup M = \{L_1,\ldots,L_l,M_1,\ldots,M_m\}$. As the notation suggests, both L and M are linearly ordered. We say that $(L; M)$ is *affinely generic* if $L \cup M$ is, which means that it is a normal crossing divisor in $\mathbb{C}^n$. In the sequel, *we will only consider affinely generic bi-arrangements of hyperplanes.*

Among the relative cohomology groups $H^\bullet(\mathbb{C}^n \setminus L, M \setminus M \cap L)$, we will focus on the middle-degree one: we set

$$H(L; M) = H^n(\mathbb{C}^n \setminus L, M \setminus M \cap L).$$

According to Deligne [Del74], $H(L; M)$ is endowed with a functorial mixed Hodge structure. It is clear (and will be re-proved in the proof of Theorem 2.2.5) that this is actually a mixed Hodge-Tate structure. This means that for all k we have $\mathrm{gr}^W_{2k+1} H(L; M) = 0$, and $\mathrm{gr}^W_{2k} H(L; M)$ is isomorphic to a direct sum of the Tate structures $\mathbb{Q}(-k)$. The graded quotient $\mathrm{gr}^W_{2k} H(L; M)$ is 0 for $k \notin \{0,\ldots,n\}$.

Theorem 2.2.5. *Let $(L; M)$ be an affinely generic bi-arrangement in $\mathbb{C}^n$. Then for all $k = 0,\ldots,n$ we have a presentation*

$$\mathrm{gr}^W_{2k} H(L; M) \cong \left(\Lambda^k(e_1,\ldots,e_l) \otimes \Lambda^{n-k}(f_1,\ldots,f_m)\right)/R_k(L; M) \tag{2.8}$$

where $R_k(L; M)$ is spanned by the elements

– $e_I \otimes f_J$ *if* $L_I \cap M_J = \varnothing$, $|I| = k$, $|J| = n-k$.

– $e_I \otimes \left(\displaystyle\sum_{j \notin J'} \mathrm{sgn}(\{j\}, J') f_{J' \cup \{j\}}\right)$ *for* $|I| = k$, $|J'| = n-k-1$.

Proof. Let us denote by $j : \mathbb{C}^n \setminus (L \cup M) \hookrightarrow \mathbb{C}^n \setminus L$ the natural open immersion. Then $H^\bullet(\mathbb{C}^n \setminus L, M \setminus M \cap L)$ is the cohomology of the sheaf $j_!\mathbb{Q}_{\mathbb{C}^n\setminus(L\cup M)}$. One readily checks that we have a resolution

$$0 \to j_!\mathbb{Q}_{\mathbb{C}^n\setminus(L\cup M)} \to \mathbb{Q}_{\mathbb{C}^n\setminus L} \to \bigoplus_i (\iota_i)_*\mathbb{Q}_{M_i\setminus M_i\cap L} \to \bigoplus_{i<j}(\iota_{i,j})_*\mathbb{Q}_{M_{ij}\setminus M_{ij}\cap L} \to \cdots$$

where $\iota_J : M_J \setminus M_J \cap L \hookrightarrow \mathbb{C}^n \setminus L$ denotes the natural closed immersion. More precisely let us set

$$\mathcal{K}^p = \bigoplus_{|J|=p} (\iota_J)_*\mathbb{Q}_{M_J\setminus M_J\cap L}$$

and $d : \mathcal{K}^p \to \mathcal{K}^{p+1}$ is given by the natural restriction morphisms

$$(\iota_J)_*\mathbb{Q}_{M_J\setminus M_J\cap L} \to (\iota_{J\cup\{j\}})_*\mathbb{Q}_{M_{J\cup\{j\}}\setminus M_{J\cup\{j\}}\cap L}$$

for $j \notin J$, multiplied by the sign $\operatorname{sgn}(\{j\}, J)$. We then have a quasi-isomorphism

$$j_!\mathbb{Q}_{\mathbb{C}^n\setminus(L\cup M)} \cong \mathcal{K}^\bullet.$$

Let w be the descending filtration on $\mathcal{K}^\bullet$ given by $w^p\mathcal{K}^\bullet = \mathcal{K}^{\geqslant p}$. The corresponding hypercohomology spectral sequence is

$$E_1^{p,q} = \bigoplus_{|J|=p} H^q(M_J \setminus M_J \cap L) \Longrightarrow E_\infty^{p,q} = \operatorname{gr}_w^p H^{p+q}(\mathbb{C}^n \setminus L, M \setminus M \cap L)$$

On the E_1-term, the differential d_1 is given by the natural restriction morphisms

$$H^q(M_J \setminus M_J \cap L) \to H^q(M_{J\cup\{j\}} \setminus M_{J\cup\{j\}} \cap L)$$

for $j \notin J$, multiplied by the sign $\operatorname{sgn}(\{j\}, J)$.
According to Deligne [Del74, 8.3.5], this spectral sequence is a spectral sequence of mixed Hodge structures. Since by Remark 2.2.3 the mixed Hodge structures $H^q(M_J \setminus M_J \cap L)$ are pure of weight $2q$, the spectral sequence degenerates at E_2: $E_\infty = E_2$. The same argument implies that on $H^\bullet(\mathbb{C}^n \setminus L, M \setminus L \cap M)$, w is (up to a shift) the canonical weight filtration.
According to Remark 2.2.4, we have $H^k(M_J \setminus M_J \cap L) = 0$ for $|J| > n - k$. Thus in degree n we get

$$\operatorname{gr}_{2k}^W H(L; M) \cong \operatorname{Coker}\left(\bigoplus_{|J'|=n-k-1} H^k(M_{J'} \setminus M_{J'} \cap L) \xrightarrow{d_1} \bigoplus_{|J|=n-k} H^k(M_J \setminus M_J \cap L)\right)$$

which is obviously 0 if $k \notin \{0, \dots, n\}$. Introducing basis elements f_J, Theorem 2.2.2 tells us that $H^k(M_J \setminus M_J \cap L)$ has a presentation given by generators $e_I \otimes f_J$, $|I| = k$, and relations $e_I \otimes f_J = 0$ if $L_I \cap M_J = \varnothing$. Since the differential is given by $d_1(e_I \otimes f_{J'}) = e_I \otimes \left(\sum_{j\notin J'} \operatorname{sgn}(\{j\}, J') f_{J'\cup\{j\}}\right)$, this implies the theorem. □

Remark 2.2.6. In order to do explicit computations, we introduce a useful acyclic model for the complex of sheaves $\mathcal{K}^\bullet_{\mathbb{C}} := \mathcal{K}^\bullet \otimes \mathbb{C}$. For $(L; M)$ an affinely generic hyperplane arrangement, let us define a double complex of sheaves on X

$$\Omega^{p,q}_{(L;M)} = \bigoplus_{|J|=p} (i^{\mathbb{C}^n}_{M_J})_*\Omega^q_{M_J}(\log L)$$

where $\Omega^\bullet_{M_J}(\log L)$ is the complex of logarithmic forms defined in [Del71, 3.1], and $i^{\mathbb{C}^n}_{M_J}$ is the inclusion of M_J inside $\mathbb{C}^n$. The horizontal differential $\Omega^{p,q}_{(L;M)} \to \Omega^{p+1,q}_{(L;M)}$ is given by the restriction morphisms $\Omega^q_{M_J}(\log L) \to (i^{M_J}_{M_J\cap M_j})_*\Omega^q_{M_{J\cup\{j\}}}(\log L)$ for $j \notin J$, multiplied by the sign $\mathrm{sgn}(\{j\}, J)$. The vertical differential $\Omega^{p,q}_{(L;M)} \to \Omega^{p,q+1}_{(L;M)}$ is the exterior differential on forms. We let $\Omega^\bullet_{(L;M)}$ denote the total complex. Using [Del71, 3.1.8]., one easily proves that we have a quasi-isomorphism

$$\mathcal{K}^\bullet \xrightarrow{\cong} \Omega^\bullet_{(L;M)}.$$

We adapt the notions of deletion and restriction to the setting of (affinely generic) bi-arrangements. Furthermore, we allow ourselves to iterate them. Thus, for a subset $I_0 \subset \{1, \ldots, l\}$ (resp. $J_0 \subset \{1, \ldots, m\}$) we consider the deletion $(L(I_0); M)$ (resp. $(L; M(J_0))$) obtained by forgetting the hyperplanes L_i, $i \notin I_0$ (resp. the hyperplanes M_j, $j \notin J_0$), and the restriction $(L_{I_0}|L(\overline{I_0}); M)$ (resp. $(M_{J_0}|L; M(\overline{J_0}))$) obtained by considering the intersections of the hyperplanes with L_{I_0} (resp. with M_{J_0}).
On the relative cohomology groups $H(L; M)$, we get natural deletion/restriction morphisms, which are computed in the next Theorem.

Theorem 2.2.7. *The isomorphism (2.8) is functorial in the following sense.*

1. *For a subset $J_0 \subset \{1, \ldots, m\}$, the deletion morphism $H(L; M) \to H(L; M(J_0))$ is given by $e_I \otimes f_J \mapsto 0$ if $J \not\subset J_0$ and $e_I \otimes f_J \mapsto e_I \otimes f_J$ if $J \subset J_0$.*

2. *For a subset $J_0 \subset \{1, \ldots, m\}$, the restriction morphism $H(M_{J_0}|L; M(\overline{J_0})) \to H(L; M)$ is given, for $J \subset \overline{J_0}$, by*

$$e_I \otimes f_J \mapsto e_I \otimes (f_{J_0} \wedge f_J) = \mathrm{sgn}(J_0, J) e_I \otimes f_{J_0 \cup J}.$$

3. *For a subset $I_0 \subset \{1, \ldots, l\}$, the deletion morphism $H(L(I_0); M) \to H(L; M)$ is given, for $I \subset I_0$, by $e_I \otimes f_J \mapsto e_I \otimes f_J$.*

4. *For a subset $I_0 \subset \{1, \ldots, l\}$ of cardinality k_0, the restriction morphism $H(L; M)(k_0) \to H(L_{I_0}|L(\overline{I_0}); M)$ is given, for $I \subset \overline{I_0}$, by $e_I \otimes f_J \mapsto 0$, and*

$$e_{I\cup I_0} \otimes f_J = \mathrm{sgn}(I, I_0)(e_I \wedge e_{I_0}) \otimes f_J \mapsto \mathrm{sgn}(I, I_0) e_I \otimes f_J.$$

5. *The Künneth morphism $H(L^{(1)}; M^{(1)}) \otimes H(L^{(2)}; M^{(2)}) \to H(L^{(1)} \times L^{(2)}; M^{(1)} \times M^{(2)})$ is given by $(e_{I_1} \otimes f_{J_1}) \otimes (e_{I_2} \otimes f_{J_2}) \to (e_{I_1 \sqcup I_2}) \otimes (f_{J_1 \sqcup J_2})$.*

Proof. 1. It is obvious.

2. Let k_0 be the cardinality of J_0. Let us denote $\mathcal{K}_0$ the complex of sheaves corresponding to $(M_{J_0}|L; M(\overline{J_0}))$ as defined in the proof of Theorem 2.2.5. The restriction morphism $H(M_{J_0}|L; M(\overline{J_0})) \to H(L; M)$ is defined by a morphism $\Phi : \mathcal{K}_0^{\bullet - k_0} \to \mathcal{K}^\bullet$.
By definition we have

$$\mathcal{K}_0^{p-k_0} = \bigoplus_{\substack{|J|=p-k_0 \\ J_0\cap J=\varnothing}} (\iota_{J_0\cup J})_* \mathbb{Q}_{M_{J_0\cup J}\setminus M_{J_0\cup J}\cap L}$$

which is obviously a sub-sheaf of $\mathcal{K}^p$. We define Φ by multiplying the natural inclusion by the sign $\mathrm{sgn}(J_0, J)$ on the component indexed by J. We check that with this sign, Φ is a morphism of complexes of sheaves, and the claim follows.

3. It is obvious.

4. It is enough to do the proof over $\mathbb{C}$ and work with the models defined in Remark 2.2.6. The residue morphism is then given by morphisms (with obvious notations)

$$\Omega^{p,q}_{(L;M)} \to (i^{\mathbb{C}^n}_{L_{I_0}})_* \Omega^{p,q-k_0}_{(L_{I_0}|L(\overline{I_0});M)}$$

which are induced by the residue morphisms

$$\Omega^{q}_{M_J}(\log L) \to (i^{M_J}_{L_{I_0}\cap M_J})_* \Omega^{q-k_0}_{L_{I_0}\cap M_J}(\log L(\overline{I_0}))$$

defined in [Del71, 3.1.5]. The formula follows.

5. We also work over $\mathbb{C}$ with the models defined in Remark 2.2.6. It is easy to check that the Künneth morphism is given by the cup-product

$$(p_1)^*\Omega^{q_1}_{M^{(1)}_{J_1}}(\log L^{(1)}) \otimes (p_2)^*\Omega^{q_2}_{M^{(2)}_{J_2}}(\log L^{(2)}) \to \Omega^{q_1+q_2}_{M^{(1)}_{J_1}\times M^{(2)}_{J_2}}(\log L^{(1)} \times L^{(2)})$$

and the formula follows.

□

2.3 Dissection polylogarithms

In this section, we focus on $\mathbb{C}$-decorated dissection diagrams, which we simply call decorated dissection diagrams.

2.3.1 The bi-arrangement attached to a decorated dissection diagram

Definition

We attach to any decorated dissection diagram D of degree n a bi-arrangement $(L;M)$ inside $\mathbb{C}^n$. The equations of the L_i's depend on the chords of D and their decorations, while the equations of the M_j's depend on the decorations of the sides of the polygon (hence not on the combinatorics of D).
Let us recall that the total directed graph $\Gamma(D)$ of D is the graph whose vertices are the $(n+1)$ vertices of Π_n, and whose $(2n+1)$ directed edges are the chords of D and the sides of Π_n, oriented clockwise.

Let us work in the complex affine space $\mathbb{C}^n$ with coordinates $(t_1,\ldots,t_n)$. To each edge in $\Gamma(D)$ we associate a hyperplane in $\mathbb{C}^n$ in the following way:

– To an edge $\overset{i}{\bullet}\xrightarrow{\alpha}\overset{j}{\bullet}$ between two non-root vertices, we associate the hyperplane $t_i - t_j - \alpha = 0$.

– To an edge $\overset{i}{\bullet}\xrightarrow{\alpha}\circ$ that goes to the root, we associate the hyperplane $t_i - \alpha = 0$.

– To an edge $\circ\xrightarrow{\alpha}\overset{i}{\bullet}$ that comes from the root, we associate the hyperplane $-t_i - \alpha = 0$.

Hence the rule is always the same: we interpret the vertex i as the coordinate t_i, and the root as 0. The third case above only occurs for the side labeled 0.

We label $L_1,\ldots,L_n$ the hyperplanes given by the chords of the decorated dissection diagram D, L_i being given by the i-th chord (which by definition is the chord starting at the vertex i). Hence L_i is defined by $t_i - t_j - a_i = 0$ if the i-th chord goes to the j-th vertex, and by $t_i - a_i = 0$ if it goes to the root.

We label $M_0, M_1, \ldots, M_n$ the hyperplanes given by the sides of the polygon Π_n, in clockwise order. They are defined by $M_0 = \{t_1 = -b_0\}$, $M_j = \{t_j = t_{j+1} + b_j\}$ for $j = 1, \ldots, n-1$, and $M_n = \{t_n = b_n\}$.

This defines a bi-arrangement

$$(L; M) = (L_1, \ldots, L_n; M_0, M_1, \ldots, M_n)$$

in $\mathbb{C}^n$.

Example 2.3.1. Let us look at the decorated dissection diagram D of degree 3 from Example 2.1.16. Then the bi-arrangement $(L_1, L_2, L_3; M_0, M_1, M_2, M_3)$ in $\mathbb{C}^3$ is defined by the equations $L_1 = \{t_1 - a_1 = 0\}$, $L_2 = \{t_2 - t_1 - a_2 = 0\}$, $L_3 = \{t_3 - t_1 - a_3 = 0\}$, $M_0 = \{t_1 = -b_0\}$, $M_1 = \{t_1 = t_2 + b_1\}$, $M_2 = \{t_2 = t_3 + b_2\}$, $M_3 = \{t_3 = b_3\}$.

The combinatorics of the bi-arrangement $(L; M)$ can be read directly off the dissection diagram, as the following Lemma shows.

Lemma 2.3.2. *Let D be a decorated dissection diagram with generic decorations. Let $I \subset \{1, \ldots, n\}$ be a set of chords of D and $J \subset \{0, \ldots, n\}$ be a set of sides of D. We view $I \cup J$ as a subgraph of the total directed graph $\Gamma(D)$.*

1. *$L_I \cap M_J = \varnothing$ if and only if the graph $I \cup J$ contains an undirected cycle.*

2. *If $L_I \cap M_J \neq \varnothing$, then* $\operatorname{codim}(L_I \cap M_J) = |I| + |J|$.

Thus the bi-arrangement $(L; M)$ is affinely generic.

Proof. If there is an undirected path of total decoration λ from the vertex i to the vertex j in $I \cup J$, then for any point $(t_1, \ldots, t_n) \in L_I \cap M_J$, we get $t_i = t_j + \lambda$. If $I \cup J$ contains an undirected cycle, then it contains some simple cycle with total decoration $\lambda \neq 0$. Let i be a non-root vertex inside this simple cycle. For a point $(t_1, \ldots, t_n)$ in $L_I \cap M_J$, we get by definition $t_i = t_i + \lambda$, which is impossible. Thus $L_I \cap M_J = \varnothing$.
Conversely, one easily sees that if $I \cup J$ does not contain an undirected cycle then $L_I \cap M_J \neq \varnothing$ and $\operatorname{codim}(L_I \cap M_J) = |I| + |J|$. □

Operations on dissection diagrams and bi-arrangements

We can now explain the conventions from § 2.1.4 on dissection diagrams.

– If we change the direction of an edge and multiply its decoration by -1, this does not change the equation given by this edge.

– The convention for the contraction of edges accounts for the restriction of hyperplanes in bi-arrangements. Indeed, let us look at a restricted bi-arrangement $(L_i | L_1, \ldots, \widehat{L_i}, \ldots, L_n; M)$. If we choose the coordinates on $L_i \cong \mathbb{C}^{n-1}$ to be $(t_1, \ldots, \widehat{t_i}, \ldots, t_n)$, then the equations of the hyperplanes in this restricted bi-arrangement are exactly given by the edges of the graph resulting from the contraction of the i-th chord, with the convention from § 2.1.4. The same is of course true for a restriction of some hyperplane M_j, $j \geqslant 1$.

This allows us to reinterpret the operations q_C and r_C in terms of restriction and deletion of bi-arrangements.

Lemma 2.3.3. *Let D be a decorated dissection diagram of degree n and $(L; M)$ the corresponding bi-arrangement in $\mathbb{C}^n$. Let $C \subset \mathscr{C}$ be a set of chords of D.*

1. *For each α, let $(L^{(\alpha)}; M^{(\alpha)})$ be the bi-arrangement corresponding to the dissection diagram $q_C^\alpha(D)$. We have an isomorphism of bi-arrangements*

$$\prod_\alpha \left(L^{(\alpha)}; M^{(\alpha)}\right) \cong (L_C | L(\overline{C}); M).$$

2. *The bi-arrangement of the dissection diagram $r_C(D)$ is*

$$\left(M_{\mathscr{S}_C^+} \middle| L(C); M_0, M\left(\overline{\mathscr{S}_C^+}\right)\right).$$

Proof. 1. Let us recall the partition $\overline{C} = \bigsqcup_\alpha \overline{C}(\alpha)$. The equations of the bi-arrangement $(L^{(\alpha)}; M^{(\alpha)})$ are written in coordinates t_i, $i \in \overline{C}(\alpha)$, hence the product $\prod_\alpha (L^{(\alpha)}; M^{(\alpha)})$ is a bi-arrangement in an affine space with coordinates t_i, $i \in \overline{C}$. The same is true of $(L_C|L(\overline{C}); M)$. We then describe the isomorphism.
Let us denote by D/C the graph obtained by contracting the chords from C; its non-root vertices are labeled by $\overline{C}$. For each α, the root of $q_C^\alpha(D)$ in D/C is either the root of D or a non-root vertex $\rho(\alpha) \in \overline{C}$. We let $t(\alpha) = 0$ in the first case, and $t(\alpha) = t_{\rho(\alpha)}$ in the second case. The isomorphism is then defined by the change of variables $t'_i = t_i + t(\alpha)$ for $i \in \overline{C}(\alpha)$.

2. It is straightforward, if we choose the coordinates t_i, $i \notin C$, on L_i.

□

Example 2.3.4. Let us look at Example 2.1.16 and illustrate the first point of the above Lemma. On L_3 with coordinates (t_1, t_2), the change of variables is defined by $t_1 = t'_1$, $t_2 = t'_2 - t'_1$. Then for instance the equation $t_2 - a_2 = 0$ becomes $t'_2 - t'_1 - a_2 = 0$.

2.3.2 Definition of the dissection polylogarithms

We fix a decorated dissection diagram D of degree n and assume that its decorations are generic. The following is a special case of the definition of Aomoto polylogarithms, see §1.2.

The differential form ω_D
For $i = 1, \ldots, n$, let φ_i be the linear equation for the hyperplane L_i defined in the previous paragraph, of the form $\varphi_i = t_i - t_j - a_i$ if the i-th chord goes to the j-th vertex, and by $\varphi_i = t_i - a_i$ if it goes to the root. We then set

$$\omega_D = \mathrm{dlog}(\varphi_1) \wedge \ldots \wedge \mathrm{dlog}(\varphi_n) = \frac{dt_1 \wedge \ldots \wedge dt_n}{\varphi_1 \cdots \varphi_n}.$$

It is a meromorphic n-form on $\mathbb{C}^n$ and its polar locus is exactly the union $L = L_1 \cup \ldots \cup L_n$.

The integration simplex Δ_D
In the previous paragraph we have defined a family of hyperplanes $M_0 = \{t_1 = -b_0\}$, $M_j = \{t_j = t_{j+1} + b_j\}$ for $j = 1, \ldots, n-1$, and $M_n = \{t_n = b_n\}$. We set $M = M_0 \cup M_1 \cup \ldots \cup M_n$. We fix a singular n-simplex Δ_D inside $\mathbb{C}^n \setminus L$ such that for all $j = 0, \ldots, n$, $\partial_j \Delta_D \subset M_j$. The existence of such a simplex is guaranteed by the fact that the decorations being generic, $L \cup M$ is a normal crossing divisor inside $\mathbb{C}^n$ (Lemma 2.3.2).

Definition 2.3.5. We set

$$I(D) = \int_{\Delta_M} \omega_D \quad \in \mathbb{C}$$

and call it the *dissection polylogarithm* attached to the dissection diagram D.

The above integral is absolutely convergent since the integration simplex Δ_D does not meet the polar locus L of the form ω_D.

As the examples in the next paragraph will show, the integral $I(D)$ really depends on Δ_D (though only via its homology class $[\Delta_D] \in H_n(\mathbb{C}^n \setminus L, M \setminus M \cap L)$). Thus, the notation $I(D)$ is abusive. We allow ourselves that abuse for at least two reasons. Firstly, there is no canonical way of choosing $[\Delta_D]$ for *all* decorated dissection diagrams; if one looks at specific families of dissection diagrams and/or decorations (see the examples in the next paragraph) then this may sometimes be achieved. Secondly, we will replace (see Definition 2.4.1) the dissection polylogarithms $I(D)$ by motivic versions $I^{\mathcal{H}}(D)$ that only depend on the decorated diagram D, and not on the homology class of Δ_D.

Remark 2.3.6. A dissection polylogarithm is a special case of an Aomoto polylogarithm in the sense of [BVGS90]. To make the connection with the setting of [BVGS90] precise, one has to work in the projective setting, adding the hyperplane at infinity L_0. One has to notice that in this case we get a pair of simplices $(L; M)$ inside $\mathbb{P}^n(\mathbb{C})$ which is not in general position: it is highly degenerate at infinity. Thus we are not in the case studied by J. Zhao in [Zha00].

2.3.3 Examples of dissection polylogarithms

We study some families of dissection polylogarithms.

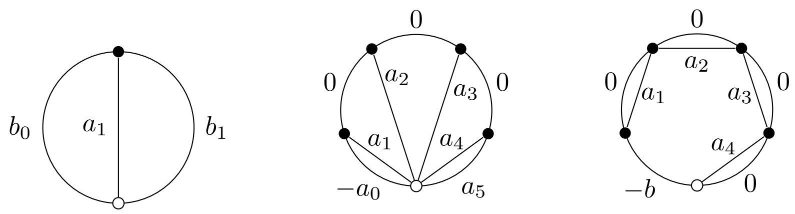


Figure 2.6: A decorated dissection diagram of degree 1; the decorated corolla corresponding to the iterated integral $\mathbb{I}(a_0; a_1, a_2, a_3, a_4; a_5)$; the decorated path tree corresponding to the $\mathbb{J}$-integral $\mathbb{J}(b; a_1, a_2, a_3, a_4)$.

Degree 1: logarithms

Let D be a decorated dissection diagram of degree 1 (see Figure 2.6). The genericity assumption on the decorations reads:

$$a_1 + b_0 \neq 0,\ a_1 - b_1 \neq 0,\ b_0 + b_1 \neq 0$$

We have $\varphi_1 = t - a_1$ so that $L_1 = \{a_1\}$ and $\omega_D = \frac{dt}{t-a_1}$. We have $M_0 = \{-b_0\}$ and $M_1 = \{b_1\}$, so that Δ_D is any continuous path from $-b_0$ to b_1 in $\mathbb{C} \setminus \{a_1\}$. We then have

$$I(D) = \int_{-b_0}^{b_1} \frac{dt}{t - a_1} = \log\left(\frac{a_1 - b_1}{a_1 + b_0}\right).$$

As is well-known, this number is well-defined up to an integer multiple of $2i\pi$, depending on the number of times that the path of integration winds around a_1.

Corollas and iterated integrals

This example generalizes the previous one. Let us consider the case where D is a corolla of degree n, which is the case when all chords of D point towards the root. In this case we have $\varphi_i = t_i - a_i$ for all $i = 1, \ldots, n$, so that

$$\omega_D = \frac{dt_1 \wedge \cdots \wedge dt_n}{(t_1 - a_1) \cdots (t_n - a_n)}.$$

By performing the change of variables

$$t'_1 = t_1, t'_2 = t_2 + b_1, t'_3 = t_3 + b_1 + b_2, \ldots, t'_n = t_n + b_1 + \ldots + b_n$$

we can always assume that the decorations on the sides of D are all 0 except for the first and the last one. We then put $a_0 = -b_0$ and $a_{n+1} = b_n$, so that the genericity condition reads: $a_i \neq a_j$ for $0 \leqslant i \neq j \leqslant n+1$ (see Figure 2.6).
We denote by $\Delta(a_0, a_{n+1})$ the integration simplex Δ_D. Its boundary is given by the hyperplanes $t_1 = a_0$, $t_j = t_{j+1}$ for $j = 1, \ldots, n$, and $t_n = a_{n+1}$. The corresponding dissection polylogarithm is the generic iterated integral

$$\mathbb{I}(a_0; a_1, \ldots, a_n; a_{n+1}) = \int_{\Delta(a_0, a_{n+1})} \frac{dt_1 \cdots dt_n}{(t_1 - a_1) \cdots (t_n - a_n)}. \tag{2.9}$$

In general, the dissection polylogarithms cannot be interpreted directly as iterated integrals in the above sense (however, see Theorem 2.3.12 for an abstract statement on a reduction to iterated integrals).

Path trees and $\mathbb{J}$-polylogarithms

Let us consider the case where D is a path tree of degree n. In this case we get

$$\omega_D = \frac{dt_1 \wedge \cdots \wedge dt_n}{(t_1 - t_2 - a_1)(t_2 - t_3 - a_2) \cdots (t_{n-1} - t_n - a_{n-1})(t_n - a_n)}.$$

As in the previous example, we can perform a change of variables so that the edge decorations are $b_0 = -b$, $b_1 = \cdots = b_{n-1} = b_n = 0$ (see Figure 2.6).
Let us write, for all $I \subset \{1, \ldots, n\}$, $a_I = \sum_{i \in I} a_i$. Then the genericity condition on the decorations reads: for all $i = 1, \ldots, n$, $a_i \neq 0$ and for all $I \subset \{1, \ldots, n\}$, $a_I \neq b$ (which includes the condition $b \neq 0$ for $I = \varnothing$).
The corresponding dissection polylogarithm is denoted

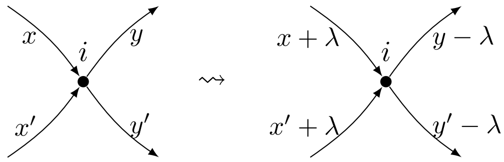
$$\mathbb{J}(b; a_1, \ldots, a_n) = \int_{\Delta(b,0)} \frac{dt_1 \cdots dt_n}{(t_1 - t_2 - a_1) \cdots (t_{n-1} - t_n - a_{n-1})(t_n - a_n)}.$$

2.3.4 Relations among dissection polylogarithms

We describe certain families of relations between dissection polylogarithms that one can describe combinatorially on the dissection diagrams.

Translations

Let D be a decorated dissection diagram of degree n; let us fix a non-root vertex $i \in \{1, \ldots, n\}$ and $\lambda \in \mathbb{C}$. Let $\tau_i(\lambda).D$ be the decorated dissection diagram obtained from D by adding λ to the decoration of every edge of $\Gamma(D)$ going to i, and substracting λ from the decoration of every edge of $\Gamma(D)$ leaving i.



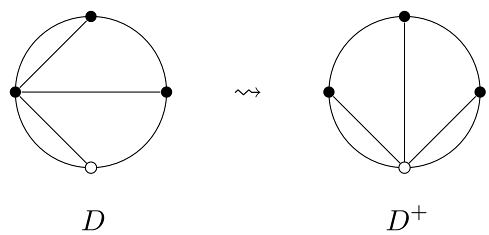
Proposition 2.3.7. *The decorations of $\tau_i(\lambda).D$ are generic if and only if the decorations of D are generic. In this case we have the equality*

$$(R1):\ I(D) = I(\tau_i(\lambda).D).$$

Proof. The first statement is straightforward. The equality follows from the change of variables $t_i \mapsto t_i - \lambda$ in the integral defining $I(D)$. Of course the simplices Δ_D and $\Delta_{\tau_i(\lambda).D}$ are chosen in a compatible way: $\Delta_{\tau_i(\lambda).D}$ is the image of Δ_D under $t_i \mapsto t_i - \lambda$. □

Rotating a dissection diagram

Let D be a decorated dissection diagram of degree n; let D^+ be the dissection diagram obtained from D by rotating the labels of the $(n+1)$ vertices of D in clockwise order. One has to flip a certain number of chords so that all the chords in D^+ point towards the root (which was formerly vertex 1). The rule for flipping chords is given in § 2.1.4.



Proposition 2.3.8. *The decorations of D^+ are generic if and only if the decorations of D are generic. In this case, let ε be the signature of the permutation relating the orders of the chords in D and in D^+. Then we have the equality*

$$(R2):\ I(D) = (-1)^n \varepsilon\, I(D^+).$$

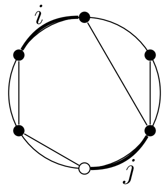
Proof. The first statement is straightforward since $\Gamma(D) = \Gamma(D^+)$ as decorated directed graphs. Let us perform the change of variables $f(t_1,\ldots,t_n) = (t_2 - t_1, t_3 - t_1, \ldots, t_{n-1} - t_1, -t_1)$ in the integral defining $I(D)$:

$$I(D) = \int_{\Delta_D} \omega_D = \int_{f^{-1}(\Delta_D)} f^*\omega_D.$$

Now Δ_{D^+} is chosen to be $f^{-1}(\Delta_D)$, but with the orientation multiplied by $(-1)^n$: indeed, we perform a cyclic permutation of the $(n+1)$ faces of the simplex. As the differential forms are concerned, we get by definition $f^*\omega_D = \varepsilon\, \omega_{D^+}$, hence the result. □

Stokes' theorem

Let us consider a set of n non-intersecting chords in Π_{n+1} such that the graph created by the chords is acyclic. For such a diagram $\widetilde{D}$ and a side $s \in \{0,\ldots,n+1\}$ of Π_{n+1}, we let $\partial_s\widetilde{D}$ be the graph obtained by contracting the side s. One easily checks that there exist exactly two sides i and j of $\widetilde{D}$ such that $\partial_i\widetilde{D}$ and $\partial_j\widetilde{D}$ are dissection diagrams.



Now let us suppose that the chords of $\widetilde{D}$ are directed and that we are given a decoration on the total directed graph of $\widetilde{D}$. Then $\widetilde{D}$ gives a bi-arrangement $(L; M) = (L_1, \ldots, L_n; M_0, M_1, \ldots, M_{n+1})$ in the same fashion as in § 2.3.2.

For a side $s \in \{0, \ldots, n+1\}$, the bi-arrangement given by $\partial_s\widetilde{D}$ is exactly the restriction $(M_s|L; M_0, \ldots, \widehat{M_s}, \ldots, M_{n+1})$, with natural coordinates $(t_1, \ldots, \widehat{t_s}, \ldots, t_{n+1})$ on M_s (the rule for contracting edges is given in § 2.1.4).

To sum up, $\partial_i\widetilde{D}$ and $\partial_j\widetilde{D}$ are decorated dissection diagrams. One way of coherently choosing the singular simplices $\Delta_{\partial_i\widetilde{D}}$ and $\Delta_{\partial_j\widetilde{D}}$ is to choose a singular simplex $\widetilde{\Delta}$ such that $\partial_s\widetilde{\Delta} \subset M_s$ for all s, and to put $\Delta_{\partial_i\widetilde{D}} = \partial_i\widetilde{\Delta}$ and $\Delta_{\partial_j\widetilde{D}} = \partial_j\widetilde{\Delta}$.

Proposition 2.3.9. *If the decorations on $\widetilde{D}$ are generic then the decorations on $\partial_i\widetilde{D}$ and $\partial_j\widetilde{D}$ are generic too. In this case, let ε_i (resp. ε_j) be the signature of the permutation relating the orders of the chords in $\widetilde{D}$ and in $\partial_i\widetilde{D}$ (resp. $\partial_j\widetilde{D}$). Then we have the equality*

$$(R3): \ (-1)^i\varepsilon_i\, I(\partial_i\widetilde{D}) + (-1)^j\varepsilon_j\, I(\partial_j\widetilde{D}) = 0.$$

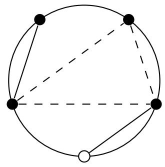
Proof. The first statement is straightforward. Let ω be the differential n-form on $\mathbb{C}^{n+1}$ given by the n decorated chords of $\widetilde{D}$ as in 2.3.2. It is a closed form so by Stokes' theorem we get

$$\sum_{s=0}^{n+1}(-1)^s\int_{\partial_s\widetilde{\Delta}}\omega_{|\partial_s\widetilde{\Delta}} = 0.$$

For $s \notin \{i, j\}$ we get $\omega_{|\partial_s\widetilde{\Delta}} = 0$; the result then follows from the equalities $\omega_{|\partial_i\widetilde{\Delta}} = \varepsilon_i\,\omega_{\partial_i\widetilde{D}}$ and $\omega_{|\partial_j\widetilde{\Delta}} = \varepsilon_j\,\omega_{\partial_j\widetilde{D}}$. □

Orlik-Solomon relations

Let us consider a set of $(n+1)$ non-intersecting chords in Π_n such that the graph created by the chords has Betti number 1. For such a diagram $\widehat{D}$, let C be the unique simple cycle. For every chord $c \in C$, we get a dissection diagram $\widehat{D} \setminus \{c\}$ by deleting c.



Now let us suppose that the chords of $\widehat{D}$ are linearly ordered by $\{1, \ldots, n+1\}$ and directed and that we are given a decoration on the total graph of $\widehat{D}$. Then for every chord $c \in C$ we get a decorated dissection diagram $\widehat{D} \setminus \{c\}$. It has to be noted that one may have to reorder the chords in $\widehat{D} \setminus \{c\}$. One can compute all the dissection polylogarithms $I(\widehat{D} \setminus \{c\})$ using the same choice of integration simplex.

Proposition 2.3.10. *Let us suppose that among all simple cycles in the total graph of $\widehat{D}$, C is the only one whose total decoration is* 0*. Then for every chord $c \in C$, the decorations on $\widehat{D} \setminus \{c\}$*

are generic. For $c \in C$, let us denote by $\varepsilon(c)$ the product of the signs $\mathrm{sgn}(\{c\}, C \setminus \{c\})$, $\mathrm{sgn}(C \setminus \{c\}, \{1, \ldots, n+1\} \setminus C)$*, and the signature of the permutation reordering the chords in $\widehat{D} \setminus \{c\}$. We then have the equality:*

$$(R4) : \sum_{c \in C} \varepsilon(c)\, I(\widehat{D} \setminus \{c\}) = 0.$$

Proof. The first statement is straightforward. Since the total decoration of C is 0, one easily sees that the hyperplanes L_c, for $c \in C$, are linearly dependent. Thus, we have the Orlik-Solomon relation [OT92, Lemma 3.119]

$$\sum_{c \in C} \mathrm{sgn}(\{c\}, C \setminus \{c\})\, \omega_{C \setminus \{c\}} = 0.$$

Multiplying on the right by $\omega_{\{1,\ldots,n+1\} \setminus C}$ we get

$$\sum_{c \in C} \mathrm{sgn}(\{c\}, C \setminus \{c\}) \mathrm{sgn}(C \setminus \{c\}, \{1, \ldots, n+1\} \setminus C)\, \omega_{\{1,\ldots,n+1\} \setminus \{c\}} = 0.$$

The result then follows from the fact that $\omega_{\{1,\ldots,n+1\} \setminus \{c\}}$ is $\omega_{\widehat{D} \setminus \{c\}}$ up to the sign implied by the reordering of the chords in $\widehat{D} \setminus \{c\}$. □

Remark 2.3.11. All the above relations are special cases of the "scissors congruence relations" between Aomoto polylogarithms [BVGS90, 2.1]. The translation relation $(R1)$ and the rotation relation $(R2)$ are special cases of projective invariance under particular subgroups of $\mathrm{PGL}_{n+1}(\mathbb{C})$. Stokes' theorem $(R3)$ is a particular case of the intersection additivity relation with respect to M, which has been shown [Zha00, Proposition 2.4] to follow from the scissors congruence relations. The Orlik-Solomon relation $(R4)$ is a particular case of the additivity relation with respect to L.

2.3.5 Reduction to iterated integrals

Theorem 2.3.12. *Let D be a generic decorated dissection diagram. Then the dissection polylogarithm $I(D)$ can be written as a linear combination with integer coefficients of generic iterated integrals $\mathbb{I}(a_0; a_1, \ldots, a_n; a_{n+1})$ where the a_i's are linear combinations with integer coefficients of the decorations of D.*

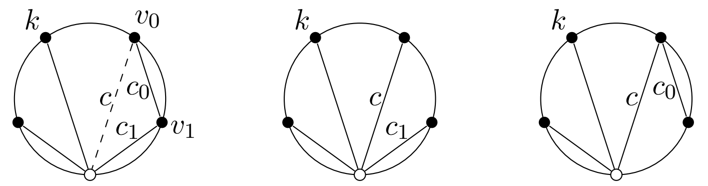
Proof. It is enough to prove that using relations $(R2)$, $(R3)$, $(R4)$, one can write $I(D)$ as a linear combination with integer coefficients of dissection polylogarithms $I(X)$ for X a corolla with generic decorations as in the statement of the Theorem. Indeed, using relation $(R1)$, one can always perform a change of variables so that any $I(X)$ is an iterated integral.
Because the chords of D do not cross, at least one chord has to connect consecutive vertices of Π_n. Thus, using relation $(R2)$, one can assume that in D there is a chord between 1 and the root.
We prove by induction on $k = 1, \ldots, n$ that using relations $(R3)$ and $(R4)$, one can write $I(D)$ as a linear combination with integer coefficients of generic dissection polylogarithms involving dissection diagrams where the first k non-root vertices are linked to the root, with decorations as in the statement of the Theorem. The case $k = 1$ has already been settled, and the case $k = n$ gives the Theorem.
Let us suppose that in D all vertices between 1 and k are linked to the root by a chord. There are two cases to consider.
Case 1: there is no chord going to the vertex k. Let us then consider the chord from the vertex $v_0 = k+1$. If its endpoint is the root then we are done. Else, its endpoint must be a vertex $v_1 \in \{k+2, \ldots, n\}$. Let us consider the sequence of chords

$$\overset{v_0}{\bullet} \xrightarrow{c_0} \overset{v_1}{\bullet} \xrightarrow{c_1} \overset{v_2}{\bullet} \xrightarrow{c_2} \cdots \overset{v_r}{\bullet} \xrightarrow{c_r} \circ$$

going to the root, where c_i has decoration α_i. Let $\widehat{D}$ be the diagram obtained by adding to D a chord c from $k+1$ to the root decorated by the sum $\alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_r$. Then we have created a simple cycle $C = (c, c_0, c_1, \ldots, c_r)$ and we are in the situation where we can apply relation $(R4)$. Since $\widehat{D} \setminus \{c\} = D$ by definition we get

$$I(D) = \sum_{i=0}^{r} \pm I(\widehat{D} \setminus \{c_i\})$$

and for every $i = 0, \ldots, r$, $\widehat{D} \setminus \{c_i\}$ is a dissection diagram in which all vertices between 1 and $k+1$ are linked to the root. Moreover its decorations are linear combinations with integer coefficients of the decorations of D.

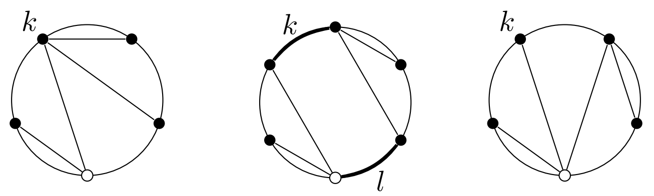


$D = \widehat{D} \setminus \{c\}$ $\qquad$ $\widehat{D} \setminus \{c_0\}$ $\qquad$ $\widehat{D} \setminus \{c_1\}$

Case 2: there are chords going to the vertex k. We are going to use relation $(R3)$ to reduce to Case 1. Let $\widetilde{D}$ be the diagram obtained by opening the angle between the chord going from k to the root and the first of the chords going to k, as in the picture below. The decoration of the new edge is 0. Then by definition we get $\partial_k \widetilde{D} = D$. The other $\partial_l \widetilde{D}$ that is a dissection diagram has no chord arriving at k. Thus, relation $(R3)$ gives

$$I(D) = \pm I(\partial_l \widetilde{D}).$$

The decorations of $\partial_l \widetilde{D}$ are linear combinations with integer coefficients of the decorations of D, and we are reduced to Case 1.



$D = \partial_k \widetilde{D}$ $\qquad$ $\widetilde{D}$ $\qquad$ $\partial_l \widetilde{D}$

□

Remark 2.3.13. The algorithm defined in the above proof is not canonical in any sense. It is worth noting that the number of iterated integrals that appear in the final sum is between 1 (for D a corolla) and $(n-1)!$ (for D a path graph).
For these reasons, Theorem 2.3.12 should be taken as a technical tool, and not as an abstract statement on the internal structure of dissection polylogarithms.

2.4 Motivic dissection polylogarithms and their coproduct

As in the previous section, the decorations on dissection diagrams are implicitly taken in $\mathbb{C}$.

2.4.1 Motivic dissection polylogarithms

Let D be a generic decorated dissection diagram. Let $(L; M) = (L_1, \ldots, L_n; M_0, M_1, \ldots, M_n)$ be the bi-arrangement in $\mathbb{C}^n$ corresponding to D (see 2.3.1). With the notations of 2.2.2, we set

$$H(D) = H(L; M).$$

According to Theorem 2.2.5, it is a mixed Hodge-Tate structure with non-negative weights between 0 and $2n$ and we have

- $\mathrm{gr}^W_{2n} H(D) \cong \Lambda^n(e_1, \ldots, e_n)$ which is one-dimensional with basis

$$v(D) = e_1 \wedge \cdots \wedge e_n.$$

- $\mathrm{gr}^W_0 H(D)$ is isomorphic to the quotient of $\Lambda^n(f_0, f_1, \ldots, f_n)$ by the vector space spanned by the elements

$$(-1)^i f_0 \wedge \cdots \wedge \widehat{f_i} \wedge \cdots \wedge f_n - (-1)^j f_0 \wedge \ldots \wedge \widehat{f_j} \wedge \ldots \wedge f_n$$

  for $0 \leqslant i < j \leqslant n$. Hence it is one-dimensional with basis $f_1 \wedge \ldots \wedge f_n$. We let

$$\varphi(D) = f_1^\vee \wedge \cdots \wedge f_n^\vee$$

  be the dual linear form in $\left(\mathrm{gr}^W_0 H(D)\right)^\vee$.

Definition 2.4.1. We set

$$I^{\mathcal{H}}(D) = (H(D), v(D), \varphi(D)) \in \mathcal{H}_n^{\mathsf{MHTS}}$$

and call it the *motivic dissection polylogarithm* corresponding to D.

More geometrically, we get

$$\mathrm{gr}^W_{2n} H(D) \cong H^n(\mathbb{C}^n \setminus L) = \mathbb{Q}[\omega_D]$$

so that $v(D)$ is the cohomology class of the the n-form ω_D. We also have a commutative diagram

$$\begin{tikzcd} H(D)^\vee \arrow[r, "\cong"] \arrow[d] & H_n(\mathbb{C}^n \setminus L, M \setminus M \cap L) \arrow[d, "\mu"] \\ \left(\mathrm{gr}^W_0 H(D)\right)^\vee \arrow[r, "\cong"] & H_n(\mathbb{C}^n, M) \end{tikzcd}$$

Let Δ_D be any integration simplex for $I(D)$, and $[\Delta_D]$ its homology class in $H_n(\mathbb{C}^n \setminus L, M \setminus M \cap L)$. Then $\varphi(D) = \mu([\Delta_D]) \in H_n(\mathbb{C}^n, M)$ is canonical and does not depend on the choice of $[\Delta_D]$. It corresponds to an oriented simplex in $\mathbb{C}^n$ whose boundary is contained in M.

To sum up, we have

$$I^{\mathcal{H}}(D) = (H(D), [\omega_D], \mu([\Delta_D])).$$

Remark 2.4.2. This is a particular case of the construction of motivic Aomoto polylogarithms explained in §1.5.

Remark 2.4.3. In a particular situation where one has a preferred choice of $[\Delta_D]$, then a more natural thing to do is to work in the algebra $\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$ and not in $\mathcal{H}^{\mathsf{MHTS}}$, as explained in §1.5. A candidate for the motivic dissection polylogarithm is then

$$I^{\mathcal{P}}(D) = (H(D), [\omega_D], [\Delta_D]) \in \mathcal{P}_n^{\mathsf{MHTS},\mathrm{eff}}.$$

Its period is

$$\mathrm{per}(I^{\mathcal{P}}(D)) = I(D)$$

computed with the same choice of Δ_D.
Via the surjection (1.12), $I^{\mathcal{P}}(D)$ is mapped to $I^{\mathcal{H}}(D)$. We stress the fact that $I^{\mathcal{H}}(D)$ only depends on the (generic) decorated diagram D, whereas $I(D)$ and $I^{\mathcal{P}}(D)$ also depend on the choice of a homology class $[\Delta_D] \in H_n(\mathbb{C}^n \setminus L, M \setminus M \cap L)$.

Theorem 2.4.4. *The relations* $(R1)$, $(R2)$, $(R3)$, $(R4)$ *from Propositions 2.3.7, 2.3.8, 2.3.9, 2.3.10 remain true if we replace the dissection polylogarithms* $I(D)$ *by their motivic versions* $I^{\mathcal{H}}(D)$*. Thus, Theorem 2.3.12 is also true in the motivic setting.*

Remark 2.4.5. Of course, Remark 2.3.13 also applies in this setting. Furthermore, it is worth noting that the reduction to iterated integrals does not tell us anything about the combinatorial shape of the coproduct of the motivic dissection polylogarithms (Theorem 2.4.9 below).

Proof. We will not use this result in the sequel so we just sketch the proof that $(R3)$ is true in the motivic setting. Let $(L; M) = (L_1, \ldots, L_n; M_0, M_1, \ldots, M_{n+1})$ be the bi-arrangement of hyperplanes given by $\widetilde{D}$. By definition $\varepsilon_i\, I^{\mathcal{H}}(\partial_i \widetilde{D})$ is the triple

$$\left(H(M_i|L; M_0, \ldots, \widehat{M_i}, \ldots, M_{n+1}), e_1 \wedge \cdots \wedge e_n, f_1^\vee \wedge \cdots \wedge \widehat{f_i^\vee} \wedge \cdots \wedge f_{n+1}^\vee\right).$$

Thus, using the natural morphism

$$H(M_i|L; M_0, \ldots, \widehat{M_i}, \ldots, M_{n+1}) \to H(L; M)$$

from Theorem 2.2.7, we see that this triple is equivalent to

$$(H(L; M), (e_1 \wedge \cdots \wedge e_n) \otimes f_i, f_i^\vee \wedge f_1^\vee \wedge \cdots \wedge \widehat{f_i^\vee} \wedge \cdots \wedge f_{n+1}^\vee),$$

hence $(-1)^i \varepsilon_i\, I^{\mathcal{H}}(\partial_i \widetilde{D})$ is the triple

$$(H(L; M), (e_1 \wedge \cdots \wedge e_n) \otimes f_i, -f_1^\vee \wedge \cdots \wedge f_{n+1}^\vee)$$

and the sum $(-1)^i \varepsilon_i\, I^{\mathcal{H}}(\partial_i \widetilde{D}) + (-1)^j \varepsilon_j\, I^{\mathcal{H}}(\partial_j \widetilde{D})$ is the triple

$$(H(L; M), (e_1 \wedge \cdots \wedge e_n) \otimes (f_i + f_j), -f_1^\vee \wedge \cdots \wedge f_{n+1}^\vee).$$

Thus it is enough to prove that $(e_1 \wedge \cdots \wedge e_n) \otimes (f_i + f_j) = 0$.
For $s \notin \{i, j\}$, $L_1 \cap \cdots \cap L_n \cap M_s = \varnothing$ because the corresponding subgraph in the total graph of $\widetilde{D}$ has a cycle. Hence the first relation of Theorem 2.2.5 gives $(e_1 \wedge \cdots \wedge e_n) \otimes f_s = 0$ and

$$(e_1 \wedge \cdots \wedge e_n) \otimes (f_i + f_j) = (e_1 \wedge \cdots \wedge e_n) \otimes \sum_{s=0}^{n+1} f_s = 0$$

using the second relation. ☐

Remark 2.4.6. The above theorem is also valid if we work in $\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$ with the elements $I^{\mathcal{P}}(D)$ (see Remark 2.4.3; in this setting the integration simplices have to be chosen coherently as in §2.3.4).

2.4.2 The computation of the coproduct

Proposition 2.4.7. *Let D be a generic decorated dissection diagram and $k \in \{0, \dots, n\}$. The classes of the elements*

$$b_C = e_C \otimes f_{\mathscr{S}_C^+}$$

for $C \subset \mathscr{C} \simeq \{1, \dots, n\}$, $|C| = k$, form a basis of $\mathrm{gr}_{2k}^W H(D)$.

Proof. From Theorem 2.2.5 we get a presentation

$$\mathrm{gr}_{2k}^W H(L; M) \cong \bigoplus_{|I|=k} \mathbb{Q}e_I \otimes \left(\Lambda^{n-k}(f_1, \dots, f_m)/R_I(L; M)\right)$$

where $R_I(L; M)$ is spanned by the elements

1. f_J if $L_I \cap M_J = \varnothing$.
2. $\sum_{j \notin J'} \mathrm{sgn}(\{j\}, J') f_{J' \cup \{j\}}$ for $|J'| = n - k - 1$.

Let us fix a subset $I \subset \{1, \dots, n\}$, viewed as a subset $C \subset \mathscr{C}$ of chords of D. We want to prove that the quotient of $\Lambda^{n-k}(f_0, f_1, \dots, f_n)$ by relations 1. and 2. above is one-dimensional with basis element $f_{\mathscr{S}_C^+}$.

1. Let us write $\{0, \dots, n\} = \mathscr{S} = \mathscr{S}_C(0) \sqcup \dots \sqcup \mathscr{S}_C(k)$ the partition (2.3) of $\mathscr{S}$ given by the dissection defined by C. From Lemma 2.3.2 and Lemma 2.1.6, we see that the only subsets $J \subset \{0, \dots, n\}$ such that $L_I \cap M_J \neq \varnothing$ are

$$J(u_0, \dots, u_k) = (\mathscr{S}_C(0) \setminus \{u_0\}) \sqcup \dots \sqcup (\mathscr{S}_C(k) \setminus \{u_k\})$$

   for some choice of $u_\alpha \in \mathscr{S}_C(\alpha)$.
   Thus the quotient of $\Lambda^{n-k}(f_0, f_1, \dots, f_n)$ by relation 1. has a natural basis consisting of the elements $f_{J(u_0, \dots, u_k)}$.

2. Let us write $f(u_0, \dots, u_k) = f_{J(u_0, \dots, u_k)}$ for simplicity. We investigate the relations between the elements $f(u_0, \dots, u_k)$ implied by relation (2). The only non-trivial ones come from subsets

$$J' = (\mathscr{S}_C(0) \setminus \{u_0\}) \sqcup \dots \sqcup (\mathscr{S}_C(i) \setminus \{a_i, b_i\}) \sqcup \dots \sqcup (\mathscr{S}_C(k) \setminus \{u_k\})$$

   with $a_i \neq b_i$, and are of the form

$$\mathrm{sgn}(\{a_i\}, J') f(u_0, \dots, a_i, \dots, u_k) + \mathrm{sgn}(\{b_i\}, J') f(u_0, \dots, b_i, \dots, u_k) = 0 \qquad (2.10)$$

   Hence in the quotient of $\Lambda^{n-k}(f_0, \dots, f_n)$ by relations 1. and 2., all the elements $f(u_0, \dots, u_k)$ are equal up to a sign, hence this quotient is spanned by any of these elements. If we choose $u_\alpha = \min(\mathscr{S}_C(\alpha))$ for each α, we get $J(u_0, \dots, u_k) = \mathscr{S}_C^+$ by definition. Thus all there is to prove is that the elements $f(u_0, \dots, u_k)$ are all non-zero in the quotient. This follows from a compatibility between the signs in formula (2.10), which is the content of the next Lemma.

□

Lemma 2.4.8. *Let us define a graph whose vertices are the tuples $(u_0, \dots, u_k)$ with $u_\alpha \in \mathscr{S}_C(\alpha)$ for every $\alpha = 0, \dots, k$. We put an edge between the pairs of the form $(u_0, \dots, a_i, \dots, u_k)$ and $(u_0, \dots, b_i, \dots, u_k)$ for $a_i \neq b_i$ in $\mathscr{S}_C(i)$. Let us decorate such an edge by the sign*

$$-\mathrm{sgn}(\{a_i\}, J')\, \mathrm{sgn}(\{b_i\}, J')$$

with $J' = (\mathscr{S}_C(0) \setminus \{u_0\}) \sqcup \dots \sqcup (\mathscr{S}_C(i) \setminus \{a_i, b_i\}) \sqcup \dots \sqcup (\mathscr{S}_C(k) \setminus \{u_k\})$.
Then for every loop in this graph, the product of the signs of the edges of the loop is 1.

Proof. See §2.6. □

The main result of this chapter is the following.

Theorem 2.4.9. *The coproduct of the motivic dissection polylogarithms is given by the formula*

$$\Delta(I^{\mathcal{H}}(D)) = \sum_{C\subset\mathscr{C}(D)} (-1)^{k_C(D)} I^{\mathcal{H}}(q_C(D)) \otimes I^{\mathcal{H}}(r_C(D)) \tag{2.11}$$

where $I^{\mathcal{H}}(q_C(D))$ is understood as the product $\prod_\alpha I^{\mathcal{H}}(q_C^\alpha(D))$.
In other words, the morphism

$$\mathcal{D}^{\mathrm{gen}}(\mathbb{C}) \to \mathcal{H}^{\mathsf{MHTS}}\ ,\ D \mapsto I^{\mathcal{H}}(D)$$

is a morphism of graded Hopf algebras.

Proof. According to formula (1.10) and Proposition 2.4.7, we get

$$\Delta_{n-k,k}(I^{\mathcal{H}}(D)) = \sum_{\substack{C\subset\{1,\ldots,n\}\\|C|=k}} (H(D)(k), v(D), b_C^\vee) \otimes (H(D), b_C, \varphi(D))$$

1. We show that $(H(D)(k), v(D), b_C^\vee) = \pm I^{\mathcal{H}}(q_C(D))$.
   First, let us look at the bi-arrangement

$$(L_C|L(\overline{C}); M).$$

   By Lemma 2.3.3 and the Künneth isomorphism, we have an isomorphism

$$H(L_S|L(\overline{S}); M) \cong \bigotimes_\alpha H(q_S^\alpha(D))$$

   hence the graded 0 part

$$\mathrm{gr}_0^W H(L_C|L(\overline{C}); M)$$

   is one-dimensional and spanned by the vector $\bigwedge_\alpha f_{\mathscr{S}_C^+(\alpha)} = \pm f_{\mathscr{S}_C^+}$.
   Let us consider the residue morphism (Theorem 2.2.7)

$$H(D)(k) \to H(L_C|L(\overline{C}); M).$$

   On the $\mathrm{gr}^W_{2(n-k)}$ part, it sends $v(D) = e_1 \wedge \cdots \wedge e_n$ to $\mathrm{sgn}(\overline{C}, C)e_{\overline{C}}$.
   On the gr_0^W part, it sends $b_C = e_C \otimes f_{\mathscr{S}_C^+}$ to $f_{\mathscr{S}_C^+}$ and all the other basis elements $b_{C'}$ to 0.
   Thus it gives an identification

$$(H(D)(k), v(D), b_C^\vee) = \mathrm{sgn}(\overline{C}, C)(H(L_C|L(\overline{C}); M), e_{\overline{C}}, f^\vee_{\mathscr{S}_C^+}).$$

   For each α, let $\nu_C^\alpha : \overline{C}(\alpha) \xrightarrow{\simeq} \mathscr{S}_C^+(\alpha)$ be the bijection (2.2) given by the dissection diagram $q_C^\alpha(D)$, and let $\nu_C : \overline{C} \xrightarrow{\simeq} \mathscr{S}_C^+$ be the bijection induced by the ν_C^α's. This bijection accounts for the reordering of the hyperplanes, and gives a sign

$$(H(L_C|L(\overline{C}); M), e_{\overline{C}}, f^\vee_{\mathscr{S}_C^+}) = \mathrm{sgn}(\nu_C) \prod_\alpha I^{\mathcal{H}}(q_C^\alpha(D))$$

   hence the equality

$$(H(D)(k), v(D), b_C^\vee) = \mathrm{sgn}(\overline{C}, C)\,\mathrm{sgn}(\nu_C) I^{\mathcal{H}}(q_C(D)).$$

2. We show that $(H(D), b_C, \varphi(D)) = \pm I^{\mathcal{H}}(r_C(D))$.
First let us consider the bi-arrangement of hyperplanes
$$\left(M_{\mathscr{S}_C^+}\middle| L(C); M_0, M\left(\overline{\mathscr{S}_C^+}\right)\right).$$
According to Lemma 2.3.3, it is exactly the one given by the dissection diagram $r_C(D)$, so we get
$$H\left(M_{\mathscr{S}_C^+}\middle| L(C); M_0, M\left(\overline{\mathscr{S}_C^+}\right)\right) \cong H(r_C(D))$$
and the graded 0 part
$$\mathrm{gr}_0^W H\left(M_{\mathscr{S}_C^+}\middle| L(C); M_0, M\left(\overline{\mathscr{S}_C^+}\right)\right)$$
is one-dimensional and spanned by the vector $f_{\mathscr{S}_C^+}$.
Let us consider the morphism (Theorem 2.2.7)
$$H\left(M_{\mathscr{S}_C^+}\middle| L(C); M_0, M\left(\overline{\mathscr{S}_C^+}\right)\right) \to H(L; M). \tag{2.12}$$
On the gr_{2k}^W part, it sends e_C to $b_C = e_C \otimes f_{\mathscr{S}_C^+}$.
On the gr_0^W part, it sends $f_{\mathscr{S}_C^+}$ to $\mathrm{sgn}(\mathscr{S}_C^+, \overline{\mathscr{S}_C^+}) f_1 \wedge \cdots \wedge f_n$.
Thus it gives an identification
$$(H(D), b_C, \varphi(D)) = \mathrm{sgn}(\mathscr{S}_C^+, \overline{\mathscr{S}_C^+}) \left(H\left(M_{\mathscr{S}_C^+}\middle| L(C); M_0, M\left(\overline{\mathscr{S}_C^+}\right)\right), e_C, f^\vee_{\mathscr{S}_C^+}\right).$$
Because of the ordering conventions, we have
$$\left(H\left(M_{\mathscr{S}_C^+}\middle| L(C); M_0, M\left(\overline{\mathscr{S}_C^+}\right)\right), e_C, f^\vee_{\mathscr{S}_C^+}\right) = \mathrm{sgn}(\eta_C) I^{\mathcal{H}}(r_C(D))$$
where $\eta_C : C \xrightarrow{\cong} \overline{\mathscr{S}_C^+}$ is the bijection (2.2) given by $r_C(D)$. Hence we have the equality
$$(H(D), b_C(D), \varphi(D)) = \mathrm{sgn}(\mathscr{S}_C^+, \overline{\mathscr{S}_C^+}) \,\mathrm{sgn}(\eta_C) I^{\mathcal{H}}(r_C(D)).$$

3. Putting the two first steps together, it only remains to check that the signs are correct. This is done in the next Lemma.

□

Lemma 2.4.10. *We have the equality between signs:*
$$\mathrm{sgn}(\overline{C}, C)\,\mathrm{sgn}(\nu_C)\,\mathrm{sgn}(\mathscr{S}_C^+, \overline{\mathscr{S}_C^+})\,\mathrm{sgn}(\eta_C) = (-1)^{k_C(D)}.$$

Proof. See §2.7. □

Remark 2.4.11. If we work in $\mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$ with the elements $I^{\mathcal{P}}(D)$ (see Remark 2.4.3) then we get a similar formula for the coaction $\rho : \mathcal{P}^{\mathsf{MHTS},\mathrm{eff}} \to \mathcal{H}^{\mathsf{MHTS}} \otimes \mathcal{P}^{\mathsf{MHTS},\mathrm{eff}}$:
$$\rho(I^{\mathcal{P}}(D)) = \sum_{C \subset \mathscr{C}(D)} (-1)^{k_C(D)} I^{\mathcal{H}}(q_C(D)) \otimes I^{\mathcal{P}}(r_C(D)).$$

We only have to define the integration simplices for the elements $I^{\mathcal{P}}(r_C(D))$ in a coherent way. If Δ_D is the integration simplex for $I^{\mathcal{P}}(D)$, then the integration simplex for $I^{\mathcal{P}}(r_C(D))$ has to be the face $\partial_{\mathscr{S}_C^+}\Delta_D$ of Δ_D. This is because the morphism (2.12) corresponds, on the singular homology groups, to a composition of face maps.

Remark 2.4.12. Let F be a number field. If we start with a generic F-decorated dissection diagram D, then we may define a bi-arrangement $(L; M)$ inside $\mathbb{A}_F^n$, the n-dimensional affine space over F. Then $H^n(\mathbb{A}_F^n \setminus L, M \setminus M \cap L)$ defines an object in the category $\mathsf{MTM}(F)$ [Gon02, Proposition 3.6]. By the same construction, we get a morphism of graded Hopf algebras
$$\mathcal{D}^{\mathrm{gen}}(F) \to \mathcal{H}^{\mathrm{MTM}(F)}.$$

2.4.3 Examples of computations

We present two special cases of Theorem 2.4.9.

In 2.3.3 we have introduced the iterated integrals $\mathbb{I}(a_0; a_1, \ldots, a_n; a_{n+1})$ as the dissection polylogarithms corresponding to corollas. The motivic counterparts $\mathbb{I}^{\mathcal{H}}(a_0; a_1, \ldots, a_n; a_{n+1}) \in \mathcal{H}_n$ have already been defined and studied by Goncharov [Gon05, Theorem 1.1], in the framework of motivic fundamental groupoids. We leave it to the reader to check that Goncharov's definition agrees with ours. The coproduct of the motivic iterated integrals has been worked out by Goncharov.

Theorem 2.4.13 ([Gon05], Theorem 1.2)**.** *The coproduct $\Delta(\mathbb{I}^{\mathcal{H}}(a_0; a_1, \ldots, a_n; a_{n+1}))$ of motivic generic iterated integrals is given by the formula*

$$\sum_{\substack{0 \leqslant k \leqslant n \\ 0=i_0<i_1<\cdots<i_k<i_{k+1}=n+1}} \left(\prod_{s=0}^{k} \mathbb{I}^{\mathcal{H}}(a_{i_s}; a_{i_s+1}, \ldots, a_{i_{s+1}-1}; a_{i_{s+1}}) \right) \otimes \mathbb{I}^{\mathcal{H}}(a_0; a_{i_1}, \ldots, a_{i_k}; a_{n+1}). \tag{2.13}$$

Proof. It is the same computation as in Example 2.1.14, 1., but taking care of the decorations. The term indexed by $0 = i_0 < i_1 < \cdots < i_k < i_{k+1} = n+1$ corresponds to the subset $C = \{i_1, \ldots, i_k\}$. □

In 2.3.3 we have introduced the $\mathbb{J}$-polylogarithms $\mathbb{J}(b; a_1, \ldots, a_n)$ as the dissection polylogarithms corresponding to path trees. We let $\mathbb{J}^{\mathcal{H}}(b; a_1, \ldots, a_n) \in \mathcal{H}_n$ be their motivic counterparts. Their coproduct is given by a simple formula.

Theorem 2.4.14. *The coproduct of motivic generic $\mathbb{J}$-polylogarithms is given by the formula*

$$\Delta(\mathbb{J}^{\mathcal{H}}(a_1, \ldots, a_n; b)) = \sum_{I \subset \{1, \ldots, n\}} \mathbb{J}^{\mathcal{H}}(a(\overline{I}); b - a_I) \otimes \mathbb{J}^{\mathcal{H}}(a(I); b). \tag{2.14}$$

Proof. It is the same computation as in Example 2.1.14, 2., but taking care of the decorations. Here we have to make a slight translation of variables on the left-hand side of the tensor product so that it looks like the above formula. The details are left to the reader. □

2.4.4 Genericity and regularization

In this paragraph we discuss the extension of our results to non-generic dissection diagrams and polylogarithms. The genericity condition on the decorations of a dissection diagram is a sufficient, but not necessary condition, for the existence of the corresponding dissection polylogarithm.

Let us take the example of the iterated integrals $\mathbb{I}(a_0; a_1, \ldots, a_n; a_{n+1})$, for which the genericity condition reads $a_i \neq a_j$ for $i \neq j$. The convergence of the corresponding integral is actually guaranteed as soon as $a_0 \neq a_1$ and $a_n \neq a_{n+1}$. For example, the multiple zeta values

$$\zeta(n_1, \ldots, n_r) = \sum_{1 \leqslant k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}$$

defined for integers $n_1, \ldots, n_{r-1} \geqslant 1$ and $n_r \geqslant 2$, are special cases of these non-generic iterated integrals, as was first noticed by Kontsevich:

$$\zeta(n_1, \ldots, n_r) = (-1)^r \ \mathbb{I}(0; \underbrace{1, 0, \ldots, 0}_{n_1}, \ldots, \underbrace{1, 0, \ldots, 0}_{n_r}; 1)$$

for $n = n_1 + \cdots + n_r$.
The point is that in the formula for the coproduct of motivic iterated integrals (Theorem 2.4.13), there may be non-convergent motivic iterated integrals on the right-hand side even if the left-hand side corresponds to a convergent one. For example, for $\mathbb{I}(0;1,0;1) = -\zeta(2)$, the formula would look like

$$\Delta_{1,1}(\mathbb{I}^{\mathcal{H}}(0;1,0;1)) = \mathbb{I}^{\mathcal{H}}(1;0;1) \otimes \mathbb{I}^{\mathcal{H}}(0;1;1) + \mathbb{I}^{\mathcal{H}}(0;1;0) \otimes \mathbb{I}^{\mathcal{H}}(0;0;1). \tag{2.15}$$

Goncharov showed that there is a regularization procedure that gives a meaning to (possibly non-convergent) iterated integrals $\mathbb{I}(a_0;a_1,\ldots,a_n;a_{n+1})$ (for all tuples $(a_0,\ldots,a_{n+1})$).
Furthermore, he defined their motivic versions $\mathbb{I}^{\mathcal{H}}(a_0;a_1,\ldots,a_n;a_{n+1})$ and proved that Theorem 2.4.13 was valid without the genericity hypothesis. Thus, formula (2.15) makes sense (and in this particular case, the right-hand side is 0).

Building upon Goncharov's construction (see also [Gon02, §4]), one should be able to regularize all dissection polylogarithms and compute the coproduct of their motivic versions. The most naive hope would be that the formula for the coproduct would remain the same, hence extending Theorem 2.4.9 to a morphism of Hopf algebras $\mathcal{D}(\mathbb{C}) \to \mathcal{H}^{\mathsf{MHTS}}$.

2.5 Appendix: proof of Lemma 2.1.12

In this Appendix we fix a dissection diagram D of degree n. We use the identifications $\mathscr{C} = \{1,\ldots,n\}$, $\mathscr{S} = \{0,\ldots,n\}$ and $\mathscr{S}^+ = \{1,\ldots,n\}$ for the sets of chords and sides of D.

Lemma 2.5.1. *Let $C \subset \mathscr{C}$ be a subset of chords of D and $c = \overset{i_0}{\bullet}\!\longrightarrow\!\overset{i_1}{\bullet}$ be a chord in C. Then c is in $\mathscr{K}_C(D)$ if and only if the three following conditions are satisfied:*

(K1) The path in C

$$\overset{i_0}{\bullet}\xrightarrow{c}\overset{i_1}{\bullet}\longrightarrow\cdots\longrightarrow\overset{i_{M-1}}{\bullet}\longrightarrow\overset{i_M}{\bullet}$$

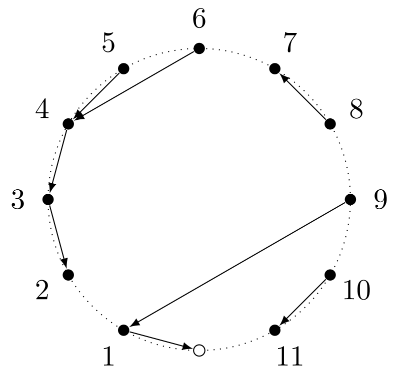
starting at i_0 does not go to the root.

(K2) This path is decreasing: for all $k = 1,\ldots,M$ we have $i_{k-1} > i_k$.

(K3) For all $k = 1,\ldots,M$, there is no chord $\overset{j}{\bullet}\!\longrightarrow\!\overset{i_k}{\bullet}$ in C such that $j > i_0$.

In particular, we have $i_0 > i_1$, so that all the chords in $\mathscr{K}_C(D)$ are decreasing.

Example 2.5.2. In the following example, we have only drawn the chords from $C = \{1,3,4,5,6,8,9,10\}$, and drawn the circle with dots for a matter of convenience.



We have $\mathscr{S}_C(D) = \{6,8,11\}$ and $\mathscr{K}_C(D)$ is made of the arrows $\overset{6}{\bullet}\!\longrightarrow\!\overset{4}{\bullet}\!\longrightarrow\!\overset{3}{\bullet}\!\longrightarrow\!\overset{2}{\bullet}$ and $\overset{8}{\bullet}\!\longrightarrow\!\overset{7}{\bullet}$.

Proof. For c a chord in C, we denote by (K) the conjunction of the three conditions $(K1)$, $(K2)$, $(K3)$ of the Lemma.
We investigate the process of contracting the edges from $\mathscr{S}_C^+$ decomposing it into steps where we contract only one edge. The number of steps is $r = n - |C|$. We label $e_1, \dots, e_r$ the elements of $\mathscr{S}_C^+$, in decreasing order.
Let $D^{(0)}$ be the diagram obtained from D by forgetting the chords from $\overline{C}$ and only keeping the chords from C. The chords form a disjoint union of rooted trees.
We define recursively diagrams $D^{(i)}$, $i = 1, \dots, r$. For $i = 1, \dots, r$, let $D^{(i)}$ be the diagram obtained from $D^{(i-1)}$ by contracting the side e_i, and possibly flipping chords so that the chords in $D^{(i)}$ still form a disjoint union of rooted trees. The number of connected components of this disjoint union decreases with i, and in the end we get a dissection diagram $D^{(r)} = r_C(D)$.
We prove the following property by induction on $i = 0, \dots, r$:

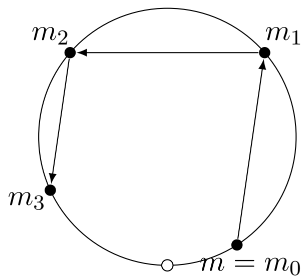
(i) In the diagram $D^{(i)}$, among the chords that are attached to the root, the ones that have been flipped have only been flipped once, and they are exactly the ones that satisfy condition (K).

(ii) For a chord that is not attached to the root, it satisfies (K) in D if and only if it satisfies (K) in $D^{(i)}$.

The case $i = 0$ is trivial, and the case $i = r$ will give the Lemma. Hence we only need to pass from $(i-1)$ to i.
Let us consider the diagram $D^{(i-1)}$ and let m be the starting vertex of the side e_i. We assume that the end vertex of e_i is the root of $D^{(i-1)}$, leaving to the reader the (very similar) case where it is another non-root vertex $(m+1)$. Let us denote

$$\overset{m=m_0}{\bullet} \longrightarrow \overset{m_1}{\bullet} \longrightarrow \cdots \longrightarrow \overset{m_{N-1}}{\bullet} \longrightarrow \overset{m_N}{\bullet}$$

the (possibly empty) path in C starting at m.



When we contract e_i, m is merged with the root and then we have to flip all these arrows. It is easy to see that they are the only ones. Hence we have to prove two things: these chords satisfy (K), and all the other chords in their connected component in $D^{(i-1)}$ do not satisfy (K).
It is trivial that $m_0 > m_1$ since m_0 is maximal in $D^{(i-1)}$. Since the chords cannot intersect each other, one easily proves by induction on k that $m_{k-1} > m_k$ for all k. If the path in C starting at m goes to the root, then we cannot have $e_i \in \mathscr{S}_C^+$, which is a contradiction. Condition $(K3)$ cannot happen because m is maximal in $D^{(i-1)}$. Hence we have proved that all the chords $\overset{m_{k-1}}{\bullet} \longrightarrow \overset{m_k}{\bullet}$ satisfy (K).
Now let $c \in C$ be another chord in the same connected component of $D^{(i-1)}$ that satisfies (K). Then c lies between m_k and m_{k-1} for some k, or between the root and m_N. Let us suppose that we are in the first case; since the path starting with c is decreasing, it has to go through m_k because the chords cannot intersect each other. But then the chord $\overset{m_{k-1}}{\bullet} \longrightarrow \overset{m_k}{\bullet}$ shows that condition $(K3)$ is not satisfied by c. In the second case, one sees that the path starting at c has to end at the root, which is also a contradiction. Thus we are done with (i). Statement (ii) is

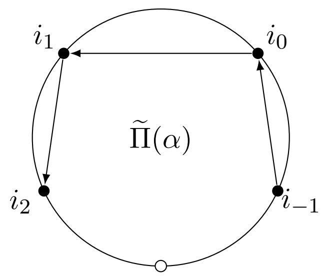
straightforward since we have not affected the other connected components of $D^{(i-1)}$. This ends the induction. □

Lemma 2.5.3. *Let $C \subset \mathscr{C}$ be a subset of chords of D and $c = \overset{i_0}{\bullet} \longrightarrow \overset{i_1}{\bullet}$ be a chord in C. Then c is in $\mathscr{K}_C(D)$ if and only if there exists a path*

$$\overset{i_{-N}}{\bullet} \longrightarrow \cdots \longrightarrow \overset{i_{-1}}{\bullet} \longrightarrow \overset{i_0}{\bullet} \xrightarrow{c} \overset{i_1}{\bullet}$$

of chords in C such that $i_{-N} \in \mathscr{S}_C^+$.

Proof. We prove the equivalence with the condition (K) of Lemma 2.5.1.
If c satisfies (K), then we define i_{-1} to be the highest vertex $> i_0$ such that there exists a chord $\overset{i_{-1}}{\bullet} \longrightarrow \overset{i_0}{\bullet}$ in C, and so on. The process stops at a vertex i_{-N} and we want to prove that $i_{-N} \in \mathscr{S}_C^+$. By construction and by condition $(K3)$, the chords $\overset{i_{k-1}}{\bullet} \longrightarrow \overset{i_k}{\bullet}$, for $k = -N + 1, \dots, M$, are sides of the same polygon $\widetilde{\Pi}(\alpha)$ in the dissection defined by C, as well as the side labeled i_{-N}.



Because of conditions $(K1)$ and $(K2)$, there is a side of this $\widetilde{\Pi}(\alpha)$ that is a side of Π_n and that is less than i_M. Hence by definition $i_{-N} \in \mathscr{S}_C^+$.
Conversely, under the assumption of the Lemma, one easily sees that if any of conditions $(K1)$, $(K2)$, $(K3)$ is satisfied, then $i_{-N} \notin \mathscr{S}_C^+$. □

For the remainder of this Appendix we use the unambiguous notation $\mathscr{S}_C^+ = \mathscr{S}_C^+(D)$ to avoid any confusion.

Lemma 2.5.4. *Let $C \subset \mathscr{C}(D)$ be a subset of chords of D and $\overline{C} = \bigsqcup_\alpha \overline{C}(\alpha)$ the partition (2.3) of $\overline{C}$ determined by C. Let us fix $C'_\alpha \subset \overline{C}(\alpha)$ for each α and $C' = C \sqcup \bigsqcup_\alpha C'_\alpha$.*

1. $\mathscr{S}_C^+(D) = \bigsqcup_\alpha \mathscr{S}_{C'_\alpha}^+(q_C^\alpha(D))$.

2. $\mathscr{S}_C^+(D) = \mathscr{S}_{C'}^+(D) \sqcup \mathscr{S}_C^+(r_{C'}(D))$.

3. $\mathscr{K}_{C'}(D) \sqcup \mathscr{K}_C(r_{C'}(D)) = \mathscr{K}_C(D) \sqcup \bigsqcup_\alpha \mathscr{K}_{C'_\alpha}(q_C^\alpha(D))$.

Proof. 1. It is straightforward, since the partition of $\mathscr{S}$ given by C' refines the one given by C.

2. The fact that $\mathscr{S}_{C'}^+(D) \subset \mathscr{S}_C^+(D)$ and $\mathscr{S}_C^+(r_{C'}(D)) \subset \mathscr{S}_C^+(D)$ are easy. Then the fact that $\mathscr{S}_{C'}^+(D) \cap \mathscr{S}_C^+(r_{C'}(D)) = \varnothing$ is straightforward since by definition $\mathscr{S}_C^+(r_{C'}(D))$ is a subset of non-root edges of $r_{C'}(D)$, which are precisely the elements from $\mathscr{S}^+(D) \backslash \mathscr{S}_{C'}^+(D)$. Then we get $\mathscr{S}_{C'}^+(D) \sqcup \mathscr{S}_C^+(r_{C'}(D)) \subset \mathscr{S}_C^+(D)$. The equality follows from a cardinality argument: $|\mathscr{S}_C^+(D)| = n - |C|$, $|\mathscr{S}_{C'}^+(D)| = n - |C'|$ and $|\mathscr{S}_C^+(r_{C'}(D))| = |C'| - |C|$.

3. Since by Lemma 2.5.1 the chords that one has to flip are all decreasing, we necessarily have $\mathscr{K}_{C'}(D) \cap \mathscr{K}_C(r_{C'}(D)) = \varnothing$.

(a) We prove that $\mathscr{K}_{C'}(D) \cap C'_\alpha = \mathscr{K}_{C'_\alpha}(q^\alpha_C(D))$. Let $c = \overset{i_0}{\bullet} \longrightarrow \overset{i_1}{\bullet}$ be a chord in $\mathscr{K}_{C'}(D) \cap C'_\alpha$, and

$$\overset{i_{-N}}{\bullet} \longrightarrow \cdots \longrightarrow \overset{i_{-1}}{\bullet} \longrightarrow \overset{i_0}{\bullet} \overset{c}{\longrightarrow} \overset{i_1}{\bullet}$$

the path in C' given by Lemma 2.5.3, with $i_{-N} \in \mathscr{S}^+_{C'}(D)$. As has been noted in the proof of Lemma 2.5.3, the chords $\overset{i_{-k+1}}{\bullet} \longrightarrow \overset{i_{-k}}{\bullet}$ are sides to the same polygon $\widetilde{\Pi}(\alpha)$. Hence $i_{-N} \in \mathscr{S}^+_{C'_\alpha}(q^\alpha_C(D))$ and then Lemma 2.5.3 implies that $c \in \mathscr{K}_{C'_\alpha}(q^\alpha_C(D))$. The converse is straightforward.

(b) We prove that $\mathscr{K}_C(D) \subset \mathscr{K}_{C'}(D) \sqcup \mathscr{K}_C(r_{C'}(D))$. Let $c = \overset{i_0}{\bullet} \longrightarrow \overset{i_1}{\bullet}$ be a chord in $\mathscr{K}_C(D)$, and

$$\overset{i_{-N}}{\bullet} \longrightarrow \cdots \longrightarrow \overset{i_{-1}}{\bullet} \longrightarrow \overset{i_0}{\bullet} \overset{c}{\longrightarrow} \overset{i_1}{\bullet}$$

the path given by Lemma 2.5.3. We know that $i_{-N} \in \mathscr{S}^+_C(D)$. According to 2., we have to possibilities: either $i_{-N} \in \mathscr{S}^+_{C'}(D)$ and then Lemma 2.5.3 implies that $c \in \mathscr{K}_{C'}(D)$, or $i_{-N} \in \mathscr{S}^+_C(r_{C'}(D))$ and then Lemma 2.5.3 implies that $c \in \mathscr{K}_S(r_T(D))$.

(c) We prove that $\mathscr{K}_{C'}(D) \cap C \subset \mathscr{K}_C(D)$. This is straightforward using the characterization of Lemma 2.5.1.

(d) We prove that $\mathscr{K}_C(r_{C'}(D)) \subset \mathscr{K}_C(D)$. We use the characterization of Lemma 2.5.1. If a chord $c \in C$ is not in $\mathscr{K}_C(D))$, then one of the conditions $(K1)$, $(K2)$, $(K3)$ is not satisfied.
If $(K1)$ is not satisfied in D, this means that the path starting from c in C goes to the root. Then no chord in this path is in $\mathscr{K}_C(D)$, and a fortiori in $\mathscr{K}_{C'}(D)$. Thus no chord is this path is flipped in $r_{C'}(D)$ and condition $(K1)$ is not satisfied in $r_{C'}(D)$.
If $(K2)$ is not satisfied in D, this means that in the path

$$\overset{i_0}{\bullet} \overset{c}{\longrightarrow} \overset{i_1}{\bullet} \longrightarrow \cdots \longrightarrow \overset{i_{M-1}}{\bullet} \longrightarrow \overset{i_M}{\bullet}$$

starting at c in C, there is an increasing arrow $i_{k-1} < i_k$. Then for $l = 1, \dots, k-1$, the chord $\overset{i_{l-1}}{\bullet} \longrightarrow \overset{i_l}{\bullet}$ is not in $\mathscr{K}_C(D)$, hence not in $\mathscr{K}_{C'}(D)$, then it is not flipped in $r_{C'}(D)$. The chord $\overset{i_{k-1}}{\bullet} \longrightarrow \overset{i_k}{\bullet}$ is increasing so it cannot be flipped in $r_{C'}(D)$ according to Lemma 2.5.3. Thus we see that condition $(K2)$ is not satisfied in $r_{C'}(D)$.
If $(K3)$ is not satisfied in D, it means that there exists a chord $c' = \overset{j}{\bullet} \longrightarrow \overset{i_k}{\bullet}$, $c' \in C$, with $j > i_0$ for some $k = 1, \dots, M$. For the same reason as above, none of the chords $\overset{i_{l-1}}{\bullet} \longrightarrow \overset{i_l}{\bullet}$ is flipped in $r_{C'}(D)$, for $l = 1, \dots, k$. Let us suppose that c' is not flipped in $r_{C'}(D)$. Then condition $(K3)$ is still not satisfied in $r_{C'}(D)$. Now let us suppose that c' is flipped in $r_{C'}(D)$. Then we necessarily have $j > i_k$, and then c' becomes decreasing in $r_{C'}(D)$, hence condition $(K2)$ is not satisfied in $r_{C'}(D)$.
In either case we have shown that $c \notin \mathscr{K}_C(r_{C'}(D))$.

□

Proof of Lemma 2.1.12. 1. The left-hand side is obtained by contracting the chords from C'; the right-hand side is obtained by contracting the chords from C, then contracting the chords from C'_α for each α. The result is thus the same since by definition $C' = C \sqcup \bigsqcup_\alpha C'_\alpha$.

2. The left-hand side is obtained by contracting the edges from $\mathscr{S}^+_T(D)$, then the chords from S; the right-hand side is obtained by contracting the chords from C, then the edges from $\mathscr{S}^+_{C_\alpha}(q^\alpha_C(D))$ for each α. The equality then follows from Lemma 2.5.4, 1.

3. The left-hand side is obtained by contracting the edges from $\mathscr{S}^+_{C'}(D)$, then the edges from $\mathscr{S}^+_C(r_{C'}(D))$; the right-hand side is obtained by contracting the edges from $\mathscr{S}^+_C(D)$. The equality then follows from Lemma 2.5.4, 2.

4. This follows from taking the cardinality in Lemma 2.5.4, 3.

$\square$

2.6 Appendix: proof of Lemma 2.4.8

We leave it to the reader to check that it is enough to do the proof for three families of loops.

1. The trivial loops

$$\begin{tikzcd} (u_0, \dots, a_i, \dots, u_k) \arrow[r, dash, bend left] \arrow[r, dash, bend right] & (u_0, \dots, b_i, \dots, u_k) \end{tikzcd}$$

The statement is trivial since the expression

$$-\mathrm{sgn}(\{a_i\}, J')\mathrm{sgn}(\{b_i\}, J')$$

is symmetric in a_i and b_i.

2. The triangles

$$\begin{tikzcd} (u_0, \dots, a_i, \dots, u_k) \arrow[rr, dash] \arrow[dr, dash] & & (u_0, \dots, b_i, \dots, u_k) \arrow[dl, dash] \\ & (u_0, \dots, c_i, \dots, u_k) & \end{tikzcd}$$

The statement follows from the following equality, valid for any linearly ordered set X and any set $\{a, b, c\}$ of pairwise disjoint elements of X:

$$\begin{aligned} &\mathrm{sgn}(\{a\}, X \setminus \{a, b\})\,\mathrm{sgn}(\{b\}, X \setminus \{a, b\}) \\ &\mathrm{sgn}(\{b\}, X \setminus \{b, c\})\,\mathrm{sgn}(\{c\}, X \setminus \{b, c\}) \\ &\mathrm{sgn}(\{c\}, X \setminus \{a, c\})\,\mathrm{sgn}(\{a\}, X \setminus \{a, c\}) \quad = -1. \end{aligned}$$

Indeed, we apply this equality to

$$X = (\mathscr{S}_C(0) \setminus \{u_0\}) \sqcup \ldots \sqcup \mathscr{S}_C(i) \sqcup \ldots \sqcup (\mathscr{S}_C(k) \setminus \{u_k\}).$$

3. The squares

$$\begin{tikzcd} (u_0, \dots, a_i, \dots, c_j, \dots, u_k) \arrow[r, dash] \arrow[d, dash] & (u_0, \dots, b_i, \dots, c_j, \dots, u_k) \arrow[d, dash] \\ (u_0, \dots, a_i, \dots, d_j, \dots, u_k) \arrow[r, dash] & (u_0, \dots, b_i, \dots, d_j, \dots, u_k) \end{tikzcd}$$

The statement follows from the following equality, valid for any linearly ordered set X and any set $\{a, b, c, d\}$ of pairwise disjoint elements of X:

$$\begin{aligned} &\mathrm{sgn}(\{a\}, X \setminus \{a, b, c\})\,\mathrm{sgn}(\{b\}, X \setminus \{a, b, c\}) \\ &\mathrm{sgn}(\{c\}, X \setminus \{b, c, d\})\,\mathrm{sgn}(\{d\}, X \setminus \{b, c, d\}) \\ &\mathrm{sgn}(\{a\}, X \setminus \{a, b, d\})\,\mathrm{sgn}(\{b\}, X \setminus \{a, b, d\}) \\ &\mathrm{sgn}(\{c\}, X \setminus \{a, c, d\})\,\mathrm{sgn}(\{d\}, X \setminus \{a, c, d\}) \quad = 1. \end{aligned}$$

Indeed, we apply this equality to

$$X = (\mathscr{S}_C(0) \setminus \{u_0\}) \sqcup \cdots \sqcup \mathscr{S}_C(i) \sqcup \cdots \sqcup \mathscr{S}_C(j) \sqcup \cdots \sqcup (\mathscr{S}_C(k) \setminus \{u_k\}).$$

2.7 Appendix: proof of Lemma 2.4.10

Let $\sigma_C : \{1, \dots, n\} \xrightarrow{\simeq} \{1, \dots, n\}$ be the permutation defined by blocks via ν_C and η_C. Then we have

$$\operatorname{sgn}(\overline{C}, C) \operatorname{sgn}(\nu_C) \operatorname{sgn}(\mathscr{S}_C^+, \overline{\mathscr{S}_C^+}) \operatorname{sgn}(\eta_C) = \operatorname{sgn}(\sigma_C)$$

thus we only have to prove that

$$\operatorname{sgn}(\sigma_C) = (-1)^{k_C(D)}.$$

This is a straightforward consequence of the next Lemma and the fact that the signature of a cyclic permutation of length $(r+1)$ is $(-1)^r$.

Lemma 2.7.1. *Let D be a dissection diagram and $C \subset \mathscr{C}$.*

1. *$\mathscr{K}_C(D)$ is a disjoint union of path graphs $\overset{i_0}{\bullet} \longrightarrow \overset{i_1}{\bullet} \longrightarrow \cdots \longrightarrow \overset{i_{r-1}}{\bullet} \longrightarrow \overset{i_r}{\bullet}$ with $i_0 > i_1 > \cdots > i_{r-1} > i_r$ and $i_0 \in \mathscr{S}_C^+$.*
2. *With these notations, σ_C is the product of the cycle permutations $(i_0\ i_1\ \cdots\ i_{r-1}\ i_r)$.*

Example 2.7.2. In the situation of Example 2.5.2, we get $\sigma_C = (6\ 4\ 3\ 2)(8\ 7)$.

Proof. 1. Let us consider $\mathscr{K}_C(D)$ as a directed graph. If it contains a vertex attached to $l \geqslant 3$ chords, then there is one outcoming chord and $l-1 \geqslant 2$ incoming chords. Hence after flipping those chords we get a vertex with $l-1 \geqslant 2$ outcoming chords, which is impossible. Hence $\mathscr{K}_C(D)$ is a disjoint union of path graphs, and the same reasoning shows that in an individual component of $\mathscr{K}_C(D)$, all chords have the same direction. The statement then follows from Lemma 2.5.3.

2. Let us denote by $\varepsilon : \mathscr{C} \xrightarrow{\simeq} \mathscr{S}$ the bijection (2.2).

   (a) Let us first consider the bijection $\eta_C : C \xrightarrow{\simeq} \overline{\mathscr{S}_C^+}$ related to $r_C(D)$. If a chord $c \in C$ starting at the vertex i is not in $\mathscr{K}_C(D)$ then it keeps the same direction in $r_C(D)$ and we get $\eta_C(c) = \varepsilon(c)$, which means $\eta(i) = i$. Now let us consider a chord $c = i_k = \overset{i_{k-1}}{\bullet} \longrightarrow \overset{i_k}{\bullet}$ with the notations of 1., $k = 1, \dots, r$. Then this chord changes direction in $r_C(D)$ and becomes $\overset{i_{k-1}}{\bullet} \longleftarrow \overset{i_k}{\bullet}$, hence $\eta_C(c)$ is the edge starting at i_k and we get $\eta_S(i_{k-1}) = i_k$.

   (b) Let us now consider the bijection $\nu_C : \overline{C} \xrightarrow{\simeq} \mathscr{S}_C^+$ related to $q_C(D)$. Let $\overset{i_0}{\bullet} \longrightarrow \overset{i_1}{\bullet} \longrightarrow \cdots \longrightarrow \overset{i_{r-1}}{\bullet} \longrightarrow \overset{i_r}{\bullet}$ be a connected component of $\mathscr{K}_C(D)$ as in 1.. According to Lemma 2.5.3, the chord starting at i_r is necessarily in $\overline{C}$. In $q_C(D)$, all the points i_k, $k = 0, \dots, r$, are identified, and thus the edge starting at i_r is the edge starting at i_0, which means that $\nu_C(i_r) = i_0$. It is easy to check that for all other chords $c \in \overline{C}$ we get $\nu(c) = \varepsilon(c)$. This concludes the proof of the Lemma.

□

Chapter 3

The Orlik-Solomon model for hypersurface arrangements

In §3.1 we recall some classical facts about the Orlik-Solomon algebra and the Brieskorn-Orlik-Solomon theorem in the framework of hyperplane arrangements, and introduce the Orlik-Solomon algebra of a hypersurface arrangement. In §3.2, we introduce the complex of logarithmic forms along a hyperplane arrangement and its weight filtration, and prove the local form (Theorem 3.2.9) of the comparison theorem 1.9.2. Then we globalize our results to the framework of hypersurface arrangements (Theorem 3.2.13). In §3.3, we use the formalism of mixed Hodge complexes to give an alternative definition of the mixed Hodge structure on the cohomology of the complemet of a hypersurface arrangement in a smooth projective variety. This allows us to prove the main result of this chapter (Theorem 3.3.8) which proves the existence of the Orlik-Solomon model. In §3.4, we study the functoriality of the Orlik-Solomon model with respect to blow-ups, giving explicit formulas (Theorem 3.4.5). In §3.5, we apply our results to configuration spaces of points on curves and prove (Theorem 3.5.2) the isomorphism between the Orlik-Solomon model and the model proposed by Kriz and Totaro and generalized by Bloch.

3.1 The Orlik-Solomon algebra of a hypersurface arrangement

We first recall some classical facts about hyperplane arrangements. The interested reader will find more details in the expository book [OT92] or the survey [Yuz01]. Then we introduce hypersurface arrangements, define their Orlik-Solomon algebras and discuss their functoriality properties.

3.1.1 The Orlik-Solomon algebra of a hyperplane arrangement

A *hyperplane arrangement* in $\mathbb{C}^n$ is a finite set L of hyperplanes of $\mathbb{C}^n$, all containing the origin. [1] For a matter of notation, we will implicitly fix a linear ordering on the hyperplanes and write $L = \{L_1, \ldots, L_l\}$. Nevertheless, the objects that we will define out of a hyperplane arrangement will be independent of such an ordering.

We will use the same letter L to denote the union of the hyperplanes:

$$L = L_1 \cup \cdots \cup L_l.$$

For a subset $I \subset \{1, \ldots, l\}$, the *stratum* of the arrangement L indexed by I is the vector space $L_I = \bigcap_{i \in I} L_i$ with the convention $L_\varnothing = \mathbb{C}^n$. We write $\mathscr{S}_\bullet(L)$ for the set of strata of L, graded by the codimension, so that $\mathscr{S}_0(L) = \{\mathbb{C}^n\}$ and $\mathscr{S}_1(L) = \{L_1, \ldots, L_l\}$. With the order

1. In many references, this would be called a central hyperplane arrangement.

given by reverse inclusion, $\mathscr{S}_\bullet(L)$ is given the structure of a graded poset, called the *poset* of the hyperplane arrangement L.

We set $\Lambda_\bullet(L) = \Lambda^\bullet(e_1, \ldots, e_l)$, the exterior algebra over $\mathbb{Q}$ with a generator e_i in degree 1 for each L_i. Let $\delta : \Lambda_\bullet(L) \to \Lambda_{\bullet-1}(L)$ be the unique derivation of $\Lambda_\bullet(L)$ such that $\delta(e_i) = 1$ for $i = 1, \ldots, l$.
For $I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, l\}$ we set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k} \in \Lambda_k(L)$ with the convention $e_\varnothing = 1$. The derivation δ is then given by the formula

$$\delta(e_I) = \sum_{s=1}^{k} (-1)^{s-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_s}} \wedge \cdots \wedge e_{i_k}.$$

A subset $I \subset \{1, \ldots, l\}$ is said to be *dependent* (resp. *independent*) if $\mathrm{codim}(L_I) < |I|$ (resp. $\mathrm{codim}(L_I) = |I|$), which is equivalent to saying that the linear forms defining the L_i's, for $i \in I$, are linearly dependent (resp. independent). Let $J_\bullet(L)$ be the homogeneous ideal of $\Lambda_\bullet(L)$ generated by the elements $\delta(e_I)$ for $I \subset \{1, \ldots, l\}$ dependent. The quotient

$$A_\bullet(L) = \Lambda_\bullet(L)/J_\bullet(L)$$

is a graded $\mathbb{Q}$-algebra called the *Orlik-Solomon algebra* of the hyperplane arrangement L. It only depends on the poset of L.

For a stratum S, let $A_S(L)$ to be the sub-vector space of $A_\bullet(L)$ spanned by the monomials e_I for I such that $L_I = S$. One easily sees that we have a direct sum decomposition

$$A_\bullet(L) = \bigoplus_{S \in \mathscr{S}_\bullet(L)} A_S(L) \tag{3.1}$$

and $A_S(L)$ only depends on the hyperplane arrangement consisting of the hyperplanes in L that contain S, and more precisely on its poset.

The product in $A_\bullet(L)$ splits with respect to the direct sum decomposition (3.1), with components

$$A_S(L) \otimes A_{S'}(L) \to A_{S \cap S'}(L) \tag{3.2}$$

which are zero if $\mathrm{codim}(S \cap S') < \mathrm{codim}(S) + \mathrm{codim} S'$.
The derivation δ induces a derivation $\delta : A_\bullet(L) \to A_{\bullet-1}(L)$ which splits with respect to the direct sum decomposition (3.1), with components

$$A_S(L) \to A_{S'}(L) \tag{3.3}$$

for $S \subset S'$, $\mathrm{codim}(S') = \mathrm{codim}(S) - 1$.

3.1.2 Deletion and restriction

Let $L = \{L_1, \ldots, L_l\}$ be a hyperplane arrangement in $\mathbb{C}^n$ such that $l \geqslant 1$. In this chapter we will only be concerned about deletion and restriction with respect to the last hyperplane L_l. The *deletion* of L (with respect to L_l) is the arrangement $L' = \{L_1, \ldots, L_{l-1}\}$ in $\mathbb{C}^n$. The *restriction* of L (with respect to L_l) is the arrangement L'' on $L_l \cong \mathbb{C}^{n-1}$ consisting of all the intersections of L_l with the L_i's, $i = 1, \ldots, l-1$. If the hyperplanes L_i are not in general position, it may happen that the cardinality l'' of L'' is less than $l - 1$.

For all k, we have a short exact sequence of $\mathbb{Q}$-vector spaces, called the *deletion-restriction short exact sequence* (see [OT92, Theorem 3.65] or [Yuz01, Corollary 2.17]):

$$0 \to A_k(L') \xrightarrow{i} A_k(L) \xrightarrow{j} A_{k-1}(L'') \to 0. \tag{3.4}$$

This exact sequence splits with respect to the direct sum decomposition (3.1). For S a stratum of L, there are three cases:

- S is not contained in L_l, then it is not a stratum of L'' but is a stratum of L', and we just get an isomorphism
$$0 \to A_S(L') \to A_S(L) \to 0 \to 0;$$
- S is contained in L_l but is not a stratum of L', and we just get an isomorphism
$$0 \to 0 \to A_S(L) \to A_S(L'') \to 0;$$
- S is contained in L_l and is a stratum of L', and we get a short exact sequence
$$0 \to A_S(L') \to A_S(L) \to A_S(L'') \to 0.$$

3.1.3 The Brieskorn-Orlik-Solomon theorem

Let $L = \{L_1, \ldots, L_l\}$ be a hyperplane arrangement in $\mathbb{C}^n$. For $i = 1, \ldots, l$ we fix a linear form f_i on $\mathbb{C}^n$ such that $L_i = \{f_i = 0\}$. Such a form is unique up to a non-zero multiplicative constant. We define holomorphic 1-forms on $\mathbb{C}^n \setminus L$:
$$\omega_i = \frac{df_i}{f_i}.$$

For a subset $I = \{i_1 < \cdots < i_k\} \subset \{1, \ldots, l\}$ we set $\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$.

Let $\Omega^\bullet(\mathbb{C}^n \backslash L)$ be the algebra of global holomorphic forms on $\mathbb{C}^n \backslash L$ and $R^\bullet(L) \subset \Omega^\bullet(\mathbb{C}^n \backslash L)$ be the subalgebra over $\mathbb{Q}$ generated by 1 and the forms $\frac{1}{2i\pi}\omega_i$ for $i = 1, \ldots, l$. We define a morphism of graded algebras $u : \Lambda_\bullet(L) \to R^\bullet(L)$ by the formula
$$u(e_i) = \frac{1}{2i\pi}\omega_i.$$

A simple computation shows that u passes to the quotient and defines a map of graded algebras
$$u : A_\bullet(L) \to R^\bullet(L).$$

Each form $\frac{1}{2i\pi}\omega_i$ is closed and its class is in the cohomology of $\mathbb{C}^n \setminus L$ with *rational* (and even integer) coefficients, thus there is a well-defined map of graded algebras
$$v : R^\bullet(L) \to H^\bullet(\mathbb{C}^n \setminus L).$$

Theorem 3.1.1 (Brieskorn-Orlik-Solomon theorem)**.** *The maps u and v are isomorphisms of graded algebras:*
$$A_\bullet(L) \xrightarrow[\simeq]{u} R^\bullet(L) \xrightarrow[\simeq]{v} H^\bullet(\mathbb{C}^n \setminus L).$$

Remark 3.1.2. The fact that v is an isomorphism was conjectured by Arnol'd [Arn69] and proved by Brieskorn [Bri73]. The fact that u is an isomorphism was proved by Orlik and Solomon [OS80]. A proof may be found in [OT92, Theorems 3.126 and 5.89].

3.1.4 The Orlik-Solomon algebra of a hypersurface arrangement

We write $\Delta = \{|z| < 1\} \subset \mathbb{C}$ for the open unit disk and $\Delta^n \subset \mathbb{C}^n$ for the unit n-dimensional polydisk. Let X be a complex manifold. The following terminology is borrowed from P. Aluffi [Alu12].

Definition 3.1.3. A finite set $L = \{L_1, \ldots, L_l\}$ of smooth hypersurfaces of X is a *hypersurface arrangement* if around each point of X we may find a system of local coordinates in which each L_i is defined by a linear equation. In other words, X is covered by charts $V \cong \Delta^n$ such that for all i, $L_i \cap V$ is the intersection of Δ^n with a linear hyperplane in $\mathbb{C}^n$.

As for hyperplane arrangements, the objects that we will define out of a hypersurface arrangement will be independent of the linear ordering on the hypersurfaces L_i. We use the same letter L to denote the union of the hypersurfaces:

$$L = L_1 \cup \cdots \cup L_l.$$

The notion of hypersurface arrangement generalizes that of (simple) normal crossing divisor: a hypersurface arrangement is a normal crossing divisor if the local linear equations defining the L_i's are everywhere linearly independent, i.e. if we can always choose local coordinates such that the irreducible components L_i are coordinate hyperplanes.

For a subset $I \subset \{1, \ldots, l\}$, we still write $L_I = \bigcap_{i \in I} L_i$, which is a disjoint union of complex submanifolds of X. A *stratum* of L is a non-empty connected component of some L_I; it is a complex submanifold of X. We write $\mathscr{S}_\bullet(L)$ for the set of strata of L, graded by the codimension. We give $\mathscr{S}_\bullet(L)$ the structure of a graded poset using reverse inclusion, and call it the *poset* of the hypersurface arrangement L.

Let p be a point in $\mathbb{C}^n$ and V a neighbourhood of p. Then any chart $V \cong \Delta^n$ as in the above definition defines a hyperplane arrangement denoted $L^{(p)}$ in $\mathbb{C}^n$. It is an abuse of notation since choosing another chart gives a different hyperplane arrangement, but it will not matter since we will only be interested in the poset of $L^{(p)}$, which is well-defined. More intrinsically, $L^{(p)}$ may be read off the tangent space of X at p. Since S is connected, the poset of the strata of $L^{(p)}$ that contain S is independent of the point $p \in S$, and we may define

$$A_S(L) = A_S(L^{(p)})$$

for any choice of point $p \in S$. Let us then define

$$A_\bullet(L) = \bigoplus_{S \in \mathscr{S}_\bullet(L)} A_S(L).$$

We now give $A_\bullet(L)$ the structure of a graded algebra. The product

$$A_S(L) \otimes A_{S'}(L) \to A_T(L) \tag{3.5}$$

is non-zero only if T is a connected component of $S \cap S'$ such that $\mathrm{codim}(T) = \mathrm{codim}(S) + \mathrm{codim}(S')$, ans is then given by (3.2) by choosing any point $p \in T$.

The graded algebra $A_\bullet(L)$ is called the *Orlik-Solomon algebra* of the hypersurface arrangement L.

For $S \subset S'$ an inclusion of strata of L such that $\mathrm{codim}(S') = \mathrm{codim}(S) - 1$, we define

$$A_S(L) \to A_{S'}(L) \tag{3.6}$$

as in the local case (3.3) by choosing any point $p \in S$. One should note that in general the map $A_\bullet(L) \to A_{\bullet-1}(L)$ induced by (3.6) is not a derivation of the Orlik-Solomon algebra.

Remark 3.1.4. Let us assume that

$$\textit{for all } I,\ L_I \textit{ is connected.} \tag{3.7}$$

The Orlik-Solomon algebra of $L = \{L_1, \ldots, L_l\}$ thus has a presentation similar to that of a hyperplane arrangement. A subset $I \subset \{1, \ldots, l\}$ is said to be *null* if $L_I = \varnothing$ and *dependent* (resp. *independent*) if $L_I \neq \varnothing$ and $\operatorname{codim}(L_I) < |I|$ (resp. $\operatorname{codim}(L_I) = |I|$). Then $A_\bullet(L)$ is the quotient of $\Lambda^\bullet(e_1, \ldots, e_l)$ by the homogeneous ideal generated by the monomials e_I for I null and $\delta(e_I)$ for I dependent. In the case of a general hyperplane arrangement (the hyperplanes do not necessarily contain the origin), we recover the classical definition [OT92, Defintion 3.45]. Without the assumption (3.7), the Orlik-Solomon algebra may not even be generated in degree 1.

3.1.5 Functoriality of the Orlik-Solomon algebra

Let $L = \{L_1, \ldots, L_l\}$ and $L' = \{L'_1, \ldots, L'_{l'}\}$ be hyperplane arrangements respectively in $\mathbb{C}^n$ and $\mathbb{C}^{n'}$. Let $\varphi : \Delta^n \to \Delta^{n'}$ be a holomorphic map such that $\varphi^{-1}(L') \subset L$, i.e. $\varphi(\Delta^n \setminus L) \subset \Delta^{n'} \setminus L'$.
Then φ induces a map $\varphi^* : H^\bullet(\Delta^{n'} \setminus L') \to H^\bullet(\Delta^n \setminus L)$ in cohomology. The inclusions $\Delta^n \setminus L \subset \mathbb{C}^n \setminus L$ and $\Delta^{n'} \setminus L' \subset \mathbb{C}^{n'} \setminus L'$ are retractions and hence induce isomorphisms in cohomology. Thus the Brieskorn-Orlik-Solomon theorem 3.1.1 implies that there is a unique map of graded algebras

$$A_\bullet(\varphi) : A_\bullet(L') \to A_\bullet(L)$$

that fits into the following commutative square.

$$\begin{tikzcd}
A_\bullet(L') \arrow[r, "A_\bullet(\varphi)"] \arrow[d, "\cong"] & A_\bullet(L) \arrow[d, "\cong"] \\
H^\bullet(\Delta^{n'} \setminus L') \arrow[r, "\varphi^*"] & H^\bullet(\Delta^n \setminus L)
\end{tikzcd}$$

For $j = 1, \ldots, l'$, there is an equality

$$f'_j \circ \varphi = u_j \prod_i f_i^{m_{ij}}$$

between germs at 0 of holomorphic functions on Δ^n, with u_j a holomorphic function such that $u_j(0) \neq 0$ and $m_{ij} \geqslant 0$. On then sees that $A_\bullet(\varphi) : A_\bullet(L') \to A_\bullet(L)$ is the unique map of graded algebras such that for $j = 1, \ldots, l'$,

$$A_1(\varphi)(e'_j) = \sum_i m_{ij} e_i.$$

We may globalize this construction; if L (resp. L') is a hypersurface arrangement in a complex manifold X (resp. X'), and $\varphi : X \to X'$ a holomorphic map such that $\varphi^{-1}(L') \subset L$, then we define

$$A_{S,S'}(\varphi) : A_{S'}(L') \to A_S(L) \tag{3.8}$$

for strata $S \in \mathscr{S}_\bullet(L)$ and $S' \in \mathscr{S}_\bullet(L')$ by looking at φ in local charts and applying the above definition. It is clear that this defines a map of graded algebras $A_\bullet(\varphi) : A_\bullet(L) \to A_\bullet(L')$ that is functorial in the sense that we have $A_\bullet(\psi \circ \varphi) = A_\bullet(\varphi) \circ A_\bullet(\psi)$ whenever this is meaningful. If $\varphi : X \to X \times X$ is the diagonal of X, then $A_\bullet(\varphi)$ is the product morphism $A_\bullet(L) \otimes A_\bullet(L) \to A_\bullet(L)$.

3.2 Logarithmic forms and the weight filtration

We define and study the forms with logarithmic poles along a hyperplane arrangement. In § 3.2.1, 3.2.2, 3.2.3, 3.2.4, we focus on hyperplane arrangements (the local case). The main results are Theorem 3.2.6 which computes its graded pieces, and Theorem 3.2.9 which states that the logarithmic complex computes the cohomology of the complement of the hyperplane arrangement. Then in § 3.2.5 we extend our constructions and results to the case of hypersurface arrangements (the global case).

If Y is a complex manifold, we write Ω^p_Y for the sheaf of holomorphic p-forms on Y and $\Omega^p(Y) = \Gamma(Y, \Omega^p_Y)$ for the vector space of global holomorphic p-forms on Y.

3.2.1 The logarithmic complex

Let $L = \{L_1, \ldots, L_l\}$ be a hyperplane arrangement in $\mathbb{C}^n$. We recall that we defined some differential forms $\omega_i = \frac{df_i}{f_i}$ for $i = 1, \ldots, l$, and $\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$ for $I = \{i_1 < \cdots < i_k\}$, which is zero if I is dependent.

Definition 3.2.1. A meromorphic form on $\mathbb{C}^n$ is said to have *logarithmic poles along* L if it is a linear combination over $\mathbb{C}$ of forms of the type $\eta \wedge \omega_I$ for some $I \subset \{1, \ldots, l\}$, where η is a holomorphic form on $\mathbb{C}^n$.

We define $\Omega^p\langle L\rangle$ to be the $\mathbb{C}$-vector space of meromorphic p-forms on $\mathbb{C}^n$ with logarithmic poles along L. These forms are stable under the exterior differential, hence we get a complex $\Omega^\bullet\langle L\rangle$ that embeds into the complex of holomorphic forms on $\mathbb{C}^n \setminus L$:

$$\Omega^\bullet\langle L\rangle \hookrightarrow \Omega^\bullet(\mathbb{C}^n \setminus L)$$

which we call the *complex of logarithmic forms* of L.

Remark 3.2.2. This definition is not standard in the theory of hyperplane arrangements. In [OT92], following Saito [Sai80], one defines a complex $\Omega^\bullet(\log L)$ in the following way. Let $Q = f_1 \cdots f_l$ be a defining polynomial for the arrangement. Then $\Omega^p(\log L)$ is the set of meromorphic p-forms ω on $\mathbb{C}^n$ such that $Q\omega$ and $Qd\omega$ are holomorphic.

We have an inclusion $\Omega^\bullet\langle L\rangle \subset \Omega^\bullet(\log L)$ which is an equality if and only if $L = \{L_1, \ldots, L_l\}$ is independent. For instance, in $\mathbb{C}^2$ with coordinates x and y, let us look at $L_1 = \{x = 0\}$, $L_2 = \{y = 0\}$, $L_3 = \{x = y\}$. Then $Q = xy(x-y)$ and the closed form $\omega = \frac{dx \wedge dy}{xy(x-y)}$ is in $\Omega^2(\log L)$ but not in $\Omega^2\langle L\rangle$.

3.2.2 Residues

We briefly recall the notion of residue of a form with logarithmic poles along a hyperplane arrangement. In the case of dimension $n = 1$, this is the usual Cauchy residue in complex analysis; the general notion of residue is due to Poincaré and Leray [Ler59]. For residues in the setting of hyperplane arrangements, see [OT92, 3.124].

We fix a hyperplane arrangement $L = \{L_1, \ldots, L_l\}$ in $\mathbb{C}^n$. Let L' (resp. L'') the deletion (resp. the restriction) of L with respect to $L_l = \{f_l = 0\}$. Let ω be a p-form on $\mathbb{C}^n$ with logarithmic poles along L. Then there exists a $(p-1)$-form α and a p-form β, both of which have logarithmic poles along L', such that

$$\omega = \alpha \wedge \omega_l + \beta.$$

The form

$$\mathrm{Res}_{L_l}(\omega) = 2i\pi\, \alpha_{|L_l}$$

is independent of the choices. It is a $(p-1)$-form on L_l with logarithmic poles along L'', called the *residue* of ω along L_l. We then have a morphism of complexes

$$\mathrm{Res}_{L_l} : \Omega^\bullet\langle L\rangle \to \Omega^{\bullet-1}\langle L''\rangle$$

where L'' is the restriction of L with respect to L_l. We then have a sequence of morphisms of complexes

$$(\mathcal{R}) : \quad 0 \to \Omega^\bullet\langle L'\rangle \overset{i}{\to} \Omega^\bullet\langle L\rangle \overset{\mathrm{Res}_{L_l}}{\longrightarrow} \Omega^{\bullet-1}\langle L''\rangle \to 0$$

where i is the natural inclusion. It is obvious from the definitions that $\mathrm{Res}_{L_l} \circ i = 0$, that i is injective and Res_{L_l} is surjective. We will prove in the next paragraph that $\ker(\mathrm{Res}_{L_l}) \subset \mathrm{Im}(i)$, so that the above sequence is a short exact sequence.

Remark 3.2.3. When taking iterated residues, one should note that they "do not commute" in general, even when this has a clear meaning. For example, if $L_1 = \{x = 0\}$, $L_2 = \{y = 0\}$, $L_3 = \{x = y\}$ in $\mathbb{C}^2$ and $\omega = \frac{dx}{x} \wedge \frac{dy}{y} \in \Omega^2\langle L\rangle$, we have $\mathrm{Res}_{L_2\cap L_3}\mathrm{Res}_{L_2}(\omega) = (2i\pi)^2$ and $\mathrm{Res}_{L_3\cap L_2}\mathrm{Res}_{L_3}(\omega) = 0$.

3.2.3 The weight filtration

We fix a hyperplane arrangement $L = \{L_1, \ldots, L_l\}$ in $\mathbb{C}^n$. The following terminology is borrowed from P. Deligne [Del71, 3.1.5].

Definition 3.2.4. For $k \geqslant 0$, we define $W_k\Omega^\bullet\langle L\rangle \subset \Omega^\bullet\langle L\rangle$ to be the subcomplex spanned by the forms that are of the type $\eta \wedge \omega_I$ with $|I| \leqslant k$, where η is a holomorphic form on $\mathbb{C}^n$. These subcomplexes define an ascending filtration

$$W_0\Omega^\bullet\langle L\rangle \subset W_1\Omega^\bullet\langle L\rangle \subset \cdots$$

on $\Omega^\bullet\langle L\rangle$ called the *weight filtration*.

We have $W_0\Omega^\bullet\langle L\rangle = \Omega^\bullet(\mathbb{C}^n)$ and $W_p\Omega^p\langle L\rangle = \Omega^p\langle L\rangle$.

By definition, the residue morphisms induce morphisms $\mathrm{Res}_{L_l} : W_k\Omega^\bullet\langle L\rangle \to W_{k-1}\Omega^{\bullet-1}\langle L''\rangle$ which are easily seen to be surjective. Thus the sequence $(\mathcal{R})$ induces sequences

$$(W_k\mathcal{R}) : \quad 0 \to W_k\Omega^\bullet\langle L'\rangle \overset{i}{\to} W_k\Omega^\bullet\langle L\rangle \overset{\mathrm{Res}_{L_l}}{\longrightarrow} W_{k-1}\Omega^{\bullet-1}\langle L''\rangle \to 0 \tag{3.9}$$

and

$$(\mathrm{gr}_k^W\mathcal{R}) : \quad 0 \to \mathrm{gr}_k^W\Omega^\bullet\langle L'\rangle \overset{i}{\to} \mathrm{gr}_k^W\Omega^\bullet\langle L\rangle \overset{\mathrm{Res}_{L_l}}{\longrightarrow} \mathrm{gr}_{k-1}^W\Omega^{\bullet-1}\langle L''\rangle \to 0. \tag{3.10}$$

We will prove that they are short exact sequences. For now, the only easy facts are that $(W_k\mathcal{R})$ is exact on the left and on the right, and that $(\mathrm{gr}_k^W\mathcal{R})$ is exact on the right.

The following lemma is easily proved by choosing appropriate coordinates on $\mathbb{C}^n$.

Lemma 3.2.5. *Let $I \subset \{1, \ldots, l\}$, $|I| = k$, be an independent subset and η a holomorphic form on $\mathbb{C}^n$. If $\eta_{|L_I} = 0$ then $\eta \wedge \omega_I \in W_{k-1}\Omega^\bullet\langle L\rangle$.*

For all k, we define

$$G_k^\bullet(L) = \bigoplus_{S\in\mathscr{S}_k(L)} \Omega^{\bullet-k}(S)\otimes A_S(L).$$

This is a complex of $\mathbb{C}$-vector spaces. We define a morphism of complexes

$$\Phi : G_k^\bullet(L) \to \mathrm{gr}_k^W \Omega^\bullet\langle L\rangle$$

in the following way. For I independent of cardinality k, for $\eta\in\Omega^{\bullet-k}(L_I)$, we set

$$\Phi(\eta\otimes e_I) = (2i\pi)^{-k}\widetilde{\eta}\wedge\omega_I$$

where $\widetilde{\eta}\in\Omega^{\bullet-k}(\mathbb{C}^n)$ is any form such that $\widetilde{\eta}_{|L_l}=\eta$. Lemma 3.2.5 implies that this does not depend on the choice of $\widetilde{\eta}$ and one immediately sees that it passes to the quotient that defines the groups $A_S(L)$. It is then easy to check that Φ is a morphism of complexes.

Theorem 3.2.6. *The morphism $\Phi : G_k^\bullet(L) \to \mathrm{gr}_k^W \Omega^\bullet\langle L\rangle$ is an isomorphism of complexes.*

Proof. The surjectivity is trivial; we prove the injectivity by induction on the cardinal l of the arrangement.
For $l=0$, the only non-trivial case is $k=0$ and Φ is just the identity of $\Omega^\bullet(\mathbb{C}^n)$.
Suppose that the statement is proved for arrangements of cardinality $\leqslant l-1$ and take an arrangement L of cardinality l. Tensoring the deletion-restriction short exact sequence from §3.1.2 with the complexes $\Omega^{\bullet-k}(S)$ we get a short exact sequence of complexes of $\mathbb{C}$-vector spaces

$$0\to G_k^\bullet(L')\to G_k^\bullet(L)\to G_{k-1}^{\bullet-1}(L'')\to 0.$$

We then have a diagram

$$\begin{tikzcd}
0 \arrow[r] & G_k^\bullet(L') \arrow[r] \arrow[d, "\Phi"] & G_k^\bullet(L) \arrow[r] \arrow[d, "\Phi"] & G_{k-1}^{\bullet-1}(L'') \arrow[r] \arrow[d, "\Phi"] & 0 \\
0 \arrow[r] & \mathrm{gr}_k^W\Omega^\bullet\langle L'\rangle \arrow[r] & \mathrm{gr}_k^W\Omega^\bullet\langle L\rangle \arrow[r] & \mathrm{gr}_{k-1}^W\Omega^{\bullet-1}\langle L''\rangle \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is the sequence (3.10). This diagram is easily seen to be commutative.
By the inductive hypothesis, the vertical arrows on the right and on the left are isomorphisms. Thus a diagram chase shows that the bottom row is exact in the middle.
Now the complexes (3.9) and (3.10) give rise to a short exact sequence of complexes

$$0\to(W_{k-1}\mathcal{R})\to(W_k\mathcal{R})\to(\mathrm{gr}_k^W\mathcal{R})\to 0.$$

The long exact sequence in cohomology tells us that if $(W_{k-1}\mathcal{R})$ is exact in the middle then it is also the case for $(W_k\mathcal{R})$. Since $(W_0\mathcal{R})$ is just the sequence

$$0\to\Omega^\bullet(\mathbb{C}^n)\xrightarrow{\mathrm{id}}\Omega^\bullet(\mathbb{C}^n)\to 0\to 0,$$

we show by induction on k shows that $(W_k\mathcal{R})$ is exact in the middle, hence a short exact sequence, for all k. Again, the long exact sequence in cohomology shows that $(\mathrm{gr}_k^W\mathcal{R})$ is also a short exact sequence for all k.
Thus, in the above commutative diagram, both rows are exact and a diagram chase (the 5-lemma) shows that the middle Φ is injective. This completes the induction and the proof of the theorem. □

Remark 3.2.7. The inverse morphism $\Psi : \mathrm{gr}_k^W \Omega^\bullet\langle L\rangle \to G_k^\bullet(L)$ is given, for η holomorphic and I independent of cardinality k, by

$$\Psi(\eta \wedge \omega_I) = (2i\pi)^k \, \eta_{|L_I} \in \Omega^{\bullet - k}(L_I)$$

For $k = 1$ this is exactly the definition of a residue, but for $k > 1$ one should note that this has nothing to do with an "iterated residue" (see Remark 3.2.3).

Since $(\mathcal{R}) = (W_k\mathcal{R})$ for k large enough, the proof of Theorem 3.2.6 implies the following.

Theorem 3.2.8. *The sequences $(\mathcal{R})$, $(W_k\mathcal{R})$ and $(\mathrm{gr}_k^W\mathcal{R})$ are short exact sequences of complexes.*

3.2.4 The comparison theorem

Theorem 3.2.9. *The inclusion $\Omega^\bullet\langle L\rangle \hookrightarrow \Omega^\bullet(\mathbb{C}^n \setminus L)$ is a quasi-isomorphism.*

Proof. Since $\mathbb{C}^n \setminus L$ is a smooth affine algebraic variety over $\mathbb{C}$, the cohomology of $\Omega^\bullet(\mathbb{C}^n \setminus L)$ is the cohomology of $\mathbb{C}^n \setminus L$ with complex coefficients. Thus we have to prove that the natural map

$$H^p(\Omega^\bullet\langle L\rangle) \to H^p(\mathbb{C}^n \setminus L, \mathbb{C})$$

is an isomorphism for all p. We proceed by induction on the cardinality l of the arrangement. For $l = 0$ the statement is trivial. To pass from $l-1$ to l we consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(\Omega^\bullet\langle L'\rangle) \arrow[r] \arrow[d] & H^p(\Omega^\bullet\langle L\rangle) \arrow[r] \arrow[d] & H^{p-1}(\Omega^\bullet\langle L''\rangle) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & H^p(\mathbb{C}^n \setminus L') \arrow[r] & H^p(\mathbb{C}^n \setminus L) \arrow[r] & H^{p-1}(L_l \setminus L'') \arrow[r] & 0
\end{tikzcd}$$

The first row is the long exact sequence in cohomology associated to $(\mathcal{R})$, the second row is induced by the deletion-restriction exact sequence via the Brieskorn-Orlik-Solomon theorem. Both rows are exact. By induction the vertical arrows on the left and on the right are isomorphisms. A classical diagram chase implies that the vertical arrow in the middle is also an isomorphism. □

Remark 3.2.10. We have the inclusions of complexes

$$\Omega^\bullet\langle L\rangle \overset{i_1}{\hookrightarrow} \Omega^\bullet(\log L) \overset{i_2}{\hookrightarrow} \Omega^\bullet(\mathbb{C}^n \setminus L)$$

where $\Omega^\bullet(\log L)$ has been defined in Remark 3.2.2.
A conjecture by H. Terao [Ter78] states that i_2 is a quasi-isomorphism. According to Theorem 3.2.9, the composite $i_2 \circ i_1$ is a quasi-isomorphism, hence Terao's conjecture is equivalent to the fact that i_1 is a quasi-isomorphism. This is equivalent to the acyclicity of the quotient complex $\Omega^\bullet(\log L)/\Omega^\bullet\langle L\rangle$.

3.2.5 Logarithmic forms along hypersurface arrangements

In this paragraph we globalize the definitions of the logarithmic complex and the weight filtration. As in the local case, we determine the weight-graded parts of the logarithmic complex and prove a comparison theorem. This generalizes the case of normal crossing divisors, studied by Deligne in [Del71, 3.1].

Let X be a complex manifold and L a hypersurface arrangement in X. A meromorphic form on X is said to have *logarithmic poles along* L if it is locally a linear combination over $\mathbb{C}$ of forms of the type

$$\eta\wedge\frac{df_{i_1}}{f_{i_1}}\wedge\cdots\wedge\frac{df_{i_r}}{f_{i_r}} \tag{3.11}$$

with η holomorphic and the f_i's local defining (linear) equations for the L_i's. The meromorphic forms on X with logarithmic poles along L form a complex of sheaves of $\mathbb{C}$-vector spaces on X, that we denote by $\Omega^\bullet_{\langle X,L\rangle}$. As in the local setting (Remark 3.2.2), we should point out that $\Omega^\bullet_{\langle X,D\rangle}$ differs from Saito's complex $\Omega^\bullet_X(\log L)$ if L is not a normal crossing divisor.

We globalize the *weight filtration* on $\Omega^\bullet_{\langle X,L\rangle}$ which gives subcomplexes of sheaves $W_k\Omega^\bullet_{\langle X,L\rangle}\subset\Omega^\bullet_{\langle X,L\rangle}$.

The complex of sheaves $\Omega^\bullet_{\langle X,L\rangle}$ is functorial in (X,L) in the following sense. If L' is another hypersurface arrangement in a complex manifold X', and if we have a holomorphic map $\varphi:X\to X'$ such that $\varphi^{-1}(L')\subset L$, then there is a pull-back map

$$\varphi^*:\varphi^{-1}\Omega^\bullet_{\langle X',L'\rangle}\to\Omega^\bullet_{\langle X,L\rangle}$$

that is compatible with composition in the usual sense. This follows from the discussion in §3.1.5. The weight filtration is also functorial.

For a stratum S we denote by $i_S:S\hookrightarrow X$ the closed immersion of S inside X. We globalize the definition of $G^\bullet_k(L)$ from §3.2.3 and define a complex of sheaves of $\mathbb{C}$-vector spaces on X:

$$\mathcal{G}^\bullet_k(X,L)=\bigoplus_{S\in\mathscr{S}_k(L)}(i_S)_*\Omega_S^{\bullet-k}\otimes A_S(L).$$

As in the local case, we may define a morphism of complexes of sheaves

$$\Phi:\mathcal{G}^\bullet_k(X,L)\to\mathrm{gr}^W_k\Omega^\bullet_{\langle X,L\rangle}$$

by putting

$$\Phi(\eta\otimes e_I)=(2i\pi)^{-k}\,\widetilde{\eta}\wedge\frac{df_{i_1}}{f_{i_1}}\wedge\ldots\wedge\frac{df_{i_k}}{f_{i_k}}$$

for $I=\{i_1<\ldots<i_k\}$, $\eta\in\Omega_S^{\bullet-k}$ a local section, $\widetilde{\eta}\in\Omega_X^{\bullet-k}$ a local extension of η, and the f_i's local equations for the L_i's. This definition is independent from the choice of the local equations f_i. The following theorem is a global version of Theorem 3.2.6.

Theorem 3.2.11. *The morphism* $\Phi:\mathcal{G}^\bullet_k(X,L)\to\mathrm{gr}^W_k\Omega^\bullet_{\langle X,L\rangle}$ *is an isomorphism.*

Proof. It is enough to prove that for every chart $V\cong\Delta^n$ on which L is a hyperplane arrangement, the morphism

$$\Gamma(V,\mathcal{G}^\bullet_k(X,L))\to\Gamma(V,\mathrm{gr}^W_k\Omega^\bullet_{\langle X,L\rangle})$$

is an isomorphism. This is exactly Theorem 3.2.6 with the ambient space $\mathbb{C}^n$ replaced by the polydisk Δ^n. One can check that the proof of Theorem 3.2.6 can be copied word for word in that local setting. □

Remark 3.2.12. The inverse morphism $\Psi:\mathrm{gr}^W_k\Omega^\bullet_{\langle X,L\rangle}\to\mathcal{G}^\bullet_k(X,L)$ is given locally by the same formula as in Remark 3.2.7. As already noted, this should not be mistaken with an iterated residue, unless L is a normal crossing divisor (in this case, Deligne calls Ψ the Poincaré residue, see [Del71, 3.1.5.2]).

Let $j : X \setminus L \hookrightarrow X$ be the open immersion of the complement of L inside X. The following theorem is a global version of Theorem 3.2.9.

Theorem 3.2.13. *The inclusion $\Omega^\bullet_{\langle X,L\rangle} \hookrightarrow j_*\Omega^\bullet_{X\setminus L}$ is a quasi-isomorphism.*

Proof. It is enough to prove that for every chart $V \cong \Delta^n$ on which L is a hyperplane arrangement, the morphism

$$\Gamma(V, \Omega^\bullet_{\langle X,L\rangle}) \to \Gamma(V, j_*\Omega^\bullet_{X\setminus L}) = \Omega^\bullet(V \setminus L)$$

is a quasi-isomorphism. This is exactly Theorem 3.2.9 with the ambient space $\mathbb{C}^n$ replaced by the polydisk Δ^n. One can check that the proof of Theorem 3.2.9 can be copied word for word in the local setting. The argument that the strata L_I are contractible has to be replaced by the fact that the local strata $\Delta^n \cap L_I$ are contractible (because they are polydisks). The Brieskorn-Orlik-Solomon theorem remains true in the local setting because the inclusion $\Delta^n \setminus L \subset \mathbb{C}^n \setminus L$ is a retraction and hence induces an isomorphism in cohomology. □

Remark 3.2.14. It has been pointed out to us by A. Dimca that the sheaves $\Omega^1_{\langle X,L\rangle}$ have been previously defined in [CHKS06] (where they are denoted $\Omega_X(\log L)$) and [Dol07] (where they are denoted $\tilde{\Omega}_X(\log L)$).

3.3 A functorial mixed Hodge structure and the Orlik-Solomon model

If X is a smooth projective variety and L is a hypersurface arrangement in X, we put a functorial mixed Hodge structure on the cohomology of the complement $X \setminus L$. Our construction mimicks Deligne's [Del71] in the case of normal crossing divisors.

3.3.1 Mixed Hodge complexes

We refer to [Del74, 7.1, 8.1] for the definitions of mixed Hodge complexes. If $\mathbb{K}$ is a field, the filtered (resp. bifiltered) derived category of (bounded from above) complexes of $\mathbb{K}$-vector spaces on a complex manifold Y is denoted by $\mathrm{D}^+\mathrm{F}(Y,\mathbb{K})$ (resp. $\mathrm{D}^+\mathrm{F}_2(Y,\mathbb{K})$). A cohomological mixed Hodge complex on Y is a triple

$$\mathcal{K} = ((\mathcal{K}_\mathbb{Q}, W), (\mathcal{K}_\mathbb{C}, W, F), \alpha)$$

with $(\mathcal{K}_\mathbb{Q}, W) \in \mathrm{D}^+\mathrm{F}(Y,\mathbb{Q})$, $(\mathcal{K}_\mathbb{C}, W, F) \in \mathrm{D}^+\mathrm{F}_2(Y,\mathbb{C})$ and $\alpha : (\mathcal{K}_\mathbb{Q}, W) \otimes \mathbb{C} \cong (\mathcal{K}_\mathbb{C}, W)$ an isomorphism in $\mathrm{D}^+\mathrm{F}(Y,\mathbb{C})$. These data must satisfy some compatibility conditions.

The following theorem [Del74, 8.1.9] is the fundamental theorem of mixed Hodge complexes. Our convention for spectral sequences uses decreasing filtrations. One passes from an increasing filtration $\{W_p\}_{p\in\mathbb{Z}}$ to a decreasing filtration $\{W^p\}_{p\in\mathbb{Z}}$ by putting $W^p = W_{-p}$.

Theorem 3.3.1. *Let Y be a complex manifold and $\mathcal{K} = ((\mathcal{K}_\mathbb{Q}, W), (\mathcal{K}_\mathbb{C}, W, F), \alpha)$ a cohomological mixed Hodge complex on Y.*

1. *For all n, the filtration $W[-n]$ and the filtration F define a mixed Hodge structure on $\mathbb{H}^n(\mathcal{K}_\mathbb{Q})$.*

2. *Let ${}_{\mathrm{W}}E$ be the cohomological spectral sequence defined by $(\mathcal{K}_\mathbb{Q}, W)$. Then for all (p,q), the filtration F induces on ${}_{\mathrm{W}}E_1^{-p,q} = \mathbb{H}^{-p+q}(\mathrm{gr}_p^W \mathcal{K}_\mathbb{Q})$ a Hodge structure of weight q and the differentials $d_1^{-p,q}$ are morphisms of Hodge structures.*

3. *The spectral sequence ${}_{\mathrm{W}}E$ degenerates at E_2: ${}_{\mathrm{W}}E_2^{-p,q} = {}_{\mathrm{W}}E_\infty^{-p,q} = \mathrm{gr}_p^W \mathbb{H}^n(\mathcal{K}_\mathbb{Q}) = \mathrm{gr}_q^{W[-n]} \mathbb{H}^n(\mathcal{K}_\mathbb{Q})$ for $n = -p+q$.*

3.3.2 A functorial mixed Hodge structure

Let X be a smooth projective variety over $\mathbb{C}$ and L a hypersurface arrangement in X. We use the previous constructions to put a functorial mixed Hodge structure on the cohomology $H^\bullet(X\setminus L)$ of the complement, using the formalism of mixed Hodge complexes. This generalizes the case of normal crossing divisors, studied by Deligne in [Del71, 3.2], and summarized in terms of mixed Hodge complexes in [Del74, 8.1.8]. We recall the notation $j : X\setminus L \hookrightarrow X$.

We define a triple

$$\mathcal{K}(X,L) = ((\mathcal{K}_{\mathbb{Q}}(X,L),W),(\mathcal{K}_{\mathbb{C}}(X,L),W,F),\alpha)$$

in the following way:

1. $\mathcal{K}_{\mathbb{Q}}(X,L) = Rj_*\mathbb{Q}_{X\setminus L}$ with the filtration $W=\tau$, the canonical filtration [Del71, 1.4.6].
2. $\mathcal{K}_{\mathbb{C}}(X,L) = \Omega^\bullet_{\langle X,L\rangle}$ with the weight filtration W defined in §3.2.5, and the Hodge filtration F defined by
$$F^p\Omega^\bullet_{\langle X,L\rangle} = \Omega^{\geqslant p}_{\langle X,L\rangle}.$$
3. We have isomorphisms in $\mathrm{D}^+(X,\mathbb{C})$:
$$Rj_*\mathbb{Q}_{X\setminus L}\otimes\mathbb{C} \cong Rj_*\mathbb{C}_{X\setminus L} \cong j_*\Omega^\bullet_{X\setminus L} \cong \Omega^\bullet_{\langle X,L\rangle}$$
the last one being the quasi-isomorphism of the comparison theorem 3.2.13.
Hence we have an isomorphism $(Rj_*\mathbb{Q}_{X\setminus L}\otimes\mathbb{C},\tau)\cong(\Omega^\bullet_{\langle X,L\rangle},\tau)$ in $\mathrm{D}^+\mathrm{F}(X,\mathbb{C})$. Finally the identity gives a filtered quasi-isomorphism $(\Omega^\bullet_{\langle X,L\rangle},\tau)\cong(\Omega^\bullet_{\langle X,L\rangle},W)$, as follows from the same proof as in [Del71, 3.1.8], in view of the comparison theorem 3.2.13. This gives the isomorphism
$$\alpha : (Rj_*\mathbb{Q}_{X\setminus L},\tau)\otimes\mathbb{C}\cong(\Omega^\bullet_{\langle X,L\rangle},W)$$
in $\mathrm{D}^+\mathrm{F}(X,\mathbb{C})$.

Theorem 3.3.2. *The triple $\mathcal{K}(X,L)$ is a cohomological mixed Hodge complex on X, which is functorial with respect to the pair (X,L). It thus defines a functorial mixed Hodge structure on $\mathbb{H}^n(Rj_*\mathbb{Q}_{X\setminus L})\cong H^n(X\setminus L)$ for all n.*

Here, functoriality has to be understood in the sense of §3.1.5.

Proof. Theorem 3.2.6 gives an isomorphism

$$\mathrm{gr}_k^W\Omega^\bullet_{\langle X,L\rangle}\cong\bigoplus_{S\in\mathscr{S}_k(L)}(i_S)_*\Omega_S^{\bullet-k}\otimes A_S(L).$$

A local computation as in [PS08, Lemma 4.9], shows that this isomorphism is defined over $\mathbb{Q}$ if we take care of the Tate twists. In other words we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{gr}_k^W\Omega^\bullet_{\langle X,L\rangle} \arrow[r, "\cong"] & \bigoplus_{S\in\mathscr{S}_k(L)}(i_S)_*\Omega^\bullet_S[-k]\otimes A_S(L) \\
\mathrm{gr}_k^\tau Rj_*\mathbb{C}_U \arrow[r, "\cong"] \arrow[u, "\cong"] & \bigoplus_{S\in\mathscr{S}_k(L)}(i_S)_*\mathbb{C}_S[-k]\otimes A_S(L) \arrow[u, "\cong"'] \\
\mathrm{gr}_k^\tau Rj_*\mathbb{Q}_U \arrow[r, "\cong"] \arrow[u] & \bigoplus_{S\in\mathscr{S}_k(L)}(i_S)_*\mathbb{Q}_S[-k](-k)\otimes A_S(L) \arrow[u]
\end{tikzcd}$$

To complete the proof it is enough to notice that the top row of this diagram is compatible with the Hodge filtrations. Hence we get

$$\mathrm{gr}_k^W \mathcal{K}(X,L) = \bigoplus_{S\in\mathscr{S}_k(L)} (i_S)_*\mathcal{K}(S)[-k](-k)\otimes A_S(L)$$

which is a cohomological Hodge complex of weight k.
The functoriality statement follows from the functoriality of the sheaves of logarithmic forms. □

The following theorem shows that the Hodge structures that we have just defined are indeed the functorial Hodge structures defined by Deligne.

Theorem 3.3.3. *Let U be a smooth quasi-projective variety over $\mathbb{C}$.*

1. *There exists a smooth projective variety X and an open immersion $U \hookrightarrow X$ such that the complement $L = X \setminus U$ is a hypersurface arrangement in X.*
2. *Given two such compactifications (X_1, L_1) and (X_2, L_2), the mixed Hodge structures on $H^\bullet(U)$ defined via (X_1, L_1) and (X_2, L_2) are the same.*
3. *The mixed Hodge structure on $H^\bullet(U)$ defined in Theorem 3.3.2 is the same as the mixed Hodge structure defined by Deligne in [Del71].*

Proof. 1. This follows from Nagata's compactification theorem and Hironaka's resolution of singularities. In fact, we can assume that L is a normal crossing divisor.

2. Using resolution of singularities, we can always embed U in a smooth projective variety X such that $X \setminus U = L$ is a simple normal crossing divisor (and hence a hypersurface arrangement), and such that there exists morphisms

$$(X_1, X_1\setminus L_1) \leftarrow (X, X\setminus L) \rightarrow (X_2, X_2 \setminus L_2)$$

that are the identity on U. Hence by functoriality the two mixed Hodge structures are isomorphic to the mixed Hodge structure defined via (X, L).

3. The claim follows from (2) and the fact that for a given U, one can always choose (X, L) such that L is a normal crossing divisor (using resolution of singularities).

□

3.3.3 The Orlik-Solomon spectral sequence

Let X be a smooth projective variety and L a hypersurface arrangement in X. In the previous paragraph we defined a cohomological mixed Hodge complex on X that defines a mixed Hodge structure on the cohomology of $X \setminus L$. The general formalism of mixed Hodge complexes (Theorem 3.3.1) tells us that the *Orlik-Solomon spectral sequence* ${}_{\mathrm{w}}E_r^{p,q}$ associated to the weight filtration degenerates at E_2. In this section we make the E_1 term explicit. We will write ${}_{\mathrm{w}}E_r^{p,q} = {}_{\mathrm{w}}E_r^{p,q}(X,L)$ when confusion might occur.

By definition we have ${}_{\mathrm{w}}E_1^{-p,q} = \mathbb{H}^{-p+q}(\mathrm{gr}_p^W \mathcal{K}_{\mathbb{Q}}(X,L))$. From the proof of Theorem 3.3.2 we get

$${}_{\mathrm{w}}E_1^{-p,q} \cong \bigoplus_{S\in\mathscr{S}_p(L)} H^{-2p+q}(S)(-p)\otimes A_S(L).$$

We first study the functoriality of the Orlik-Solomon spectral sequence.

Proposition 3.3.4. *Let L (resp. L') be a hypersurface arrangement in a smooth projective variety X (resp. X'), and $\varphi : X \to X'$ a holomorphic map such that $\varphi^{-1}(L') \subset L$. Let S and S' be strata of codimension p respectively of L and L' such that $\varphi(S) \subset S'$ and let us denote by $\varphi_{S,S'} : S \to S'$ the restriction of φ. Then the component of the morphism*

$$ {}_{\mathrm{w}}E_1^{-p,q}(\varphi) : {}_{\mathrm{w}}E_1^{-p,q}(X', L') \to {}_{\mathrm{w}}E_1^{-p,q}(X, L) $$

indexed by strata S and S' is obtained by tensoring the morphism (3.8)

$$ A_{S,S'}(\varphi) : A_{S'}(L') \to A_S(L) $$

with the pull-back morphism

$$ \varphi^*_{S,S'} : H^{-2p+q}(S') \to H^{-2p+q}(S). $$

The other components of ${}_{\mathrm{w}}E_1^{-p,q}(\varphi)$ are zero.

Proof. It is enough to do the proof over $\mathbb{C}$ and work with the complexes $\Omega^\bullet_{\langle X,L\rangle}$. There is a pull-back morphism

$$ \varphi^{-1}\Omega^\bullet_{\langle X',L'\rangle} \to \Omega^\bullet_{\langle X,L\rangle} $$

that is compatible with the weight filtrations. Via the isomorphisms of Theorem 3.2.11, one sees by local computation that this pull-back is as described in the Proposition at the level of holomorphic forms. $\square$

When applied to the diagonal morphism $X \to X \times X$, one gets an algebra structure on the E_1 term of the Orlik-Solomon spectral sequence, as follows.

Proposition 3.3.5. *The product*

$$ {}_{\mathrm{w}}E_1^{-p,q} \otimes {}_{\mathrm{w}}E_1^{-p',q'} \to {}_{\mathrm{w}}E_1^{-(p+p'),q+q'} \tag{3.12} $$

is obtained by tensoring the product morphisms (3.5)

$$ A_S(L) \otimes A_{S'}(L) \to A_T(L) $$

with the morphisms

$$ H^{-2p+q}(S) \otimes H^{-2p'+q'}(S') \to H^{-2p+q}(T) \otimes H^{-2p'+q'}(T) \xrightarrow{\cup} H^{-2(p+p')+(q+q')}(T) $$

multiplied by the sign $(-1)^{pq'}$. The above morphism is the composition of the restriction morphisms for the inclusion of T inside S and S', followed by the cup-product on T.

Note the sign $(-1)^{pq'}$, which is a Koszul sign associated to the interchanging of the terms $A_S(L)$ and $H^{-2p'+q'}(S')$.

We now turn to the description of the differential of the E_1 term of the Orlik-Solomon spectral sequence.

Proposition 3.3.6. *Let $S \subset S'$ be an inclusion of strata of L with $\operatorname{codim}(S) = p$ and $\operatorname{codim}(S') = p-1$. Then the component of the differential*

$$ d_1 : {}_{\mathrm{w}}E_1^{-p,q} \to {}_{\mathrm{w}}E_1^{-p+1,q} $$

indexed by S and S' is obtained by tensoring the natural morphism (3.6)

$$ A_S(L) \to A_{S'}(L) $$

with the Gysin morphism

$$ H^{-2p+q}(S)(-p) \to H^{-2p+q+2}(S')(-p+1) $$

multiplied by the sign $(-1)^{q-1}$. The other components of d_1 are zero.

Proof. First step: If $L = D = \{D_1, \ldots, D_l\}$ is a normal crossing divisor, this is Proposition 8.34 in [Voi02] (see also [PS08, Proposition 4.7]). Indeed in this case we have for every subset $I \subset \{1, \ldots, l\}$, $A_{D_I}(D) = \mathbb{Q}\, e_I$ a one-dimensional vector space.
Second step: We deduce the general case from the functoriality of the Orlik-Solomon spectral sequenceand the fact that $A_\bullet(L)$ is spanned by monomials e_I with I independent. Let e_I be such a monomial and let us write $L(I) = \bigcup_{i\in I} L_i$, which is a normal crossing divisor in X. From the functoriality of the spectral sequence, there is a map of spectral sequences

$$ {}_{\mathrm{w}}E_1^{-p,q}(X, L(I)) \to {}_{\mathrm{w}}E_1^{-p,q}(X, L) $$

which is easily seen to be injective (this follows from the injectivity in the deletion-restriction short exact sequence). Thus the differential of an element in $H^{-2p+q}(S) \otimes \mathbb{Q}e_I$ can be read off ${}_{\mathrm{w}}E_1^{p,q}(X, L(I))$. We are then reduced to the first step. □

Remark 3.3.7. If X is any complex manifold, then we can also consider the Orlik-Solomon spectral sequence converging to the cohomology of $X \setminus L$, and the above discussion for the E_1 term remains valid. The only thing that we gain when assuming that X is a projective variety is the degeneracy of this spectral sequence at the E_2 term, by Theorem 3.3.1.

3.3.4 The Orlik-Solomon model and the main theorem

We restate the results of the previous paragraph. Let X be a smooth projective variety and L a hypersurface arrangement in X. Let us define

$$ M_q^n(X, L) = \bigoplus_{S \in \mathscr{S}_{q-n}(L)} H^{2n-q}(S)(n-q) \otimes A_S(L) $$

viewed as a Hodge structure of weight q.

1. We have a *product*
$$ M_q^n(X, L) \otimes M_{q'}^{n'}(X, L) \to M_{q+q'}^{n+n'}(X, L). \tag{3.13} $$
obtained by tensoring the product morphisms (3.5)
$$ A_S(L) \otimes A_{S'}(L) \to A_T(L) $$
with the morphisms
$$ H^{2n-q}(S) \otimes H^{2n'-q'}(S') \to H^{2n-q}(T) \otimes H^{2n'-q'}(T) \overset{\cup}{\to} H^{2(n+n')-(q+q')}(T) $$
multiplied by the sign $(-1)^{(q-n)q'}$. The above morphism is the composition of the restriction morphisms for the inclusion of T inside S and S', followed by the cup-product on T.

2. We have a *differential*
$$ d : M_q^n(X, L) \to M_q^{n+1}(X, L). \tag{3.14} $$
Let $S \subset S'$ be an inclusion of strata of L with $\mathrm{codim}(S) = q-n$ and $\mathrm{codim}(S') = q-(n+1)$. Then the component of the differential (3.14) indexed by S and S' is obtained by tensoring the natural morphism (3.6)
$$ A_S(L) \to A_{S'}(L) $$
with the Gysin morphism
$$ H^{2n-q}(S)(n-q) \to H^{2n-q+2}(S')(n-q+1) $$
multiplied by the sign $(-1)^q$. The other components of the differential (3.14) are zero.

3. Let X' be another smooth projectiver variety, L' be a hypersurface arrangement in X' and $\varphi : X \to X'$ a holomorphic map such that $\varphi^{-1}(L') \subset L$. Then we define a map

$$M^\bullet(\varphi) : M^\bullet(X', L') \to M^\bullet(X, L). \tag{3.15}$$

Let S and S' be strata of codimension $q-n$ respectively of L and L' such that $\varphi(S) \subset S'$, and let $\varphi_{S,S'} : S \to S'$ be the restriction of φ. Then the component of $M_q^n(\varphi)$ indexed by S and S' is obtained by tensoring the morphism (3.8)

$$A_{S,S'}(\varphi) : A_{S'}(L') \to A_S(L)$$

with the pull-back morphism

$$\varphi_{S,S'}^* : H^{2n-q}(S') \to H^{2n-q}(S).$$

The other components of $M^\bullet(\varphi)$ are zero.

In the next theorem, a *split mixed Hodge structure* is a mixed Hodge structure that is a direct sum of pure Hodge structures.

Recall that a graded algebra $B = \oplus_{n \geqslant 0} B_n$ is said to be graded-commutative if for homogeneous elements x and x' in B we have $xx' = (-1)^{|x||x'|} x'x$.

Theorem 3.3.8. *Let X be a smooth projective variety over $\mathbb{C}$ and L a hypersurface arrangement in X.*

1. *The direct sum $M^\bullet(X, L) = \bigoplus_q M_q^\bullet(X, L)$ is a graded-commutative differential graded algebra in the category of split mixed Hodge structures. It is functorial with respect to (X, L), using (3.15).*

2. *We have isomorphisms of algebras in the category of split mixed Hodge structures:*

$$\mathrm{gr}^W H^\bullet(X \setminus L) \cong H^\bullet(M^\bullet(X, L)).$$

They are functorial with respect to (X, L).

We call $M^\bullet(X, L)$ the *Orlik-Solomon model* of the pair (X, L).

Proof of Theorem 3.3.8. 1. Note that we have multiplied the differential by -1 for a matter of convenience; this gives an isomorphic differential graded algebra. The assertion is a consequence of the previous paragraph (Propositions 3.3.5, 3.3.6 and 3.3.4).

2. The isomorphism is simply, after the change of variables $n = -p + q$, the fact that the spectral sequence ${}_{\mathrm{w}}E_r^{p,q}$ degenerates at E_2 and converges to the cohomology of $X \setminus L$:

$$H^p({}_{\mathrm{w}}E_1^{-\bullet,q}) \cong \mathrm{gr}_q^W H^{-p+q}(X \setminus L).$$

□

Remark 3.3.9. Under the assumption (3.7), we may give a presentation of the Orlik-Solomon model that is more suitable in certain situations. For S a stratum of L and $I \subset \{1, \ldots, l\}$ an independent subset such that $L_I = S$, we have a monomial $e_I \in A_S(L)$. If we identify $H^{2n-q}(S) \otimes \mathbb{Q}e_I = H^{2n-q}(L_I)$, then we see that $M_q^n(X, L)$ is the quotient of

$$\bigoplus_{\substack{|I|=q-n \\ I \text{ indep.}}} H^{2n-q}(L_I)(n-q)$$

by the sub-vector space spanned by the images of the morphisms

$$H^{2n-q}(L_{I'}) \to \bigoplus_{\substack{i\in I' \\ I'\setminus\{i\} \text{ indep.}}} H^{2n-q}(L_{I'\setminus\{i\}})$$

for I' dependent. The above morphism the alternate sum of identity morphisms (if I' is dependent and $I' \setminus \{i\}$ is independent, then $L_{I'\setminus\{i\}} = L_{I'}$ for dimension reasons).

3.4 Wonderful compactifications and the Orlik-Solomon model

3.4.1 Hypersurface arrangements and wonderful compactifications

Definition 3.4.1. Let $L = \{L_1, \ldots, L_l\}$ be a hyperplane arrangement in $\mathbb{C}^n$, Z a stratum of L. We say that Z is a *good stratum* if there exists coordinates $(z_1, \ldots, z_n)$ on $\mathbb{C}^n$ such that $Z = \{z_1 = \ldots = z_r = 0\}$ for some r, and for each $i = 1, \ldots, l$, L_i is either of the type $\{a_1z_1 + \ldots + a_rz_r = 0\}$ or of the type $\{a_{r+1}z_{r+1} + \ldots + a_nz_n = 0\}$.

Example 3.4.2. In $\mathbb{C}^3$, let $L_1 = \{x = 0\}$, $L_2 = \{y = 0\}$, $L_3 = \{z = 0\}$, $L_4 = \{x = y\}$. Then the stratum $\{x = y = 0\}$ is good, but the stratum $\{x = z = 0\}$ is not.

Let $L = \{L_1, \ldots, L_l\}$ be a hypersurface arrangement in a complex manifold X, Z a stratum of L. We say that Z is a *good stratum* if in every local chart where the L_i's are hyperplanes, it is a good stratum in the sense of the above definition. A stratum of dimension 0 (a point) is always good. In the case of a normal crossing divisor, all non-empty strata are good.

Lemma 3.4.3. *Let $L = \{L_1, \ldots, L_l\}$ be a hypersurface arrangement in a complex manifold X, Z a good stratum of L, and*

$$\pi : \widetilde{X} \to X$$

the blow-up of X along Z. Let $E = \pi^{-1}(Z)$ be the exceptional divisor, and for all i, let $\widetilde{L}_i$ be the strict transform of L_i. Then $\widetilde{L} = \{E, \widetilde{L}_1, \ldots, \widetilde{L}_l\}$ is a hypersurface arrangement in $\widetilde{X}$.

Proof. It is enough to do the proof for $X = \Delta^n$ and the L_i's hyperplanes. We choose coordinates $(z_1, \ldots, z_n)$ as in Definition 3.4.1.

We have r natural local charts $\widetilde{X}_k \cong \Delta^n$ on $\widetilde{X}$, $k = 1, \ldots, r$. On the chart $\widetilde{X}_k$, the blow-up morphism is given by

$$\pi(z_1, \ldots, z_n) = (z_1z_k, \ldots, z_{k-1}z_k, z_k, z_{k+1}z_k, \ldots, z_rz_k, z_{r+1}, \ldots, z_n)$$

In this chart, E is defined by the equation $z_k = 0$. The strict transform of a hyperplane of the type $\{a_1z_1 + \ldots + a_rz_r = 0\}$ is given by the equation $a_1z_1 + \ldots + a_{k-1}z_{k-1} + a_k + a_{k+1}z_{k+1} + \ldots + a_rz_r = 0$. The strict transform of a hyperplane of the type $\{a_{r+1}z_{r+1} + \ldots + a_nz_n = 0\}$ is defined by the same equation.

To sum up, in the chart $\widetilde{X}_k$, all the hypersurfaces of $\widetilde{L}$ are given by affine equations. Up to some translations, we can then find smaller charts where all the equations are linear. This completes the proof. □

With the notations of the above lemma, we will simply write that

$$\pi : (\widetilde{X}, \widetilde{L}) \to (X, L)$$

is the blow-up of the pair (X, L) along the good stratum Z. We stress the fact that $\widetilde{L}$ is the hypersurface arrangement consisting of the exceptional divisor E and all the proper transforms $\widetilde{L}_i$ of the hypersurfaces L_i.

The blow-ups along good strata are enough to resolve the singularities of a hypersurface arrangement, as the following theorem shows. It is simply a reformulation of classical results on "wonderful compactifications" (see [FM94, DCP95, Hu03, Li09]).

Theorem 3.4.4. *Let L be a hypersurface arrangement in a complex manifold X. There exists a sequence*

$$(\widetilde{X},\widetilde{L})=(X^{(N)},L^{(N)})\xrightarrow{\pi_N}(X^{(N-1)},L^{(N-1)})\xrightarrow{\pi_{N-1}}\cdots\xrightarrow{\pi_1}(X^{(0)},L^{(0)})=(X,L)$$

where

1. *for all k, $X^{(k)}$ is a complex manifold and $L^{(k)}$ a hypersurface arrangement in $X^{(k)}$*
2. *for all k, $\pi_k:(X^{(k)},L^{(k)})\to(X^{(k-1)},L^{(k-1)})$ is the blow-up of $(X^{(k-1)},L^{(k-1)})$ along a good stratum of $L^{(k-1)}$*
3. *$\widetilde{L}$ is a normal crossing divisor in $\widetilde{X}$.*

Proof. An arrangement of hypersurfaces defines an arrangement of subvarieties in the sense of [Li09]. Let us fix a building set $\mathcal{G}$ and let $\pi:\widetilde{X}\to X$ be the corresponding wonderful compactification, with $\widetilde{L}=\pi^{-1}(L)$. Then according to [Li09], π is a composition of blow-ups along a minimal element of a building set. It simply remains to prove that a minimal element of a building set is a good stratum. We work in the cotangent spaces, hence reducing to a statement of linear algebra.

Let $\mathcal{G}$ be a building set of an arrangement of subspaces $\mathcal{C}$ in the context of [DCP95], and let us write $M=\sum_{C\in\mathcal{C}}C$. We have a $\mathcal{G}$-decomposition

$$M=G_1\oplus\cdots\oplus G_r$$

where the $G_i\in\mathcal{G}$ are the maximal elements. Let $X\in\mathcal{C}$ be any element, then $X\subset M$ and by definition of a building set $X\subset G_i$ for some unique $i=1,\ldots,r$. Hence if we write $U_i=\bigoplus_{j\neq i}G_j$, we then have, for all $X\in\mathcal{C}$, $X\not\subset G_i\Rightarrow X\subset U_i$. □

3.4.2 Functoriality of the Orlik-Solomon model with respect to blow-ups

Let us consider a sequence of blow-ups along good strata as in Theorem 3.4.4:

$$(\widetilde{X},\widetilde{L})=(X^{(N)},L^{(N)})\xrightarrow{\pi_N}(X^{(N-1)},L^{(N-1)})\xrightarrow{\pi_{N-1}}\cdots\xrightarrow{\pi_1}(X^{(0)},L^{(0)})=(X,L).$$

Then by the functoriality of the Orlik-Solomon model we get a sequence of morphisms of differential graded algebras (in the category of split mixed Hodge structures):

$$M^\bullet(X,L)=M^\bullet(X^{(0)},L^{(0)})\overset{M^\bullet(\pi_1)}{\xrightarrow{\sim}}\cdots\overset{M^\bullet(\pi_N)}{\xrightarrow{\sim}}M^\bullet(X^{(N)},L^{(N)})=M^\bullet(\widetilde{X},\widetilde{L}).$$

For each k, $M^\bullet(\pi_k)$ is a quasi-isomorphism since π_k induces an isomorphism $X^{(k)}\setminus L^{(k)}\xrightarrow{\cong}X^{(k-1)}\setminus L^{(k-1)}$. Thus we get a natural quasi-isomorphism between the Orlik-Solomon model of (X,L) and that of $(\widetilde{X},\widetilde{L})$.

In the following theorem, we give explicit formulas in the case of a single blow-up. For simplicity, we work under the assumption (3.7) and use the presentation of the Orlik-Solomon model given in Remark 3.3.9.

Theorem 3.4.5. *Let X be a smooth projective variety over $\mathbb{C}$ and L a hypersurface arrangement in X such that the assumption (3.7) is satisfied. Let Z be a good stratum of L and*

$$\pi:(\widetilde{X},\widetilde{L})\to(X,L)$$

the blow-up of (X, L) along Z. Let

$$M^\bullet(\pi) : M^\bullet(X, L) \to M^\bullet(\widetilde{X}, \widetilde{L})$$

be the morphism induced by π on the Orlik-Solomon models. Then

1. *$M^\bullet(\pi)$ is a quasi-isomorphism.*
2. *the components of $M_q^n(\pi)$ are given, for $I = \{i_1 < \cdots < i_{q-n}\}$ independent, by*
   (a) *the pull-back morphism $H^{2n-q}(L_I) \xrightarrow{\pi^*} H^{2n-q}(\widetilde{L}_I)$.*
   (b) *for all s such that $Z \subset L_{i_s}$, the morphism $H^{2n-q}(L_I) \to H^{2n-q}(E \cap \widetilde{L}_{I\setminus\{i_s\}})$ which is the pull-back morphism corresponding to $E \cap \widetilde{L}_{I\setminus\{i_s\}} \xrightarrow{\pi} Z \cap L_{I\setminus\{i_s\}} = Z \cap L_I \hookrightarrow L_I$, multiplied by the sign $(-1)^{s-1}$.*

Proof. 1. This is obvious by Theorem 3.3.8, since π induces an isomorphism $\widetilde{X} \setminus \widetilde{L} \xrightarrow{\cong} X \setminus L$.

2. It is a consequence of the general formula for functoriality given in §3.3.4. Using the notation $E = \widetilde{L}_0$, a local computation shows that we have the following formula for $A_\bullet(\pi) : A_\bullet(L) \to A_\bullet(\widetilde{L})$.

$$A_1(\pi)(e_i) = \begin{cases} e_i & \text{if } L_i \text{ does not contain } S \\ e_0 + e_i & \text{if } L_i \text{ contains } S \end{cases}$$

Thus we get

$$A_\bullet(\pi)(e_I) = e_I + \sum_{\substack{1 \leqslant s \leqslant q-n \\ Z \subset L_{i_s}}} (-1)^{s-1} e_0 \wedge e_{I\setminus\{i_s\}}$$

and the claim follows. □

3.5 Configuration spaces of points on curves

3.5.1 Configuration spaces associated to graphs

Let Y be a compact Riemann surface, i.e. a smooth projective complex curve. Let Γ be a finite unoriented graph with no multiple edges and no self-loops, with V its set of vertices and E its set of edges. Let Y^V be the cartesian power of Y indexed by V, with coordinates y_v. For $v \in V$, we have a projection

$$p_v : Y^V \to Y.$$

Every edge $e \in E$ with endpoints v and v' defines a diagonal $\Delta_e = \{y_v = y_{v'}\} \subset Y^V$ which is the locus where the coordinates corresponding to the two endpoints of e are equal. We define $\Delta_\Gamma = \bigcup_{e \in E} \Delta_e$ and then the configuration space of points on Y associated to Γ:

$$C(Y, \Gamma) = Y^V \setminus \Delta_\Gamma.$$

In the case where $\Gamma = K_n$ is the complete graph on n vertices, we recover the configuration space

$$C(Y, n) = \{(y_1, \ldots, y_n) \in Y^n \mid y_i \neq y_j \text{ for } i \neq j\} = Y^n \setminus \bigcup_{i<j} \Delta_{i,j}.$$

3.5.2 A model for the cohomology

In [Kri94] and [Tot96], I. Kriz and B. Totaro independently found a model for the cohomology of $C(Y,n)$. Their result has been recently generalized to $C(Y,\Gamma)$ by S. Bloch in [Blo12] (even though Bloch's framework is slightly more general, with external edges in Γ labeled by points of Y). We recall the definition of this model. Here Y has dimension 1, but the general definition is similar.

If $B=\oplus_{n\geqslant 0}B_n$ is a graded-commutative graded algebra and $\{x_\alpha\}$ are indeterminates with prescribed degrees $\{d_\alpha\}$, then there is a well-defined notion of graded-commutative algebra generated by the x_α's over B. This is a graded-commutative graded algebra which is the quotient of $B[\{x_\alpha\}]$ by the relations $bx_\alpha=(-1)^{|b|d_\alpha}x_\alpha b$ for b homogeneous, and $x_\beta x_\alpha=(-1)^{d_\alpha d_\beta}x_\alpha x_\beta$ for all α and β. For example, if B is a field concentrated in degree 0 then we recover the exterior algebra generated by the x_α's. We use the wedge notation $x_\alpha\wedge x_\beta$ to remember the graded-commutativity property.

Let us define, following [Blo12], a graded-commutative differential graded algebra $N^\bullet(Y,\Gamma)$ in the following way. It is generated (as a graded-commutative algebra) by the cohomology $H^\bullet(Y^V)$ and elements G_e in degree 1 for every edge $e\in E$, modulo the relations:

(R1) $p_v^*(c)G_e=p_{v'}^*(c)G_e$ for every class $c\in H^\bullet(Y)$, where v and v' are the endpoints of e in Γ.

(R2) $\sum_{i=1}^r(-1)^{i-1}G_{e_1}\wedge\cdots\wedge\widehat{G_{e_i}}\wedge\cdots\wedge G_{e_r}=0$ if $\{e_1,\ldots,e_r\}\subset E$ contains a loop.

We now define a differential d on $N^\bullet(Y,\Gamma)$ as zero on $H^\bullet(Y^V)$ and given on the elements G_e by the formula

$$d(G_e)=[\Delta_e]\in H^2(Y^V).$$

One shows that d is well-defined and makes $N^\bullet(Y,\Gamma)$ into a graded-commutative differential graded algebra.

3.5.3 The isomorphism with the Orlik-Solomon model

By choosing charts on Y, one easily sees that $L=\Delta_\Gamma$ is a hypersurface arrangement in $X=Y^V$. Thus theorem 3.3.8 can be applied to the pair (Y^V,Δ_Γ) and gives a model for the cohomology of $C(Y,\Gamma)=Y^V\setminus\Delta_\Gamma$. We fix an linear order on the set E of edges of Γ, hence on the irreducible components Δ_e of Δ_Γ. This allows us to consider the Orlik-Solomon model $M^\bullet(Y^V,\Delta_\Gamma)$, with its presentation given by Remark 3.3.9. Thus $M_q^n(Y^V,\Delta_\Gamma)$ is a quotient of

$$\bigoplus_{\substack{I\subset E\\ |I|=q-n\\ I\text{ indep.}}} H^{2n-q}(\Delta_I)(n-q).$$

We note that a subset $I\subset E$ is dependent if and only if it contains a loop, and is a circuit if and only if it is a simple loop.

We define a morphism of differential graded algebras

$$\alpha: N^\bullet(Y,\Gamma)\to M^\bullet(Y^V,\Delta_\Gamma)$$

in the following way.
First we note that for all n we have $M_n^n(Y^V,\Delta_\Gamma)=H^n(Y^V)$, and we easily see that the resulting (injective) map $H^\bullet(Y^V)\to M^\bullet(Y^V,\Delta_\Gamma)$ is a map of graded algebras. Then we define $\alpha(G_e)$ to be a generator g_e of $H^0(\Delta_e)(-1)\subset M_2^1(Y^V,\Delta_\Gamma)$.

Lemma 3.5.1. *The morphism α is well-defined and compatible with the differentials. It is thus a map of differential graded algebras.*

Proof. First we show that α respects the relations (R1) and (R2). For the relation (R1) we see that by definition

$$\alpha(p_v^*(c)G_e) = p_v^*(c)g_e = p_v^*(c)_{|\Delta_e} \in H^\bullet(\Delta_e).$$

This equals $i_e^*(p_v^*(c)) = (p_v \circ i_e)^*(c)$ where $i_e : \Delta_e \hookrightarrow Y^V$ is the inclusion of Δ_e. The relation then follows from the equality $p_v \circ i_e = p_{v'} \circ i_e$.
For the relation (R2) we can assume that we have $e_1 < \cdots < e_r$. Then if R is the expression in the relation (R2) we have

$$\alpha(R) = \sum_{i=1}^{r} (-1)^{i-1} g_{e_1} \cdots \widehat{g_{e_i}} \cdots g_{e_r}$$

and $g_{e_1} \cdots \widehat{g_{e_i}} \cdots g_{e_r}$ is a generator of $H^0(\Delta_{e_1} \cap \cdots \cap \widehat{\Delta_{e_i}} \cap \cdots \cap \Delta_{e_r})(-r+1)$. Since $\{\Delta_{e_1}, \ldots, \Delta_{e_r}\}$ is dependent, $\alpha(R)$ is thus killed by the quotient that defines $M^\bullet(Y^V, \Delta_\Gamma)$.
We then show that α is compatible with the differentials. By definition, the differential is zero on $H^\bullet(Y^V) \subset M^\bullet(Y^V, \Delta_\Gamma)$. Furthermore, $d\alpha(G_e) = d(g_e)$ is, by definition of the Gysin morphism, the class of Δ_e in $H^2(Y^V)$. This completes the proof. □

Theorem 3.5.2. *The morphism $\alpha : N^\bullet(Y, \Gamma) \to M^\bullet(Y^V, \Delta_\Gamma)$ is an isomorphism of differential graded algebras.*

Proof. We sketch the proof and leave the details to the reader. We define the inverse morphism β in the following way. Let $I \subset E$ be an independent set of edges of Γ of cardinality $|I| = q - n$, let $i_I : \Delta_I \hookrightarrow Y^V$ be the inclusion of the corresponding stratum. Let $f_I : Y^V \to \Delta_I$ be any natural splitting of i_I defined out of projections p_v's. Then we define the component of β:

$$\beta_q^n : H^{2n-q}(\Delta_I) \to H^{2n-q}(Y^V)G_I$$

to be the pull-back f_I^*. The degrees match since $H^{2n-q}(Y^V)G_I$ is in degree $2n - q + |I| = n$. It remains to prove that β passes to the quotient that defines $M^\bullet(Y^V, \Delta_\Gamma)$, and defines an inverse to α. □

Remark 3.5.3. It is striking that Kriz and Totaro's model works for configuration spaces of points on any smooth projective variety Y, where the diagonals can have any codimension. It is then tempting to ask for a generalization of the Orlik-Solomon model to the cohomology of $X \setminus L$ where $L \subset X$ locally looks like a union of sub-vector spaces of any codimension inside $\mathbb{C}^n$. In [Tot96], B. Totaro suggests a particular case of the previous question, focusing on vector spaces V_i of a fixed codimension c such that all intersections $V_{i_1} \cap \ldots \cap V_{i_r}$ have codimension a multiple of c (this chapter handles the case $c = 1$).

3.5.4 Comparison with Kriz's quasi-isomorphism

In this paragraph we sketch the proof that Kriz's quasi-isomorphism φ from [Kri94] can be recovered as a consequence of the functoriality of the Orlik-Solomon model.
For the sake of comfort we use the notations from [Kri94] and write $E^\bullet(n)$ for $N^\bullet(Y, K_n)$ where K_n is the complete graph on n vertices. We write $\Delta = \Delta_{K_n}$ for the union of all diagonals of Y^n. According to Theorem 3.5.2, we have an isomorphism of differential graded algebras

$$\alpha : E^\bullet(n) \overset{\cong}{\to} M^\bullet(Y^n, \Delta).$$

Let $\pi : Y[n] \to Y^n$ be the Fulton-MacPherson wonderful compactification [FM94]. Then $D = \pi^{-1}(\Delta)$ is a simple normal crossing divisor whose irreducible components $D(S)$ are indexed by subsets $S \subset \{1, \ldots, n\}$ with $|S| \geqslant 2$. We now describe the model $F^\bullet(n)$ defined by Kriz. By its very definition [FM94, §6], we have a natural isomorphism of differential graded algebras

$$\varepsilon : F^\bullet(n) \xrightarrow{\cong} M^\bullet(Y[n], D)$$

between $F^\bullet(n)$ and the Orlik-Solomon model $M^\bullet(Y[n], D)$. To make this isomorphism precise, let us mention that

- on $H^\bullet(Y^n)$, ε is the pull-back $H^\bullet(\pi) : H^\bullet(Y^n) \to H^\bullet(Y[n])$;
- $\varepsilon(S)$ is the generator $g_S \in H^0(D(S))(-1)$ and $\varepsilon(D_S)$ is the class $[D(S)] \in H^2(Y[n])$.

Theorem 3.5.4. *We have a commutative square*

$$\begin{tikzcd} F^\bullet(n) \arrow[r, "\varepsilon"] & M^\bullet(Y[n], D) \\ E^\bullet(n) \arrow[u, "\varphi"] \arrow[r, "\alpha"'] & M^\bullet(Y^n, \Delta) \arrow[u, "M^\bullet(\pi)"'] \end{tikzcd}$$

where φ is defined in [Kri94, §3], the horizontal arrows are isomorphisms of differential graded algebras and the vertical arrows are quasi-isomorphisms of differential graded algebras.

Proof. It only remains to prove that we have

$$M^1(\pi)(g_{a,b}) = \sum_{S \supset \{a,b\}} g_S.$$

We do the proof in the case $n = 3$ (the cases $n < 3$ being trivial) and leave the general case to the reader. We may assume that $\{a, b\} = \{1, 2\}$. Then π is simply the blow-up along $\Delta_{1,2,3}$, $D(1, 2, 3)$ is the exceptional divisor, and the equality $M^1(\pi)(g_{1,2}) = g_{1,2} + g_{1,2,3}$ is a consequence of Theorem 3.4.5. $\square$

Chapter 4

The motive of a bi-arrangement

In §4.1 we introduce the formalism of bi-arrangements and Orlik-Solomon bi-complexes as a generalization of the Orlik-Solomon algebra of an arrangement. In §4.2 we define deletion and restriction of bi-arrangements and prove the existence of a deletion-restriction short exact sequence in the context of exact bi-arrangements. In §4.3 we introduce bi-arrangements of hypersurfaces in a complex manifold. We define the motive of a bi-arrangement of hypersurfaces and study the behaviour of the Orlik-Solomon bi-complexes with respect to blow-ups. In §4.4 we define the geometric Orlik-Solomon bi-complex of a bi-arrangement of hypersurfaces, and study its behaviour with respect to blow-up. We state the main theorem of this chapter (Theorem 4.4.11). In §4.5 we study the particular case of projective bi-arrangements, with an emphasis on bi-arrangements coming from multiple zeta values. In §4.6, which is quite technical, we prove Theorem 4.4.11. The last two sections are appendices. In §4.7 we recall some more or less well-known facts on relative cohomology in the case of normal crossing divisors. In §4.8 we state a collection of cohomological identities related to Chern classes and blow-ups. They are used in the proof of the main theorem.

4.1 The Orlik-Solomon bi-complex of a bi-arrangement of hyperplanes

4.1.1 The Orlik-Solomon algebra of an arrangement of hyperplanes

Here we recall a few definitions and notations from the theory of arrangements of hyperplanes. We refer the reader to the classical book [OT92] for more details.

Definitions and notations

An *arrangement of hyperplanes* (or simply an *arrangement*) $\mathscr{A}$ in $\mathbb{C}^n$ is a finite set of hyperplanes of $\mathbb{C}^n$ that pass through the origin. Let us write $\mathscr{A} = \{K_1, \ldots, K_k\}$. For $i = 1, \ldots, k$, we may write $K_i = \{f_i = 0\}$ where f_i is a non-zero linear form on $\mathbb{C}^n$.

If $\mathscr{A}'$ is an arrangement in $\mathbb{C}^{n'}$ and $\mathscr{A}''$ is an arrangement in $\mathbb{C}^{n''}$, then we define their *product* $\mathscr{A} = \mathscr{A}' \times \mathscr{A}''$, which is the arrangement in $\mathbb{C}^{n'+n''}$ consisting of the hyperplanes $K' \times \mathbb{C}^{n''}$, for $K' \in \mathscr{A}'$, and $\mathbb{C}^{n'} \times K''$, for $K'' \in \mathscr{A}''$.

A *stratum* of $\mathscr{A}$ is an intersection $K_I = \bigcap_{i \in I} K_i$ of some of the K_i's, for $I \subset \{1, \ldots, k\}$. By convention, we have $K_\varnothing = \mathbb{C}^n$, and all other strata are called *strict*. We write $\mathscr{S}_m(\mathscr{A})$ for the set of strata of $\mathscr{A}$ of codimension m, $\mathscr{S}(\mathscr{A}) = \bigsqcup_{m \geqslant 0} \mathscr{S}_m(\mathscr{A})$ for the set of all strata of $\mathscr{A}$ and $\mathscr{S}_+(\mathscr{A}) = \bigsqcup_{m > 0} \mathscr{S}_m(\mathscr{A})$ for the set of strict strata of $\mathscr{A}$.

It is classical to view the set of strata as a poset ordered by reverse inclusion. For S a stratum of $\mathscr{A}$, we write $\mathscr{A}^{\leqslant S}$ for the arrangement consisting of the hyperplanes that contain S.

Let us write $S^\perp \subset (\mathbb{C}^n)^\vee$ for the space of linear forms on $\mathbb{C}^n$ that vanish on a stratum S; it is spanned by the f_i's for i such that $S \subset K_i$. We say that a family of strata $S_1, \ldots, S_r$ *intersect transversely* and write $S_1 \pitchfork \cdots \pitchfork S_r$ if $S_1^\perp, \ldots, S_r^\perp$ form a direct sum in $\mathbb{C}^n$.

If S is a stratum of $\mathscr{A}$, a *decomposition* of S is an equality $S = S_1 \pitchfork \cdots \pitchfork S_r$ with the S_j's strata of $\mathscr{A}$, and such that for every hyperplane K_i that contains S, K_i contains some S_j. Dually, this amounts to saying that we can write $S^\perp = (S_1)^\perp \oplus \cdots \oplus (S_r)^\perp$ such that every $f_i \in S^\perp$ is in some $(S_j)^\perp$. Equivalently, we have a product decomposition $\mathscr{A}^{\leqslant S} \cong \mathscr{A}^{\leqslant S_1} \times \cdots \times \mathscr{A}^{\leqslant S_r}$. We say that S is *reducible* if it has a non-trivial decomposition, i.e. with all S_j's strict strata, and *irreducible* otherwise. Every $K \in \mathscr{A}$ is irreducible. A stratum S has a unique decomposition $S = S_1 \pitchfork \cdots \pitchfork S_r$ with the S_j's irreducible.

The Orlik-Solomon algebra

Let $\mathscr{A} = \{K_1, \ldots, K_k\}$ be an arrangement of hyperplanes in $\mathbb{C}^n$. Let $E_\bullet(\mathscr{A}) = \Lambda^\bullet(e_1, \ldots, e_k)$ be the exterior algebra on generators e_i, $i = 1, \ldots, k$ in degree 1. For $I = \{i_1 < \cdots < i_r\} \subset \{1, \ldots, k\}$ we write $e_I = e_{i_1} \wedge \cdots \wedge e_{i_r}$ for the corresponding basis element of $E_r(\mathscr{A})$, with the convention $e_\varnothing = 1$. Let $d : E_\bullet(\mathscr{A}) \to E_{\bullet-1}(\mathscr{A})$ be the unique derivation of $E_\bullet(\mathscr{A})$ such that $d(e_i) = 1$ for all i. It is given by

$$d(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{j=1}^{r} (-1)^{j-1} e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_r}.$$

A subset $I \subset \{1, \ldots, k\}$ is said to be *dependent* if the hyperplanes K_i, for $i \in I$, are linearly dependent, and *independent* otherwise. A *circuit* of $\mathscr{A}$ is a minimally dependent subset. Let $R_\bullet(\mathscr{A})$ be the homogeneous ideal of $E_\bullet(\mathscr{A})$ generated by the elements $d(e_I)$ for I dependent. The Leibniz rule implies that it is generated by the elements $d(e_I)$ for I a circuit.

The *Orlik-Solomon algebra* of $\mathscr{A}$ is the quotient $A_\bullet(\mathscr{A}) = E_\bullet(\mathscr{A})/R_\bullet(\mathscr{A})$. It is a differential graded algebra which is easily seen to be exact if $\mathscr{A}$ is non-empty, a contracting homotopy $h : A_\bullet(\mathscr{A}) \to A_{\bullet+1}(\mathscr{A})$ being given by $h(x) = e_1 \wedge x$. An important feature of the Orlik-Solomon algebra is the following direct sum decomposition with respect to the set of strata:

$$A_r(\mathscr{A}) = \bigoplus_{S \in \mathscr{S}_r(\mathscr{A})} A_r^S(\mathscr{A})$$

where $A_r^S(\mathscr{A})$ is spanned by the classes of the elements e_I for I such that $K_I = S$.

We will write $S \overset{m}{\hookrightarrow} T$ for an inclusion of strata of codimension m; for an inclusion $S \overset{1}{\hookrightarrow} T$, we then have a component $d_{S,T} : A_r^S(\mathscr{A}) \to A_{r-1}^T(\mathscr{A})$ for the differential d.

If Σ is a strict stratum of $\mathscr{A}$ of codimension r, then the complex

$$0 \to A_r^\Sigma(\mathscr{A}) \xrightarrow{d} \bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A_{r-1}^S(\mathscr{A}) \xrightarrow{d} \bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A_{r-2}^T(\mathscr{A}) \xrightarrow{d} \cdots \xrightarrow{d} A_0^{\mathbb{C}^n}(\mathscr{A}) \to 0$$

is the Orlik-Solomon algebra of the arrangement $\mathscr{A}^{\leqslant \Sigma}$, hence is exact. This property allows one to uniquely define [Loo93, Lemma 2.2] the groups $A_r^S(\mathscr{A})$ and the differentials $d_{S,T}$ by induction on the codimension, starting with $A_0^{\mathbb{C}^n}(\mathscr{A}) = \mathbb{Q}$. We will use this inductive point of view to generalize this construction to bi-arrangements.

4.1.2 Bi-arrangements of hyperplanes

Definition 4.1.1. A *bi-arrangement of hyperplanes* (or simply a *bi-arrangement*) $\mathscr{B} = (\mathscr{A}, \chi)$ in $\mathbb{C}^n$ is the data of an arrangement of hyperplanes $\mathscr{A}$ in $\mathbb{C}^n$ along with a *coloring function*

$$\chi : \mathscr{S}_+(\mathscr{A}) \to \{\lambda, \mu\}$$

on the strict strata of $\mathscr{A}$, such that the *Künneth condition* is satisfied:

$$\text{for any non-trivial decomposition } S = S' \pitchfork S'',\ \chi(S) = \chi(S') \text{ or } \chi(S) = \chi(S''). \tag{4.1}$$

Remark 4.1.2. The Künneth condition is empty if $\chi(S') \neq \chi(S'')$. More generally, let $S = S_1 \pitchfork \cdots \pitchfork S_r$ be the decomposition of a strict stratum S into irreducible strata S_k. If $\chi(S_1) = \cdots = \chi(S_r)$, then the Künneth condition forces $\chi(S) = \chi(S_1) = \cdots = \chi(S_r)$. Otherwise, $\chi(S)$ is not constrained by the definition of a bi-arrangement of hyperplanes. To sum up, a coloring function that satisfies the Künneth condition is uniquely determined by

– the colors of the irreducible strata;

– the colors of the strata $S = S_1 \pitchfork \cdots \pitchfork S_r$ with the S_k's irreducible which do not all have the same color.

For all our purposes, only the colors of the irreducible strata will matter, thus we make the following definition.

Definition 4.1.3. Two bi-arrangements are *equivalent* if their underlying arrangements are the same and if their coloring functions agree on the irreducible strata.

In most of the article, we will implicitly consider bi-arrangements up to this equivalence relation. In particular, we will allow ourselves to define a bi-arrangement by only specifying the colors of the irreducible strata.

Remark 4.1.4. The hyperplanes $L \in \mathscr{A}$ such that $\chi(L) = \lambda$ (resp. the hyperplanes $M \in \mathscr{A}$ such that $\chi(M) = \mu$) form an arrangement denoted by $\mathscr{L}$ (resp. $\mathscr{M}$). In most geometric situations (see §1.5.1) these two arrangements play very different roles, hence the union $\mathscr{A} = \mathscr{L} \sqcup \mathscr{M}$ is an artificial object. In other words, one should not view a bi-arrangement as an arrangement with some coloring datum, but as *two* arrangements with some coloring datum. To emphasize this point, we will use the following notational conventions.

Notation 4.1.5. We will sometimes denote a bi-arrangement $\mathscr{B}$ in $\mathbb{C}^n$ by a triple $(\mathscr{L}, \mathscr{M}, \chi)$, where $\mathscr{L}$ and $\mathscr{M}$ are two disjoint arrangements in $\mathbb{C}^n$, and $\chi : \mathscr{S}_+(\mathscr{L} \sqcup \mathscr{M}) \to \{\lambda, \mu\}$ is a function that satisfies $\chi(L) = \lambda$ for $L \in \mathscr{L}$, $\chi(M) = \mu$ for $M \in \mathscr{M}$, and the Künneth condition (4.1).

Notation 4.1.6. For $\mathscr{B} = (\mathscr{A}, \chi)$ a bi-arrangement, we will often make an abuse of notation and simply write $K \in \mathscr{B}$ for $K \in \mathscr{A}$, $S \in \mathscr{S}(\mathscr{B})$ for $S \in \mathscr{S}(\mathscr{A})$, and so on.

We will make great use of a natural involution on bi-arrangements.

Definition 4.1.7. The *dual* of a bi-arrangement $\mathscr{B} = (\mathscr{A}, \chi)$ is the bi-arrangement $\mathscr{B}^\vee = (\mathscr{A}, \chi^\vee)$ where $\chi^\vee$ is the composition of χ with the involution $\lambda \leftrightarrow \mu$. Equivalently, the dual of $\mathscr{B} = (\mathscr{L}, \mathscr{M}, \chi)$ is $\mathscr{B}^\vee = (\mathscr{M}, \mathscr{L}, \chi^\vee)$. We have $(\mathscr{B}^\vee)^\vee = \mathscr{B}$.

We may also take product of bi-arrangements. This operation is only well-defined if we work up to equivalence (Definition 4.1.3).

Definition 4.1.8. If $\mathscr{B}' = (\mathscr{A}', \chi')$ is a bi-arrangement of hyperplanes in $\mathbb{C}^{n'}$ and $\mathscr{B}'' = (\mathscr{A}'', \chi'')$ is a bi-arrangement of hyperplanes in $\mathbb{C}^{n''}$, then we define their *product* $\mathscr{B} = \mathscr{B}' \times \mathscr{B}'' = (\mathscr{A}, \chi)$, whose underlying arrangement of hyperplanes is $\mathscr{A} = \mathscr{A}' \times \mathscr{A}''$. Its irreducible strata have the form $S' \times \mathbb{C}^{n''}$ or $\mathbb{C}^{n'} \times S''$ for S' (resp. S'') an irreducible stratum of $\mathscr{A}'$ (resp. $\mathscr{A}''$). We thus define the coloring by $\chi(S' \times \mathbb{C}^{n''}) = \chi'(S')$ and $\chi(\mathbb{C}^{n'} \times S'') = \chi''(S'')$.

Example 4.1.9. There are two (dual) ways in which an arrangement $\mathscr{A}$ may be viewed as a bi-arrangement: by defining the coloring χ to be constant equal to λ or μ. We will simply denote these bi-arrangements by $(\mathscr{A}, \lambda)$ and $(\mathscr{A}, \mu)$.

Example 4.1.10. By taking products, we can define bi-arrangements $(\mathscr{L},\lambda)\times(\mathscr{M},\mu)$. They are somewhat trivial examples since the arrangements $\mathscr{L}$ and $\mathscr{M}$ "do not mix".

Example 4.1.11. Let $\mathscr{L}$ and $\mathscr{M}$ be two disjoint arrangements in $\mathbb{C}^n$. We define the λ-*extreme* coloring e_λ and the μ-*extreme* coloring e_μ so that $(\mathscr{L},\mathscr{M},e_\lambda)$ and $(\mathscr{L},\mathscr{M},e_\mu)$ are bi-arrangements.

$$e_\lambda(S)=\begin{cases}\lambda & \text{if } S\subset L \text{ for some } L\in\mathscr{L}\\ \mu & \text{otherwise.}\end{cases}\qquad e_\mu(S)=\begin{cases}\mu & \text{if } S\subset M \text{ for some } M\in\mathscr{M}\\ \lambda & \text{otherwise.}\end{cases}$$

To understand the terminology, let us anticipate and note (see for instance Lemma 4.1.24 below) that we will be interested mostly in the bi-arrangements such that for every stratum S, there exists a hyperplane $K\supset S$ with the same color as S. The λ-extreme coloring (resp. the μ-extreme coloring) is extreme in the sense that we give the color λ (resp. the color μ) to as many strata as possible while staying in that class of bi-arrangements.

4.1.3 The formalism of Orlik-Solomon bi-complexes

The definition

Lemma 4.1.12. *Let $\mathscr{B}$ be a bi-arrangement in $\mathbb{C}^n$. There exists a unique datum of*

– *for all $i,j\geqslant 0$, for every stratum $S\in\mathscr{S}_{i+j}(\mathscr{B})$, a finite-dimensional $\mathbb{Q}$-vector space $A^S_{i,j}$;*

– *for every inclusion $S\overset{1}{\hookrightarrow}T$ of strata of codimension 1, linear maps*

$$d'_{S,T}:A^S_{i,j}\to A^T_{i-1,j}\quad\text{and}\quad d''_{S,T}:A^T_{i,j-1}\to A^S_{i,j};$$

such that the following conditions are satisfied:

– $A^{\mathbb{C}^n}_{0,0}=\mathbb{Q}$;

– *for every stratum Σ,*

$$A^{\leqslant\Sigma}_{\bullet,\bullet}=\left(\bigoplus_{S\supset\Sigma}A^S_{\bullet,\bullet},d',d''\right)$$

is a bi-complex, where d' and d'' respectively denote the collection of the maps $d'_{S,T}$ and $d''_{S,T}$ for $S\supset\Sigma$;

– *for every strict stratum $\Sigma\in\mathscr{S}_{i+j}(\mathscr{B})$ such that $\chi(\Sigma)=\lambda$, we have exact sequences*

$$0\to A^\Sigma_{i,j}\xrightarrow{d'}\bigoplus_{\Sigma\overset{1}{\hookrightarrow}S}A^S_{i-1,j}\xrightarrow{d'}\bigoplus_{\Sigma\overset{2}{\hookrightarrow}T}A^T_{i-2,j};$$

– *for every strict stratum $\Sigma\in\mathscr{S}_{i+j}(\mathscr{B})$ such that $\chi(\Sigma)=\mu$, we have exact sequences*

$$0\leftarrow A^\Sigma_{i,j}\xleftarrow{d''}\bigoplus_{\Sigma\overset{1}{\hookrightarrow}S}A^S_{i,j-1}\xleftarrow{d''}\bigoplus_{\Sigma\overset{2}{\hookrightarrow}T}A^T_{i,j-2}.$$

Proof. We define the bi-complexes $A^{\leqslant\Sigma}_{\bullet,\bullet}$ by induction on the codimension of Σ. The case of codimension 0 is forced by the equality $A^{\mathbb{C}^n}_{0,0}=\mathbb{Q}$. If Σ is a strict stratum and $\chi(\Sigma)=\lambda$ then one is forced to define

$$A^\Sigma_{i,j}=\ker\left(\bigoplus_{\Sigma\overset{1}{\hookrightarrow}S}A^S_{i-1,j}\xrightarrow{d'}\bigoplus_{\Sigma\overset{2}{\hookrightarrow}T}A^T_{i-2,j}\right)$$

and the differentials $d'_{\Sigma,S} : A^{\Sigma}_{i,j} \to A^{S}_{i-1,j}$ to be the components of the natural inclusion. This uniquely defines the differentials $d''_{\Sigma,S} : A^{S}_{i,j-1} \to A^{\Sigma}_{i,j}$ by filling the dotted arrow in the following commutative diagram.

$$\begin{tikzcd}
\bigoplus\limits_{\Sigma \overset{1}{\hookrightarrow} S} A^{S}_{i,j-1} \arrow[r] \arrow[d, dotted] & \bigoplus\limits_{\Sigma \overset{2}{\hookrightarrow} T} A^{T}_{i-1,j-1} \arrow[r] \arrow[d] & \bigoplus\limits_{\Sigma \overset{3}{\hookrightarrow} U} A^{U}_{i-2,j-1} \arrow[d] \\
A^{\Sigma}_{i,j} \arrow[r] & \bigoplus\limits_{\Sigma \overset{1}{\hookrightarrow} S} A^{S}_{i-1,j} \arrow[r] & \bigoplus\limits_{\Sigma \overset{2}{\hookrightarrow} T} A^{T}_{i,j-1}
\end{tikzcd}$$

The case $\chi(\Sigma) = \mu$ is dual, with the definition

$$A^{\Sigma}_{i,j} = \operatorname{coker}\left(\bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A^{T}_{i,j-2} \overset{d''}{\to} \bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A^{S}_{i,j-1} \right).$$

□

Definition 4.1.13. The above datum is called the *Orlik-Solomon bi-complex* of the bi-arrangement $\mathscr{B}$ and denoted by $A_{\bullet,\bullet}(\mathscr{B})$, or simply $A_{\bullet,\bullet}$ when the situation is clear.

Visually, we get a bi-complex that is defined inductively, starting in the top right corner and going in the bottom left direction.

$$\begin{tikzcd}
{} \arrow[r, dotted] & A_{3,0} \arrow[r] \arrow[d, dotted] & A_{2,0} \arrow[r] \arrow[d] & A_{1,0} \arrow[r] \arrow[d] & A_{0,0} \arrow[d] \\
& {} \arrow[r, dotted] & A_{2,1} \arrow[r] \arrow[d, dotted] & A_{1,1} \arrow[r] \arrow[d] & A_{0,1} \arrow[d] \\
& & {} \arrow[r, dotted] & A_{1,2} \arrow[r] \arrow[d, dotted] & A_{0,2} \arrow[d] \\
& & & {} \arrow[r, dotted] & A_{0,3} \arrow[d, dotted] \\
& & & & {}
\end{tikzcd}$$

Remark 4.1.14. The Orlik-Solomon bi-complex is a local object: the bi-complex $A^{\leqslant\Sigma}_{\bullet,\bullet}(\mathscr{B})$ is the Orlik-Solomon bi-complex of the bi-arrangement $\mathscr{B}^{\leqslant\Sigma}$ consisting of the hyperplanes that contain Σ.

Lemma 4.1.15. *Let $\mathscr{B}$ be an arrangement and $A_{\bullet,\bullet}$ be its Orlik-Solomon bi-complex. The fact that all $A^{\leqslant\Sigma}_{\bullet,\bullet}$ are bi-complexes is equivalent to the following identities.*

1. *For an inclusion $S \overset{2}{\hookrightarrow} U$ we have*

$$\sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookrightarrow} U} d'_{T,U} \circ d'_{S,T} = 0 \quad \textit{and} \quad \sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookrightarrow} U} d''_{S,T} \circ d''_{T,U} = 0.$$

2. (a) *Let $S \neq U$ be two strata of the same codimension such that there is no diagram $S \overset{1}{\hookrightarrow} T \overset{1}{\hookleftarrow} U$. Then for every diagram $S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} U$ we have*

$$d'_{R,U} \circ d''_{R,S} = 0.$$

(b) *Let $S \neq U$ be two strata of the same codimension such that there is a diagram $S \overset{1}{\hookrightarrow} T \overset{1}{\hookleftarrow} U$. Then we necessarily have $T = S + U$ and there is a unique diagram $S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} U$, which is $R = S \cap U$. We then have*

$$d'_{R,U} \circ d''_{R,S} = d''_{U,T} \circ d'_{S,T}.$$

(c) *For every stratum S, we have*

$$\sum_{S \overset{1}{\hookrightarrow} T} d''_{S,T} \circ d'_{S,T} = 0.$$

For every inclusion $R \overset{1}{\hookrightarrow} S$ we have

$$d'_{R,S} \circ d''_{R,S} = 0.$$

Proof. 1. It expresses the fact that $d' \circ d' = 0$ and $d'' \circ d'' = 0$ in $A^{\leqslant S}_{\bullet,\bullet}$.

2. (a) It expresses the fact that the components $A^S_{i,j-1} \to A^U_{i-1,j}$ of $d' \circ d''$ and $d'' \circ d'$ in $A^{\leqslant R}_{\bullet,\bullet}$ are equal.

(b) Same.

(c) There is no $d''_{R,S}$ in $A^{\leqslant S}_{\bullet,\bullet}$, hence the component $A^S_{i,j-1} \to A^S_{i-1,j}$ of $d'' \circ d'$ is zero. This gives the first equality. Now for some $R \overset{1}{\hookrightarrow} S$, the second equality follows from the first equality and the fact that the components $A^S_{i,j-1} \to A^S_{i-1,j}$ of $d' \circ d''$ and $d'' \circ d'$ in $A^{\leqslant R}_{\bullet,\bullet}$ are equal.

□

Definition 4.1.16. Let $\mathscr{B}$ be an arrangement and $A_{\bullet,\bullet}$ be its Orlik-Solomon bi-complex. We say that a strict stratum Σ of $\mathscr{B}$ is *exact* if the following condition, depending on the color of Σ, is satisfied:

– $\chi(\Sigma) = \lambda$ and all the rows

$$0 \to A^{\Sigma}_{i,j} \xrightarrow{d'} \bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A^S_{i-1,j} \xrightarrow{d'} \bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A^T_{i-2,j} \xrightarrow{d'} \cdots \xrightarrow{d'} \bigoplus_{\Sigma \overset{i}{\hookrightarrow} Z} A^Z_{0,j} \to 0$$

of the bi-complex $A^{\leqslant \Sigma}_{\bullet,\bullet}$ are exact;

– $\chi(\Sigma) = \mu$ and all the columns

$$0 \leftarrow A^{\Sigma}_{i,j} \xleftarrow{d''} \bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A^S_{i,j-1} \xleftarrow{d''} \bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A^T_{i,j-2} \xleftarrow{d''} \cdots \xleftarrow{d''} \bigoplus_{\Sigma \overset{j}{\hookrightarrow} Z} A^Z_{i,0} \leftarrow 0$$

of the bi-complex $A^{\leqslant \Sigma}_{\bullet,\bullet}$ are exact.

We say that $\mathscr{B}$ is *exact* if all its strict strata are exact.

The next easy Lemma expresses the fact that the definition of the Orlik-Solomon bi-complex is self-dual.

Lemma 4.1.17. *The Orlik-Solomon bi-complexes of $\mathscr{B}$ and $\mathscr{B}^\vee$ are dual to each other: we have $A^S_{i,j}(\mathscr{B}^\vee) = \left(A^S_{j,i}(\mathscr{B})\right)^\vee$, d' being the transpose of d'' and d'' the transpose of d'. Furthermore, $\mathscr{B}$ is exact if and only if $\mathscr{B}^\vee$ is exact.*

The Künneth formula

Up to now, we haven't used the Künneth condition (4.1). This condition is actually crucial since it implies that the Orlik-Solomon bi-complexes behave well with respect to decompositions.

Proposition 4.1.18. *Let $\mathscr{B}$ be a bi-arrangement and Σ a stratum of $\mathscr{B}$. Let us assume that Σ has a decomposition $\Sigma = \Sigma' \pitchfork \Sigma''$. Then we have an isomorphism of bi-complexes ("Künneth formula")*

$$A_{\bullet,\bullet}^{\leqslant\Sigma} \cong A_{\bullet,\bullet}^{\leqslant\Sigma'} \otimes A_{\bullet,\bullet}^{\leqslant\Sigma''}.$$

More precisely, a stratum $S \supset \Sigma$ of codimension r has a unique decomposition $S = S' \pitchfork S''$ with $S' \supset \Sigma'$ of codimension r' and $S'' \supset \Sigma''$ of codimension r'' with $r = r' + r''$; we then have isomorphisms

$$A_{i,j}^{S} \cong \bigoplus A_{i',j'}^{S'} \otimes A_{i'',j''}^{S''}$$

that are compatible with the differentials (the above sum is restricted to the indices such that $i' + i'' = i$, $j' + j'' = j$, $i' + j' = r'$, $i'' + j'' = r''$).

Proof. We proceed by induction on the codimension of Σ. The case of codimension 0 is just the isomorphism $\mathbb{Q} \cong \mathbb{Q} \otimes \mathbb{Q}$. More generally, the result is trivial if Σ' or Σ'' is the whole space $\mathbb{C}^n$. We thus assume that Σ' and Σ'' are strict strata. Let us assume that $\chi(\Sigma) = \lambda$, the case $\chi(\Sigma) = \mu$ being dual. Then by the Künneth condition (4.1), we necessarily have $\chi(\Sigma') = \lambda$ or $\chi(\Sigma'') = \lambda$. We consider the complexes

$$0 \to A_{i',j'}^{\Sigma'} \overset{d'}{\to} \bigoplus_{\Sigma' \overset{1}{\hookrightarrow} S'} A_{i'-1,j'}^{S'} \overset{d'}{\to} \bigoplus_{\Sigma' \overset{2}{\hookrightarrow} T'} A_{i'-2,j'}^{T'} \tag{4.2}$$

and

$$0 \to A_{i'',j''}^{\Sigma''} \overset{d'}{\to} \bigoplus_{\Sigma'' \overset{1}{\hookrightarrow} S''} A_{i''-1,j''}^{S''} \overset{d'}{\to} \bigoplus_{\Sigma'' \overset{2}{\hookrightarrow} T''} A_{i''-2,j''}^{T''}. \tag{4.3}$$

The tensor product of these two complexes is necessarily exact, since one of the two is exact. Summing over all possible indices (i', j', i'', j'') and using the induction hypothesis leads to an exact complex

$$\begin{aligned} 0 \to \bigoplus A_{i',j'}^{\Sigma'} \otimes A_{i'',j''}^{\Sigma''} \to \bigoplus_{\Sigma' \overset{1}{\hookrightarrow} S'} A_{i-1,j}^{S' \pitchfork \Sigma''} \oplus \bigoplus_{\Sigma'' \overset{1}{\hookrightarrow} S''} A_{i-1,j}^{\Sigma' \pitchfork S''} \to \\ \to \bigoplus_{\Sigma' \overset{2}{\hookrightarrow} T'} A_{i-2,j}^{T' \pitchfork \Sigma''} \oplus \bigoplus_{\substack{\Sigma' \overset{1}{\hookrightarrow} S' \\ \Sigma'' \overset{1}{\hookrightarrow} S''}} A_{i-2,j}^{S' \pitchfork S''} \oplus \bigoplus_{\Sigma'' \overset{2}{\hookrightarrow} T''} A_{i-2,j}^{\Sigma' \pitchfork T''}. \end{aligned}$$

This gives the desired isomorphism. One easily checks the compatibilities with the differentials. □

Corollary 4.1.19.
1. *The Orlik-Solomon bi-complex $A_{\bullet,\bullet}(\mathscr{B})$ of a bi-arrangement $\mathscr{B}$ only depends on its equivalence class (Definition 4.1.3).*
2. *A bi-arrangement $\mathscr{B}$ is exact if and only if all its irreducible strata of codimension $\geqslant 2$ are exact. Thus, the exactness of $\mathscr{B}$ only depends on its equivalence class.*

Proof. 1. Proposition 4.1.18 implies that for a decomposition into irreducibles $S = S_1 \pitchfork \cdots \pitchfork S_r$, $A_{\bullet,\bullet}^{\leqslant S}$ is the tensor product of the bi-complexes $A_{\bullet,\bullet}^{\leqslant S_k}$, hence it does not depend on the color $\chi(S)$.

2. Let us assume that all the S_k's are exact, and that $\chi(S) = \lambda$ (the case $\chi(S) = \mu$ being dual). By definition, we can then assume that $\chi(S_1) = \lambda$, and hence the rows of $A_{\bullet,\bullet}^{\leqslant S_1}$ are exact. The Künneth formula implies that the rows of $A_{\bullet,\bullet}^{\leqslant S}$ are exact, hence S is exact. The claim then follows from the fact that all hyperplanes $K \in \mathscr{B}$ are exact.

□

Another way of stating the Künneth formula is the following.

Corollary 4.1.20. *The Orlik-Solomon bi-complex of a product $\mathscr{B}' \times \mathscr{B}''$ is the tensor product*

$$A_{\bullet,\bullet}(\mathscr{B}' \times \mathscr{B}'') \cong A_{\bullet,\bullet}(\mathscr{B}') \otimes A_{\bullet,\bullet}(\mathscr{B}'').$$

Furthermore, $\mathscr{B}' \times \mathscr{B}''$ is exact if and only if $\mathscr{B}'$ and $\mathscr{B}''$ are exact.

Examples

Example 4.1.21. The notion of an Orlik-Solomon bi-complex generalizes the construction of the Orlik-Solomon algebra. Indeed, if $\mathscr{A}$ is an arrangement then the Orlik-Solomon bi-complex of the bi-arrangement $(\mathscr{A}, \lambda)$ is concentrated in bi-degrees $(k, 0)$ and agrees with the Orlik-Solomon algebra of $\mathscr{A}$: $A_{k,0}^S(\mathscr{A}, \lambda) = A_k^S(\mathscr{A})$ for all $S \in \mathscr{S}_k(\mathscr{A})$, and $d'_{S,T} = d_{S,T}$ the classical differential of the Orlik-Solomon algebra. Dually, the Orlik-Solomon bi-complex of $(\mathscr{A}, \mu)$ is concentrated in bi-degrees $(0, k)$ and is the linear dual of the Orlik-Solomon algebra of $\mathscr{A}$: $A_{0,k}^S(\mathscr{A}, \mu) = \left(A_k^S(\mathscr{A})\right)^\vee$. The bi-arrangements $(\mathscr{A}, \lambda)$ and $(\mathscr{A}, \mu)$ are thus always exact.

Example 4.1.22. More generally, for a bi-arrangement $\mathscr{B} = (\mathscr{L}, \mathscr{M}, \chi)$, if all strata of $\mathscr{L}$ are colored λ then we have an isomorphism $A_{0,\bullet}(\mathscr{L}, \mathscr{M}, \chi) \cong A_\bullet(\mathscr{L})$. Dually, if all strata of $\mathscr{M}$ are colored μ then we have an isomorphism $A_{0,\bullet}(\mathscr{L}, \mathscr{M}, \chi) \cong (A_\bullet(\mathscr{M}))^\vee$.

Example 4.1.23. By Example 4.1.21 and Corollary 4.1.20, a product $(\mathscr{L}, \lambda) \times (\mathscr{M}, \mu)$ is always exact, with its Orlik-Solomon bi-complex

$$A_{\bullet,\bullet}((\mathscr{L}, \lambda) \times (\mathscr{M}, \mu)) = A_\bullet(\mathscr{L}) \otimes (A_\bullet(\mathscr{M}))^\vee.$$

The first obstruction to exactness

Let $\mathscr{B} = (\mathscr{L}, \mathscr{M}, \chi)$ be a bi-arrangement. By the definition of an Orlik-Solomon bi-complex, we have for each $L \in \mathscr{L}$ an isomorphism $A_{1,0}^L \xrightarrow{\cong} \mathbb{Q}$, and $A_{0,1}^L = 0$. Dually, we get for each $M \in \mathscr{M}$ an isomorphism $\mathbb{Q} \xrightarrow{\cong} A_{0,1}^M$, and $A_{1,0}^M = 0$. This remark gives us the first obstruction to the exactness of a bi-arrangement.

Lemma 4.1.24. *If a bi-arrangement $\mathscr{B} = (\mathscr{L}, \mathscr{M}, \chi)$ is exact, then for every strict stratum S,*

1. *if $\chi(S) = \lambda$ then $S \subset L$ for some $L \in \mathscr{L}$;*
2. *if $\chi(S) = \mu$ then $S \subset M$ for some $M \in \mathscr{M}$.*

Proof. Let us assume that $\chi(S) = \lambda$, the case $\chi(S) = \mu$ being dual. Then the first row of the bi-complex $A_{\bullet,\bullet}^{\leqslant S}$ is exact, which means that we have a surjection

$$\bigoplus_{L \in \mathscr{L} \mid S \subset L} A_{1,0}^L \to \mathbb{Q} \to 0$$

hence $S \subset L$ for some $L \in \mathscr{L}$. □

Example 4.1.25. The simplest bi-arrangement of hyperplanes that is not exact is made of three lines L_1, L_2, L_3 in $\mathbb{C}^2$ that meet at the origin Z, with $\chi(L_1) = \chi(L_2) = \chi(L_3) = \lambda$, and $\chi(Z) = \mu$.

4.1.4 The Orlik-Solomon bi-complex of a tame bi-arrangement

Tame bi-arrangements

Let $\mathscr{L} = \{L_1, \ldots, L_l\}$ and $\mathscr{M} = \{M_1, \ldots, M_m\}$ be two arrangements of hyperplanes in $\mathbb{C}^n$. We say that a pair (I, J) formed by a subset $I \subset \{1, \ldots, l\}$ and a subset $J \subset \{1, \ldots, m\}$ is *dependent* if the hyperplanes L_i, for $i \in I$, and M_j, for $j \in J$, are linearly dependent, and *independent* otherwise. A *circuit* is a minimally dependent pair (I, J) in the sense that if $I' \subset I$ and $J' \subset J$ are two subsets such that (I', J') is dependent, then $I' = I$ and $J' = J$. We note that if (I, J) is a circuit, then $L_I \cap M_J$ is an irreducible stratum.

Definition 4.1.26. Let $\mathscr{B} = (\mathscr{L}, \mathscr{M}, \chi)$ be a bi-arrangement. A strict stratum S of $\mathscr{B}$ is *tame* if the following condition, depending on the color of S, is satisfied:

1. $\chi(S) = \lambda$ and there exists a hyperplane L_i that contains S and such that i does not belong to any circuit (I, J) with $S \subset L_I \cap M_J$ and $\chi(L_I \cap M_J) = \mu$;

2. $\chi(S) = \mu$ and there exists a hyperplane M_j that contains S and such that j does not belong to any circuit (I, J) with $S \subset L_I \cap M_J$ and $\chi(L_I \cap M_J) = \lambda$.

A bi-arrangement of hyperplanes is *tame* if all its strict strata are tame.

Remark 4.1.27. The tameness is a local condition in the sense that the tameness of a stratum S of $\mathscr{B}$ only depends on the bi-arrangement $\mathscr{B}^{\leqslant S}$ consisting of the hyperplanes that contain S.

Lemma 4.1.28. *A bi-arrangement is tame if and only if all its irreducible strata of codimension $\geqslant 2$ are tame. Thus, the tameness of a bi-arrangement only depends on its equivalence class.*

Proof. We note that the hypersurfaces $K \in \mathscr{B}$ are necessarily exact. Let us assume that all irreducible strata of $\mathscr{B}$ are tame. Let S be a reducible stratum of $\mathscr{B}$ with a decomposition $S = S_1 \pitchfork \cdots \pitchfork S_r$ into irreducibles S_j. Let us assume that $\chi(S) = \lambda$, the case $\chi(S) = \mu$ being dual. Then by the Künneth condition (4.1) we can assume that $\chi(S_1) = \lambda$. Thus, there is a hyperplane $L_i \supset S_1$ such that i does not belong to any circuit (I, J) with $S_1 \subset L_I \cap M_J$ and $\chi(L_I \cap M_J) = \mu$. Then L_i contains S; furthermore, a circuit (I, J) containing i and such that $S \subset L_I \cap M_J$ necessarily satisfies $S \subset S_1 \subset L_I \cap M_J$, hence S is tame. □

Remark 4.1.29. Let us say that a stratum S of $\mathscr{B}$ is *hamiltonian* if it can be written $S = L_I \cap M_J$ with (I, J) a circuit. A hamiltonian stratum is irreducible, but the converse is false in general. If $\mathscr{B}$ is tame, then the color of the hamiltonian strata determine the colors of all irreducible strata, using the following basic fact about connected (=irreducible) matroids.

Lemma 4.1.30 ([Oxl11], Proposition 4.1.3). *Let $\mathscr{A} = \{K_1, \ldots, K_k\}$ be an arrangement of hyperplanes, S an irreducible stratum of $\mathscr{A}$, $K_i, K_j \in \mathscr{A}$ hyperplanes containing S. Then there exists a circuit I containing i, j such that $S \subset K_I$.*

Example 4.1.31.

1. If $\mathscr{A}$ is an arrangement, then the bi-arrangements $(\mathscr{A}, \lambda)$ and $(\mathscr{A}, \mu)$ are tame.

2. The class of tame bi-arrangements is closed under products (this is a consequence of Lemma 4.1.28).

3. As a consequence, any product $(\mathscr{L}, \lambda) \times (\mathscr{M}, \mu)$ is tame.

4. The tameness condition implies the necessary condition of Lemma 4.1.24. For bi-arrangements in $\mathbb{C}^2$, these conditions are equivalent.

Lemma 4.1.32. *Let $\mathscr{L}$ and $\mathscr{M}$ be disjoint arrangements in $\mathbb{C}^n$. Then the bi-arrangements $(\mathscr{L},\mathscr{M},e_\lambda)$ and $(\mathscr{L},\mathscr{M},e_\mu)$, equipped with the λ-extreme and μ-extreme colorings, are tame.*

Proof. By duality, it is enough to do the proof for $(\mathscr{L},\mathscr{M},e_\lambda)$.

– Let S be a stratum such that $e_\lambda(S)=\lambda$, then there exists a hyperplane L_i such that $S\subset L_i$. Let (I,J) be a circuit such that $i\in I$, $S\subset L_I\cap M_J$, and $e_\lambda(L_I\cap M_J)=\mu$. Then by definition, $I=\varnothing$, which is a contradiction.

– Let S be a stratum such that $e_\lambda(S)=\mu$, then there exists a hyperplane M_j such that $S\subset M_j$. Let (I,J) be a circuit such that $j\in J$, $S\subset L_I\cap M_J$, and $e_\lambda(L_I\cap M_J)=\lambda$. Then there exists a hyperplane L_i such that $L_I\cap M_J\subset L_i$. Then $S\subset L_i$ and $e_\lambda(S)=\lambda$, which is a contradiction.

□

The Orlik-Solomon bi-complex

The goal of this section is to give an explicit formula for the Orlik-Solomon bi-complex of a tame bi-arrangement, and to prove at the same time that tame bi-arrangements are exact. Let us fix a tame bi-arrangement $\mathscr{B}=(\mathscr{L},\mathscr{M},\chi)$ with $\mathscr{L}=\{L_1,\ldots,L_l\}$ and $\mathscr{M}=\{M_1,\ldots,M_m\}$. We first set

$$E_{\bullet,\bullet}(\mathscr{B})=E_\bullet(\mathscr{L})\otimes E_\bullet(\mathscr{M})^\vee=\Lambda^\bullet(e_1,\ldots,e_l)\otimes\Lambda^\bullet(f_1^\vee,\ldots,f_m^\vee).$$

Thus, $E_{i,j}(\mathscr{B})$ has a basis consisting of monomials $e_I\otimes f_J^\vee$ for $|I|=i$ and $|J|=j$. We define

$$d'=d\otimes\mathrm{id}:E_{\bullet,\bullet}(\mathscr{B})\to E_{\bullet-1,\bullet}(\mathscr{B})$$

and

$$d''=\mathrm{id}\otimes d^\vee:E_{\bullet,\bullet}(\mathscr{B})\to E_{\bullet,\bullet+1}(\mathscr{B})$$

so that $E_{\bullet,\bullet}(\mathscr{B})$ is a bi-complex.

We consider on $E_{\bullet,\bullet}(\mathscr{B})$ the following homogeneous relations (subspaces of $E_{\bullet,\bullet}(\mathscr{B})$) and co-relations (subspaces of the dual space $E_{\bullet,\bullet}(\mathscr{B})^\vee$):

– for a circuit (I,J) such that $\chi(L_I\cap M_J)=\lambda$, for all $J'\supset J$, we consider the relation

$$(d(e_I))\otimes f_{J'}^\vee$$

where $(d(e_I))$ is the ideal of $\Lambda^\bullet(e_1,\ldots,e_l)$ generated by $d(e_I)$.

– for a circuit (I,J) such that $\chi(L_I\cap M_J)=\mu$, for all $I'\supset I$, we consider the co-relation

$$e_{I'}^\vee\otimes(d(f_J))$$

where $(d(f_J))$ is the ideal of $\Lambda^\bullet(f_1,\ldots,f_m)$ generated by $d(f_J)$.

Definition 4.1.33. Let $A_{\bullet,\bullet}(\mathscr{B})$ be the subquotient of $E_{\bullet,\bullet}(\mathscr{B})$ defined by the above relations and co-relations.

The notation will be justified by the fact that $A_{\bullet,\bullet}(\mathscr{B})$ is the Orlik-Solomon bi-complex of $\mathscr{B}$, see Theorem 4.1.38 below. It is worth noting that the definition of $A_{\bullet,\bullet}(\mathscr{B})$ only uses the colors of the hamiltonian strata, which is not surprising in view of Remark 4.1.29.

Lemma 4.1.34. *The differentials d' and d'' pass to the subquotient and give $A_{\bullet,\bullet}(\mathscr{B})$ the structure of a bi-complex.*

Proof. By duality, it is enough to prove that d' and d'' pass to the quotient by the relations. It follows easily from the definitions:

$$d'((e_K \wedge d(e_I)) \otimes f_{J'}^\vee) = (d(e_K) \wedge d(e_I)) \otimes f_{J'}^\vee$$

and

$$d''((e_K \wedge d(e_I)) \otimes f_{J'}^\vee) = \sum_{j \notin J'} \pm (e_K \wedge d(e_I)) \otimes f_{J' \cup \{j\}}^\vee.$$

□

For integers $i, j \geqslant 0$ and a stratum $S \in \mathscr{S}_{i+j}(\mathscr{B})$, let us denote by $E_{i,j}^S(\mathscr{B})$ the direct summand of $E_{i,j}(\mathscr{B})$ spanned by the $e_I \otimes f_J^\vee$ such that $L_I \cap M_J = S$. Note that this implies that (I, J) is independent. Then we have a direct sum decomposition

$$E_{i,j}(\mathscr{B}) = \bigoplus_{S \in \mathscr{S}_{i+j}(\mathscr{B})} E_{i,j}^S(\mathscr{B}) \oplus \bigoplus_{\substack{|I|=i \\ |J|=j \\ (I,J) \text{ dependent}}} \mathbb{Q}\, e_I \otimes f_J^\vee. \tag{4.4}$$

Lemma 4.1.35. *The direct sum decomposition (4.4) passes to the subquotient and induces*

$$A_{i,j}(\mathscr{B}) = \bigoplus_{S \in \mathscr{S}_{i+j}(\mathscr{B})} A_{i,j}^S(\mathscr{B}).$$

Proof. We first prove that if (I, J) is dependent then in the definition of $A_{\bullet,\bullet}(\mathscr{B})$ we either have the relation $e_I \otimes f_J^\vee = 0$ or the co-relation $e_I^\vee \otimes f_J = 0$, so that the second direct summand of (4.4) disappears.
Let (I, J) be dependent. There exists $I' \subset I$, $J' \subset J$ such that (I', J') is a circuit. We assume that $\chi(L_{I'} \cap M_{J'}) = \lambda$, and show that the relation $e_I \otimes f_J^\vee = 0$ holds in $A_{\bullet,\bullet}(\mathscr{B})$ (dually, if $\chi(L_{I'} \cap M_{J'}) = \mu$ we would get the co-relation $e_I^\vee \otimes f_J = 0$). There are two cases to consider.
First case: $I' \neq \varnothing$. For any $i \in I'$, the Leibniz rule implies that $e_{I'} = \pm e_i \wedge d(e_{I'})$, hence $e_{I'}$ and then e_I are in the ideal of $\Lambda^\bullet(e_1, \ldots, e_l)$ generated by $d(e_{I'})$. Thus the relation $(d(e_{I'})) \otimes f_J^\vee$ entails $e_I \otimes f_J^\vee = 0$ in $A_{i,j}(\mathscr{B})$.
Second case: $I' = \varnothing$. Let L_i be a hyperplane containing $M_{J'}$ and satisfying the condition given in the definition of a tame arrangement. Then one easily shows that there exists a subset $J'' \subset J'$ such that $(\{i\}, J'')$ is a circuit. Since $M_{J'} \subset L_i \cap M_{J''}$, we necessarily have $\chi(L_i \cap M_{J''}) = \lambda$, and we are reduced to the first case.

We next prove that the relations and co-relations are homogeneous with respect to the grading by $\mathscr{S}(\mathscr{B})$. Let (I, J) be a circuit such that $\chi(L_I \cap M_J) = \lambda$, and let $J' \supset J$. Then the corresponding relation reads

$$\sum_{i \in I} \pm e_{I \setminus \{i\}} \otimes f_{J'}^\vee = 0.$$

For all $i \in I$, $(I \setminus \{i\}, J)$ is independent, hence $L_{I \setminus \{i\}} \cap M_J = L_I \cap M_J$ does not depend on i, and $L_{I \setminus \{i\}} \cap M_{J'}$ does not depend on i. Hence the relations are homogeneous with respect to the grading by $\mathscr{S}(\mathscr{B})$. Dually, the same is true for the co-relations. □

Remark 4.1.36. By definition, the component $A_{\bullet,\bullet}^S(\mathscr{B})$ only depends on the arrangement $\mathscr{B}^{\leqslant S}$, which is tame according to Remark 4.1.27. For a strict stratum Σ, we then have $A_{\bullet,\bullet}^{\leqslant \Sigma}(\mathscr{B}) \cong A_{\bullet,\bullet}(\mathscr{B}^{\leqslant \Sigma})$.

Example 4.1.37. Let $\mathscr{L} = \{L_1, L_2\}$ and $\mathscr{M} = \{M_1\}$ be three distinct lines in $\mathbb{C}^2$. Let Z be the origin, we set $\chi(Z) = \lambda$. This defines a tame bi-arrangement $\mathscr{B} = (\mathscr{L}, \mathscr{M}, \chi)$. The only circuit is $(\{1,2\},\{1\})$. Then $A_{\bullet,\bullet}(\mathscr{B})$ is the quotient of $\Lambda^\bullet(e_1,e_2)\otimes\Lambda^\bullet(f_1^\vee)$ by the relations $(e_2-e_1)f_1^\vee=0$ and $e_{12}f_1^\vee=0$. It can be pictured as

$$\begin{tikzcd}
\mathbb{Q}\,e_{12} \arrow[r] & \mathbb{Q}\,e_1\oplus\mathbb{Q}\,e_2 \arrow[r]\arrow[d] & \mathbb{Q}\,1 \arrow[d] \\
 & \left(\mathbb{Q}\,e_1f_1^\vee\oplus\mathbb{Q}\,e_2f_1^\vee\right)/\left(e_1f_1^\vee=e_2f_2^\vee\right) \arrow[r] & \mathbb{Q}\,f_1^\vee
\end{tikzcd}$$

and its rows are exact.

Theorem 4.1.38. *Let $\mathscr{B}$ be a tame bi-arrangement. Then $A_{\bullet,\bullet}(\mathscr{B})$ is the Orlik-Solomon bi-complex of $\mathscr{B}$, and $\mathscr{B}$ is exact.*

Proof. Firstly, $A_{0,0}^{\mathbb{C}^n}(\mathscr{B})$ is indeed one-dimensional with basis $1\otimes 1$. Secondly, for every strict stratum Σ, $A_{\bullet,\bullet}^{\leqslant\Sigma}(\mathscr{B})=A_{\bullet,\bullet}(\mathscr{B})$ is a bi-complex by Remark 4.1.36 and Lemma 4.1.34. Thirdly, let Σ be a strict stratum of $\mathscr{B}$ such that $\chi(\Sigma)=\lambda$ (the case $\chi(\Sigma)=\mu$ being dual). We want to show that all the rows of $A_{\bullet,\bullet}^{\leqslant\Sigma}(\mathscr{B})$ are exact. By the same remark as above, we can assume that Σ is the intersection of all the hyperplanes of $\mathscr{B}$ and show that all the rows of $A_{\bullet,\bullet}(\mathscr{B})$ are exact.

By the definition of a tame bi-arrangement, there exists a hyperplane L_i such that i does not belong to any circuit (I,J) with $\chi(L_I\cap M_J)=\mu$. We define $h:E_{\bullet,\bullet}(\mathscr{B})\to E_{\bullet+1,\bullet}(\mathscr{B})$ by the formula $h(x\otimes y)=(e_i\wedge x)\otimes y$. Then the Leibniz rule implies that $d'\circ h+h\circ d'=\mathrm{id}$, hence h is a contracting homotopy for all the rows of $E_{\bullet,\bullet}(\mathscr{B})$. Hence we are done if we prove that h passes to the subquotient and induces $h:A_{\bullet,\bullet}(\mathscr{B})\to A_{\bullet+1,\bullet}(\mathscr{B})$.
The fact that h respects the relations is trivial. Let (I,J) be a circuit such that $\chi(L_I\cap M_J)=\mu$. Then by assumption $i\notin I$. Thus, any subset $I'\supset I$ that contains i is of the form $I'=\{i\}\sqcup I''$ with $I''\supset I$. Hence we have $h^\vee(e_{I'}\otimes(f_K\wedge d(f_J)))=\pm e_{I''}\otimes(f_K\wedge d(f_J))$ and h respects the co-relations. □

Remark 4.1.39. The definition of $A_{\bullet,\bullet}(\mathscr{B})$ is automatically self-dual, viewing $A_{\bullet,\bullet}(\mathscr{B}^\vee)$ as a subquotient of $\Lambda^\bullet(e_1^\vee,\ldots,e_l^\vee)\otimes\Lambda^\bullet(f_1,\ldots,f_m)\cong\Lambda^\bullet(f_1,\ldots,f_m)\otimes\Lambda^\bullet(e_1^\vee,\ldots,e_l^\vee)$.

Remark 4.1.40. There is a natural structure of graded module over $E_\bullet(\mathscr{L})$ on $E_{\bullet,\bullet}(\mathscr{L},\mathscr{M})$. Let $(\mathscr{L},\mathscr{M},e_\lambda)$ be a bi-arrangement equipped with the λ-extreme coloring; then this structure passes to the subquotient and induces on $A_{\bullet,\bullet}(\mathscr{L},\mathscr{M},e_\lambda)$ a structure of graded module over the Orlik-Solomon algebra $A_\bullet(\mathscr{L})$. Dually, $A_{\bullet,\bullet}(\mathscr{L},\mathscr{M},e_\mu)$ is a graded comodule over $(A_\bullet(\mathscr{M}))^\vee$, which is the same as a graded module over $A_\bullet(\mathscr{M})$.

4.1.5 Examples

A non-tame bi-arrangement which is not exact

To find a non-tame non-exact bi-arrangement, we may choose trivial examples that do not satisfy the necessary condition of Lemma 4.1.24. Here we present a less trivial example.

Let us consider, in $\mathbb{C}^3$, a bi-arrangement $\mathscr{B}=(\mathscr{L},\mathscr{M},\chi)$ with $\mathscr{L}=\{L_1,L_2,L_3\}$ and $\mathscr{M}=\{M_1,M_2\}$ defined by the equations $L_1=\{x_1=0\}$, $L_2=\{x_2=0\}$, $L_3=\{x_3=0\}$, $M_1=\{x_1+x_3=0\}$, $M_2=\{x_2+x_3=0\}$. Apart from the hyperplanes, the irreducible strata are the lines $D_{12}=\{x_1=x_2=0\}$, $D_{13}=\{x_1=x_3=0\}$ and the point $P=\{x_1=x_2=x_3=0\}$. We define $\chi(D_{12})=\chi(D_{13})=\mu$ and $\chi(P)=\lambda$. The circuits are $(\{1,3\},\{1\})$, $(\{2,3\},\{2\})$ with color λ, and $(\{1,2\},\{1,2\})$ with color μ. The stratum P is not tame, thus $\mathscr{B}$ is not tame.

It is easy to check that $\mathscr{B}$ is not exact. This follows from looking at the first row $(\bullet,0)$ of its Orlik-Solomon bi-complex. The only non-zero terms are $A_{1,0}^{L_i}=\mathbb{Q}$ for $i=1,2,3$, and $A_{2,0}^{L_{23}}=\ker\left(A_{1,0}^{L_2}\oplus A_{1,0}^{L_3}\to\mathbb{Q}\right)\cong\mathbb{Q}$. The first row is then

$$0\to 0\to\mathbb{Q}\to\mathbb{Q}\oplus\mathbb{Q}\oplus\mathbb{Q}\to\mathbb{Q}\to 0$$

which is not exact.

A non-tame bi-arrangement which is exact

Let us consider the same bi-arrangement as in the previous example, but with the coloring $\chi(D_{12}) = \chi(D_{13}) = \lambda$ and $\chi(P) = \mu$ (it is not its dual, since we have not exchanged $\mathscr{L}$ and $\mathscr{M}$). This bi-arrangement $\mathscr{B}$ is not tame. Nevertheless, it can be checked that the bi-complex $A_{\bullet,\bullet}(\mathscr{B})$ defined in §4.1.4 is indeed the Orlik-Solomon bi-complex of $\mathscr{B}$ and that $\mathscr{B}$ is exact.

Another non-tame bi-arrangement of hyperplanes which is exact

Let us consider a bi-arrangement $\mathscr{B} = (\mathscr{L}, \mathscr{M}, \chi)$ in $\mathbb{C}^3$ with $\mathscr{L} = \{L_1, L_2, L_3\}$ and $\mathscr{M} = \{M_1, M_2, M_3\}$ defined by $L_1 = \{x_1 = 0\}$, $L_2 = \{x_2 = 0\}$, $L_3 = \{x_3 = 0\}$, $M_1 = \{x_2 + x_3 = 0\}$, $M_2 = \{x_1 + x_3 = 0\}$, $M_3 = \{x_1 + x_2 = 0\}$. The irreducible strata are the lines $D_{12} = \{x_1 = x_2 = 0\}$, $D_{13} = \{x_1 = x_3 = 0\}$, $D_{23} = \{x_2 = x_3 = 0\}$ and the point $P = \{x_1 = x_2 = x_3 = 0\}$. We set $\chi(D_{12}) = \chi(D_{13}) = \chi(D_{23}) = \lambda$ and $\chi(P) = \mu$. The circuits are $(\{1,2\},\{3\})$, $(\{1,3\},\{2\})$, $(\{2,3\},\{1\})$ with color λ, and $(\{1,2\},\{1,2\})$, $(\{1,3\},\{1,3\})$, $(\{2,3\},\{2,3\})$ with color μ. The stratum P is not tame, thus $\mathscr{B}$ is not tame.

It is easy to check that $\mathscr{B}$ is exact, but that its Orlik-Solomon bi-complex is not the one defined in § 4.1.4.

4.2 Deletion and restriction

In this section we introduce deletion and restriction of bi-arrangements of hyperplanes. The main result is Theorem 4.2.9 which proves that there is a deletion-restriction short exact sequence in the formalism of Orlik-Solomon bi-complexes. It implies that the exactness property is “stable by extension” in the sense that a bi-arrangement is exact if its deletion and its restriction with respect to some hyperplane are exact. When we apply our results to bi-arrangements $\mathscr{B} = (\mathscr{A}, \lambda)$, we recover the classical deletion and restriction short exact sequence for Orlik-Solomon algebras [OT92, Theorem 3.65].

4.2.1 Deletion

Let $\mathscr{A}$ be an arrangement in $\mathbb{C}^n$, and $K_0 \in \mathscr{A}$. The *deletion* of $\mathscr{A}$ with respect to K_0 is the arrangement $\delta\mathscr{A} = \mathscr{A} \setminus \{K_0\}$.

Lemma 4.2.1. *1. We have an inclusion $\mathscr{S}_\bullet(\delta\mathscr{A}) \subset \mathscr{S}_\bullet(\mathscr{A})$. The complement consists in those strata S such that there exits a decomposition $S = K_0 \pitchfork T$.*

2. Any irreducible stratum of $\delta\mathscr{A}$ is irreducible as a stratum of $\mathscr{A}$.

Proof. 1. The inclusion is trivial. For a stratum S of $\mathscr{A}$, let T be the intersection of all the hyperplanes $K \in \mathscr{A}$, $K \neq K_0$, that contain S. If $S = T$ then S is a stratum of $\delta\mathscr{A}$. Otherwise, S is not a stratum of $\delta\mathscr{A}$ and we have a decomposition $S = K_0 \pitchfork T$.

2. Let $S \in \mathscr{S}(\delta\mathscr{A})$ such that S is reducible as a stratum of $\mathscr{A}$. Let $S = S' \pitchfork S''$ be a decomposition for S, with S' and S'' strict strata of $\mathscr{A}$. If S' is not a stratum of $\delta\mathscr{A}$ then the first point implies that we have a decomposition $S' = K_0 \pitchfork T'$, and hence a decompositon $S = K_0 \pitchfork T' \pitchfork S''$, and the first point implies that S is not a stratum of $\delta\mathscr{A}$, which is a contradiction. Hence S' and S'' are strata of $\delta\mathscr{A}$, and S is reducible as a stratum of $\delta\mathscr{A}$.

□

If now $\mathscr{B} = (\mathscr{A}, \chi)$ is a bi-arrangement, we can define a bi-arrangement $(\delta\mathscr{A}, \chi')$ by defining χ' on the irreducible strata of $\delta\mathscr{A}$ as the resriction of χ. Lemma 4.2.1 implies that this operation is well-defined among equivalence classes of bi-arrangements (Definition 4.1.3).

Definition 4.2.2. We write $\delta\mathscr{B} = (\delta\mathscr{A}, \chi')$ and call it the *deletion* of $\mathscr{B}$ with respect to K_0.

Definition 4.2.3. The deletion of $\mathscr{B}$ with respect to K_0 is *color-consistent* if there exists in the respective equivalence classes of $\mathscr{B}$ and $\delta\mathscr{B}$ representatives such that $\chi' : \mathscr{S}_+(\delta\mathscr{B}) \to \{\lambda, \mu\}$ is obtained as the restriction of $\chi : \mathscr{S}_+(\mathscr{B}) \to \{\lambda, \mu\}$. In this case, we then implicitly work with these representatives.

Example 4.2.4. Let $\mathscr{B} = \{K_0, K_1, K_2\}$ consist of three distinct lines in $\mathbb{C}^2$, and Z be the origin. The irreducible strata of $\mathscr{B}$ are $\{K_0, K_1, K_2, Z\}$. We let $\chi(K_0) = \lambda$, $\chi(K_1) = \chi(K_2) = \mu$, $\chi(Z) = \lambda$. Then $\delta\mathscr{B} = \{K_1, K_2\}$ and the irreducible strata of $\delta\mathscr{B}$ are $\{K_1, K_2\}$. The Künneth condition for $\delta\mathscr{B}$ implies that $\chi(Z) = \mu$ for $\delta\mathscr{B}$. Thus, this deletion is not color-consistent.

Proposition 4.2.5. *Let us assume that the deletion $\delta\mathscr{B}$ of $\mathscr{B}$ with respect to K_0 is color-consistent, and that $\chi(K_0) = \lambda$. Then there exists a unique collection of morphisms*

$$\delta_S : A^S_{i,j}(\delta\mathscr{B}) \to A^S_{i,j}(\mathscr{B})$$

for $S \in \mathscr{S}_{i+j}(\delta\mathscr{B})$ such that

– $\delta_{\mathbb{C}^n} : A^{\mathbb{C}^n}_{0,0}(\delta\mathscr{B}) \to A^{\mathbb{C}^n}_{0,0}(\mathscr{B})$ *is the identity of $\mathbb{Q}$;*

– *for every strict stratum $\Sigma \in \mathscr{S}_+(\delta\mathscr{B})$, the morphisms δ_S induce a morphism of bi-complexes*

$$\delta : A^{\leqslant\Sigma}_{\bullet,\bullet}(\delta\mathscr{B}) \to A^{\leqslant\Sigma}_{\bullet,\bullet}(\mathscr{B}).$$

If $\chi(K_0) = \mu$, the dual statement is true, with morphisms $\delta_S : A^S_{i,j}(\mathscr{B}) \to A^S_{i,j}(\delta\mathscr{B})$.

Proof. We treat the case $\chi(K_0) = \lambda$, the case $\chi(K_0) = \mu$ being dual. We define the morphisms δ_S by induction on the codimension of S, the case of codimension 0 being imposed by the definition. Let Σ be a strict stratum of $\delta\mathscr{B}$ such that all the morphisms δ_S have been defined for strata S that strictly contain Σ. Since the deletion is color-consistent, Σ has the same color in $\mathscr{B}$ and $\delta\mathscr{B}$. There are two cases.

1. $\chi(\Sigma) = \lambda$. We have the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & A^{\Sigma}_{i,j}(\delta\mathscr{B}) \arrow[r, "d'"] \arrow[d, dotted, "\delta_\Sigma"] & \displaystyle\bigoplus_{\substack{\Sigma \overset{1}{\hookrightarrow} S \\ S \in \mathscr{S}(\delta\mathscr{B})}} A^S_{i-1,j}(\delta\mathscr{B}) \arrow[r, "d'"] \arrow[d, "\bigoplus \delta_S"] & \displaystyle\bigoplus_{\substack{\Sigma \overset{2}{\hookrightarrow} T \\ T \in \mathscr{S}(\delta\mathscr{B})}} A^T_{i-2,j}(\delta\mathscr{B}) \arrow[d, "\bigoplus \delta_T"] \\
0 \arrow[r] & A^{\Sigma}_{i,j}(\mathscr{B}) \arrow[r, "d'"'] & \displaystyle\bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A^S_{i-1,j}(\mathscr{B}) \arrow[r, "d'"'] & \displaystyle\bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A^T_{i-2,j}(\mathscr{B})
\end{tikzcd} \tag{4.5}$$

The right square commutes by the induction hypothesis applied to a stratum $S \in \mathscr{S}(\delta\mathscr{B})$. Since the bottom row of the diagram is exact, there is a unique way of completing the diagram with the dotted arrow δ_Σ. This morphism is then automatically compatible with the differentials d'. To see that it is compatible with the differentials d'', a diagram chase shows that it is enough to show that the following diagram commutes.

$$\begin{tikzcd}
\bigoplus\limits_{\substack{\Sigma \overset{2}{\hookrightarrow} T \\ T \in \mathscr{S}(\delta\mathscr{B})}} A^T_{i-1,j-1}(\delta\mathscr{B}) \arrow[r, "\bigoplus \delta_T"] \arrow[d, "d''"'] & \bigoplus\limits_{\Sigma \overset{2}{\hookrightarrow} T} A^T_{i-1,j-1}(\mathscr{B}) \arrow[d, "d''"] \\
\bigoplus\limits_{\substack{\Sigma \overset{1}{\hookrightarrow} S \\ S \in \mathscr{S}(\delta\mathscr{B})}} A^S_{i-1,j}(\delta\mathscr{B}) \arrow[r, "\bigoplus \delta_S"'] & \bigoplus\limits_{\Sigma \overset{1}{\hookrightarrow} S} A^S_{i-1,j}(\mathscr{B})
\end{tikzcd} \tag{4.6}$$

According to the induction hypothesis applied to a stratum $S \in \mathscr{S}(\delta\mathscr{B})$, one has for $S \overset{1}{\hookrightarrow} T$ with $T \in \mathscr{S}(\delta\mathscr{B})$, $d''_{S,T} \circ \delta_T = \delta_S \circ d''_{S,T}$. Thus, all we need to prove is that if $S \notin \mathscr{S}(\delta\mathscr{B})$ and $T \in \mathscr{S}(\delta\mathscr{B})$, the composite $d''_{S,T} \circ \delta_T$ is zero. We prove that the differential $d''_{S,T}$ is zero. According to Lemma 4.2.1, there is a decomposition $S = K_0 \pitchfork T$, hence the Künneth formula (Proposition 4.1.18) implies that $d''_{S,T}$ factors as

$$A^T_{i-1,j-1}(\mathscr{B}) \to A^{K_0}_{0,1}(\mathscr{B}) \otimes A^T_{i-1,j-1}(\mathscr{B}) \hookrightarrow A^S_{i-1,j}(\mathscr{B}).$$

Since $\chi(K_0) = \lambda$ we have $A^{K_0}_{0,1}(\mathscr{B}) = 0$, hence $d''_{S,T} = 0$ and we are done.

2. $\chi(\Sigma) = \mu$. It is similar to the first case. The existence of δ_Σ follows from the same argument using the commutativity of the square (4.6). The compatibility with the differentials d'' is then automatic. To check the compatibility with the differentials d', one has to use the commutativity of the right square in diagram (4.5).

□

4.2.2 Restriction

Let $\mathscr{A}$ be an arrangement in $\mathbb{C}^n$, and $K_0 \in \mathscr{A}$. The *restriction* of $\mathscr{A}$ with respect to K_0 is the arrangement $\rho\mathscr{A}$ on $K_0 \cong \mathbb{C}^{n-1}$ consisting in the intersections of all hyperplanes of $\mathscr{A} \setminus \{K_0\}$ with K_0.

Lemma 4.2.6. *1. We have an inclusion $\mathscr{S}_\bullet(\rho\mathscr{A}) \subset \mathscr{S}_{\bullet+1}(\mathscr{A})$ corresponding to those strata of $\mathscr{A}$ that are contained in K_0.*

2. An irreducible stratum for $\rho\mathscr{A}$ is either an irreducible stratum for $\mathscr{A}$ or has a decomposition $S = K_0 \pitchfork T$ with T an irreducible stratum of $\mathscr{A}$.

Proof. 1. This is trivial.

2. Let us work in the dual space $(\mathbb{C}^n)^\vee$. Let $f_0 \in (\mathbb{C}^n)^\vee$ be a linear form that defines K_0. Then the restriction corresponds to the quotient $(\mathbb{C}^n)^\vee \twoheadrightarrow (\mathbb{C}^n)^\vee/\mathbb{C}f_0$. Let $S \subset K_0$ be a stratum and $S^\perp = S_1^\perp \oplus S_2^\perp \oplus \cdots \oplus S_r^\perp$ be its decomposition into irreducibles. Then we can assume that $f_0 \in S_1^\perp$. We then have a decomposition in $(\mathbb{C}^n)^\vee/\mathbb{C}f_0$: $S^\perp/\mathbb{C}f_0 = (S_1^\perp/\mathbb{C}f_0) \oplus S_2^\perp \oplus \cdots \oplus S_r^\perp$. If $S^\perp/\mathbb{C}f_0$ is irreducible in $\mathbb{C}^n/\mathbb{C}f_0$, we have two cases. Either $r = 1$, which means that $S^\perp$ is irreducible, or $r = 2$ and $S_1^\perp/\mathbb{C}f_0 = 0$, which means that $S^\perp$ has a decomposition $S^\perp = \mathbb{C}f_0 \oplus S_2^\perp$ with $S_2^\perp$ irreducible.

□

If now $\mathscr{B} = (\mathscr{A}, \chi)$ is a bi-arrangement of hyperplanes, we can define an arrangement of hyperplanes $(\rho\mathscr{A}, \chi')$ by defining the coloring χ' on the irreducible strata of $\rho\mathscr{A}$. For an irreducible stratum S of $\rho\mathscr{A}$ which is an irreducible stratum of $\mathscr{A}$, we set $\chi'(S) = \chi(S)$. For an irreducible stratum S of $\rho\mathscr{A}$ of the form $S = K_0 \pitchfork T$ with T irreducible, we set $\chi'(S) = \chi(T)$. Lemma 4.2.6 implies that this operation is well-defined among equivalence classes of bi-arrangements (Definition 4.1.3).

Definition 4.2.7. We write $\rho\mathscr{B} = (\rho\mathscr{A}, \chi')$ and call it the *restriction* of $\mathscr{B}$ with respect to K_0.

If we choose a representative of $\mathscr{B}$ in its equivalence class such that for a decomposition $K_0 \pitchfork T$ we have $\chi(K_0 \pitchfork T) = \chi(T)$, then we can choose a representative of $\rho\mathscr{B}$ in its equivalence class such that $\chi' : \mathscr{S}_+(\rho\mathscr{B}) \to \{\lambda, \mu\}$ is the restriction of $\chi : \mathscr{S}_+(\mathscr{B}) \to \{\lambda, \mu\}$. In the sequel, we implicitly work with these representatives.

Proposition 4.2.8. *Let us assume that $\chi(K_0) = \lambda$. Then there exists a unique collection of morphisms*

$$\rho_S : A^S_{i,j}(\mathscr{B}) \to A^S_{i-1,j}(\rho\mathscr{B})$$

for $S \in \mathscr{S}_{i+j-1}(\rho\mathscr{B})$ such that

— $\rho_{K_0} : A^{K_0}_{1,0}(\mathscr{B}) \to A^{K_0}_{0,0}(\rho\mathscr{B})$ *is the identity of* $\mathbb{Q}$;

— *for every strict stratum $\Sigma \in \mathscr{S}_+(\rho\mathscr{B})$, the morphisms ρ_S induce a morphism of bi-complexes*[1]

$$\rho : A^{\leqslant\Sigma}_{\bullet,\bullet}(\mathscr{B}) \to A^{\leqslant\Sigma}_{\bullet-1,\bullet}(\rho\mathscr{B}).$$

If $\chi(K_0) = \mu$, the dual statement is true, with morphisms $\rho_S : A^S_{i,j-1}(\rho\mathscr{B}) \to A^S_{i,j}(\mathscr{B})$.

Proof. We treat the case $\chi(K_0) = \lambda$, the case $\chi(K_0) = \mu$ being dual. We define the morphisms ρ_S by induction on the codimension of S in K_0, the case of codimension 0 being imposed by the definition. Let Σ be a strict stratum of $\rho\mathscr{B}$ such that all the morphisms ρ_S have been defined for strata S that strictly contain Σ. There are two cases.

1. $\chi(\Sigma) = \lambda$. The proof is similar to the proof of Proposition 4.2.5. The existence of ρ_Σ is obtained by filling the dotted arrow in the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & A^\Sigma_{i,j}(\mathscr{B}) \arrow[r, "d'"] \arrow[d, dotted, "\rho_\Sigma"] & \displaystyle\bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A^S_{i-1,j}(\mathscr{B}) \arrow[r, "d'"] \arrow[d, "\bigoplus \rho_S"] & \displaystyle\bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A^T_{i-2,j}(\mathscr{B}) \arrow[d, "\bigoplus \rho_T"] \\
0 \arrow[r] & A^\Sigma_{i-1,j}(\rho\mathscr{B}) \arrow[r, "-d'"'] & \displaystyle\bigoplus_{\substack{\Sigma \overset{1}{\hookrightarrow} S \\ S \in \mathscr{S}(\rho\mathscr{B})}} A^S_{i-2,j}(\rho\mathscr{B}) \arrow[r, "-d'"'] & \displaystyle\bigoplus_{\substack{\Sigma \overset{2}{\hookrightarrow} T \\ T \in \mathscr{S}(\rho\mathscr{B})}} A^T_{i-3,j}(\rho\mathscr{B})
\end{tikzcd}$$

The fact that the right square commutes is implied by the induction hypothesis applied to a stratum $S \in \mathscr{S}(\rho\mathscr{B})$. One has to note that for an inclusion $S \overset{1}{\hookrightarrow} T$, if $T \in \mathscr{S}(\rho\mathscr{B})$ then $T \subset K_0$ and hence $S \subset K_0$ and $S \in \mathscr{S}(\rho\mathscr{B})$. The above diagram guarantees the compatibility of ρ_Σ with the differentials d'. To see that ρ_Σ is compatible with the

1. According to the Koszul sign rule, the differential d' gets a minus sign in $A^{\leqslant\Sigma}_{\bullet-1,\bullet}(\rho\mathscr{B})$.

differentials d'', a diagram chase shows that it is enough to show that the following diagram commutes.

$$\begin{tikzcd}
\bigoplus\limits_{\Sigma \overset{2}{\hookrightarrow} T} A^T_{i-1,j-1}(\mathscr{B}) \arrow[r, "\bigoplus \rho_T"] \arrow[d, "d''"'] & \bigoplus\limits_{\substack{\Sigma \overset{2}{\hookrightarrow} T \\ T \in \mathscr{S}(\rho\mathscr{B})}} A^T_{i-2,j-1}(\rho\mathscr{B}) \arrow[d, "d''"] \\
\bigoplus\limits_{\Sigma \overset{1}{\hookrightarrow} S} A^S_{i-1,j}(\mathscr{B}) \arrow[r, "\bigoplus \rho_S"'] & \bigoplus\limits_{\substack{\Sigma \overset{1}{\hookrightarrow} S \\ S \in \mathscr{S}(\rho\mathscr{B})}} A^S_{i-2,j}(\rho\mathscr{B})
\end{tikzcd}$$

It is a consequence of the induction hypothesis applied to a stratum $S \in \mathscr{S}(\rho\mathscr{B})$.

2. $\chi(\Sigma) = \mu$. This is similar to the first case and left to the reader.

□

4.2.3 The deletion-restriction short exact sequence

Theorem 4.2.9. *Let $\mathscr{B}$ be a bi-arrangement, and $K_0 \in \mathscr{B}$. We assume that the deletion $\delta\mathscr{B}$ with respect to K_0 is color-consistent, and let $\rho\mathscr{B}$ be the restriction of $\mathscr{B}$ with respect to K_0. We also assume that $\delta\mathscr{B}$ and $\rho\mathscr{B}$ are exact.*

1. *If $\chi(K_0) = \lambda$ then the deletion and restriction morphisms induce a short exact sequence*

$$0 \to A_{\bullet,\bullet}(\delta\mathscr{B}) \xrightarrow{\delta} A_{\bullet,\bullet}(\mathscr{B}) \xrightarrow{\rho} A_{\bullet-1,\bullet}(\rho\mathscr{B}) \to 0.$$

Dually, if $\chi(K_0) = \mu$, then we get a short exact sequence

$$0 \to A_{\bullet,\bullet-1}(\rho\mathscr{B}) \xrightarrow{\rho} A_{\bullet,\bullet}(\mathscr{B}) \xrightarrow{\delta} A_{\bullet,\bullet}(\delta\mathscr{B}) \to 0.$$

2. *$\mathscr{B}$ is exact.*

More precisely, the statement of the Theorem above includes three special cases. Let S be a stratum of $\mathscr{B}$.

– If S is a stratum of $\delta\mathscr{B}$ but not a stratum of $\rho\mathscr{B}$, then we get an isomorphism

$$0 \to A^S_{\bullet,\bullet}(\delta\mathscr{B}) \xrightarrow{\delta_S} A^S_{\bullet,\bullet}(\mathscr{B}) \to 0 \to 0.$$

– If S is a stratum of $\rho\mathscr{B}$ but not a stratum of $\delta\mathscr{B}$, then we get an isomorphism

$$0 \to 0 \to A^S_{\bullet,\bullet}(\mathscr{B}) \xrightarrow{\rho_S} A^S_{\bullet-1,\bullet}(\rho\mathscr{B}) \to 0.$$

– If S is a stratum of both $\delta\mathscr{B}$ and $\rho\mathscr{B}$, then we get a short exact sequence

$$0 \to A^S_{\bullet,\bullet}(\delta\mathscr{B}) \xrightarrow{\delta_S} A^S_{\bullet,\bullet}(\mathscr{B}) \xrightarrow{\rho_S} A^S_{\bullet-1,\bullet}(\rho\mathscr{B}) \to 0.$$

Proof. We treat the case $\chi(K_0) = \lambda$, the case $\chi(K_0) = \mu$ being dual.

1. We first deal with the special cases separately.

– If S is a stratum of $\delta\mathscr{B}$ but not a stratum of $\rho\mathscr{B}$, this means that S is not contained in K_0, hence it is also the case for all strata T containing S. One then easily proves by induction on the codimension of S that $\delta_S : A^S_{i,j}(\delta\mathscr{B}) \to A^S_{i,j}(\mathscr{B})$ is an isomorphism. The case of codimension 0 is easy by definition.

– If S is a stratum of $\rho\mathscr{B}$ but not a stratum of $\delta\mathscr{B}$, then Lemma 4.2.1 implies that we have a decomposition $S = K_0 \pitchfork S'$. Since $\chi(K_0) = \lambda$, we have $A^{K_0}_{1,0}(\mathscr{B}) = \mathbb{Q}$ and $A^{K_0}_{0,1}(\mathscr{B}) = 0$, hence the Künneth formula (Proposition 4.1.18) implies that we have an isomorphism $A^S_{i,j}(\mathscr{B}) \cong A^{S'}_{i-1,j}(\mathscr{B})$. Now by our convention on the restriction, we have $\chi(K_0 \pitchfork S') = \chi(S')$; more generally, for every stratum T' containing S' we have $\chi(K_0 \pitchfork T') = \chi(T')$. It is then easy to show by induction on the codimension of S' that we have an isomorphism $A^{S'}_{i-1,j}(\mathscr{B}) \cong A^{K_0\pitchfork S'}_{i-1,j}(\rho\mathscr{B})$. The case of codimension 0 is easy by definition. One easily sees that the isomorphism $A^S_{i,j}(\mathscr{B}) \cong A^S_{i-1,j}(\rho\mathscr{B})$ is indeed ρ_S.

Now we deal with the third case by induction on the codimension of the strata. Let Σ be a strict stratum of $\mathscr{B}$ which is a stratum of both $\delta\mathscr{B}$ and $\rho\mathscr{B}$. Let us assume the result for all strata that strictly contain Σ. We deal with the case $\chi(\Sigma) = \lambda$, the case $\chi(\Sigma) = \mu$ being similar. We have the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A^{\Sigma}_{i,j}(\delta\mathscr{B}) \arrow[r] \arrow[d] & \displaystyle\bigoplus_{\substack{\Sigma \overset{1}{\hookrightarrow} S \\ S\in\mathscr{S}(\delta\mathscr{B})}} A^S_{i-1,j}(\delta\mathscr{B}) \arrow[r] \arrow[d] & \displaystyle\bigoplus_{\substack{\Sigma \overset{2}{\hookrightarrow} T \\ T\in\mathscr{S}(\delta\mathscr{B})}} A^T_{i-2,j}(\delta\mathscr{B}) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A^{\Sigma}_{i,j}(\mathscr{B}) \arrow[r] \arrow[d] & \displaystyle\bigoplus_{\Sigma \overset{1}{\hookrightarrow} S} A^S_{i-1,j}(\mathscr{B}) \arrow[r] \arrow[d] & \displaystyle\bigoplus_{\Sigma \overset{2}{\hookrightarrow} T} A^T_{i-2,j}(\mathscr{B}) \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & A^{\Sigma}_{i-1,j}(\rho\mathscr{B}) \arrow[r] \arrow[d] & \displaystyle\bigoplus_{\substack{\Sigma \overset{1}{\hookrightarrow} S \\ S\in\mathscr{S}(\rho\mathscr{B})}} A^S_{i-2,j}(\rho\mathscr{B}) \arrow[r] \arrow[d] & \displaystyle\bigoplus_{\substack{\Sigma \overset{2}{\hookrightarrow} T \\ T\in\mathscr{S}(\rho\mathscr{B})}} A^T_{i-3,j}(\rho\mathscr{B}) \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We use the following facts:

– the top row is exact, since $\delta\mathscr{B}$ is exact;

– the middle row and the bottom row are exact at the two first places by definition of the Orlik-Solomon bi-complex;

– the second and third columns are exact by the induction hypothesis and the special cases.

Then a diagram chase shows that the first column is exact. Note that in the case $\chi(\Sigma) = \mu$, we use the fact that $\rho\mathscr{B}$ is exact.

2. It is a direct consequence of the first point and the long exact sequence in cohomology.

□

4.2.4 The case of tame bi-arrangements of hyperplanes

Let $\mathscr{B} = (\mathscr{L}, \mathscr{M}, \chi)$ be a tame bi-arrangement of hyperplanes with $\mathscr{L} = \{L_1, \ldots, L_l\}$ and $\mathscr{M} = \{M_1, \ldots, M_m\}$. For the sake of convenience, we discuss the deletion and restriction with respect to L_l (resp. M_m).

Proposition 4.2.10. *1. We assume that the deletion $\delta\mathscr{B}$ of $\mathscr{B}$ with respect to L_l is color-consistent and tame. The deletion morphism $\delta : A_{\bullet,\bullet}(\delta\mathscr{B}) \to A_{\bullet,\bullet}(\mathscr{B})$ is then induced by the natural inclusion $\Lambda^\bullet(e_1, \ldots, e_{l-1}) \otimes \Lambda^\bullet(f_1^\vee, \ldots, f_m^\vee) \hookrightarrow \Lambda^\bullet(e_1, \ldots, e_l) \otimes \Lambda^\bullet(f_1^\vee, \ldots, f_m^\vee)$.*

2. We assume that the restriction $\rho\mathscr{B}$ of $\mathscr{B}$ with respect to L_l is tame. The restriction morphism $\rho : A_{\bullet,\bullet}(\mathscr{B}) \to A_{\bullet-1,\bullet}(\rho\mathscr{B})$ is then induced, for $I \subset \{1, \ldots, l-1\}$ and $J \subset \{1, \ldots, m\}$, by $\rho(e_I \otimes f_J^\vee) = 0$ and $\rho((e_I \wedge e_l) \otimes f_J^\vee) = e_I \otimes f_J^\vee$.

3. We assume that the deletion $\delta\mathscr{B}$ of $\mathscr{B}$ with respect to M_m is color-consistent and tame. The deletion morphism $\delta : A_{\bullet,\bullet}(\mathscr{B}) \to A_{\bullet,\bullet}(\delta\mathscr{B})$ is then induced by the natural projection $\Lambda^\bullet(e_1, \ldots, e_l) \otimes \Lambda^\bullet(f_1^\vee, \ldots, f_m^\vee) \twoheadrightarrow \Lambda^\bullet(e_1, \ldots, e_l) \otimes \Lambda^\bullet(f_1^\vee, \ldots, f_{m-1}^\vee)$.

4. We assume that the restriction $\rho\mathscr{B}$ of $\mathscr{B}$ with respect to M_m is tame. The restriction morphism $\rho : A_{\bullet,\bullet-1}(\rho\mathscr{B}) \to A_{\bullet,\bullet}(\mathscr{B})$ is then induced, for $I \subset \{1, \ldots, l\}$ and $J \subset \{1, \ldots, m-1\}$, by $\rho(e_I \otimes f_J^\vee) = e_I \otimes (f_J^\vee \wedge f_m^\vee)$.

Proof. The details are left to the reader. In every case, one only needs to show that the morphisms described pass to the subquotient and are compatible with the differentials. □

4.3 Bi-arrangements of hypersurfaces

4.3.1 Arrangements of hypersurfaces and resolution of singularities

We fix a complex manifold X. An *arrangement of hypersurfaces* in X is a finite set $\mathscr{A}$ of smooth hypersurfaces of X which is locally an arrangement of hyperplanes. More precisely, it means that around every point $p \in X$ we can find a system of local coordinates centered at p such that all hypersurfaces $K \in \mathscr{A}$ are defined by a linear equation.

Example 4.3.1. A (simple) normal crossing divisor in X is a special case of an arrangement of hypersurfaces. In this case, we can find local coordinates around every point such that all hypersurfaces are defined by the vanishing of a coordinate.

Example 4.3.2. 1. An arrangement of hyperplanes in $\mathbb{C}^n$ is an arrangement of hypersurfaces. More generally, a finite set of hyperplanes of $\mathbb{C}^n$ that do not necessarily pass through the origin is an arrangement of hypersurfaces.

2. A finite set of hyperplanes of $\mathbb{P}^n(\mathbb{C})$ is an arrangement of hypersurfaces.

3. If Y is a Riemann surface and $X = Y^n$ is the n-fold cartesian power of Y, then there are distinguished hypersurfaces in X: the diagonals $\{y_i = y_j\}$, and the hypersurfaces $\{y_i = a\}$ where $a \in Y$ is a point. Any finite set of such hypersurfaces is an arrangement of hypersurfaces. In the context of motivic periods, these arrangements of hypersurfaces have been studied by S. Bloch [Blo12].

A *stratum* of $\mathscr{A}$ is a connected component of a non-empty intersection $K_I = \bigcap_{i\in I} K_i$ of some hypersurfaces $K_i \in \mathscr{A}$. It is a submanifold of X. For instance, the whole space $X = K_\varnothing$ is always a stratum of $\mathscr{A}$, and the other strata are called *strict*. A stratum S is *reducible* (resp. *irreducible*) if it is reducible (resp. irreducible) locally around every point $p \in S$. Every hypersurface $K \in \mathscr{A}$ is irreducible; if they are the only irreducible strata, then $\mathscr{A}$ is a normal crossing divisor.

The class of arrangements of hypersurfaces is closed under blow-ups along a certain class of strata, that we now introduce.

Definition 4.3.3. Let $\mathscr{A}$ be an arrangement of hyperplanes in $\mathbb{C}^n$. A strict stratum Z of $\mathscr{A}$ is *good* if there exists a stratum U and a decomposition $Z \pitchfork U$ such that for every hyperplane $K \in \mathscr{A}$, K contains Z or U.

Let $\mathscr{A}$ be an arrangement of hypersurfaces in X. A strict stratum Z of $\mathscr{A}$ is *good* if it is good in the above sense locally around every point $p \in Z$.

For instance, a stratum of dimension 0 (a point) is always good.

Lemma 4.3.4. *Let $\mathscr{A}$ be an arrangement of hypersurfaces in X, and S be a minimal (for the usual inclusion order) irreducible stratum of $\mathscr{A}$. Then S is good.*

Proof. The statement is local, so we can assume that $X = \mathbb{C}^n$ and $\mathscr{A}$ is an arrangement of hyperplanes. Let $M = \bigcap_{K\in\mathscr{A}} K$ be the minimal stratum of $\mathscr{A}$ and $M = S_1 \pitchfork \cdots \pitchfork S_r$ be its decomposition into irreducibles. Then the S_i's are exactly the minimal irreducible strata. We can then assume that $S = S_1$. Let us define $U = S_2 \pitchfork \cdots \pitchfork S_r$. Then we have a decomposition $S \pitchfork U$ and every hyperplane $K \in \mathscr{A}$ contains S or U, hence S is a good stratum. □

Lemma 4.3.5. *Let $\mathscr{A}$ be an arrangement of hypersurfaces in X and Z a good stratum of $\mathscr{A}$ of codimension $\geqslant 2$. Let $\pi : \widetilde{X} \to X$ be the blow-up of X along Z and $E = \pi^{-1}(Z)$ the exceptional divisor. We write $\widetilde{Y}$ for the strict transform of a submanifold $Y \subset X$. Then*

1. *The set $\widetilde{\mathscr{A}} = \{E\} \cup \{\widetilde{K}, K \in \mathscr{A}\}$ is an arrangement of hypersurfaces in $\widetilde{X}$.*

2. *The strata of $\widetilde{\mathscr{A}}$ are of the form $\widetilde{S}$ or $E \cap \widetilde{S}$, for strata S of $\mathscr{A}$ that are not contained in Z.*

3. *The irreducible strata of $\mathscr{A}$ are E and the strict transforms $\widetilde{S}$ of the irreducible strata S of $\mathscr{A}$ that are not contained in Z.*

Definition 4.3.6. We call $\widetilde{\mathscr{A}} = \{E\} \cup \{\widetilde{K}, K \in \mathscr{A}\}$ the *blow-up* of $\mathscr{A}$ along Z.

Proof. The statement is local, so we assume that $X = \mathbb{C}^n$ and $\mathscr{A}$ is an arrangement of hyperplanes. Since Z is a good stratum, we can choose coordinates $(z_1, \ldots, z_n)$ such that $Z = \{z_1 = \cdots = z_r = 0\}$ for some integer r, and such that the hyperplanes $K \in \mathscr{A}$ are given by equations of the form $\alpha_1 z_1 + \cdots + \alpha_r z_r = 0$ or $\alpha_{r+1} z_{r+1} + \cdots + \alpha_n z_n = 0$.

1. We have r local charts for the blow-up $\pi : \widetilde{X} \to X$, given for $k = 1, \ldots, r$ by

$$\pi_k(z_1, \ldots, z_r) = (z_k z_1, \ldots, z_k z_{k-1}, z_k, z_k z_{k+1}, \ldots, z_k z_r, z_{z+1}, \ldots, z_n).$$

In such a chart, the exceptional divisor is $E = \{z_k = 0\}$; the strict transform of $K = \{\alpha_1 z_1 + \cdots + \alpha_r z_r = 0\}$ is $\widetilde{K} = \{\alpha_1 z_1 + \cdots + \alpha_{k-1} z_{k-1} + \alpha_k + \alpha_{k+1} z_{k+1} + \cdots + \alpha_r z_r = 0\}$; the strict transform of $K = \{\alpha_{r+1} z_{r+1} + \cdots + \alpha_n z_n = 0\}$ is $\widetilde{K} = \{\alpha_{r+1} z_{r+1} + \cdots + \alpha_n z_n = 0\}$. All these equations are linear, hence the result.

2. For S a stratum of $\mathscr{A}$, it is easy to show using the above local charts that we have

$$\widetilde{S} = \varnothing \;\Leftrightarrow\; E \cap \widetilde{S} = \varnothing \;\Leftrightarrow\; S \subset Z$$

hence the result.

3. The exceptional divisor E is obviously irreducible. Now let us fix a stratum S of $\mathscr{A}$ not contained in Z. Then it is easy to see using the above local charts that $E \pitchfork \widetilde{S}$ and that for every $K \in \mathscr{A}$, $E \cap \widetilde{S} \subset \widetilde{K} \Rightarrow S \subset K$; thus, $E \cap \widetilde{S}$ is reducible if S is not the whole space $\mathbb{C}^n$. We are left with proving that $\widetilde{S}$ is irreducible if and only if S is irreducible. It it easy to see that a decomposition $S = A \pitchfork B$ gives a decomposition $\widetilde{S} = \widetilde{A} \pitchfork \widetilde{B}$ and vice versa, hence the result.

□

Blow-ups along good strata are enough to resolve the singularities of hypersurface arrangements, as the next theorem shows.

Theorem 4.3.7. *Let $\mathscr{A}$ be an arrangement of hypersurfaces in X. We inductively define a sequence of complex manifolds $X^{(k)}$ and arrangements of hypersurfaces $\mathscr{A}^{(k)}$ inside $X^{(k)}$, via the following process.*

(a) $X^{(0)} = X$ and $\mathscr{A}^{(0)} = \mathscr{A}$;

(b) for $k \geqslant 0$, let $Z^{(k)}$ be a minimal irreducible stratum of $\mathscr{A}^{(k)}$ of codimension $\geqslant 2$, $X^{(k+1)} \to X^{(k)}$ the blow-up of $X^{(k)}$ along $Z^{(k)}$. We let $\mathscr{A}^{(k+1)} = \widetilde{\mathscr{A}^{(k)}}$ be the blow-up of $\mathscr{A}^{(k)}$ along $Z^{(k)}$.

After a finite number of steps, we get a normal crossing divisor $\mathscr{A}^{(\infty)}$ inside $X^{(\infty)}$.

Proof. The process is well-defined according to Lemma 4.3.4 and Lemma 4.3.5. For $k \geqslant 0$, let $\mathscr{I}^{(k)}$ be the set of irreducible strata of $\mathscr{A}^{(k)}$ of codimension $\geqslant 2$. Then $Z^{(k)}$ is a minimal element of $\mathscr{I}^{(k)}$, and $\mathscr{I}^{(k+1)}$ consists of the strict transforms of the other elements of $\mathscr{I}^{(k)}$. Thus, we get $|\mathscr{I}^{(k+1)}| = |\mathscr{I}^{(k)}| - 1$. After a finite number of steps, we end up with an arrangement $\mathscr{A}^{(\infty)}$ inside $X^{(\infty)}$ such that $\mathscr{I}^{(\infty)}$ is empty, hence $\mathscr{A}^{(\infty)}$ is a normal crossing divisor. □

4.3.2 The motive of a bi-arrangement of hypersurfaces

Definition 4.3.8. Let X be a complex manifold. A *bi-arrangement of hypersurfaces* $\mathscr{B} = (\mathscr{A}, \chi)$ in X is the data of an arrangement of hypersurfaces $\mathscr{A}$ in X along with a coloring function

$$\chi : \mathscr{S}_+(\mathscr{A}) \to \{\lambda, \mu\}$$

such that the Künneth condition (4.1) is satisfied locally around every point of X.

As for bi-arrangements of hyperplanes, only the colors of the irreducible strata will matter, and thus we will consider bi-arrangements of hypersurfaces up to equivalence (see Definition 4.1.3).

We will also use the notational conventions 4.1.5 and 4.1.6 in the context of bi-arrangements of hypersurfaces. When the underlying arrangement of hypersurfaces is a normal crossing divisor, then χ is only determined (up to equivalence) by the colors $\chi(K)$ of the hypersurfaces $K \in \mathscr{B}$, hence we can simply write $\mathscr{B} = (\mathscr{L}, \mathscr{M})$.

We also define the dual $\mathscr{B}^\vee$ of a bi-arrangement of hypersurfaces.

Let $\mathscr{B} = (\mathscr{A}, \chi)$ be a bi-arrangement of hypersurfaces in a complex manifold X, and Z be a good stratum of $\mathscr{B}$ of codimension $\geqslant 2$. Let $\pi : \widetilde{X} \to X$ be the blow-up of X along Z, and $E = \pi^{-1}(Z)$ be the exceptional divisor. Let $\widetilde{\mathscr{A}} = \{E\} \cup \{\widetilde{K}\,,\, K \in \mathscr{A}\}$ be the blow-up of $\mathscr{A}$ along Z. Then we define a bi-arrangement of hypersurfaces $\widetilde{\mathscr{B}} = (\widetilde{\mathscr{A}}, \widetilde{\chi})$ in $\widetilde{X}$ whose underlying arrangement of hypersurfaces is $\widetilde{\mathscr{A}} = \{E\} \cup \{\widetilde{K}\,,\, K \in \mathscr{A}\}$. We define the coloring $\widetilde{\chi}$ only on the irreducible strata: we set $\widetilde{\chi}(E) = \chi(Z)$, and for an irreducible stratum S not contained in Z, we set $\widetilde{\chi}(\widetilde{S}) = \chi(S)$.

Definition 4.3.9. We call $\widetilde{\mathscr{B}} = (\widetilde{\mathscr{A}}, \widetilde{\chi})$ the *blow-up* of $\mathscr{B}$ along Z.

If Z is irreducible (which will be our main case of interest) then the blow-up is a well-defined operation among equivalence classes of bi-arrangements of hypersurfaces.

Let $\mathscr{B}$ be a bi-arrangement of hypersurfaces in a complex manifold X. We inductively define a sequence of complex manifolds $X^{(k)}$ and bi-arrangements of hypersurfaces $\mathscr{B}^{(k)}$ inside $X^{(k)}$, via the following process.

(a) $X^{(0)} = X$ and $\mathscr{B}^{(0)} = \mathscr{B}$;

(b) for $k \geqslant 0$, let $Z^{(k)}$ be a minimal irreducible stratum of $\mathscr{B}^{(k)}$ of codimension $\geqslant 2$, $X^{(k+1)} \to X^{(k)}$ the blow-up of $X^{(k)}$ along $Z^{(k)}$. We let $\mathscr{B}^{(k+1)} = \widetilde{\mathscr{B}^{(k)}}$ be the blow-up of $\mathscr{B}^{(k)}$ along $Z^{(k)}$.

As in the case of arrangements of hypersurfaces, we get after a finite number of steps a bi-arrangement of hypersurfaces $\mathscr{B}^{(\infty)}$ inside $X^{(\infty)}$, whose underlying arrangement of hypersurfaces is a normal crossing divisor. We write $\mathscr{B}^{(\infty)} = (\mathscr{L}^{(\infty)}, \mathscr{M}^{(\infty)})$, with $\mathscr{L}^{(\infty)} \cup \mathscr{M}^{(\infty)}$ a normal crossing divisor. By an abuse of notation, we write $\mathscr{L}^{(\infty)}$ (resp. $\mathscr{M}^{(\infty)}$) for the union of all the hypersurfaces $K \in \mathscr{L}^{(\infty)}$ (resp. $K \in \mathscr{M}^{(\infty)}$).

Definition 4.3.10. The *motive* of the bi-arrangement of hypersurfaces $\mathscr{B}$ is the collection of relative cohomology groups (see (4.14))

$$H^\bullet(\mathscr{B}) = H^\bullet(X^{(\infty)} \setminus \mathscr{L}^{(\infty)}, \mathscr{M}^{(\infty)} \setminus \mathscr{M}^{(\infty)} \cap \mathscr{L}^{(\infty)}).$$

If X is a smooth complex variety, then $H^\bullet(\mathscr{B})$ is endowed with a mixed Hodge structure.

At each step of the blow-up, we choose a minimal irreducible stratum of codimension $\geqslant 2$. One could worry that the resulting cohomology group $H^\bullet(\mathscr{B})$ depends on the resulting order of the blow-ups. It is not the case, as the work of Li shows [Li09, Theorem 1.3].

Example 4.3.11. 1. If $\mathscr{A}$ is a hypersurface arrangement in X, we have

$$H^\bullet(\mathscr{A}, \lambda) \cong H^\bullet(X \setminus \mathscr{A}) \quad \text{and} \quad H^\bullet(\mathscr{A}, \mu) \cong H^\bullet(X, \mathscr{A}).$$

2. For $\mathscr{B} = (\mathscr{L}, \mathscr{M})$ a normal crossing divisor, then there is no blow-up and we simply have

$$H^\bullet(\mathscr{L}, \mathscr{M}) = H^\bullet(X \setminus \mathscr{L}, \mathscr{M} \setminus \mathscr{M} \cap \mathscr{L}).$$

Remark 4.3.12. There is also the compactly-supported version (see (4.18))

$$H_c^\bullet(\mathscr{B}) = H_c^\bullet(X^{(\infty)} \setminus \mathscr{L}^{(\infty)}, \mathscr{M}^{(\infty)} \setminus \mathscr{M}^{(\infty)} \cap \mathscr{L}^{(\infty)}).$$

Putting $n = \dim_{\mathbb{C}}(X)$, the duality of bi-arrangements is viewed as a Poincaré-Verdier duality isomorphism (Proposition 4.7.4)

$$H^k(\mathscr{B}^\vee) \cong \left(H_c^{2n-k}(\mathscr{B})\right)^\vee.$$

4.3.3 The Orlik-Solomon bi-complex, and blow-ups

Let $\mathscr{B}$ be a bi-arrangement of hypersurfaces in a complex manifold X. The definition of the *Orlik-Solomon bi-complex* of $\mathscr{B}$ may be repeated word for word from the local case: we start with $A^X_{0,0}(\mathscr{B}) = \mathbb{Q}$ and define the bi-complexes $A^{\leqslant\Sigma}_{\bullet,\bullet}(\mathscr{B})$ by induction on the codimension of Σ. Note that $A^{\leqslant\Sigma}_{\bullet,\bullet}(\mathscr{B})$ only depends on the hypersurfaces that contain Σ and can be computed in a local chart around any point of Σ. We say that a bi-arrangement of hypersurfaces is *exact* if all its strict strata are exact in the sense of Definition 4.1.16.

It is worth noting that although every $A^{\leqslant\Sigma}_{\bullet,\bullet}(\mathscr{B})$ is a bi-complex, the direct sum $\bigoplus_S A^S_{\bullet,\bullet}$ is not in general. For instance, if $\mathscr{B}$ is made of two non-intersecting hypersurfaces, one colored λ and the other colored μ, we get the following non-commutative square.

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{Q} \arrow[d, "\mathrm{id}"] \\ 0 \arrow[r] & \mathbb{Q} \end{tikzcd}$$

Let now Z be a good stratum of $\mathscr{B}$, $\pi : \widetilde{X} \to X$ be the blow-up along Z, $E = \pi^{-1}(Z)$ be the exceptional divisor, and $\widetilde{\mathscr{B}}$ be the blow-up of $\mathscr{B}$ along Z. The following Proposition, which will be crucial in the sequel, expresses the Orlik-Solomon bi-complex of $\widetilde{\mathscr{B}}$ in terms of that of $\mathscr{B}$.

Proposition 4.3.13. *Let us assume that $\chi(Z) = \lambda$. We have isomorphisms, for S a stratum of $\mathscr{B}$ that is not contained in Z:*

$$A^{\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}}) \cong A^S_{i,j}(\mathscr{B}) \quad \textit{and} \quad A^{E\cap\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}}) \cong A^S_{i-1,j}(\mathscr{B}).$$

They are compatible with the differentials in that we have the following commutative diagrams.

1. *For the inclusions $\widetilde{S} \overset{1}{\hookrightarrow} \widetilde{T}$:*

$$\begin{tikzcd} A^{\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}}) \arrow[r, "d'_{\widetilde{S},\widetilde{T}}"] \arrow[d, leftrightarrow, "\cong"'] & A^{\widetilde{T}}_{i-1,j}(\widetilde{\mathscr{B}}) \arrow[d, leftrightarrow, "\cong"] \\ A^S_{i,j}(\mathscr{B}) \arrow[r, "d'_{S,T}"'] & A^T_{i-1,j}(\mathscr{B}) \end{tikzcd} \qquad \begin{tikzcd} A^{\widetilde{T}}_{i,j-1}(\widetilde{\mathscr{B}}) \arrow[r, "d''_{\widetilde{S},\widetilde{T}}"] \arrow[d, leftrightarrow, "\cong"'] & A^{\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}}) \arrow[d, leftrightarrow, "\cong"] \\ A^T_{i,j-1}(\mathscr{B}) \arrow[r, "d''_{S,T}"'] & A^S_{i,j}(\mathscr{B}) \end{tikzcd}$$

2. *For the inclusions $E\cap\widetilde{S} \overset{1}{\hookrightarrow} E\cap\widetilde{T}$:*

$$\begin{tikzcd} A^{E\cap\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}}) \arrow[r, "d'_{E\cap\widetilde{S},E\cap\widetilde{T}}"] \arrow[d, leftrightarrow, "\cong"'] & A^{E\cap\widetilde{T}}_{i-1,j}(\widetilde{\mathscr{B}}) \arrow[d, leftrightarrow, "\cong"] \\ A^S_{i-1,j}(\mathscr{B}) \arrow[r, "-d'_{S,T}"'] & A^T_{i-2,j}(\mathscr{B}) \end{tikzcd} \qquad \begin{tikzcd} A^{E\cap\widetilde{T}}_{i,j-1}(\widetilde{\mathscr{B}}) \arrow[r, "d''_{E\cap\widetilde{S},E\cap\widetilde{T}}"] \arrow[d, leftrightarrow, "\cong"'] & A^{E\cap\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}}) \arrow[d, leftrightarrow, "\cong"] \\ A^T_{i-1,j-1}(\mathscr{B}) \arrow[r, "d''_{S,T}"'] & A^S_{i-1,j}(\mathscr{B}) \end{tikzcd}$$

3. *For the inclusions $E\cap\widetilde{S} \overset{1}{\hookrightarrow} \widetilde{S}$:*

$$\begin{tikzcd} A^{E\cap\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}}) \arrow[r, "d'_{E\cap\widetilde{S},\widetilde{S}}"] \arrow[d, leftrightarrow, "\cong"'] & A^{\widetilde{S}}_{i-1,j}(\widetilde{\mathscr{B}}) \arrow[d, leftrightarrow, "\cong"] \\ A^S_{i-1,j}(\mathscr{B}) \arrow[r, equal, "\mathrm{id}"'] & A^S_{i-1,j}(\mathscr{B}) \end{tikzcd} \qquad \begin{tikzcd} A^{\widetilde{S}}_{i,j-1}(\widetilde{\mathscr{B}}) \arrow[r, "d''_{E\cap\widetilde{S},\widetilde{S}}"] \arrow[d, leftrightarrow, "\cong"'] & A^{E\cap\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}}) \arrow[d, leftrightarrow, "\cong"] \\ A^S_{i,j-1}(\mathscr{B}) \arrow[r, "0"'] & A^S_{i-1,j}(\mathscr{B}) \end{tikzcd}$$

The case $\chi(Z) = \mu$ is dual.

Proof. We have an isomorphism $\pi : \widetilde{X} \setminus E \stackrel{\cong}{\to} X \setminus Z$. Let us recall that the construction of the Orlik-Solomon bi-complex is local. Let S be a stratum of $\mathscr{B}$ that is not contained in Z, $p \in S \setminus S \cap Z$, $\widetilde{p} = \pi^{-1}(p) \in \widetilde{S}$. Around the point $\widetilde{p}$, the local situation is the same as the one around the point p, hence the first isomorphism.
For the second isomorphism, we see using local coordinates as in the proof of Lemma 4.3.5 that the local situation around a point of $E \cap \widetilde{S}$ is that of a decomposition $E \pitchfork \widetilde{S}$. Thus, the Künneth formula (Proposition 4.1.18) implies that we have

$$A_{i,j}^{E\cap\widetilde{S}}(\widetilde{\mathscr{B}}) \cong \left(A_{1,0}^{E}(\widetilde{\mathscr{B}}) \otimes A_{i-1,j}^{\widetilde{S}}(\widetilde{\mathscr{B}})\right) \oplus \left(A_{0,1}^{E}(\widetilde{\mathscr{B}}) \otimes A_{i,j-1}^{\widetilde{S}}(\widetilde{\mathscr{B}})\right).$$

Since we have $\chi(E) = \lambda$, we have $A_{1,0}^{E}(\widetilde{\mathscr{B}}) = \mathbb{Q}$ and $A_{0,1}^{E}(\widetilde{\mathscr{B}}) = 0$. Hence the second isomorphism follows from the first isomorphism $A_{i-1,j}^{\widetilde{S}}(\widetilde{\mathscr{B}}) \cong A_{i-1,j}^{S}(\mathscr{B})$.
The compatibility with the differentials is easy. One only has to note the minus sign in front of $d'_{S,T}$ which follows from the Koszul sign rule in a tensor product of two (bi-)complexes. □

Corollary 4.3.14. *If $\mathscr{B}$ is exact, then $\widetilde{\mathscr{B}}$ is exact.*

Proof. According to Corollary 4.1.19 it is enough to check the exactness of the strata $\widetilde{S}$, for S an irreducible stratum of $\mathscr{B}$ not contained in Z. Proposition 4.3.13 implies that we have $A_{\bullet,\bullet}^{\widetilde{S}}(\widetilde{\mathscr{B}}) \cong A_{\bullet,\bullet}^{S}(\mathscr{B})$, hence the result. □

4.4 The geometric Orlik-Solomon bi-complex and the main theorem

4.4.1 The geometric Orlik-Solomon bi-complex

We fix a complex manifold X and a bi-arrangement of hypersurfaces $\mathscr{B}$ in X. We fix an integer q. Let us write, for $S \in \mathscr{S}_{i+j}(\mathscr{B})$,

$${}^{(q)}D_{i,j}^{S}(\mathscr{B}) = H^{q-2i}(S)(-i) \otimes A_{i,j}^{S}(\mathscr{B}).$$

If X is a smooth complex variety and the hypersurfaces $K \in \mathscr{B}$ are smooth divisors (we call this the "algebraic case"), then this is endowed with a mixed Hodge structure. If furthermore X is projective, it is a pure Hodge structure of weight q.

Let $\iota_S^T : S \stackrel{1}{\hookrightarrow} T$ be an inclusion of strata of $\mathscr{B}$, with $S \in \mathscr{S}_{i+j}(\mathscr{B})$ and $T \in \mathscr{S}_{i+j-1}(\mathscr{B})$. We refer the reader to Appendix 4.8 for details on Gysin morphisms and pull-backs.

– We have the Gysin morphism $(\iota_S^T)_* : H^{q-2i}(S)(-i) \to H^{q-2i+2}(T)(-i+1)$. We then define a morphism [2]

$$d'_{S,T} : {}^{(q)}D_{i,j}^{S}(\mathscr{B}) \to {}^{(q)}D_{i-1,j}^{T}(\mathscr{B})$$

by the formula

$$d'_{S,T}(s \otimes X) = (\iota_S^T)_*(s) \otimes d'_{S,T}(X)$$

for $s \in H^{q-2i}(S)(-i)$ and $X \in A_{i,j}^{S}(\mathscr{B})$.

2. We make an abuse of notation by denoting by the same symbols $d'_{S,T}$ and $d''_{S,T}$ the differentials in the Orlik-Solomon bi-complex and in the geometric Orlik-Solomon bi-complex; no confusion should arise.

– We have the pull-back morphism $(\iota_S^T)^* : H^{q-2i}(T)(-i) \to H^{q-2i}(S)(-i)$. We then define a morphism
$$d''_{S,T} : {}^{(q)}D^T_{i,j-1}(\mathscr{B}) \to {}^{(q)}D^S_{i,j}(\mathscr{B})$$
by the formula
$$d''_{S,T}(t \otimes X) = (\iota_S^T)^*(t) \otimes d''_{S,T}(X)$$
for $t \in H^{q-2i}(T)(-i)$ and $X \in A^T_{i,j-1}(\mathscr{B})$.

Let us now set
$${}^{(q)}D_{i,j}(\mathscr{B}) = \bigoplus_{S \in \mathscr{S}_{i+j}(\mathscr{B})} {}^{(q)}D^S_{i,j}(\mathscr{B}).$$

The above morphisms induce
$$d' : {}^{(q)}D_{\bullet,\bullet}(\mathscr{B}) \to {}^{(q)}D_{\bullet-1,\bullet}(\mathscr{B})$$
and
$$d'' : {}^{(q)}D_{\bullet,\bullet-1}(\mathscr{B}) \to {}^{(q)}D_{\bullet,\bullet}(\mathscr{B}).$$

If X is a smooth complex variety, d' and d'' are morphisms of mixed Hodge structures. We will prove the following theorem in the rest of this section.

Theorem 4.4.1. *The differentials d' and d'' make ${}^{(q)}D_{\bullet,\bullet}(\mathscr{B})$ into a bi-complex.*

Definition 4.4.2. We call ${}^{(q)}D_{\bullet,\bullet}(\mathscr{B})$ the *geometric Orlik-Solomon bi-complex* of index q of $\mathscr{B}$. We will denote by ${}^{(q)}D_{\bullet}(\mathscr{B})$ its total complex, and call it the *geometric Orlik-Solomon complex* of index q:
$${}^{(q)}D_n(\mathscr{B}) = \bigoplus_{i-j=n} {}^{(q)}D_{i,j}(\mathscr{B}).$$

Example 4.4.3.

1. Let $\mathscr{A}$ be an arrangement of hypersurfaces in X. Then the geometric Orlik-Solomon bi-complexes for $(\mathscr{A}, \lambda)$ are concentrated in bi-degrees $(n, 0)$ with
$${}^{(q)}D_{n,0}(\mathscr{A}, \lambda) = \bigoplus_{S \in \mathscr{S}_n(\mathscr{A})} H^{q-2n}(S)(-n) \otimes A_n(\mathscr{A}).$$
Up to a shift, it is the same as the Gysin complex defined in Chapter 3. Dually, the geometric Orlik-Solomon bi-complexes for $(\mathscr{A}, \mu)$ are concentrated in bi-degrees $(0, n)$ with
$${}^{(q)}D_{0,n}(\mathscr{A}, \mu) = \bigoplus_{S \in \mathscr{S}_n(\mathscr{A})} H^q(S) \otimes (A_n(\mathscr{A}))^\vee .$$

2. If $\mathscr{B} = (\mathscr{L}, \mathscr{M})$ is a normal crossing divisor with $\mathscr{L} = \{L_1, \ldots, L_l\}$ and $\mathscr{M} = \{M_1, \ldots, M_m\}$, then we get
$${}^{(q)}D_{i,j}(\mathscr{L}, \mathscr{M}) = \bigoplus_{\substack{|I|=i \\ |J|=j}} H^{q-2i}(L_I \cap M_J)(-i)$$
and the Orlik-Solomon complexes ${}^{(q)}D_{\bullet}(\mathscr{L}, \mathscr{M})$ form the E_1 page of the spectral sequence (4.15) described in Appendix 4.7.

In the rest of this section, we prove Theorem 4.4.1 by showing that in ${}^{(q)}D_{\bullet,\bullet}(\mathscr{B})$ we have the equalities $d' \circ d' = 0$, $d'' \circ d'' = 0$ (Lemma 4.4.4) and $d' \circ d'' = d'' \circ d'$ (Lemma 4.4.5).

Lemma 4.4.4. *We have $d' \circ d' = 0$ and $d'' \circ d'' = 0$ in ${}^{(q)}D_{\bullet,\bullet}(\mathscr{B})$.*

Proof. By duality, it is enough to prove that $d' \circ d' = 0$. Let $s \otimes X \in H^{q-2i}(S)(-i) \otimes A^S_{i,j}(\mathscr{B})$, we get

$$(d' \circ d')(s \otimes X) = \sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookrightarrow} U} (\iota^U_T)_*(\iota^T_S)_*(s) \otimes d'_{T,U} d'_{S,T}(X).$$

Since $(\iota^U_T)_* \circ (\iota^T_S)_* = (\iota^U_S)_*$, the above sum decomposes as

$$\sum_{S \overset{2}{\hookrightarrow} U} (\iota^U_S)_*(s) \otimes \left(\sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookrightarrow} U} d'_{T,U} d'_{S,T}(X) \right).$$

For U fixed, the right-hand side of the tensor product is zero because $A^{\leqslant S}_{\bullet,\bullet}(\mathscr{B})$ is a bi-complex (Lemma 4.1.15). The result follows. □

The task of proving that $d' \circ d'' = d'' \circ d'$ is more intricate. We fix a stratum S and an element $s \otimes X \in H^{q-2i}(S)(-i) \otimes A^S_{i,j}(\mathscr{B})$. Let us write

$$d' \circ d''(s \otimes X) = \sum_{S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} U} (\iota^U_R)_*(\iota^S_R)^*(s) \otimes d'_{R,U} d''_{R,S}(X) = \Sigma_1 + \Sigma_1'$$

where Σ_1 is the sum over diagrams $S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} U$ with $S \neq U$ and Σ_1' is the sum over diagrams $S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} S$. In the same fashion we write

$$d'' \circ d'(s \otimes X) = \sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookleftarrow} U} (\iota^T_U)^*(\iota^T_S)_*(s) \otimes d''_{U,T} d'_{S,T}(X) = \Sigma_2 + \Sigma_2'$$

where Σ_2 is the sum over diagrams $S \overset{1}{\hookrightarrow} T \overset{1}{\hookleftarrow} U$ with $S \neq U$ and Σ_2' is the sum over diagrams $S \overset{1}{\hookrightarrow} T \overset{1}{\hookleftarrow} S$.

Lemma 4.4.5. *We have the following equalities:*

1. $\Sigma_1 = \Sigma_2$;
2. $\Sigma_1' = 0$;
3. $\Sigma_2' = 0$.

Thus, $d' \circ d'' = d'' \circ d'$ in ${}^{(q)}D_{\bullet,\bullet}(\mathscr{B})$.

Proof. 1. We fix strata $S \neq U$. There are three cases to consider.

First case: $S \cap U = \varnothing$. Then $\Sigma_1 = 0$. For any diagram $S \overset{1}{\hookrightarrow} T \overset{1}{\hookleftarrow} U$, S and U intersect transversely in T, hence by (4.33) the composite $(\iota^T_U)^* \circ (\iota^T_S)_*$ is zero, hence $\Sigma_2 = 0$.

Second case: $S \cap U \neq \varnothing$, and there is no diagram $S \overset{1}{\hookrightarrow} T \overset{1}{\hookleftarrow} U$. Then $\Sigma_2 = 0$. For every diagram $S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} U$ we have $d'_{R,U} d''_{S,R}(X) = 0$ because $A^{\leqslant R}_{\bullet,\bullet}(\mathscr{B})$ is a bi-complex (Lemma 4.1.15), hence $\Sigma_1 = 0$.

Third case: $S \cap U \neq \varnothing$, and there is a diagram $S \overset{1}{\hookrightarrow} T \overset{1}{\hookleftarrow} U$. Then T is unique for dimension reasons (locally around a point of $S \cap U$, T is the sum $S + U$). The diagrams $S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} U$ correspond to the connected components of $S \cap U$. For such a connected component R we have

$$d'_{R,U} d''_{R,S}(X) = d''_{U,T} d'_{S,T}(X)$$

because $A_{\bullet,\bullet}^{\leqslant R}(\mathscr{B})$ is a bi-complex (Lemma 4.1.15). Thus

$$\Sigma_1 = \left(\sum_{S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} U} (\iota_R^U)_* (\iota_R^S)^*(s) \right) \otimes d''_{U,T} d'_{S,T}(X).$$

Using (4.33) we have

$$(\iota_U^T)^* (\iota_S^T)_*(s) = \sum_{S \overset{1}{\hookleftarrow} R \overset{1}{\hookrightarrow} U} (\iota_R^U)_* (\iota_R^S)^*(s)$$

hence $\Sigma_1 = (\iota_U^T)^*(\iota_S^T)_*(s) \otimes d''_{U,T} d'_{S,T}(X) = \Sigma_2$.

2. For an inclusion $R \overset{1}{\hookrightarrow} S$, the fact that $A_{\bullet,\bullet}^{\leqslant R}(\mathscr{B})$ is a bi-complex implies that we have $d'_{R,S} \circ d''_{R,S} = 0$ (Lemma 4.1.15). The result then follows.

3. We have

$$\Sigma_2 = \sum_{S \overset{1}{\hookrightarrow} T} (\iota_S^T)^* (\iota_S^T)_*(s) \otimes d''_{S,T} d'_{S,T}(X).$$

By (4.31), $(\iota_S^T)^*(\iota_S^T)_*(s) =$ is the cup-product $c_1(N_{S/T})\,.\,s$ where $c_1(N_{S/T}) \in H^2(S)(-1)$ is the first Chern class of the normal bundle of the inclusion $S \hookrightarrow T$. We first consider a special case.

Special case: We assume that the stratum S is irreducible. For an inclusion $S \overset{1}{\hookrightarrow} T$, there exists a hypersurface $K \in \mathscr{B}$ such that S is a connected component of the intersection $T \cap K$. According to (4.29), we get $c_1(N_{S/T}) \cong c_1(N_{K/X})_{|S}$. Now Lemma 4.4.6 below implies that $c_1(N_{K/X})_{|S} = c$ is independent of K, hence we can write

$$\Sigma_2 = (c\,.\,s) \otimes \left(\sum_{S \overset{1}{\hookrightarrow} T} d''_{S,T} d'_{S,T}(X) \right).$$

Now the fact that $A_{\bullet,\bullet}^{\leqslant S}(L; M; \chi)$ is a bi-complex (Lemma 4.1.15) implies that the right-hand side of the tensor product is zero, hence the result .

General case: In general there is a (local) decomposition of S into irreducible strata. Let us assume for simplicity that this decomposition has two terms, i.e. we have a (local) decomposition into irreducibles $S = S' \pitchfork S''$. Then an inclusion $S \overset{1}{\hookrightarrow} T$ is (locally) either of the form $T = S' \pitchfork T''$ for $S'' \overset{1}{\hookrightarrow} T''$ or of the form $T = T' \pitchfork S''$ for $S' \overset{1}{\hookrightarrow} T'$. Using the Künneth formula (Proposition 4.1.18) for the Orlik-Solomon bi-complex, we can then split Σ_2 into two sums. One gets the result by applying the same reasoning as in the first case to each of these two sums.

□

We have used the following Lemma.

Lemma 4.4.6. *Let $\mathscr{A}$ be an arrangement of hypersurfaces in a complex manifold X, and S an irreducible stratum of $\mathscr{A}$. Then the line bundles $\left(N_{K/X}\right)_{|S}$, for $K \in \mathscr{A}$ such that $K \supset S$, are all isomorphic.*

Proof. Let us write $\mathscr{A}^{\leqslant S} = \{K_1, \ldots, K_r\}$ for the hypersurfaces of $\mathscr{A}$ that contain S. Let $i, j \in \{1, \ldots, r\}$. We first consider a special case.

Special case: Let us first assume that $\{1, \ldots, r\}$ is a circuit. Let T be the connected component of $K_1 \cap \cdots \cap \widehat{K_i} \cap \cdots \cap \widehat{K_j} \cap \cdots \cap K_r$ that contains S. We then have an inclusion $S \overset{1}{\hookrightarrow} T$, S

being at the same time a connected component of $K_i \cap T$ and $K_j \cap T$. From (4.29) we deduce isomorphisms

$$\left(N_{K_i/X}\right)_{|S} \cong N_{S/T} \cong \left(N_{K_j/X}\right)_{|S}.$$

General case: One can reduce to the special case above by using Lemma 4.1.30. □

4.4.2 Blow-ups and the geometric Orlik-Solomon bi-complex

We now define a morphism between the geometric Orlik-Solomon bi-complex of a bi-arrangement of hypersurfaces $\mathscr{B}$ and that of its blow-up $\widetilde{\mathscr{B}}$. For all the rest of this article, we make the following assumption on bi-arrangements of hypersurfaces:

$$\textit{any intersection of strata is connected} \tag{4.7}$$

(this includes the empty case). Equivalently, this means that the intersection of any number of hypersurfaces $K \in \mathscr{B}$ is connected.

This assumption is not necessary, and we will sketch in §4.6.5 how to deal with the general case. However, working under the assumption (4.7) makes the discussion and the computations more accessible to the reader by keeping the notations light. One can note that (4.7) is satisfied by all the examples of arrangements of hypersurfaces introduced in Example 4.3.2, and is stable by blow-up.

The framework

We fix a bi-arrangement of hypersurfaces $\mathscr{B}$ in a complex manifold X, and a good stratum Z for $\mathscr{B}$.

Definition 4.4.7. An inclusion $S \overset{1}{\hookrightarrow} T$ of strata of $\mathscr{B}$ has *parallel type* with respect to Z if $Z \cap S \neq \varnothing$ and $Z \cap S = Z \cap T$. In this case we write $S \underset{\parallel}{\overset{1}{\hookrightarrow}} T$.

In view of assumption (4.7), $Z \cap S$ is connected and this can be checked locally. Around any point of $Z \cap S$, there is a decomposition $Z \pitchfork W$, hence one has a decomposition $S = S_{\parallel} \pitchfork S_{\perp}$ with $S_{\parallel} \supset Z$ and $S_{\perp} \supset W$. For an inclusion $S \overset{1}{\hookrightarrow} T$ of strata, we then have two mutually exclusive cases.

- $T = T_{\parallel} \pitchfork S_{\perp}$ with $S_{\parallel} \overset{1}{\hookrightarrow} T_{\parallel}$;
- $T = S_{\parallel} \pitchfork T_{\perp}$ with $S_{\perp} \overset{1}{\hookrightarrow} T_{\perp}$.

The parallel type corresponds to the first case: $Z \cap S = Z \cap T = Z \pitchfork S_{\perp}$.

Let $\pi : \widetilde{X} \to X$ be the blow-up along Z. For every stratum S, it restricts to $\pi_S^{\widetilde{S}} : \widetilde{S} \to S$ the blow-up along $Z \cap S$.

In the case $S \underset{\parallel}{\overset{1}{\hookrightarrow}} T$, we have $Z \cap S = Z \cap T$, hence π induces a morphism

$$\pi_{Z\cap S}^{E\cap\widetilde{T}} : E \cap \widetilde{T} \to Z \cap T = Z \cap S.$$

Definition of Φ

Let us assume that we have $\chi(Z) = \lambda$. We recall that we have made explicit the Orlik-Solomon bi-complex of a blow-up in Proposition 4.3.13. Having this in mind, we define a morphism

$$\Phi : {}^{(q)}D_{i,j}(\mathscr{B}) \to {}^{(q)}D_{i,j}(\widetilde{\mathscr{B}}). \tag{4.8}$$

Let $S \in \mathscr{S}_{i+j}(\mathscr{B})$ be a stratum. We define, for $s \otimes X \in H^{q-2i}(S)(-i) \otimes A^S_{i,j}(\mathscr{B}) = {}^{(q)}D^S_{i,j}(\mathscr{B})$,

$$\Phi(s \otimes X) = (\pi^{\widetilde{S}}_S)^*(s) \otimes X + \sum_{\substack{S \overset{1}{\hookrightarrow} T \\ \shortparallel}} (\pi^{E\cap\widetilde{T}}_{Z\cap S})^*(\iota^S_{Z\cap S})^*(s) \otimes d'_{S,T}(X). \tag{4.9}$$

Let us explain more precisely the meaning of this formula:

– the term $(\pi^{\widetilde{S}}_S)^*(s) \otimes X$ lives in $H^{q-2i}(\widetilde{S})(-i) \otimes A^S_{i,j} = {}^{(q)}D^{\widetilde{S}}_{i,j}(\widetilde{\mathscr{B}})$; if $S \subset Z$ then $\widetilde{S} = \varnothing$ and this is zero by convention;

– the term $(\pi^{E\cap\widetilde{T}}_{Z\cap S})^*(\iota^S_{Z\cap S})^*(s) \otimes d'_{S,T}(X)$ lives in $H^{q-2i}(E\cap\widetilde{T})(-i) \otimes A^T_{i-1,j}(\mathscr{B}) = {}^{(q)}D^{E\cap\widetilde{T}}_{i,j}(\widetilde{\mathscr{B}})$.

In the algebraic case, Φ is a morphism of mixed Hodge structures.

The motivation for formula (4.9) comes from the case of normal crossing divisors, as the next Lemma shows.

Lemma 4.4.8. *If $\mathscr{L}\cup\mathscr{M}$ is a normal crossing divisor, then the total complex ${}^{(q)}D_\bullet$ is then $E_1^{-\bullet,q}$ term of the natural spectral sequence (4.15) that computes the relative cohomology groups $H^\bullet(X\setminus \mathscr{L}, \mathscr{M}\setminus\mathscr{M}\cap\mathscr{L})$. In this case, the morphism ${}^{(q)}D_\bullet(\mathscr{L},\mathscr{M}) \to {}^{(q)}D_\bullet(\widetilde{\mathscr{L}},\widetilde{\mathscr{M}})$ induced by Φ is the natural morphism (4.24) that expresses the isomorphism $\pi^* : H^\bullet(X\setminus\mathscr{L}, \mathscr{M}\setminus\mathscr{M}\cap\mathscr{L}) \xrightarrow{\cong} H^\bullet(\widetilde{X}\setminus\widetilde{\mathscr{L}}, \widetilde{\mathscr{M}}\setminus\widetilde{\mathscr{M}}\cap\widetilde{\mathscr{L}})$.*

Proof. It follows from a direct comparison of the formulas since by definition

$$d'(e_I \otimes f_J^\vee) = \sum_{i\in I} \mathrm{sgn}(\{i\}, I\setminus\{i\})\, e_{I\setminus\{i\}} \otimes f_J^\vee.$$

$\square$

If we now assume that $\chi(Z) = \mu$, then we are in the dual situation and we can define a morphism

$$\Psi : {}^{(q)}D_{i,j}(\widetilde{\mathscr{B}}) \to {}^{(q)}D_{i,j}(\mathscr{B}). \tag{4.10}$$

by the formulas

$$\Psi(\widetilde{s}\otimes X) = (\pi^{\widetilde{S}}_S)_*(\widetilde{s}) \otimes X \quad \text{and} \quad \Psi(\widetilde{e}\otimes X) = \sum_{\substack{S \overset{1}{\hookrightarrow} T \\ \shortparallel}} (\iota^S_{Z\cap S})_*(\pi^{E\cap\widetilde{T}}_{Z\cap S})_*(e) \otimes d''_{S,T}(X)$$

for $\widetilde{s}\otimes X \in H^{q-2i}(\widetilde{S})(-i) \otimes A^S_{i,j}(\mathscr{B})$ and $e \otimes X \in H^{q-2i}(E\cap\widetilde{T})(-i) \otimes A^T_{i,j-1}(\mathscr{B})$.

The essential case

Definition 4.4.9. Let $\mathscr{B}$ be a bi-arrangement of hypersurfaces in a complex manifold X. We say that $\mathscr{B}$ is *essential* if the intersection $\bigcap_{K\in\mathscr{B}} K$ of all hypersurfaces in $\mathscr{B}$ is non-empty.

According to assumption (4.7), the intersection $Z = \bigcap_{K\in\mathscr{B}} K$ is the minimal stratum of $\mathscr{B}$. It is necessarily a good stratum. In this case, formula (4.9) takes a simpler form. Indeed, we always have $Z\cap S = Z$, and all inclusions $S \overset{1}{\hookrightarrow} T$ are of the form $S \underset{\|}{\overset{1}{\hookrightarrow}} T$. Hence we get

$$\Phi(s\otimes X) = (\pi_S^{\widetilde{S}})^*(s)\otimes X + \sum_{S\overset{1}{\hookrightarrow}T} (\pi_Z^{E\cap\widetilde{T}})^*(\iota_Z^S)^*(s)\otimes d'_{S,T}(X).$$

If $S = Z$ the formula simply reads, for $z\otimes X \in H^{q-2i}(Z)(-i)\otimes A^Z_{i,j}(\mathscr{B}) = {}^{(q)}D^Z_{i,j}(\mathscr{B})$:

$$\Phi(z\otimes X) = \sum_{Z\overset{1}{\hookrightarrow}T} (\pi_Z^{E\cap\widetilde{T}})^*(z)\otimes d'_{Z,T}(X).$$

4.4.3 The main theorem

The following theorem will be proved in §4.6.

Theorem 4.4.10. *Let $\mathscr{B}$ be a bi-arrangement of hypersurfaces in a complex manifold X, let Z be a good stratum of $\mathscr{B}$ such that $\chi(Z) = \lambda$, and let $\widetilde{\mathscr{B}}$ be the blow-up of $\mathscr{B}$ along Z.*

1. *Formula (4.9) defines a morphism of bi-complexes $\Phi : {}^{(q)}D_{\bullet,\bullet}(\mathscr{B}) \to {}^{(q)}D_{\bullet,\bullet}(\widetilde{\mathscr{B}})$.*

2. *If Z is exact, then the morphism $\Phi : {}^{(q)}D_{\bullet}(\mathscr{B}) \to {}^{(q)}D_{\bullet}(\widetilde{\mathscr{B}})$ induced on the total complexes is a quasi-isomorphism.*

If $\chi(Z) = \mu$, the dual statements are true, with Φ replaced by Ψ defined in (4.10).

It implies the main theorem of this article.

Theorem 4.4.11. *Let $\mathscr{B}$ be an exact bi-arrangement of hypersurfaces in a complex manifold X.*

1. *There is a spectral sequence*
$$E_1^{-p,q}(\mathscr{B}) = {}^{(q)}D_p(\mathscr{B}) \implies H^{-p+q}(\mathscr{B}). \tag{4.11}$$

2. *If X is a smooth complex variety and all hypersurfaces of $\mathscr{B}$ are smooth divisors in X, then this is a spectral sequence in the category of mixed Hodge structures.*

3. *If X is a smooth and projective complex variety, then this spectral sequence degenerates at the E_2 term and we have*
$$E_\infty^{-p,q} \cong E_2^{-p,q} \cong \mathrm{gr}_q^W H^{-p+q}(\mathscr{B}).$$

Proof. 1. Let $X^{(\infty)} = X^{(N)} \to X^{(N-1)} \to \cdots \to X^{(1)} \to X^{(0)} = X$ be the sequence of blow-ups used to define the motive of $\mathscr{B}$ (Definition 4.3.10) and $\mathscr{B}^{(\infty)} = \mathscr{B}^{(N)}$, $\mathscr{B}^{(N-1)}$, …, $\mathscr{B}^{(1)}$, $\mathscr{B}^{(0)} = \mathscr{B}$ be the corresponding bi-arrangements of hypersurfaces, with $\mathscr{B}^{(\infty)} = (\mathscr{L}^{(\infty)},\mathscr{M}^{(\infty)})$ a normal crossing divisor. According to Proposition 4.7.1, there is a spectral sequence
$$E_1^{-p,q} = {}^{(q)}D_p(\mathscr{B}^{(\infty)}) \implies H^{-p+q}(\mathscr{B}).$$
Since $\mathscr{B}$ is exact, Corollary 4.3.14 implies that for each k, $\mathscr{B}^{(k)}$ is exact. Then for each k, Theorem 4.4.10 implies that there is a quasi-isomorphism ${}^{(q)}D_\bullet(\mathscr{B}^{(k)}) \sim {}^{(q)}D_\bullet(\mathscr{B}^{(k+1)})$. Thus we have a quasi-isomorphism ${}^{(q)}D_\bullet(\mathscr{B}) \sim {}^{(q)}D_\bullet(\mathscr{B}^{(\infty)})$ and the result follows.

2. This follows from the analogous statement for normal crossing divisors (Proposition 4.7.1) and the fact that the morphisms (4.8) are morphisms of mixed Hodge structures.

3. This follows from the analogous statement for normal crossing divisors (Proposition 4.7.1). □

The independence of the spectral sequence (4.11) with respect to the choice of the order of blow-ups, and its functoriality with respect to a choice of wonderful compactification [Li09] will be proven in a subsequent article.

4.5 Application to projective bi-arrangements

4.5.1 The setup

Let $\mathscr{A}$ be an arrangement in $\mathbb{C}^{n+1}$ with $n \geqslant 1$. We let $\mathbb{P}\mathscr{A}$ be the corresponding projective arrangement in $\mathbb{P}^n(\mathbb{C})$; it is an arrangement of hypersurfaces consisting of the images $\mathbb{P}K$ of the hyperplanes $K \in \mathscr{A}$ by the projection $\mathbb{C}^{n+1} \setminus 0 \to \mathbb{P}^n(\mathbb{C})$. The strata of $\mathbb{P}\mathscr{A}$ are the images $\mathbb{P}S$ of the strata $S \neq 0$ of $\mathscr{A}$. We implicitly assume that 0 is a stratum of $\mathscr{A}$.

A partial coloring function $\chi : \mathscr{S}_+(\mathscr{A}) \setminus \{0\} \to \{\lambda, \mu\}$ that satisfies the Künneth condition (4.1) gives rise to a *projective bi-arrangement* $\mathbb{P}\mathscr{B} = (\mathbb{P}\mathscr{A}, \chi)$ where we put $\chi(\mathbb{P}S) = \chi(S)$. It is a bi-arrangement of hypersurfaces in $X = \mathbb{P}^n(\mathbb{C})$.

This projective bi-arrangement does not necessarily come from a bi-arrangement $\mathscr{B} = (\mathscr{A}, \chi)$ since the color $\chi(0)$ is not defined. We will write $\mathscr{B}_\lambda$ (resp. $\mathscr{B}_\mu$) for the bi-arrangements $(\mathscr{A}, \chi)$ with $\chi(0) = \lambda$ (resp. $\chi(0) = \mu$), if they are well-defined (i.e. if they satisfy the Künneth condition for the stratum 0).

There is a partial Orlik-Solomon bi-complex $A_{\bullet,\bullet}(\mathscr{B})$ where we have vector spaces $A^S_{i,j}(\mathscr{B})$ for strata $S \neq 0$. If $\mathscr{B}_\lambda$ (resp. $\mathscr{B}_\mu$) are well-defined, then it can be completed to an Orlik-Solomon bi-complex $A_{\bullet,\bullet}(\mathscr{B}_\lambda)$ (resp. $A_{\bullet,\bullet}(\mathscr{B}_\mu)$).

Remark 4.5.1. For a projective space $\mathbb{P}^r(\mathbb{C})$ we have canonical isomorphisms $H^{2k}(\mathbb{P}^r(\mathbb{C})) \cong \mathbb{Q}(-k)$ for $k = 0, \ldots, r$, and $H^{2k+1}(\mathbb{P}^r(\mathbb{C})) = 0$ for all k. Furthermore, for the inclusion $\iota : \mathbb{P}^{r-1}(\mathbb{C}) \hookrightarrow \mathbb{P}^r(\mathbb{C})$ of a projective hyperplane:

– the Gysin morphism $\iota_* : H^{2(k-1)}(\mathbb{P}^{r-1}(\mathbb{C}))(-1) \to H^{2k}(\mathbb{P}^r(\mathbb{C}))$ is the identity of $\mathbb{Q}(-k)$ for $k = 1, \ldots, r$;

– the pull-back morphism $\iota^* : H^{2k}(\mathbb{P}^r(\mathbb{C})) \to H^{2k}(\mathbb{P}^{r-1}(\mathbb{C}))$ is the identity of $\mathbb{Q}(-k)$ for $k = 0, \ldots, r-1$.

The next Proposition expresses the (geometric) Orlik-Solomon bi-complex of $\mathbb{P}\mathscr{B}$ in terms of that of $\mathscr{B}$.

Proposition 4.5.2. *1. We have isomorphisms*

$$A^{\mathbb{P}S}_{i,j}(\mathbb{P}\mathscr{B}) \cong A^S_{i,j}(\mathscr{B})$$

for $S \in \mathscr{S}_{i+j}(\mathscr{B})$, $S \neq 0$, which induce isomorphisms of bi-complexes

$$A^{\leqslant \mathbb{P}\Sigma}_{\bullet,\bullet}(\mathbb{P}\mathscr{B}) \cong A^{\leqslant \Sigma}_{\bullet,\bullet}(\mathscr{B})$$

for $\Sigma \neq 0$ a strict stratum of $\mathscr{B}$.

2. *We have* ${}^{(q)}D^{\mathbb{P}S}_{i,j}(\mathbb{P}\mathscr{B}) = 0$ *for* q *odd. For* $k = 0, \ldots, n$ *we have isomorphisms of pure Hodge structures of weight* $2k$*:*

$$ {}^{(2k)}D^{\mathbb{P}S}_{i,j}(\mathbb{P}\mathscr{B}) \cong \begin{cases} A^S_{i,j}(\mathscr{B})(-k) & \text{if } 0 \leqslant i \leqslant k \text{ and } 0 \leqslant j \leqslant n-k; \\ 0 & \text{otherwise.} \end{cases} $$

Furthermore, these isomorphisms are compatible with the differentials d' *and* d''*.*

Proof. 1. It is trivial.

2. The first statement comes from the fact (Remark 4.5.1) that the projective spaces do not have cohomology in odd degree. For $k = 0, \ldots, n$, we have ${}^{(2k)}D^{\mathbb{P}S}_{i,j} = H^{2(k-i)}(\mathbb{P}S)(-i) \otimes A^{\mathbb{P}S}_{i,j}$. The cohomology group $H^{2(k-i)}(\mathbb{P}S)(-i)$ is non-zero if and only if $0 \leqslant k - i \leqslant n - \mathrm{codim}(S) = n - i - j$, which amounts to $i \leqslant k$ and $j \leqslant n - k$. In this range, we have a canonical isomorphism $H^{2(k-i)}(\mathbb{P}S)(-i) \cong \mathbb{Q}(-k)$, hence the result. The compatibility with the differentials come from Remark 4.5.1.

□

According to Proposition 4.5.2, $\mathbb{P}\mathscr{B}$ is exact if all strict strata $\Sigma \neq 0$ of $\mathscr{B}$ are exact. It is actually convenient to ask for more and make the following definition.

Definition 4.5.3. We say that $\mathbb{P}\mathscr{B}$ is λ*-exact* (resp. μ*-exact*) if $\mathscr{B}_\lambda$ (resp. $\mathscr{B}_\mu$) is well-defined and exact. We say that $\mathbb{P}\mathscr{B}$ is *strongly exact* if it is λ-exact and μ-exact.

We define in the same fashion the concepts of λ-tame, μ-tame and strongly tame projective bi-arrangements; for such bi-arrangements, Theorem 4.1.38 provides an explicit presentation of the Orlik-Solomon bi-complex.

Let us then write ${}^{(k)}A_{\bullet,\bullet}(\mathscr{B})$ for the bi-complex obtained from $A_{\bullet,\bullet}(\mathscr{B})$ by keeping only the rectangle $0 \leqslant i \leqslant k$, $0 \leqslant j \leqslant n - k$. We write ${}^{(k)}A_\bullet(\mathscr{B})$ for the total complex, with

$$ {}^{(k)}A_r(\mathscr{B}) = \bigoplus_{\substack{i-j=r \\ 0\leqslant i\leqslant k \\ 0\leqslant j\leqslant n-k}} A_{i,j}(\mathscr{B}). $$

Theorem 4.5.4. *Let* $\mathbb{P}\mathscr{B}$ *be a projective bi-arrangement in* $\mathbb{P}^n(\mathbb{C})$*.*

1. *If* $\mathbb{P}\mathscr{B}$ *is exact then we have isomorphisms, for* $r = 0, \ldots, 2n$*:*

$$ \mathrm{gr}^W_{2k} H^r(\mathbb{P}\mathscr{B}) \cong H_{2k-r}({}^{(k)}A_\bullet(\mathscr{B})). $$

2. *If* $\mathbb{P}\mathscr{B}$ *is* λ*-exact then we have* $H^r(\mathbb{P}\mathscr{B}) = 0$ *for* $r > n$*. For* $r = 0, \ldots, n$ *we have isomorphisms, for* $k = 0, \ldots, r$*:*

$$ \mathrm{gr}^W_{2k} H^r(\mathbb{P}\mathscr{B}) \cong \mathrm{coker}\left(A_{k+1,r-k-1}(\mathscr{B}_\lambda) \xrightarrow{d''} A_{k+1,r-k}(\mathscr{B}_\lambda) \right). $$

3. *If* $\mathbb{P}\mathscr{B}$ *is* μ*-exact then we have* $H^r(\mathbb{P}\mathscr{B}) = 0$ *for* $r < n$*. For* $r = n, \ldots, 2n$ *we have isomorphisms, for* $k = n - r, \ldots, n$*:*

$$ \mathrm{gr}^W_{2k} H^r(\mathbb{P}\mathscr{B}) \cong \ker\left(A_{r-n+k,n-k+1}(\mathscr{B}_\mu) \xrightarrow{d'} A_{r-n+k-1,n-k+1}(\mathscr{B}_\mu) \right). $$

4. *If $\mathbb{P}\mathscr{B}$ is strongly exact then we have $H^r(\mathbb{P}\mathscr{B})=0$ for $r\neq n$, and we have isomorphisms, for $k=0,\ldots,n$:*

$$\begin{aligned}\mathrm{gr}^W_{2k}H^n(\mathbb{P}\mathscr{B}) &\cong \mathrm{coker}\left(A_{k+1,n-k-1}(\mathscr{B}_\lambda)\xrightarrow{d''}A_{k+1,n-k}(\mathscr{B}_\lambda)\right)\\ &\cong \ker\left(A_{k,n-k+1}(\mathscr{B}_\mu)\xrightarrow{d'}A_{k-1,n-k+1}(\mathscr{B}_\mu)\right).\end{aligned}$$

Proof. 1. This is a consequence of Theorem 4.4.11 and Proposition 4.5.2.

2. The differential $d':A_{k+1,r-k}(\mathscr{B}_\lambda)\to A_{k,r-k}(\mathscr{B}_\lambda)=A_{k,i-k}(\mathscr{B})$ induces a morphism $A_{k+1,r-k}(\mathscr{B}_\lambda)\to H_{2k-r}({}^{(k)}A_\bullet(\mathscr{B}))$. A diagram chase shows that it induces an isomorphism as in the statement.

3. This is the dual of 3.

4. This is a consequence of 2 and 3.

□

4.5.2 Multizeta bi-arrangements

Let $r\geqslant 1$ and $n_1,\ldots,n_r$ be integers with $n_1,\ldots,n_{r-1}\geqslant 1$ and $n_r\geqslant 2$. We let $n=n_1+\cdots+n_r$ and define a n-uple

$$(a_1,\ldots,a_n)=(\underbrace{1,0,\ldots,0}_{n_1},\ldots,\underbrace{1,0,\ldots,0}_{n_r}).$$

In $\mathbb{P}^n(\mathbb{C})$ with projective coordinates $(z_0,z_1,\ldots,z_n)$, we define two arrangements of hyperplanes $\mathscr{L}=\{L_0,\ldots,L_n\}$ and $\mathscr{M}=\{M_0,\ldots,M_n\}$:

– we let $L_0=\{z_0=0\}$ be the hyperplane at infinity, and for $k=1,\ldots,n$, let $L_k=\{z_k=a_kz_0\}$;

– we let $M_0=\{z_1=0\}$, $M_n=\{z_n=z_0\}$, and for $k=1,\ldots,n-1$, $M_k=\{z_k=z_{k+1}\}$.

We define the *multizeta bi-arrangement* $\mathscr{Z}(n_1,\ldots,n_r)=(\mathscr{L},\mathscr{M},\chi)$ where χ is defined to be μ on all $\mathscr{M}$-strata, and λ on all other strata. According to the general formalism of Aomoto polylogarithms, the multiple zeta value $\zeta(n_1,\ldots,n_r)$ is a period of the motive $H^n(\mathscr{Z}(n_1,\ldots,n_r))$, which is the Hodge realization of a mixed Tate motive over $\mathbb{Z}$.

The following Proposition is easily proved by direct inspection.

Proposition 4.5.5. *The multizeta bi-arrangements $\mathscr{Z}(n_1,\ldots,n_r)$ are all λ-tame, hence λ-exact.*

When studying multiple zeta values, the motives $\mathscr{Z}(n_1,\ldots,n_r)$ are alternatives to the approach of [Del89, DG05] via the motivic fundamental group of $\mathbb{P}^1\setminus\{\infty,0,1\}$ (see also [Ter02]). One advantage of such an alternative is that it generalizes to a larger family of integrals. More specifically, let us look at the periods of the moduli spaces $\overline{\mathcal{M}}_{0,n}$ considered by Brown in [Bro09]. They are integrals of a rational function over a simplex $0<t_1<\cdots<t_n<1$, such as

$$\iiint_{0<x<y<z<1}\frac{dx\,dy\,dz}{(1-x)y(z-x)}. \tag{4.12}$$

The main result of [Bro09] is that these integrals are all linear combinations (with rational coefficients) of multiple zeta values, although not in an explicit way. It so happens that the projective bi-arrangement of hyperplanes corresponding to the integral (4.12) is also λ-exact, hence the corresponding motive may be computed explicitly via an Orlik-Solomon bi-complex. This will be studied in more detail in a subsequent article.

4.6 Proof of the main theorem

The goal of this section is to prove the two points of Theorem 4.4.10. We first deal with the essential case (§4.6.1 and §4.6.2) then with the general case (§4.6.3 and §4.6.4). The reader is encouraged to focus on the essential case, since the general case reduces to the essential case at the cost of a few technical trivialities.

4.6.1 The essential case: Φ is a morphism of bi-complexes

In this paragraph, we assume that $\mathscr{B}$ is essential and that Z is the minimal stratum. We prove the first point of Theorem 4.4.10 by showing that Φ is compatible with d' (Proposition 4.6.1) and with d'' (Proposition 4.6.2).

Proposition 4.6.1. *We have $\Phi \circ d' = d' \circ \Phi$.*

Proof. For $s \otimes X \in H^{q-2i}(S)(-i) \otimes A^S_{i,j}(\mathscr{B})$, we compute

$$
\begin{aligned}
(\Phi d')(s \otimes X) = & \sum_{S \overset{1}{\hookrightarrow} T} (\pi^{\widetilde{T}}_T)^*(\iota^T_S)_*(s) \otimes d'_{S,T}(X) \\
& + \sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookrightarrow} U} (\pi^{E\cap\widetilde{U}}_Z)^*(\iota^T_Z)^*(\iota^T_S)_*(s) \otimes d'_{T,U}d'_{S,T}(X).
\end{aligned}
$$

and

$$
\begin{aligned}
(d'\Phi)(s \otimes X) = & \sum_{S \overset{1}{\hookrightarrow} T} (\iota^{\widetilde{T}}_{\widetilde{S}})_*(\pi^{\widetilde{S}}_S)^*(s) \otimes d'_{S,T}(X) \\
& + \sum_{S \overset{1}{\hookrightarrow} T} (\iota^{\widetilde{T}}_{E\cap\widetilde{T}})_*(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(s) \otimes d'_{S,T}(X) \\
& - \sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookrightarrow} U} (\iota^{E\cap\widetilde{U}}_{E\cap\widetilde{T}})_*(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(s) \otimes d'_{T,U}d'_{S,T}(X).
\end{aligned}
$$

The terms $(\ldots) \otimes d'_{S,T}(X)$ cancel because of the equality

$$(\pi^{\widetilde{T}}_T)^*(\iota^T_S)_*(s) = (\iota^{\widetilde{T}}_{\widetilde{S}})_*(\pi^{\widetilde{S}}_S)^*(s) + (\iota^{\widetilde{T}}_{E\cap\widetilde{T}})_*(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(s)$$

which is a special case of (4.34) (set $L = S$ and $X = T$). Thus, it remains to show that we have, for every U fixed,

$$\sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookrightarrow} U} \Delta_T \otimes d'_{T,U}d'_{S,T}(X) = 0$$

where we have set

$$\Delta_T = (\pi^{E\cap\widetilde{U}}_Z)^*(\iota^T_Z)^*(\iota^T_S)_*(s) + (\iota^{E\cap\widetilde{U}}_{E\cap\widetilde{T}})_*(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(s).$$

For S and U fixed, the fact that $A^{\leqslant S}_{\bullet,\bullet}(\mathscr{B})$ is a bi-complex (Lemma 4.1.15) implies that we have

$$\sum_{S \overset{1}{\hookrightarrow} T \overset{1}{\hookrightarrow} U} d'_{T,U}d'_{S,T}(X) = 0.$$

Thus, we are done if we prove that Δ_T is independent of T. On the one hand we have

$$(\iota^T_Z)^*(\iota^T_S)_*(s) = (\iota^S_Z)^*(\iota^T_S)^*(\iota^T_S)_*(s) = (\iota^S_Z)^*(s \, . \, c_1(N_{S/T}))$$

where we have used (4.31). On the other hand, we can use (4.36) to get

$$(\iota^{E\cap\widetilde{U}}_{E\cap\widetilde{T}})_*(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(s) = (\pi^{E\cap\widetilde{U}}_Z)^*(\iota^S_Z)^*(s)\,.\,\Big((\pi^{E\cap\widetilde{U}}_Z)^*(c_1(N_{T/U})_{|Z}) - c_1(N_{E\cap\widetilde{U}/\widetilde{U}})\Big)\,.$$

Thus, we can rewrite

$$\Delta_T = (\pi^{E\cap\widetilde{U}}_Z)^*(\iota^S_Z)^*(s)\,.\,\Delta'_T$$

with

$$\Delta'_T = (\pi^{E\cap\widetilde{U}}_Z)^*(c_1(N_{S/T})_{|Z} + c_1(N_{T/U})_{|Z}) - c_1(N_{E\cap\widetilde{U}/\widetilde{U}}).$$

Using (4.27) we get $c_1(N_{S/T}) + c_1(N_{T/U})_{|S} = c_1(N_{S/U})$ and hence

$$\Delta'_T = (\pi^{E\cap\widetilde{U}}_Z)^*(c_1(N_{S/U})_{|Z}) - c_1(N_{E\cap\widetilde{U}/\widetilde{U}})$$

which is independent of T, hence Δ_T is independent of T and we are done. □

Proposition 4.6.2. *We have $\Phi \circ d'' = d'' \circ \Phi$.*

Proof. For $s \otimes X \in H^{q-2i}(S)(-i) \otimes A^S_{i,j}(\mathscr{B})$, we compute

$$\begin{aligned}(\Phi d'')(s\otimes X) = &\sum_{S\overset{1}{\hookleftarrow}R} (\pi^{\widetilde{R}}_R)^*(\iota^S_R)^*(s)\otimes d''_{R,S}(X)\\ &+\sum_{S\overset{1}{\hookleftarrow}R\overset{1}{\hookrightarrow}U} (\pi^{E\cap\widetilde{U}}_Z)^*(\iota^R_Z)^*(\iota^S_R)^*(s)\otimes d'_{R,U}d''_{R,S}(X).\end{aligned}$$

and

$$\begin{aligned}(d''\Phi)(s\otimes X) = &\sum_{S\overset{1}{\hookleftarrow}R} (\iota^{\widetilde{S}}_{\widetilde{R}})^*(\pi^{\widetilde{S}}_S)^*(s)\otimes d''_{R,S}(X)\\ &+\sum_{S\overset{1}{\hookrightarrow}T\overset{1}{\hookleftarrow}U} (\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{U}})^*(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(s)\otimes d''_{U,T}d'_{S,T}(X).\end{aligned}$$

The terms $(\ldots)\otimes d''_{R,S}(X)$ cancel because of the equality $(\pi^{\widetilde{R}}_R)^*(\iota^S_R)^*(s) = (\iota^{\widetilde{S}}_{\widetilde{R}})^*(\pi^{\widetilde{S}}_S)^*(s)$, which follows from $(\iota^S_R)\circ(\pi^{\widetilde{R}}_R) = (\pi^{\widetilde{S}}_S)\circ(\iota^{\widetilde{S}}_{\widetilde{R}})$. Thus it remains to show that, for U fixed, we have

$$\sum_{S\overset{1}{\hookleftarrow}R\overset{1}{\hookrightarrow}U} (\pi^{E\cap\widetilde{U}}_Z)^*(\iota^R_Z)^*(\iota^S_R)^*(s)\otimes d'_{R,U}d''_{R,S}(X) = \sum_{S\overset{1}{\hookrightarrow}T\overset{1}{\hookleftarrow}U} (\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{U}})^*(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(s)\otimes d''_{U,T}d'_{S,T}(X).$$

Now $(\pi^{E\cap\widetilde{U}}_Z)^*(\iota^R_Z)^*(\iota^S_R)^*(s) = (\pi^{E\cap\widetilde{U}}_Z)^*(\iota^S_Z)^*(s) = (\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{U}})^*(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(s)$ which is independent of R and T. Thus, the claim follows from the equality

$$\sum_{S\overset{1}{\hookleftarrow}R\overset{1}{\hookrightarrow}U} d'_{R,U}d''_{R,S}(X) = \sum_{S\overset{1}{\hookrightarrow}T\overset{1}{\hookleftarrow}U} d''_{U,T}d'_{S,T}(X)$$

which is a consequence of the fact that $A^{\leqslant Z}_{\bullet,\bullet}(\mathscr{B})$ is a bi-complex. □

4.6.2 The essential case: Φ is a quasi-isomorphism

In this paragraph, we still assume that $\mathscr{B}$ is essential and that Z is the minimal stratum. We further assume that Z is exact and prove the second point of Theorem 4.4.10.

The strategy

We start with a basic fact of homological algebra.

Lemma 4.6.3. *Let $f : C_\bullet \to C'_\bullet$ be a morphism of complexes, let $(F_pC_\bullet)_p$ and $(F_pC'_\bullet)_p$ be finite increasing filtrations on $C_\bullet$ and $C'_\bullet$ such that $f(F_pC_\bullet) \subset F_pC'_\bullet$. Then f is a quasi-isomorphism if for every p, the induced morphism $\mathrm{gr}_p^F f : \mathrm{gr}_p^F C'_\bullet \to \mathrm{gr}_p^F C'_\bullet$ is a quasi-isomorphism.*

Proof. By induction on the length of the filtration, using the long exact sequence in cohomology and the 5-lemma. □

Let $\Phi : {}^{(q)}D_\bullet(\mathscr{B}) \to {}^{(q)}D_\bullet(\widetilde{\mathscr{B}})$ be the morphism of complexes induced on the total complexes. Using the filtration on the lines and the above lemma, one sees that Φ is a quasi-isomorphism if for every j, the morphism

$$\Phi_{\bullet,j} : {}^{(q)}D_{\bullet,j}(\mathscr{B}) \to {}^{(q)}D_{\bullet,j}(\widetilde{\mathscr{B}})$$

induced on the j-th lines is a quasi-isomorphism. In the rest of §4.6.2, we fix an index j. We are reduced to proving that the cone $C_{\bullet,j}$ of $\Phi_{\bullet,j}$ is exact.

We have

$$C_{i,j} = {}^{(q)}D_{i,j}(\mathscr{B}) \oplus {}^{(q)}D_{i+1,j}(\widetilde{\mathscr{B}})$$

and the differential $d' : C_{i,j} \to C_{i-1,j}$ is given by

$$d'(x, \widetilde{x}) = (d'(x), \Phi(x) - d'(\widetilde{x})).$$

The strategy is as follows. We define an complex $B_{\bullet,j}$ and morphisms $\alpha : B_{i,j} \to C_{i,j}$; the second point of Theorem 4.4.10 then follows from the following facts:

– $B_{\bullet,j}$ is exact (Lemma 4.6.4);

– α is a morphism of complexes (Proposition 4.6.6);

– α is a quasi-isomorphism (Proposition 4.6.7).

The exact complex $B_{\bullet,j}$

Let r be the codimension of Z inside X. For $S \in \mathscr{S}_{i+j}(\mathscr{B})$, let us set

$$B_{i,j}^S = H^{q-2r+2j}(Z)(r-j) \otimes A_{i,j}^S(\mathscr{B}).$$

For an inclusion $S \overset{1}{\hookrightarrow} T$, we define $d'_{S,T} : B_{i,j}^S \to B_{i-1,j}^T$. For $z \otimes X \in H^{q-2r+2j}(Z)(r-j) \otimes A_{i,j}^S(\mathscr{B})$, it is given by

$$d'_{S,T}(z \otimes X) = z \otimes d'_{S,T}(X).$$

If we now set $B_{i,j} = \bigoplus_{S \in \mathscr{S}_{i+j}(\mathscr{B})} B_{i,j}^S$, we get a complex $B_{\bullet,j}$.

Lemma 4.6.4. *$B_{\bullet,j}$ is an exact complex.*

Proof. $B_{\bullet,j}$ is nothing but the tensor product of $H^{q-2r+2j}(Z)(r-j)$ with the complex $(A_{\bullet,j}^{\leq Z}(\mathscr{B}), d')$, which is exact since Z is exact. □

Remark 4.6.5. The complexes $(B_{\bullet,j}, d')$ are the lines of a bi-complex whose differentials $d'_{S,T}$ are given by

$$d''_{S,T}(z \otimes X) = (z \,.\, c_1(N_{S/T})_{|Z}) \otimes d''_{S,T}(X).$$

A quasi-isomorphism $\alpha : B_{\bullet,j} \to C_{\bullet,j}$

A morphism $\alpha : B_{\bullet,j} \to C_{\bullet,j}$ is determined by two morphisms $f : B_{\bullet,j} \to {}^{(q)}D_{\bullet,j}(\mathscr{B})$ and $g : B_{\bullet,j} \to {}^{(q)}D_{\bullet+1,j}(\widetilde{\mathscr{B}})$.

We define $f : B^S_{i,j} \to {}^{(q)}D^S_{i,j}(\mathscr{B})$ by the formula

$$f(z \otimes X) = (\iota^S_Z)_*(z) \otimes X$$

and $g : B^S_{i,j} \to {}^{(q)}D^{E\cap\widetilde{S}}_{i+1,j}(\widetilde{\mathscr{B}})$ by the formula

$$g(z \otimes X) = (\pi^{E\cap\widetilde{S}}_Z)^*(z) \,.\, \gamma_S \otimes X$$

where γ_S is the excess class of the blow-up $\pi^{\widetilde{S}}_S : \widetilde{S} \to S$ along Z, defined in §4.8.4.

Proposition 4.6.6. *We have $f \circ d' = d' \circ f$ and $g \circ d' + d' \circ g = \Phi \circ f$. Thus, f and g define a morphism of complexes $\alpha : B_{\bullet,j} \to C_{\bullet,j}$.*

Proof. The first equality is trivial. For the second equality, we compute

$$\begin{aligned}
(\Phi \circ f)(z \otimes X) =& (\pi^{\widetilde{S}}_S)^*(\iota^S_Z)_*(z) \otimes X + \sum_{S \overset{1}{\hookrightarrow} T} (\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(\iota^S_Z)_*(z) \otimes d'_{S,T}(X);\\
(gd'_{S,T})(z \otimes X) =& (\pi^{E\cap\widetilde{T}}_Z)^*(z) \,.\, \gamma_T \otimes d'_{S,T}(X);\\
(d'_{E\cap\widetilde{S},\widetilde{S}}g)(z \otimes X) =& (\iota^{\widetilde{S}}_{E\cap\widetilde{S}})_*((\pi^{E\cap\widetilde{S}}_Z)(z) \,.\, \gamma_S) \otimes X;\\
(d'_{E\cap\widetilde{S},E\cap\widetilde{T}}g)(z \otimes X) =& -(\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{S}})_*((\pi^{E\cap\widetilde{S}}_Z)^*(z) \,.\, \gamma_S) \otimes d'_{S,T}(X).
\end{aligned}$$

The terms $(\dots) \otimes X$ cancel because of the equality

$$(\pi^{\widetilde{S}}_S)^*(\iota^S_Z)_*(z) = (\iota^{\widetilde{S}}_{E\cap\widetilde{S}})_*((\pi^{E\cap\widetilde{S}}_Z)(z) \,.\, \gamma_S)$$

which is a special case of (4.41). For the terms $(\dots) \otimes d'_{S,T}(X)$, we have to prove the equality

$$(\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{S}})_*((\pi^{E\cap\widetilde{S}}_Z)^*(z) \,.\, \gamma_S) = (\pi^{E\cap\widetilde{T}}_Z)^*(z) \,.\, \gamma_T - (\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(\iota^S_Z)_*(z).$$

We have $(\pi^{E\cap\widetilde{S}}_Z)^* = (\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{S}})^* \circ (\pi^{E\cap\widetilde{T}}_Z)^*$, hence the projection formula (4.30) gives

$$(\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{S}})_*((\pi^{E\cap\widetilde{S}}_Z)^*(z) \,.\, \gamma_S) = (\pi^{E\cap\widetilde{T}}_Z)^*(z) \,.\, (\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{S}})_*(\gamma_S).$$

Let us write $r(T)$ for the codimension of Z inside T. Then $(\pi^{E\cap\widetilde{T}}_Z)^*(\iota^S_Z)^*(\iota^S_Z)_*(z) = (\pi^{E\cap\widetilde{T}}_Z)^*(c_{r(T)-1}(N_{Z/S}) \,.\, z)$. To sum up, we are reduced to proving the equality

$$(\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{S}})_*(\gamma_S) = \gamma_T - (\pi^{E\cap\widetilde{T}}_Z)^*(c_{r(T)-1}(N_{Z/S}))$$

which is a special case of (4.43). □

Proposition 4.6.7. $\alpha : B_{\bullet,j} \to C_{\bullet,j}$ *is a quasi-isomorphism. Thus, $C_{\bullet,j}$ is exact, and Φ is a quasi-isomorphism.*

Proof. We use Lemma 4.6.3, defining the filtration $F_p\alpha : F_pB_{\bullet,j} \to F_pC_{\bullet,j}$ which corresponds to the terms involving strata S, $\widetilde{S}$ and $E \cap \widetilde{S}$ with $\operatorname{codim}(S) \leqslant p+j$. All we have to prove is that $\operatorname{gr}_p^F \alpha : \operatorname{gr}_p^F B_{\bullet,j} \to \operatorname{gr}_p^F C_{\bullet,j}$ is a quasi-isomorphism for every p. On the one hand, $\operatorname{gr}_p^F B_{\bullet,j}$ is concentrated in degree p with

$$\operatorname{gr}_p^F B_{p,j} = \bigoplus_{S \in \mathscr{S}_{p+j}(\mathscr{B})} B_{p,j}^S$$

and differential 0. On the other hand, $\operatorname{gr}_p^F C_{\bullet,j}$ is concentrated in degrees $\{p, p-1\}$ with

$$\operatorname{gr}_p^F C_{p,j} = \bigoplus_{S \in \mathscr{S}_{p+j}(\mathscr{B})} D_{p,j}^S(\mathscr{B}) \oplus D_{p+1,j}^{E\cap\widetilde{S}}(\widetilde{\mathscr{B}});$$

$$\operatorname{gr}_p^F C_{p-1,j} = \bigoplus_{S \in \mathscr{S}_{p+j}(\mathscr{B})} D_{p,j}^{\widetilde{S}}(\widetilde{\mathscr{B}}).$$

The differential $D_{p,j}^S(\mathscr{B}) \to D_{p,j}^{\widetilde{S}}(\widetilde{\mathscr{B}})$ is $s \otimes X \mapsto (\pi_S^{\widetilde{S}})^*(s) \otimes X$; the differential $D_{p+1,j}^{E\cap\widetilde{S}}(\widetilde{\mathscr{B}}) \to D_{p,j}^{\widetilde{S}}(\widetilde{\mathscr{B}})$ is given by $e \otimes X \mapsto -(\iota_{E\cap\widetilde{S}}^{\widetilde{S}})_*(e) \otimes X$. We are left with proving that for a fixed stratum $S \in \mathscr{S}_{p+j}(\mathscr{B})$ we have a quasi-isomorphism

$$\begin{tikzcd}
0 \arrow[r] & D_{p,j}^S(\mathscr{B}) \oplus D_{p+1,j}^{E\cap\widetilde{S}}(\widetilde{\mathscr{B}}) \arrow[r] & D_{p,j}^{\widetilde{S}}(\widetilde{\mathscr{B}}) \arrow[r] & 0 \\
0 \arrow[r] & B_{p,j}^S \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

The above diagram is, up to a Tate twist, the tensor product of $A_{i,j}^S$ with

$$\begin{tikzcd}
0 \arrow[r] & H^{q-2p}(S) \oplus H^{q-2p-2}(E\cap\widetilde{S})(-1) \arrow[r] & H^{q-2p}(\widetilde{S}) \arrow[r] & 0 \\
0 \arrow[r] & H^{q-2r+2j}(Z)(p+j-r) \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0.
\end{tikzcd}$$

The fact that this is a quasi-isomorphism is a reformulation of the short exact sequence (4.42). □

4.6.3 The general case: Φ is a morphism of bi-complexes

In this paragraph we prove the general case of the first point of Theorem 4.4.10.

Proposition 4.6.8. *We have $\Phi \circ d' = d' \circ \Phi$.*

Proof. Here are the details to add in the proof of Proposition 4.6.1. We write $S \underset{\perp}{\overset{1}{\hookrightarrow}} T$ for an inclusion which is not of parallel type.

– The terms $(\cdots) \otimes d'_{S,T}(X)$ still cancel, but there are two cases to consider. For the terms corresponding to an inclusion $S \underset{\parallel}{\overset{1}{\hookrightarrow}} T$, the argument is the same as in the essential case, replacing Z by $Z \cap S = Z \cap T$. For the terms corresponding to an inclusion $S \underset{\perp}{\overset{1}{\hookrightarrow}} T$, the cancelation follows from the formula

$$(\pi_T^{\widetilde{T}})^*(\iota_S^T)_*(s) = (\iota_{\widetilde{S}}^{\widetilde{T}})_*(\pi_S^{\widetilde{S}})^*(s)$$

which is a special case of (4.37).

– The terms corresponding to chains $S \overset{1}{\underset{\parallel}{\hookrightarrow}} T \overset{1}{\underset{\parallel}{\hookrightarrow}} U$ cancel thanks to the same argument as in the essential case, replacing Z by $Z \cap S = Z \cap T = Z \cap U$.

– We are left with proving the equality, for U fixed:

$$\sum_{S \overset{1}{\underset{\perp}{\hookrightarrow}} Q \overset{1}{\underset{\parallel}{\hookrightarrow}} U} (\pi^{E\cap\widetilde{U}}_{Z\cap Q})^*(\iota^{Q}_{Z\cap Q})^*(\iota^{Q}_{S})^*(s)\otimes d'_{Q,U}d'_{S,Q}(X) = -\sum_{S \overset{1}{\underset{\parallel}{\hookrightarrow}} T \overset{1}{\underset{\perp}{\hookrightarrow}} U} (\iota^{E\cap\widetilde{U}}_{E\cap\widetilde{T}})_*(\pi^{E\cap\widetilde{T}}_{Z\cap S})^*(\iota^{S}_{Z\cap S})^*(s)\otimes d'_{T,U}d'_{S,T}(X).$$

Let us start with a local decomposition $S = S_\parallel \pitchfork S_\perp$, and $U = U_\parallel \pitchfork U_\perp$ with $S_\parallel \overset{1}{\hookrightarrow} U_\parallel$ and $S_\perp \overset{1}{\hookrightarrow} U_\perp$. There is thus a unique diagram $S \overset{1}{\underset{\perp}{\hookrightarrow}} Q \overset{1}{\underset{\parallel}{\hookrightarrow}} U$ and a unique diagram $S \overset{1}{\underset{\parallel}{\hookrightarrow}} T \overset{1}{\underset{\perp}{\hookrightarrow}} U$, i.e. $Q = U_\parallel \pitchfork S_\perp$ and $T = S_\parallel \pitchfork U_\perp$. Using the Künneth formula (4.1.18) for $A^{\leqslant S}_{\bullet,\bullet}(\mathscr{B})$ with respect to the decomposition $S = S_\parallel \pitchfork S_\perp$, the fact that $d' \circ d' = 0$ implies that $d'_{Q,U}d'_{S,Q}(X) = -d'_{T,U}d'_{S,T}(X)$. Thus, we are left with proving the equality

$$(\pi^{E\cap\widetilde{U}}_{Z\cap Q})^*(\iota^{Q}_{Z\cap Q})^*(\iota^{Q}_{S})^*(s) = (\iota^{E\cap\widetilde{U}}_{E\cap\widetilde{T}})_*(\pi^{E\cap\widetilde{T}}_{Z\cap S})^*(\iota^{S}_{Z\cap S})^*(s).$$

Since $Z \cap Q$ and S are transverse in Q, (4.33) implies the identity

$$(\iota^{Q}_{Z\cap Q})^*(\iota^{Q}_{S})^*(s) = (\iota^{Z\cap Q}_{Z\cap S})_*(\iota^{S}_{Z\cap S})^*(s).$$

Thus, writing $z = (\iota^{S}_{Z\cap S})^*(s)$ and remembering that $Z \cap S = Z \cap T$, we only need to prove that

$$(\pi^{E\cap\widetilde{U}}_{Z\cap Q})^*(\iota^{Z\cap Q}_{Z\cap T})_*(z) = (\iota^{E\cap\widetilde{U}}_{E\cap\widetilde{T}})_*(\pi^{E\cap\widetilde{T}}_{Z\cap\widetilde{T}})^*(z)$$

which is a special case of (4.39) since $Z \cap U$ and T are transverse in U.

□

Proposition 4.6.9. *We have $\Phi \circ d'' = d'' \circ \Phi$.*

Proof. Here are the details to add in the proof of Proposition 4.6.2.

– The terms $(\cdots) \otimes d''_{R,S}(X)$ cancel by the same argument as in the essential case. Thus it remains to show that for U fixed we have

$$\sum_{S \overset{1}{\hookleftarrow} R \overset{1}{\underset{\parallel}{\hookrightarrow}} U} (\pi^{E\cap\widetilde{U}}_{Z})^*(\iota^{R}_{Z})^*(\iota^{S}_{R})^*(s)\otimes d'_{R,U}d''_{R,S}(X) = \sum_{S \overset{1}{\underset{\parallel}{\hookrightarrow}} T \overset{1}{\hookleftarrow} U} (\iota^{E\cap\widetilde{T}}_{E\cap\widetilde{U}})^*(\pi^{E\cap\widetilde{T}}_{Z})^*(\iota^{S}_{Z})^*(s)\otimes d''_{U,T}d'_{S,T}(X).$$

– If $S\cap\widetilde{U} = \varnothing$ then the left-hand side is zero. For a diagram $S \overset{1}{\underset{\parallel}{\hookrightarrow}} T \overset{1}{\hookleftarrow} U$ we have $Z\cap U \subset Z\cap S$, hence $Z \cap U = \varnothing$ and $E \cap \widetilde{U} = \varnothing$, thus the corresponding term in the right-hand side is zero.

– If $S \cap \widetilde{U} \neq \varnothing$, the same argument as in the essential case works. To prove the identity

$$\sum_{S \overset{1}{\hookleftarrow} R \overset{1}{\underset{\parallel}{\hookrightarrow}} U} d'_{R,U}d''_{R,S}(X) = \sum_{S \overset{1}{\underset{\parallel}{\hookrightarrow}} T \overset{1}{\hookleftarrow} U} d''_{U,T}d'_{S,T}(X)$$

one has to use the Künneth formula (Proposition 4.1.18) in addition of the fact that $A^{\leqslant S\cap U}_{\bullet,\bullet}(\mathscr{B})$ is a bi-complex.

□

4.6.4 The general case: Φ is a quasi-isomorphism

In this paragraph we prove the general case of the second point of Theorem 4.4.10 by reducing to the essential case, already proved in §4.6.2.

Definition 4.6.10. Let P be a stratum of $\mathscr{B}$ that is transverse to Z; in particular, $Z \pitchfork P \neq \varnothing$. Let S be a stratum such that $Z \cap S \neq \varnothing$. Then by looking at a local chart around any point of $Z \cap S$, one sees that we have a decomposition $S = S_Z \pitchfork P$ with $S_Z \supset Z$ and P transverse to Z. We call P the *transverse direction* of S.

We let $\mathscr{B}_P$ be the arrangement of hypersurfaces on P consisting in the intersections of P and the hypersurfaces $K \in \mathscr{B}^{\leqslant Z}$. It is essential, with minimal stratum $Z \pitchfork P$. The strata of $\mathscr{B}_P$ are exactly the strata of $\mathscr{B}$ with transverse direction P. As the coloring is concerned, we ask that $\chi(S_Z \pitchfork P) = \chi(S_Z)$ for every $S_Z \supset Z$.

The Orlik-Solomon bi-complex of $\mathscr{B}_P$ is related to the one of $\mathscr{B}$ by

$$A_{\bullet,\bullet}^{\leqslant S_Z \pitchfork P}(\mathscr{B}_P) \cong A_{\bullet,\bullet}^{\leqslant S_Z}(\mathscr{B}). \tag{4.13}$$

In particular, if Z is exact in $\mathscr{B}$ then $Z \pitchfork P$ is exact in $\mathscr{B}_P$.

Let $S = S_Z \pitchfork P$ be a stratum with transverse direction P. Combining the Künneth formula (Proposition 4.1.18) and (4.13), we get an isomorphism

$$A_{i,j}^S(\mathscr{B}) \cong \bigoplus_{k+l=\mathrm{codim}(P)} A_{i-k,j-l}^{S_Z \pitchfork P}(\mathscr{B}_P) \otimes A_{k,l}(\mathscr{B}).$$

and hence an isomorphism at the level of the Orlik-Solomon bi-complexes:

$${}^{(q)}D_{i,j}^S(\mathscr{B}) \cong \bigoplus_{k+l=\mathrm{codim}(P)} {}^{(q-2k)}D_{i-k,j-l}^{S_Z \pitchfork P}(\mathscr{B}_P)(-k) \otimes A_{k,l}^P(\mathscr{B}).$$

Summing over all strata $S \in \mathscr{S}_{i+j}(\mathscr{B})$ and grouping together the strata having the same transverse direction P, we get a decomposition:

$${}^{(q)}D_{i,j}(\mathscr{B}) = \left(\bigoplus_{\substack{S \in \mathscr{S}_{i+j}(\mathscr{B}) \\ Z \cap S = \varnothing}} {}^{(q)}D_{i,j}^S(\mathscr{B}) \right) \oplus \left(\bigoplus_{\substack{P \perp Z \\ k+l=\mathrm{codim}(P)}} {}^{(q-2k)}D_{i-k,j-l}(\mathscr{B}_P)(-k) \otimes A_{k,l}^P(\mathscr{B}) \right)$$

where $P \perp Z$ means that we sum over all strata P that are transverse to Z.

Now it is clear that in the blown-up situation we have

$${}^{(q)}D_{i,j}(\widetilde{\mathscr{B}}) = \left(\bigoplus_{\substack{S \in \mathscr{S}_{i+j}(\mathscr{B}) \\ Z \cap S = \varnothing}} {}^{(q)}D_{i,j}^{\widetilde{S}}(\widetilde{\mathscr{B}}) \right) \oplus \left(\bigoplus_{\substack{P \perp Z \\ k+l=\mathrm{codim}(P)}} {}^{(q-2k)}D_{i-k,j-l}(\widetilde{\mathscr{B}_P})(-k) \otimes A_{k,l}^P(\mathscr{B}) \right)$$

where $\widetilde{\mathscr{B}_P}$ is the blow-up of $\mathscr{B}_P$ along $Z \pitchfork P$.

These decompositions are compatible with Φ in the following sense:

- for $S \in \mathscr{S}_{i+j}(\mathscr{B})$ such that $Z \cap S = \varnothing$, Φ is an isomorphism ${}^{(q)}D_{i,j}^S(\mathscr{B}) \cong {}^{(q)}D_{i,j}^{\widetilde{S}}(\widetilde{\mathscr{B}})$.
- for every $P \perp Z$, $\Phi : {}^{(q)}D_{i,j}(\mathscr{B}) \to {}^{(q)}D_{i,j}(\widetilde{\mathscr{B}})$ restricts to

$$D_{i-k,j-l}(\mathscr{B}_P)(-k) \otimes A_{k,l}^P(\mathscr{B}) \to D_{i-k,j-l}(\widetilde{\mathscr{B}_P})(-k) \otimes A_{k,l}^P(\mathscr{B})$$

which is nothing but $\Phi \otimes \mathrm{id}$.

Proposition 4.6.11. *If Z is exact, then $\Phi : {}^{(q)}D_\bullet(\mathscr{B}) \to {}^{(q)}D_\bullet(\widetilde{\mathscr{B}})$ is a quasi-isomorphism*

Proof. As in §4.6.2, it is enough to prove that for every line j, the morphism $\Phi_{\bullet,j} : {}^{(q)}D_{\bullet,j}(\mathscr{B}) \to {}^{(q)}D_{\bullet,j}(\widetilde{\mathscr{B}})$ is a quasi-isomorphism.

With j fixed, we define an increasing filtration $F_p\Phi_{\bullet,j} : F_p{}^{(q)}D_{\bullet,j}(\mathscr{B}) \to F_p{}^{(q)}D_{\bullet,j}(\widetilde{\mathscr{B}})$. By definition, $F_p{}^{(q)}D_{\bullet,j}(\mathscr{B})$ is the sum of the terms corresponding to $\operatorname{codim}(P) \leqslant p$. We add the convention $F_p{}^{(q)}D_{\bullet,j}(\mathscr{B}) = {}^{(q)}D_{\bullet,j}(\mathscr{B})$ for $p = \dim(X)+1$ to include the terms corresponding to $Z \cap S = \varnothing$. We make the analogous definition for ${}^{(q)}D_{\bullet,j}(\widetilde{\mathscr{B}})$.

In view of Lemma 4.6.3, it is enough to show that for every p, the morphism $\operatorname{gr}_p^F\Phi_{\bullet,j} : \operatorname{gr}_p^F{}^{(q)}D_{\bullet,j}(\mathscr{B}) \to \operatorname{gr}_p^F{}^{(q)}D_{\bullet,j}(\widetilde{\mathscr{B}})$ is a quasi-isomorphism. For $p = \dim(X)+1$, $\operatorname{gr}_p^F\Phi_{\bullet,j}$ is an isomorphism. For $p \leqslant \dim(X)$ we get

$$\operatorname{gr}_p^F{}^{(q)}D_{\bullet,j}(\mathscr{B}) = \bigoplus_{\substack{P\perp Z\\ \operatorname{codim}(P)=p\\ k+l=p}} {}^{(q-2k)}D_{\bullet-k,j-l}(\mathscr{B}_P)(-k) \otimes A^P_{k,l}(\mathscr{B})$$

and the differential on $D_{\bullet-k,j-l}(\mathscr{B}_P)(-k) \otimes A^P_{k,l}(\mathscr{B})$ is $d' \otimes \mathrm{id}$. The same is true for

$$\operatorname{gr}_p^F{}^{(q)}D_{\bullet,j}(\widetilde{\mathscr{B}}) = \bigoplus_{\substack{P\perp Z\\ \operatorname{codim}(P)=p\\ k+l=p}} {}^{(q-2k)}D_{\bullet-k,j-l}(\widetilde{\mathscr{B}_P})(-k) \otimes A^P_{k,l}(\mathscr{B}).$$

Thus, $\operatorname{gr}_p^F\Phi_{\bullet,j}$ is a quasi-isomorphism if and only if every $\Phi : {}^{(q-2k)}D_{\bullet-k,j-l}(\mathscr{B}_P) \to {}^{(q-2k)}D_{\bullet-k,j-l}(\widetilde{\mathscr{B}_P})$ is a quasi-isomorphism. Since the arrangements $\mathscr{B}_P$ are essential with $Z \pitchfork P$ exact, this follow from the essential case, already proved in §4.6.2. $\square$

4.6.5 Working withouth the connectedness assumption

Let $\mathscr{B}$ be a bi-arrangement of hypersurfaces in a complex manifold X, and Z a good stratum of X. If we do not assume (4.7) that the intersection of strata are all connected, then it is still possible to define the morphisms Φ as in 4.4.2.

Let us fix a stratum S of $\mathscr{B}$. For every $S \overset{1}{\hookrightarrow} T$, we have a decomposition into connected components

$$Z \cap T = \bigsqcup_{\alpha\in I_{\parallel}(T)} (Z\cap T)_\alpha \sqcup \bigsqcup_{\beta\in I_\perp(T)} (Z\cap T)_\beta$$

where for each $\alpha \in I_\parallel(T)$, $(Z\cap T)_\alpha \subset S$, and for each $\beta \in I_\perp(T)$, $(Z\cap T)_\beta \not\subset S$. In the same fashion, we have a decomposition into connected components

$$E \cap \widetilde{T} = \bigsqcup_{\alpha\in I_{\parallel}(T)} (E\cap \widetilde{T})_\alpha \sqcup \bigsqcup_{\beta\in I_\perp(T)} (E\cap \widetilde{T})_\beta$$

and for each α we have a morphism $\pi_{T,\alpha} : (E\cap\widetilde{T})_\alpha \to (Z\cap T)_\alpha$.

We then define

$$\Phi(s\otimes X) = (\pi_S^{\widetilde{S}})^*(s) \otimes X + \sum_{\substack{S \overset{1}{\hookrightarrow} T\\ \alpha\in I_\parallel(T)}} (\pi_{T,\alpha})^*(\iota^S_{(Z\cap T)_\alpha})^*(s) \otimes d'_{S,T}(X).$$

We leave it to the reader to check that the proof of Theorem 4.4.10 can be adapted in that setting.

4.7 Appendix: normal crossing divisors and relative cohomology

In this Appendix, we fix X a complex manifold, $\mathscr{L}$ and $\mathscr{M}$ two simple normal crossing divisors in X that do not share an irreducible component and such that $\mathscr{L} \cup \mathscr{M}$ is a normal crossing divisor. We will denote by $L_1, \ldots, L_l$ (resp. $M_1, \ldots, M_m$) the irreducible components of $\mathscr{L}$ (resp. $\mathscr{M}$). For $I \subset \{1, \ldots, l\}$ (resp. $J \subset \{1, \ldots, m\}$), we will write $L_I = \bigcap_{i \in I} L_i$ (resp. $M_J = \bigcap_{j \in J} M_j$), with the convention $L_\varnothing = M_\varnothing = X$. For every I and J, $L_I \cap M_J$ is a disjoint union of submanifolds of X.

4.7.1 The spectral sequence

We let

$$H^\bullet(\mathscr{L}, \mathscr{M}) = H^\bullet(X \setminus \mathscr{L}, \mathscr{M} \setminus \mathscr{M} \cap \mathscr{L}) \tag{4.14}$$

be the corresponding relative cohomology group. It is endowed with a canonical mixed Hodge structure if X is a complex variety and $\mathscr{L}$, $\mathscr{M}$ are complex subvarieties of X.

Proposition 4.7.1. *1. There is a spectral sequence*

$$E_1^{-p,q}(\mathscr{L}, \mathscr{M}) = \bigoplus_{\substack{i-j=p \\ |I|=i \\ |J|=j}} H^{q-2i}(L_I \cap M_J)(-i) \implies H^{-p+q}(\mathscr{L}, \mathscr{M}). \tag{4.15}$$

The differential $d_1 : E_1^{-p,q} \to E_1^{-p+1,q}$ is that of the total complex of a double complex, where

– *the horizontal differential is the collection of the morphisms*

$$H^{q-2i}(L_I \cap M_J)(-i) \to H^{q-2i+2}(L_{I \setminus \{r\}} \cap M_J)(-i+1)$$

for every $r \in I$, which are the Gysin morphisms of the inclusions $L_I \cap M_J \hookrightarrow L_{I\setminus\{r\}} \cap M_J$, multiplied by the signs $\mathrm{sgn}(\{r\}, I \setminus \{r\})$*;*

– *the vertical differential is the collection of the morphisms*

$$H^{q-2i}(L_I \cap M_J)(-i) \to H^{q-2i}(L_I \cap M_{J \cup \{s\}})(-i)$$

for every $s \notin J$, which are the pull-back morphisms of the inclusions $L_I \cap M_{J \cup \{s\}} \hookrightarrow L_I \cap M_J$, multiplied by the signs $\mathrm{sgn}(\{s\}, J \setminus \{s\})$*.*

2. If X is a smooth complex variety and $\mathscr{L}$, $\mathscr{M}$ are complex subvarieties of X, then this is a spectral sequence in the category of mixed Hodge structures.

3. If furthermore X is projective, then this spectral sequence degenerates at the E_2 term and we have

$$E_\infty^{-p,q} \cong E_2^{-p+q} \cong \mathrm{gr}_q^W H^{-p+q}(\mathscr{L}, \mathscr{M}).$$

Proof. 1. We will use the notation $j_U^Y : U \hookrightarrow Y$ for open immersions. Let us write

$$\mathscr{F}(\mathscr{L}, \mathscr{M}) = (j^X_{X \setminus \mathscr{L}})_* (j^{X \setminus \mathscr{L}}_{X \setminus \mathscr{L} \cup \mathscr{M}})_! \mathbb{Q}_{X \setminus \mathscr{L} \cup \mathscr{M}}$$

seen as an object of the (bounded) derived category of sheaves on X, where $\mathbb{Q}_Y$ stands for the constant sheaf with stalk $\mathbb{Q}$ on a space Y. Then we have

$$H^\bullet(\mathscr{L}, \mathscr{M}) = H^\bullet(X, \mathscr{F}(\mathscr{L}, \mathscr{M})).$$

The sheaf $(j^{X\setminus\mathscr{L}}_{X\setminus\mathscr{L}\cup\mathscr{M}})_!\mathbb{Q}_{X\setminus\mathscr{L}\cup\mathscr{M}}$ is resolved by the complex of sheaves

$$0 \to \mathbb{Q}_{X\setminus\mathscr{L}} \to \bigoplus_{|J|=1}(\iota^{X\setminus\mathscr{L}}_{M_J\setminus M_J\cap\mathscr{L}})_*\mathbb{Q}_{M_J\setminus M_J\cap\mathscr{L}} \to \bigoplus_{|J|=2}(\iota^{X\setminus\mathscr{L}}_{M_J\setminus M_J\cap\mathscr{L}})_*\mathbb{Q}_{M_J\setminus M_J\cap\mathscr{L}} \to \cdots$$

where the arrows are the alternating sums of the natural restriction morphisms. Thus, $\mathscr{F}(\mathscr{L},\mathscr{M})$ is isomorphic to the complex of sheaves

$$0 \to (j^X_{X\setminus\mathscr{L}})_*\mathbb{Q}_{X\setminus\mathscr{L}} \to \bigoplus_{|J|=1}(\iota^X_{M_J})_*(j^{M_J}_{M_J\setminus M_J\cap\mathscr{L}})_*\mathbb{Q}_{M_J\setminus M_J\cap\mathscr{L}} \to \bigoplus_{|J|=2}(\iota^X_{M_J})_*(j^{M_J}_{M_J\setminus M_J\cap\mathscr{L}})_*\mathbb{Q}_{M_J\setminus M_J\cap\mathscr{L}} \to \cdots$$

For a subset $J \subset \{1,\dots,m\}$ fixed, the sheaf

$$(j^{M_J}_{M_J\setminus M_J\cap\mathscr{L}})_*\mathbb{Q}_{M_J\setminus M_J\cap\mathscr{L}} = (j^{M_J}_{M_J\setminus M_J\cap\mathscr{L}})_*(j^{M_J}_{M_J\setminus M_J\cap\mathscr{L}})^*\mathbb{Q}_{M_J}$$

is isomorphic to the complex of sheaves

$$\cdots \to \bigoplus_{|I|=2}(\iota^{M_J}_{L_I\cap M_J})_!(\iota^{M_J}_{L_I\cap M_J})^!\mathbb{Q}_{M_J} \to \bigoplus_{|I|=1}(\iota^{M_J}_{L_I\cap M_J})_!(\iota^{M_J}_{L_I\cap M_J})^!\mathbb{Q}_{M_J} \to \mathbb{Q}_{M_J} \to 0$$

For $\iota^B_A : A \hookrightarrow B$ a closed immersion of complex manifolds of codimension r, we have an isomorphism

$$(\iota^B_A)^!\mathbb{Q}_B \cong \mathbb{Q}_A[-2r]$$

and hence a morphism

$$(\iota^B_A)_*\mathbb{Q}_A[-2r] \to \mathbb{Q}_B \tag{4.16}$$

which corresponds in cohomology to the Gysin morphism $H^{k-2r}(A) \to H^k(B)$. Having this in mind, we have for $|I| = i$ an isomorphism

$$(\iota^{M_J}_{L_I\cap M_J})^!\mathbb{Q}_{M_J} \cong \mathbb{Q}_{L_I\cap M_J}[-2i].$$

We have thus proved that we have an isomorphism $\mathscr{F}(\mathscr{L},\mathscr{M}) \cong \mathscr{K}(\mathscr{L},\mathscr{M})$ where $\mathscr{K}(\mathscr{L},\mathscr{M})$ is the total complex of the double complex

$$\mathscr{K}_{i,j}(\mathscr{L},\mathscr{M}) = \bigoplus_{\substack{|I|=i\\|J|=j}}(\iota^X_{L_I\cap M_J})_*\mathbb{Q}_{L_I\cap M_J}[-2i].$$

Here the differential $d' : \mathscr{K}_{i,j}(\mathscr{L},\mathscr{M}) \to \mathscr{K}_{i-1,j}(\mathscr{L},\mathscr{M})$ is the alternation sum of the Gysin morphisms (4.16), and $d'' : \mathscr{K}_{i,j-1}(\mathscr{L},\mathscr{M}) \to \mathscr{K}_{i,j}(\mathscr{L},\mathscr{M})$ is the alternating sum of the restriction morphisms. The spectral sequence that we are looking for is simply the hypercohomology spectral sequence for $\mathscr{K}(\mathscr{L},\mathscr{M})$.

2. If we work in the category of mixed Hodge modules [PS08, §14], then the above proof works and gives the compatibility of the spectral sequence with the mixed Hodge structures.

3. If X is smooth and projective, then all $L_I \cap M_J$ are (disjoint union of) smooth projective varieties. Thus, $E_1^{-p,q}$ is a pure Hodge structure of weight q. The degeneration then comes from the fact that in the category of mixed Hodge structures, a morphism between two pure Hodge structures of different weights is zero.

□

Remark 4.7.2. In the case $\mathscr{M}=\varnothing$, one recovers the spectral sequence

$$E_1^{-p,q}=\bigoplus_{|I|=p}H^{q-2p}(L_I)(-p)\implies H^{-p+q}(X\setminus\mathscr{L})$$

where the differential is the alternating sum of the Gysin morphisms of the inclusions $L_I\hookrightarrow L_{I\setminus\{r\}}$. This spectral sequence was first studied by Deligne in the smooth and projective case [Del71, Corollary 3.2.13]. If $\mathscr{L}$ is a smooth submanifold of X (i.e. $l=1$), then this spectral sequence is nothing but the residue/Gysin long exact sequence:

$$\cdots\to H^{k-2}(\mathscr{L})(-1)\to H^k(X)\to H^k(X\setminus\mathscr{L})\to\cdots$$

In the case $\mathscr{L}=\varnothing$, one recovers the spectral sequence

$$E_1^{p,q}=\bigoplus_{|J|=p}H^q(M_J)\implies H^{p+q}(X,\mathscr{M})$$

where the differential is the alternating sum of the pull-back morphisms of the inclusions $M_{J\cup\{s\}}\hookrightarrow M_J$. If $\mathscr{M}$ is a smooth submanifold of X (i.e. $m=1$), then this spectral sequence is nothing but the long exact sequence in relative cohomology:

$$\cdots\to H^k(X,\mathscr{M})\to H^k(X)\to H^k(\mathscr{M})\to\cdots$$

Remark 4.7.3. There is a way of proving the first and third points of Proposition 4.7.1 which does not make use of mixed Hodge modules, but only of mixed Hodge theory *à la* Deligne [Del71, Del74], i.e. with complexes of holomorphic differential forms. After tensoring with $\mathbb{C}$, $\mathscr{F}(\mathscr{L},\mathscr{M})$ is isomorphic to the total complex of the double complex

$$0\to\Omega_X^\bullet(\log\mathscr{L})\to\bigoplus_{|J|=1}(\iota_{M_J}^X)_*\Omega_{M_J}^\bullet(\log\mathscr{L}\cap M_J)\to\bigoplus_{|J|=2}(\iota_{M_J}^X)_*\Omega_{M_J}^\bullet(\log\mathscr{L}\cap M_J)\to\cdots\quad(4.17)$$

On each component $\Omega_{M_J}^\bullet(\log\mathscr{L}\cap M_J)$ there is the filtration P by the order of the pole [Del71] such that we have the Poincaré residue isomorphisms

$$\mathrm{gr}_k^P\Omega_{M_J}^\bullet(\log\mathscr{L}\cap M_J)\cong\bigoplus_{|I|=k}(\iota_{L_I\cap M_J}^{M_J})_*\Omega_{L_I\cap M_J}^{\bullet-k}.$$

Suitably shifted, this gives a filtration W on (4.17) whose hypercohomology spectral sequence is the spectral sequence of Proposition 4.7.1 tensored with $\mathbb{C}$. If X is projective, the formalism of mixed Hodge complexes [Del74] allows one to prove that it is defined over $\mathbb{Q}$ and compatible with the mixed Hodge structures.

4.7.2 Duality

There is also the compactly-supported version of (4.14)

$$H_c^\bullet(\mathscr{L},\mathscr{M})=H_c^\bullet(X\setminus\mathscr{L},\mathscr{M}\setminus\mathscr{L}\cap\mathscr{M}).\quad(4.18)$$

This has to be understood as the compactly supported cohomology groups of the sheaf $\mathscr{F}(\mathscr{L},\mathscr{M})$ defined in the proof of Proposition 4.7.1. If X is compact, then it is the same as (4.14).

Proposition 4.7.4. *Let $n=\dim_{\mathbb{C}}(X)$. Then $H^\bullet(\mathscr{L},\mathscr{M})$ and $H_c^\bullet(\mathscr{M},\mathscr{L})$ are dual to each other in the sense that we have a Poincaré-Verdier duality*

$$\left(H^k(\mathscr{L},\mathscr{M})\right)^\vee\cong H_c^{2n-k}(\mathscr{M},\mathscr{L})$$

that is compatible with the mixed Hodge structures in the algebraic case. The corresponding spectral sequences of Proposition 4.14 are also dual to each other.

Proof. Let D denote the Verdier duality operator. We have, using the notations of the proof of Proposition 4.7.1:

$$\mathrm{D}\,(\iota^X_{L_I\cap M_J})_*\mathbb{Q}_{L_I\cap M_J}[-2i] = (\iota^{M_J}_{L_I\cap M_J})_*\mathbb{Q}_{L_I\cap M_J}[2(n-j)].$$

Thus, we have a duality

$$\mathrm{D}\,\mathscr{K}_{i,j}(\mathscr{L},\mathscr{M}) = \mathscr{K}_{j,i}(\mathscr{M},\mathscr{L})[2n]$$

that is easily seen to be compatible with the differentials, hence the result. The compatibility with the mixed Hodge structures follows from the same computation using mixed Hodge modules. □

4.7.3 Deletion and restriction

Let $\mathscr{L}' = L_1\cup\cdots\cup L_{l-1}\subset\mathscr{L}$, we then have $(\mathscr{L}',\mathscr{M})$ the deletionof $(\mathscr{L},\mathscr{M})$ with respect to L_l. We write $(L_l|\mathscr{L}',\mathscr{M})$ for the intersection of $(\mathscr{L}',\mathscr{M})$ with L_l. It is a normal crossing divisor on L_l, called the restriction of $(\mathscr{L},\mathscr{M})$ with respect to L_l.

We have a natural deletion-restriction long exact sequence

$$\cdots\to H^\bullet(\mathscr{L}',\mathscr{M})\to H^\bullet(\mathscr{L},\mathscr{M})\to H^{\bullet-1}(L_l|\mathscr{L}',\mathscr{M})(-1)\to\cdots \tag{4.19}$$

In the same fashion, for $\mathscr{M}' = M_1\cup\cdots\cup M_{m-1}$, we have

$$\cdots\to H^{\bullet-1}(M_m|\mathscr{L},\mathscr{M}')\to H^\bullet(\mathscr{L},\mathscr{M})\to H^\bullet(\mathscr{L},\mathscr{M}')\to\cdots \tag{4.20}$$

Proposition 4.7.5. *The spectral sequences (4.15) are functorial with respect to the deletion and restriction morphisms (4.19) and (4.20) via morphisms of spectral sequences. In particular, we have on the E_1 term natural short exact sequences*

$$0\to E_1^{-p,q}(\mathscr{L}',\mathscr{M})\to E_1^{-p,q}(\mathscr{L},\mathscr{M})\to E_1^{-p+1,q-2}(L_l|\mathscr{L}',\mathscr{M})(-1)\to 0 \tag{4.21}$$

and

$$0\to E_1^{-p+1,q}(M_m|\mathscr{L},\mathscr{M}')\to E_1^{-p,q}(\mathscr{L},\mathscr{M})\to E_1^{-p,q}(\mathscr{L},\mathscr{M}')\to 0. \tag{4.22}$$

To be precise:

- $E_1^{-p,q}(\mathscr{L}',\mathscr{M})\to E_1^{-p,q}(\mathscr{L},\mathscr{M})$ is the natural inclusion corresponding to subsets I that do not contain l;

- $E_1^{-p,q}(\mathscr{L},\mathscr{M})\to E_1^{-p+1,q-2}(L_l|\mathscr{L}',\mathscr{M})(-1)$ is the natural projection corresponding to subsets I that contain l;

- $E_1^{-p+1,q}(M_m|\mathscr{L},\mathscr{M}')\to E_1^{-p,q}(\mathscr{L},\mathscr{M})$ is the natural inclusion corresponding to subsets J that contain m;

- $E_1^{-p,q}(\mathscr{L},\mathscr{M})\to E_1^{-p,q}(\mathscr{L},\mathscr{M}')$ is the natural projection corresponding to subsets J that do not contain m.

Proof. The deletion-restriction long exact sequences (4.19) and (4.20) are defined on the complexes of sheaves $\mathscr{K}_{i,j}$ (see the proof of Proposition 4.7.1) via the natural inclusions and projections. □

4.7.4 Blow-ups

We study the functoriality of the spectral sequence (4.15) with respect to the blow-up of a stratum. For simplicity we assume that all $L_I \cap M_J$'s are connected. Let $Z = L_{I_0} \cap M_{J_0}$ be a stratum, with $I_0 \neq \varnothing$ so that $Z \subset \mathscr{L}$. Let $\pi : \widetilde{X} \to X$ be the blow-up along Z, $E = \pi^{-1}(Z)$ be the exceptional divisor. We set $\widetilde{\mathscr{L}} = E \cup \widetilde{L_1} \cup \cdots \cup \widetilde{L_l}$ and $\widetilde{\mathscr{M}} = \widetilde{M_1} \cup \cdots \cup \widetilde{M_m}$. We then have a natural isomorphism

$$\pi^* : H^\bullet(X \setminus \mathscr{L}, \mathscr{M} \setminus \mathscr{M} \cap \mathscr{L}) \xrightarrow{\cong} H^\bullet(\widetilde{X} \setminus \widetilde{\mathscr{L}}, \widetilde{\mathscr{M}} \setminus \widetilde{\mathscr{M}} \cap \widetilde{\mathscr{L}}). \tag{4.23}$$

Proposition 4.7.6. *The spectral sequence (4.15) is functorial with respect to the blow-up morphism (4.23) via a morphism of spectral sequences*

$$E_1^{-p,q}(\pi) : E_1^{-p,q}(\mathscr{L}, \mathscr{M}) \to E_1^{-p,q}(\widetilde{\mathscr{L}}, \widetilde{\mathscr{M}}) \tag{4.24}$$

given for $s \in H^{q-2p}(L_I \cap M_J)(-p)$ by

$$s \mapsto \left(\pi_{L_I \cap M_J}^{\widetilde{L}_I \cap \widetilde{M}_J}\right)^* (s) + \sum_{i \in I \cap I_0} \operatorname{sgn}(\{i\}, I \setminus \{i\}) \left(\pi_{Z \cap L_{I\setminus\{i\}} \cap M_J}^{E \cap \widetilde{L}_{I\setminus\{i\}} \cap \widetilde{M}_J}\right)^* \left(\iota_{Z \cap L_I \cap M_J}^{L_I \cap M_J}\right)^* (s).$$

Proof. We sketch the proof for the case $\mathscr{M} = \varnothing$, the general case being similar. In this special case, the spectral sequence is Deligne's spectral sequence

$$E_1^{-p,q} = \bigoplus_{|I|=p} H^{q-2p}(L_I)(-p) \implies H^{-p+q}(X \setminus \mathscr{L}). \tag{4.25}$$

One works over $\mathbb{C}$ with the complex of logarithmic forms $\Omega_X^\bullet(\log \mathscr{L})$. By definition, we have a pull-back morphism

$$\pi^* : \Omega_X^\bullet(\log \mathscr{L}) \to \Omega_{\widetilde{X}}^\bullet(\log \widetilde{\mathscr{L}}).$$

The claim follows from the following local statement. In $\mathbb{C}^n$ with coordinates $(z_1, \ldots, z_n)$, let us write $\omega_i = \frac{dz_i}{z_i}$ and for $I = \{i_1 < \cdots < i_k\}$, $\omega_I = \omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$. In any standard affine chart $\pi : \mathbb{C}^n \to \mathbb{C}^n$ of the blow-up of the linear space $Z = \{z_1 = \cdots = z_r = 0\}$, one write z_E for the coordinate corresponding to the exceptional divisor, and $\omega_E = \frac{dz_E}{z_E}$. One then has the formula

$$\pi^*(\omega_I) = \omega_I + \sum_{\substack{i \in I \\ 1 \leqslant i \leqslant r}} \operatorname{sgn}(\{i\}, I \setminus \{i\}) \, \omega_E \wedge \omega_{I \setminus \{i\}}.$$

□

Dually, if $Z = L_{I_0} \cap M_{J_0}$ with $J_0 \neq \varnothing$, we get an isomorphism

$$\pi_* : H^\bullet(\widetilde{X} \setminus \widetilde{\mathscr{L}}, \widetilde{\mathscr{M}} \setminus \widetilde{\mathscr{M}} \cap \widetilde{\mathscr{L}}) \xrightarrow{\cong} H^\bullet(X \setminus \mathscr{L}, \mathscr{M} \setminus \mathscr{M} \cap \mathscr{L})$$

where $\widetilde{\mathscr{L}} = \widetilde{L_1} \cup \cdots \cup \widetilde{L_l}$ and $\widetilde{\mathscr{M}} = E \cup \widetilde{M_1} \cup \cdots \cup \widetilde{M_m}$, that can be described explicitly in terms of a morphism of spectral sequences.

4.8 Appendix: Chern classes, blow-ups, and some cohomological identities

4.8.1 Chern classes of normal bundles

Let $\iota_Z^X : Z \hookrightarrow X$ be the inclusion of a closed submanifold Z of codimension r of a complex manifold X. We denote by $N_{Z/X}$ the normal bundle of ι_Z^X and by $c_k(N_{Z/X}) \in H^{2k}(Z)$, $k = 0, \ldots, r$ its Chern classes.

For inclusions $A \hookrightarrow B \hookrightarrow C$ we have a short exact sequence

$$0 \to N_{A/B} \to N_{A/C} \to \left(N_{B/C}\right)_{|A} \to 0 \tag{4.26}$$

which implies the following transitivity property of Chern classes:

$$c_k(N_{A/C}) = \sum_{j=0}^{k} c_j(N_{A/B}) \,.\, c_{k-j}(N_{B/C})_{|A}. \tag{4.27}$$

If A and B are two closed submanifolds of a complex manifold X that are transverse, and R a connected component of the intersection $A \cap B$, we also have an isomorphism

$$N_{R/X} \cong \left(N_{A/X}\right)_{|R} \oplus \left(N_{B/X}\right)_{|R}. \tag{4.28}$$

By combining it with (4.26) for $R \hookrightarrow A \hookrightarrow X$, one gets an isomorphism

$$N_{R/A} \cong \left(N_{B/X}\right)_{|R}. \tag{4.29}$$

4.8.2 Gysin morphisms and pull-backs

Let $\iota_Z^X : Z \hookrightarrow X$ be the inclusion of a closed submanifold Z of codimension r of a complex manifold X. We have a pull-back morphism $(\iota_Z^X)^* : H^\bullet(X) \to H^\bullet(Z)$ and a Gysin morphism $(\iota_Z^X)_* : H^\bullet(Z) \to H^{\bullet+2r}(X)$. We have the projection formula

$$(\iota_Z^X)_*(z.(\iota_Z^X)^*(x)) = (\iota_Z^X)_*(z) \,.\, x. \tag{4.30}$$

We have the following compatibilities:

$$(\iota_Z^X)^*(\iota_Z^X)_*(z) = z \,.\, c_r(N_{Z/X}); \tag{4.31}$$

$$(\iota_Z^X)_*(\iota_Z^X)^*(x) = x \,.\, [Z]_X. \tag{4.32}$$

Here $N_{Z/X}$ is the normal bundle of Z inside X, and $c_k(N_{Z/X}) \in H^{2k}(Z)$, $k = 0, \ldots, r$ are its Chern classes; $[Z]_X \in H^{2r}(Z)$ is the cohomology class of Z in X.
If A and B are two closed submanifolds of a complex manifold X that are transverse, then we have

$$(\iota_A^X)^* \circ (\iota_B^X)_* = (\iota_{A\cap B}^A)_* \circ (\iota_{A\cap B}^B)^*. \tag{4.33}$$

This includes the case $A \cap B = \varnothing$ for which the right-hand side is 0, and the case where $A \cap B$ is not connected for which the right-hand side is the sum of $(\iota_R^A)_* \circ (\iota_R^B)^*$ for R a connected component of $A \cap B$.

4.8.3 Blow-ups

Let X be a complex manifold and Z a closed submanifold of X, of codimension r. We let $\pi : \widetilde{X} \to X$ be the blow-up of X along Z. We let $\pi_Z^E : E \to Z$ be the morphism induced by π, it is the projectified normal bundle of Z inside X. For S a submanifold of X, we denote by $\widetilde{S}$ its strict transform along π, and $\pi_S^{\widetilde{S}} : \widetilde{S} \to S$ the morphism induced by π. It is the blow-up of S along $Z \cap S$.

Let L be a smooth hypersurface of X that contains Z. We have the identity

$$\pi^* \circ (\iota_L^X)_* = (\iota_{\widetilde{L}}^{\widetilde{X}})_* \circ (\pi_L^{\widetilde{L}})^* + (\iota_E^{\widetilde{X}})_* \circ (\pi_Z^E)^* \circ (\iota_Z^L)^* \tag{4.34}$$

between morphisms $H^\bullet(L) \to H^{\bullet+2}(\widetilde{X})$. When applied to the element $1 \in H^0(L)$, one recovers

$$\pi^*([L]) = [\widetilde{L}] + [E]. \tag{4.35}$$

We also have the following identity, for any $z \in H^\bullet(Z)$:

$$(\iota^E_{E\cap\widetilde{L}})_*(\pi_Z^{E\cap\widetilde{L}})^*(z) = (\pi_Z^E)^*(z)\,.\,\left((\pi_Z^E)^*c_1(N_{L/X})_{|Z} - c_1(N_{E/\widetilde{X}})\right). \tag{4.36}$$

Proof (of (4.36)). We have $(\pi_Z^{E\cap\widetilde{L}})^* = (\iota^E_{E\cap\widetilde{L}})^* \circ (\pi_Z^E)^*$, hence using (4.32) we get

$$(\iota^E_{E\cap\widetilde{L}})_*(\pi_Z^{E\cap\widetilde{L}})^*(z) = (\pi_Z^E)^*(z)\,.\,[E\cap\widetilde{L}]_E$$

where $[E\cap\widetilde{L}]_E$ denotes the class of $E\cap\widetilde{L}$ in the cohomology of E. Since $\widetilde{L}$ and E are transverse in $\widetilde{X}$, we can use (4.33) to get

$$[E\cap\widetilde{L}]_E = (\iota^E_{E\cap\widetilde{L}})_*(\iota^{\widetilde{L}}_{E\cap\widetilde{L}})^*(1) = (\iota^{\widetilde{X}}_E)^*(\iota^{\widetilde{X}}_{\widetilde{L}})_*(1) = (\iota^{\widetilde{X}}_E)^*([\widetilde{L}]_{\widetilde{X}}).$$

Now using (4.35) we get

$$[\widetilde{L}]_{\widetilde{X}} = \pi^*([L]_X) - [E]$$

and thus

$$[E\cap\widetilde{L}]_E = (\iota^{\widetilde{X}}_E)^*\pi^*[L] - (\iota^{\widetilde{X}}_E)^*[E] = (\pi_Z^E)^*(\iota^X_Z)^*[L] - c_1(N_{E/\widetilde{X}}).$$

The claim then follows from the computation $(\iota^X_Z)^*[L] = (\iota^L_Z)^*(\iota^X_L)^*(\iota^X_L)_*(1) = c_1(N_{L/X})_{|Z}$ where we have used (4.31). ☐

Now if L is a smooth hypersurface of X such that Z and L are transverse in X, we have the simpler identities

$$\pi^* \circ (\iota^X_L)_* = (\iota^{\widetilde{X}}_{\widetilde{L}})_* \circ (\pi^{\widetilde{L}}_L)^*; \tag{4.37}$$

$$\pi^*([L]) = [\widetilde{L}]. \tag{4.38}$$

We also have

$$(\pi_Z^E)^* \circ (\iota^Z_{Z\cap L})_* = (\iota^E_{E\cap\widetilde{L}})_* \circ (\pi^{E\cap\widetilde{L}}_{Z\cap L})^*. \tag{4.39}$$

4.8.4 The excess class γ

Let X be a complex manifold and Z a closed submanifold of X, of codimension r. We let $\pi : \widetilde{X} \to X$ be the blow-up of X along Z. The excess class of π is by definition

$$\gamma = c_{r-1}\left((\pi_Z^E)^*(N_{Z/X})/N_{E/\widetilde{X}}\right) \in H^{2(r-1)}(E). \tag{4.40}$$

It appears in the formula

$$\pi^*(\iota^X_Z)_*(z) = (\iota^{\widetilde{X}}_E)_*((\pi_Z^E)^*(z)\,.\,\gamma). \tag{4.41}$$

We have a short exact sequence

$$0 \to H^{k-2r}(Z)(-r) \xrightarrow{\alpha} H^{k-2}(E)(-1) \oplus H^k(X) \xrightarrow{\beta} H^k(\widetilde{X}) \to 0 \tag{4.42}$$

where α and β are defined by $\alpha(z) = \left((\pi_Z^E)^*(z)\,.\,\gamma, (\iota^X_Z)_*(z)\right)$ and $\beta(e,x) = -(\iota^{\widetilde{X}}_E)_*(e) + \pi^*(x)$.

Let L be a smooth hypersurface of X that contains Z. We let γ_L be the excess class of $\pi^{\widetilde{L}}_L : \widetilde{L} \to L$, and is an element of $H^{2(r-2)}(E\cap\widetilde{L})$. We have the identity

$$(\iota^E_{E\cap\widetilde{L}})_*(\gamma_L) = \gamma - (\pi_Z^E)^*c_{r-1}(N_{Z/L}); \tag{4.43}$$

Proof (of (4.43)). Let us write $\xi = -c_1(N_{E/\widetilde{X}})$ and $\xi_L = -c_1(N_{E\cap\widetilde{L}/E})$. We then have

$$\gamma = \sum_{k=0}^{r-1} (\pi_Z^E)^*(c_{r-1-k}(N_{Z/X})) \,.\, \xi^k \text{ and } \gamma_L = \sum_{k=0}^{r-2} (\pi_Z^{E\cap\widetilde{L}})^*(c_{r-2-k}(N_{Z/L})) \,.\, \xi_L^k.$$

Using (4.32) and (4.33) (E and $\widetilde{L}$ are transverse in X) we get

$$(\iota_{E\cap\widetilde{L}}^E)^*\xi = -(\iota_{E\cap\widetilde{L}}^E)^*(\iota_E^{\widetilde{X}})^*(\iota_E^{\widetilde{X}})_*(1) = -(\iota_{E\cap\widetilde{L}}^{\widetilde{L}})^*(\iota_{\widetilde{L}}^{\widetilde{X}})^*(\iota_E^{\widetilde{X}})_*(1) = -(\iota_{E\cap\widetilde{L}}^E)^*(\iota_{E\cap\widetilde{L}}^E)_*(\iota_{E\cap\widetilde{L}}^{\widetilde{L}})^*(1) = \xi_L.$$

Repeated applications of the projection formula (4.30) then give

$$(\iota_{E\cap\widetilde{L}}^E)_*\gamma = \sum_{k=0}^{r-2} (\iota_{E\cap\widetilde{L}}^E)_*(\pi_Z^{E\cap\widetilde{L}})^*(c_{r-2-k}(N_{Z/L})) \,.\, \xi^k.$$

Using (4.36) we get

$$(\iota_{E\cap\widetilde{L}}^E)_*(\pi_Z^{E\cap\widetilde{L}})^*(c_{r-2-k}(N_{Z/L})) = (\pi_Z^E)^*(c_{r-2-k}(N_{Z/L}) \,.\, c_1(N_{L/X})_{|Z}) + (\pi_Z^E)^*(c_{r-2-k}(N_{Z/L})) \,.\, \xi.$$

Replacing in the above sum and doing a change of summation index, one gets

$$(\iota_{E\cap\widetilde{L}}^E)_*\gamma = \sum_{k=0}^{r-1} (\pi_Z^E)^*(c_{r-2-k}(N_{Z/L}) \,.\, c_1(N_{L/X})_{|Z} + c_{r-1-k}(N_{Z/L})) \,.\, \xi^k - (\pi_Z^E)^*(c_{r-1}(N_{Z/L}))$$

Now using (4.27) we get $c_{r-2-k}(N_{Z/L}) \,.\, c_1(N_{L/X})_{|Z} + c_{r-1-k}(N_{Z/L}) = c_{r-1-k}(N_{Z/X})$, hence the claim. $\square$

Bibliography

[Alu12] P. Aluffi. Chern classes of free hypersurface arrangements. *J. Singul.*, 5:19–32, 2012.

[And04] Y. André. *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, volume 17 of *Panoramas et Synthèses [Panoramas and Syntheses]*. Société Mathématique de France, Paris, 2004.

[And09] Y. André. Galois theory, motives and transcendental numbers. In *Renormalization and Galois theories*, volume 15 of *IRMA Lect. Math. Theor. Phys.*, pages 165–177. Eur. Math. Soc., Zürich, 2009.

[Aom77] K. Aomoto. On the structure of integrals of power product of linear functions. *Sci. Papers College Gen. Ed. Univ. Tokyo*, 27(2):49–61, 1977.

[Aom82] K. Aomoto. Addition theorem of Abel type for hyper-logarithms. *Nagoya Math. J.*, 88:55–71, 1982.

[Arn69] V. I. Arnol'd. The cohomology ring of the group of dyed braids. *Mat. Zametki*, 5:227–231, 1969.

[BEK06] S. Bloch, H. Esnault, and D. Kreimer. On motives associated to graph polynomials. *Comm. Math. Phys.*, 267(1):181–225, 2006.

[Blo12] S. Bloch. Motives, the fundamental group, and graphs. *preprint*, 2012.

[BMS87] A. Beĭlinson, R. MacPherson, and V. Schechtman. Notes on motivic cohomology. *Duke Math. J.*, 54(2):679–710, 1987.

[Bri73] E. Brieskorn. Sur les groupes de tresses [d'après V. I. Arnol'd]. In *Séminaire Bourbaki, 24ème année (1971/1972), Exp. No. 401*, pages 21–44. Lecture Notes in Math., Vol. 317. Springer, Berlin, 1973.

[Bro09] F. Brown. Multiple zeta values and periods of moduli spaces $\overline{\mathfrak{M}}_{0,n}$. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(3):371–489, 2009.

[Bro12] F. Brown. Mixed Tate motives over $\mathbb{Z}$. *Ann. of Math. (2)*, 175(2):949–976, 2012.

[Bro13] F. Brown. Single-valued periods and multiple zeta values. *preprint*, 2013.

[BS12] F. Brown and O. Schnetz. A K3 in ϕ^4. *Duke Math. J.*, 161(10):1817–1862, 2012.

[BVGS90] A. A. Beĭlinson, A. N. Varchenko, A. B. Goncharov, and V. V. Shekhtman. Projective geometry and K-theory. *Algebra i Analiz*, 2(3):78–130, 1990.

[CHKS06] F. Catanese, S. Hosten, A. Khetan, and B. Sturmfels. The maximum likelihood degree. *Am. J. Math.*, 128(3):671–697, 2006.

[CK99] A. Connes and D. Kreimer. Hopf algebras, renormalization and noncommutative geometry. In *Quantum field theory: perspective and prospective (Les Houches, 1998)*, volume 530 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 59–108. Kluwer Acad. Publ., Dordrecht, 1999.

[DCP95] C. De Concini and C. Procesi. Wonderful models of subspace arrangements. *Selecta Math. (N.S.)*, 1(3):459–494, 1995.

[Del71] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Del74] P. Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[Del89] P. Deligne. Le groupe fondamental de la droite projective moins trois points. In *Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 79–297. Springer, New York, 1989.

[Del10] P. Deligne. Le groupe fondamental unipotent motivique de $\mathbb{G}_m - \mu_N$, pour $N = 2, 3, 4, 6$ ou 8. *Publ. Math. Inst. Hautes Études Sci.*, (112):101–141, 2010.

[Den10] G. Denham. Homological aspects of hyperplane arrangements. In *Arrangements, local systems and singularities*, volume 283 of *Progr. Math.*, pages 39–58. Birkhäuser Verlag, Basel, 2010.

[DG05] P. Deligne and A. B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.

[Dol07] I. V. Dolgachev. Logarithmic sheaves attached to arrangements of hyperplanes. *J. Math. Kyoto Univ.*, 47(1):35–64, 2007.

[Dor10] D. Doryn. Cohomology of graph hypersurfaces associated to certain Feynman graphs. *Commun. Number Theory Phys.*, 4(2):365–415, 2010.

[DS82] J. L. Dupont and C. H. Sah. Scissors congruences. II. *J. Pure Appl. Algebra*, 25(2):159–195, 1982.

[Dup13] C. Dupont. Hypersurface arrangements and a global Brieskorn-Orlik-Solomon theorem. *Preprint: arXiv:1302.2103*, 2013.

[Dup14a] C. Dupont. The combinatorial Hopf algebra of motivic dissection polylogarithms. *Adv. Math.*, 264:646–699, 2014.

[Dup14b] C. Dupont. The motive of a bi-arrangement. *Preprint*, 2014.

[FM94] W. Fulton and R. MacPherson. A compactification of configuration spaces. *Ann. of Math. (2)*, 139(1):183–225, 1994.

[Get99] E. Getzler. Resolving mixed Hodge modules on configuration spaces. *Duke Math. J.*, 96(1):175–203, 1999.

[GM04] A. B. Goncharov and Yu. I. Manin. Multiple ζ-motives and moduli spaces $\overline{\mathcal{M}}_{0,n}$. *Compos. Math.*, 140(1):1–14, 2004.

[Gon95] A. B. Goncharov. Polylogarithms in arithmetic and geometry. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 374–387, Basel, 1995. Birkhäuser.

[Gon00] A. B. Goncharov. Geometry of the trilogarithm and the motivic Lie algebra of a field. In *Regulators in analysis, geometry and number theory*, volume 171 of *Progr. Math.*, pages 127–165. Birkhäuser Boston, Boston, MA, 2000.

[Gon01] A. B. Goncharov. The dihedral Lie algebras and Galois symmetries of $\pi_1^{(\ell)}(\mathbb{P}^1 - (\{0,\infty\} \cup \mu_N))$. *Duke Math. J.*, 110(3):397–487, 2001.

[Gon02] A. B. Goncharov. Periods and mixed motives. *Preprint: arXiv:math/0202154*, 2002.

[Gon05] A. B. Goncharov. Galois symmetries of fundamental groupoids and noncommutative geometry. *Duke Math. J.*, 128(2):209–284, 2005.

[Gon13] A. B. Goncharov. A simple construction of Grassmannian polylogarithms. *Adv. Math.*, 241:79–102, 2013.

[Gro66] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

[GZ01] A. B. Goncharov and J. Zhao. Grassmannian trilogarithms. *Compositio Math.*, 127(1):83–108, 2001.

[Hai94] R. M. Hain. Classical polylogarithms. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 3–42. Amer. Math. Soc., Providence, RI, 1994.

[Hu03] Y. Hu. A compactification of open varieties. *Trans. Amer. Math. Soc.*, 355(12):4737–4753, 2003.

[Hub00] A. Huber. Realization of Voevodsky's motives. *J. Algebraic Geom.*, 9(4):755–799, 2000.

[Hub04] A. Huber. Corrigendum to: "Realization of Voevodsky's motives" [J. Algebraic Geom. 9 (2000), no. 4, 755–799; mr1775312]. *J. Algebraic Geom.*, 13(1):195–207, 2004.

[Kri94] I. Kriz. On the rational homotopy type of configuration spaces. *Ann. of Math. (2)*, 139(2):227–237, 1994.

[KZ01] M. Kontsevich and D. Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[Ler59] J. Leray. Le calcul différentiel et intégral sur une variété analytique complexe. (Problème de Cauchy. III). *Bull. Soc. Math. France*, 87:81–180, 1959.

[Lev93] M. Levine. Tate motives and the vanishing conjectures for algebraic K-theory. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 167–188. Kluwer Acad. Publ., Dordrecht, 1993.

[Lev98] M. Levine. *Mixed motives*, volume 57 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[Li09] L. Li. Wonderful compactification of an arrangement of subvarieties. *Michigan Math. J.*, 58(2):535–563, 2009.

[Loo93] E. Looijenga. Cohomology of $\mathscr{M}_3$ and $\mathscr{M}_3^1$. In *Mapping class groups and moduli spaces of Riemann surfaces (Göttingen, 1991/Seattle, WA, 1991)*, volume 150 of *Contemp. Math.*, pages 205–228. Amer. Math. Soc., Providence, RI, 1993.

[LR10] J.-L. Loday and M. Ronco. Combinatorial Hopf algebras. In *Quanta of maths*, volume 11 of *Clay Math. Proc.*, pages 347–383. Amer. Math. Soc., Providence, RI, 2010.

[Mor78] J. W. Morgan. The algebraic topology of smooth algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (48):137–204, 1978.

[OS80] P. Orlik and L. Solomon. Combinatorics and topology of complements of hyperplanes. *Invent. Math.*, 56(2):167–189, 1980.

[OT92] P. Orlik and H. Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[Oxl11] J. Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[PS08] C. A. M. Peters and J. H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.

[Sai80] K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[Sta99] R. P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Ter78] H. Terao. Forms with logarithmic pole and the filtration by the order of the pole. In *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, pages 673–685, Tokyo, 1978. Kinokuniya Book Store.

[Ter02] T. Terasoma. Mixed Tate motives and multiple zeta values. *Invent. Math.*, 149(2):339–369, 2002.

[Tot96] B. Totaro. Configuration spaces of algebraic varieties. *Topology*, 35(4):1057–1067, 1996.

[Voi02] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.

[VSF00] V. Voevodsky, A. Suslin, and E. M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.

[Yuz01] S. Yuzvinskiĭ. Orlik-Solomon algebras in algebra and topology. *Uspekhi Mat. Nauk*, 56(2(338)):87–166, 2001.

[Zha00] J. Zhao. Remarks on a Hopf algebra for defining motivic cohomology. *Duke Math. J.*, 103(3):445–458, 2000.

[Zha04] J. Zhao. Motivic cohomology of pairs of simplices. *Proc. London Math. Soc. (3)*, 88(2):313–354, 2004.