Singular Hamilton-Jacobi equation for the tail problem

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Abstract

In this paper we study the long time-long range behavior of reaction diffusion equations with negative square root -type reaction terms. In particular we investigate the exponential behavior of the solutions after a standard hyperbolic scaling. This leads to a Hamilton-Jacobi variational inequality with an obstacle that depends on the solution itself and defines the open set where the limiting solution does not vanish. Counter-examples show a nontrivial lack of uniqueness for the variational inequality depending on the conditions imposed on the boundary of this open set. Both Dirichlet and state constraints boundary conditions play a role. When the competition term does not change sign, we can identify the limit, while, in general, we find lower and upper bounds for the limit.

Although models of this type are rather old and extinction phenomena are as important as blow-up, our motivation comes from the so-called "tail problem" in population biology. One way to avoid meaningless exponential tails, is to impose extra-mortality below a given survival threshold. Our study shows that the precise form of this extra-mortality term is asymptotically irrelevant and that, in the survival zone, the population profile is impacted by the survival threshold (except in the very particular case when the competition term is non-positive).

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1 Introduction

We consider the following reaction-diffusion equations with a singular reaction term

$$\begin{cases} n_{\varepsilon,t} - \varepsilon\Delta n_\varepsilon = \frac{1}{\varepsilon} n_\varepsilon R - \frac{1}{\varepsilon}\sqrt{\beta_\varepsilon n_\varepsilon} & \text{in } \mathbb{R}^d \times (0,+\infty), \\ n_\varepsilon = e^{u_\varepsilon^0/\varepsilon} \quad \text{on} \quad \mathbb{R}^d \times \{0\}, \end{cases} \tag{1}$$

with threshold

$$\beta_\varepsilon = e^{u_m/\varepsilon} \quad \text{for some} \quad u_m < 0. \tag{2}$$

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The parameter $\varepsilon > 0$ is introduced by a hyperbolic scaling with the aim to describe the long time and long range behavior of the unscaled problem (corresponding to $\varepsilon = 1$).

The limiting behavior of scaled reaction-diffusion equations with KPP-type reaction has been studied extensively in, among other places, the theory of front propagation ([3, 17, 10]) using the so called WKB-(exponential) change of the unknown.

The novelty of the problem we are considering here is the negative square root term in the right-hand side. To the best of our knowledge the first study of such nonlinearity goes back to [9] where it is proved that local extinction occurs, i.e., the solution can vanish in a domain and stay positive in another region. That a solution of a parabolic problem can vanish locally is a surprising effect and as singular as the blow-up phenomena for supercritical reactions terms ([15]).

In population biology such behavior prevents the so-called "tail problem" where very small (and thus meaningless) populations can generate artifacts ([12]). Although the mathematical analysis of the limit in (1) turns out to be a full subject in itself, our primary motivation comes from qualitative questions in population dynamics.

Along the same lines, in the context of front propagation, one may consider the modified Fisher–KPP equation

$$n_{\varepsilon,t} - \varepsilon\Delta n_\varepsilon = \frac{1}{\varepsilon} n_\varepsilon(1 - n_\varepsilon) - \frac{1}{\varepsilon}\sqrt{\beta_\varepsilon n_\varepsilon} \quad \text{in } \mathbb{R}^d \times (0, +\infty),$$

and ask the question whether the square root term changes fundamentally the study in [10] and [12] of the propagation of the invading/combustion fronts.

An elementary model in adaptive evolution is the non-local reaction-diffusion equation

$$n_{\varepsilon,t} - \varepsilon\Delta n_\varepsilon = \frac{1}{\varepsilon} n_\varepsilon R(x, I_\varepsilon) - \frac{1}{\varepsilon}\sqrt{\beta_\varepsilon n_\varepsilon} \quad \text{in } \mathbb{R}^d \times (0, +\infty),$$

where n_ε is the population density of individuals with phenotypical trait x and, for $t > 0$,

$$I_\varepsilon(t) = \int \psi(x) n_\varepsilon(x, t) dx.$$

Here x denotes the trait, R represents the net growth rate while ψ is the consumption rate of individuals and $I(t)$ is the total consumption of the resource at time t. The survival threshold is as in [12]. Finally ε may represent large time and rare mutations as introduced in [4, 5, 14].

It is known that under some assumptions the density concentrates as an evolving Dirac mass for the fittest trait. In biological terms this means that one or several dominant traits survive while other become extinct. Some phenomena as the discontinuous jumps of the fittest trait, non smooth branching and fast dynamics compared to stochastic simulations, motivated [12] to improve the model by including a survival threshold. Numerical results confirm that this modification gives dynamics comparable to stochastic models. It is interesting to investigate rigorously whether the dynamics of the Dirac concentration points are really changed by the survival threshold and to explain why its specific form (square root versus n_ε^α with $0 < \alpha < 1$) seems irrelevant.

A way to approach these questions is through the asymptotic analysis of n_ε. Since, as in the classical case (i.e., the Fisher-KPP equation without the square root term (see [10]), n_ε decays exponentially, the limit is better described using the Hopf-Cole transformation

$$u_\varepsilon = \varepsilon \ln n_\varepsilon, \tag{3}$$

which, for $u_\varepsilon^0 = \varepsilon \ln n_\varepsilon^0$, leads to

$$
\begin{cases}
u_{\varepsilon,t} - \varepsilon\Delta u_\varepsilon - |Du_\varepsilon|^2 = R - \exp\big((2\varepsilon)^{-1}(u_m - u_\varepsilon)\big) \quad \text{in} \quad \mathbb{R}^d \times (0,+\infty), \\
u_\varepsilon = u_\varepsilon^0 \quad \text{in } \mathbb{R}^d \times \{0\}.
\end{cases} \tag{4}
$$

Throughout the paper we assume that there exist $C > 0$ and $u^0 \in C^{0,1}(\mathbb{R}^d)$ such that

$$
\|R\|_{C^{0,1}} \leq C \quad \text{and} \quad \|u^0\| \leq C, \tag{5}
$$

and

$$
u_\varepsilon^0 \in C(\mathbb{R}^d), \quad u_\varepsilon^0 \leq C \quad \text{and, as } \varepsilon \to 0, \quad u_\varepsilon^0 \to u^0 \quad \text{in} \quad C(\mathbb{R}^d). \tag{6}
$$

It is easy to see, at least formally, that the u_ε's converge, as $\varepsilon \to 0$, to some u satisfying, in the viscosity sense ([8]), the Hamilton-Jacobi problem

$$
\begin{cases}
u_t = |Du|^2 + R \quad \text{in} \quad \Omega \subset \mathbb{R}^d \times (0,\infty), \\
u = -\infty \quad \text{in} \quad \overline{\Omega}^c \cap (\mathbb{R}^d \times (0,\infty)), \\
u \geq u_m \quad \text{in} \quad \overline{\Omega}, \\
u = u^0 \quad \text{in} \quad \overline{\Omega} \cap \{0\},
\end{cases} \tag{7}
$$

with the space-time open set Ω is defined by

$$
\Omega = \mathcal{I}nt \left\{(x,t) \in \mathbb{R}^d \times (0,\infty) : \lim_{\varepsilon \to 0} u_\varepsilon(x,t) > -\infty\right\}.
$$

Note that (7) resembles an obstacle problem where the obstacle depends on the solution itself. As a matter of fact the open set Ω plays an important role and, hence, the problem may be better stated in terms of the pair (u, Ω). The difficulty is that there are many viscosity solutions (see Appendix A for examples) to this problem and the boundary conditions on $\partial\Omega$ are of real importance.

The "natural" boundary conditions for (7) are the Dirichlet and state constraint ones. The former is

$$
\lim_{(x,t)\to(t_0,x_0)\in\partial\Omega} u(x,t) = u_m, \tag{8}
$$

while the latter says (see [16]) that

$$
u \quad \text{is a super-solution in} \quad \overline{\Omega} \text{ and a sub-solution in} \quad \Omega\ . \tag{9}
$$

The basic questions are:

• What boundary condition should be satisfied by the limit of the u_ε's on $\partial\Omega$? Dirichlet or state constraint ? It is most probably the latter, since not all solutions to (7) satisfy the former. To the best of our knowledge, however, there are no available results for state constraint problems with time varying and non smooth domains. Most of the work in this paper is geared towards going around this difficulty.

• What stability is induced by the selection of the correct solution to (7)? Is it the maximal subsolution?

• Does the limit of the u_ε's depend on the specific form of the survival threshold, i.e., can we replace $(\beta_\varepsilon n_\varepsilon)^{1/2}$ by $(\beta_\varepsilon n_\varepsilon)^\alpha$ with $\alpha \in (0,1)$ without affecting the limit?

To study (1) it is useful to consider the problem

$$\begin{cases} n^1_{\varepsilon,t} - \varepsilon \Delta n^1_\varepsilon = \frac{n^1_\varepsilon}{\varepsilon} R \quad \text{in} \quad \mathbb{R}^d \times (0,+\infty), \\ \\ n^1_\varepsilon = \exp(\varepsilon^{-1} u^0_\varepsilon) \quad \text{in } \mathbb{R}^d \times \{0\}. \end{cases} \tag{10}$$

In view of the above assumptions the theory of viscosity solutions yields that the u^1_ε's defined by

$$u^1_\varepsilon = \varepsilon \ln n^1_\varepsilon, \tag{11}$$

converge, as $\varepsilon \to 0$, locally uniformly to $u^1 \in C(\mathbb{R}^d \times (0,\infty))$, the unique solution of the eikonal -type equation

$$\begin{cases} u^1_t = |Du^1|^2 + R \quad \text{in} \quad \mathbb{R}^d \times (0,+\infty), \\ \\ u^1 = u^0 \quad \text{in} \quad \mathbb{R}^d \times \{0\}. \end{cases} \tag{12}$$

The maximum principle yields the comparison $n_\varepsilon \le n^1_\varepsilon$, which in turn implies that $u_\varepsilon \le u^1_\varepsilon$ and, in the limit (this has to be stated more carefully), $u \le u^1$. It also follows from (4), at least formally, that, as $\varepsilon \to 0$,

$$u_\varepsilon \to -\infty \qquad \text{in} \ \ \mathbb{R}^d \backslash \overline{\Omega^1},$$

where

$$\Omega^1 = \{(x,t) \mid u^1(x,t) > u_m\}. \tag{13}$$

It turns out that the case of nonpositive rate is particularly illuminating and the above questions can be answered completely and positively using u^1 (see Section 2).

The problem is considerably more complicated when R takes positive values. In this case we introduce an iterative procedure to build successive sub- and super-solutions (Section 3). This construction gives the complete limit of u_ε in the case where R is constant (Section 4). It follows that the limit is not the maximal subsolution of (7) and the Dirichlet condition is not enough to select it. In Section 5, we analyze the case of strictly positive spatially dependent R and provide a complete answer in terms of the iterative procedure. The relative roles of the Dirichlet and state constraint boundary conditions appear clearly in this case. In the Appendix we present some examples as well as the proofs of few technical facts used earlier.

We conclude the Introduction with the definition and the notation of the half-relaxed limits that we will be using throughout the paper. To this end, if $(w_\varepsilon)_{\varepsilon>0}$ is a family of bounded functions, the upper and lower limits, which are denoted by $\bar{w}$ and $\underline{w}$ respectively, are given by

$$\overline{w}(x) = \limsup_{\varepsilon\to 0, y \to x} w_\varepsilon(y) \qquad \text{and} \qquad \underline{w}(x) = \liminf_{\varepsilon\to 0, y\to x} w_\varepsilon(y). \tag{14}$$

2 Nonpositive growth rate

In this section we assume that

$$R \le 0 \quad \text{in} \quad \mathbb{R}^d. \tag{15}$$

When (15) holds, the behavior of the u_ε's, in the limit $\varepsilon \to 0$, can be described completely and the solution u^1 of (12) carries all the information.

We have:

Theorem 2.1. *Assume (5), (6) and (15). Then, as $\varepsilon \to 0$, the u_ε's converge, locally uniformly in Ω^1 and pointwise in $\mathbb{R}^d \backslash \overline{\Omega^1}$, to*

$$u(x,t) = \begin{cases} u^1(x,t) & \text{for} \quad (x,t) \in \Omega^1, \\ -\infty & \text{for} \quad (x,t) \in \mathbb{R}^d \backslash \overline{\Omega^1}, \end{cases} \tag{16}$$

with u^1 and Ω^1 defined by (12) and (13) respectively. In particular, $u(x,t) \to u_m$ as $(x,t) \to \partial\Omega^1$.

Before we begin with the proof we discuss below several observations which are important to explain the meaning of the results.

The u associated with the open set Ω^1 is the maximal solution to (7). Indeed any other solution $\widetilde{u}$, with the corresponding open set $\widetilde{\Omega}$, satisfies $\widetilde{u} \leq u^1$ and thus $\widetilde{\Omega} \subset \Omega^1$ and $\widetilde{u} \leq u$. It also satisfies the Dirichlet and state constraint boundary conditions. To verify the latter we notice, using the standard optimal control formula ([13, 11, 1]), that

$$u^1(x,t) = \sup_{\substack{(x(s),s) \in \mathbb{R}^d \times [0,\infty) \\ x(t)=x}} \left\{ \int_0^t \left(-\frac{|\dot{x}(s)|^2}{4} + R(x(s)) \right) ds + u_0(x(0)) \right\}.$$

If $\widetilde{x}(\cdot)$ is an optimal trajectory, the Dynamic Programming Principle implies that, for any $0 < \tau < t$,

$$u^1(x,t) = \int_\tau^t \left(-\frac{|\dot{\widetilde{x}}(s)|^2}{4} + R(\widetilde{x}(s)) \right) ds + u^1(\widetilde{x}(\tau), \tau).$$

Since R is nonpositive, u^1 is decreasing along the optimal trajectory. It follows that, if $u^1(x,t) > u_m$, then, for all $0 \leq \tau < t$, $u^1(\widetilde{x}(\tau), \tau) > u_m$.

Hence, for all $(x,t) \in \Omega^1$,

$$u(x,t) = \sup_{\substack{(x(s),s) \in \Omega^1 \\ x(t)=x}} \left\{ \int_0^t \left(-\frac{|\dot{x}(s)|^2}{4} + R(x(s)) \right) ds + u_0(x(0)) \right\},$$

and, therefore, u verifies the state constraint condition. Moreover, the limit u does not depend on the details on the extra death term. In particular it is same for a term like $-n_\varepsilon^\gamma \exp(\varepsilon^{-1}\gamma u_m)$ with $0 < \gamma < 1$.

We continue with the

Proof of Theorem 2.1. As already discussed, we know that $u_\varepsilon \leq u_\varepsilon^1$ but we cannot obtain directly the other inequality. It is therefore necessary to introduce a pair of auxiliary functions v_ε^A and $v_\varepsilon^{A,1}$ which converge, as $\varepsilon \to 0$, in $C(\mathbb{R}^d \times (0,\infty))$ to $\max(u^1, -A)$. Using this information for appropriate values of the parameter A, we then prove that, as $\varepsilon \to 0$, $u_\varepsilon \to u^1$ locally uniformly in the open set

$$\mathcal{A} = \{(x,t) : \; u^1(x,t) > u_m\}, \tag{17}$$

and $u_\varepsilon \to -\infty$ locally uniformly in the open set

$$\mathcal{B} = \{(x,t) : \; u^1(x,t) < u_m\}. \tag{18}$$

To this end, for A such that

$$0 < A < -u_m, \tag{19}$$

we consider the functions v_ε^A and $v_\varepsilon^{A,1}$ given by

$$n_\varepsilon + \exp(\frac{-A}{\varepsilon}) = \exp(\frac{v_\varepsilon^A}{\varepsilon}) \quad \text{and} \quad n_\varepsilon^1 + \exp(\frac{-A}{\varepsilon}) = \exp(\frac{v_\varepsilon^{A,1}}{\varepsilon}). \tag{20}$$

We have:

Proposition 2.2. *Assume (5), (6), (19) and (15). The* $v_\varepsilon^{A,1}$*'s and* v_ε^A*'s converge, as* $\varepsilon \to 0$*, in* $C(\mathbb{R}^d \times [0,\infty))$ *to the unique solution* $v^{A,1}$ *of*

$$\begin{cases} \min\big(v^{A,1} + A, v_t^{A,1} - |Dv^{A,1}|^2 - R\big) = 0 \quad \text{in} \ \ \mathbb{R}^d \times (0,\infty), \\ v^{A,1} = \max(u^0, -A) \quad \text{on} \ \ \mathbb{R}^d \times \{0\}. \end{cases} \tag{21}$$

Consequently, as $\varepsilon \to 0$*, the* $v_\varepsilon^A \to v^{A,1} = \max(u^1, -A)$ *in* $C(\mathbb{R}^d \times [0,\infty))$.

We present the proof at the end of this section and next we prove the convergence of the u_ε's in the sets $\mathcal{A}$ and $\mathcal{B}$. We begin with the former.

To this end fix $(x_0, t_0) \in \mathcal{A}$. By the definition of $\mathcal{A}$ we have $u^1(x_0, t_0) > u_m$ and, hence, we can choose A such that $u^1(x_0, t_0) > -A > u_m$. It follows from Proposition 2.2 that, as $\varepsilon \to 0$ and uniformly in any neighborhood of (x_0, t_0),

$$v_\varepsilon^A \to v^{A,1} = \max(-A, u^1) = u^1.$$

Using the latter, the choice of A and the fact that

$$u_\varepsilon = v_\varepsilon^A + \varepsilon \ln(1 - \exp(\varepsilon^{-1}(-A - v_\varepsilon^A))),$$

we deduce that, as $\varepsilon \to 0$, $u_\varepsilon \to u^1$ uniformly in any neighborhood of (x_0, t_0).

Next we consider the limiting behavior in the set $\mathcal{B}$. To this end, observe that, using (3), (11) and a sub-solution argument, we find $u_\varepsilon \le u_\varepsilon^1$ and, thus, $\overline{u} \le u^1$, and

$$\overline{u} < u_m \quad \text{in} \quad \mathcal{B}.$$

Next assume that, for some $(x_0, t_0) \in \mathcal{B}$, $\overline{u}(x_0, t_0) > -\infty$. Since $\overline{u}$ is upper semi-continuous, there exists a family $(\phi_\alpha)_{\alpha > 0}$ of smooth functions such that $\overline{u} - \phi_\alpha$ attains a strict local maximum at some (x_α, t_α) and, as $\alpha \to 0$,

$$(x_\alpha, t_\alpha) \to (x_0, t_0), \quad \overline{u}(y_\alpha, x_\alpha) \ge \overline{u}(x_0, t_0) \quad \text{and} \quad \overline{u}(t_\alpha, x_\alpha) \to \overline{u}(x_0, t_0).$$

It follows that there exists points $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$ such that $u_\varepsilon - \phi_\alpha$ attains a local maximum at $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$, $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \to (x_\alpha, t_\alpha)$, as $\varepsilon \to 0$, and, in view of (4), at $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$,

$$\phi_{\alpha,t} - \varepsilon \Delta \phi_\alpha - |D\phi_\alpha|^2 - R \le -\exp((2\varepsilon)^{-1}(u_m - u_\varepsilon)).$$

Letting $\varepsilon \to 0$ we obtain, at (x_α, t_α),

$$\phi_{\alpha,t} - |D\phi_\alpha|^2 - R \le \limsup_{\varepsilon \to 0} [-\exp((2\varepsilon)^{-1}(u_m - u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})))].$$

The definition of $\overline{u}$ yields

$$\limsup_{\varepsilon \to 0} u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \le \overline{u}(x_\alpha, t_\alpha)$$

and, since, for α sufficiently small, $\overline{u}(x,t) < u_m$, we have $\overline{u}(t_\alpha, x_\alpha) < u_m$,

$$\limsup_{\varepsilon\to 0}[-\exp((2\varepsilon)^{-1}(u_m - u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})))] = -\infty$$

and, finally, at (x_α, t_α),

$$\phi_{\alpha,t} - |D\phi_\alpha|^2 - R \leq -\infty,$$

which is not possible because ϕ_α is a smooth function.

The claim about the uniform convergence on compact is an immediate consequence of the upper semicontinuity of $\overline{u}$ and the previous argument.

□

We conclude the section with the proof of Proposition 2.2. Since it is long, before entering in the details, we briefly describe the main steps. Then we establish that the half relaxed upper and lower limits $\overline{v}^\alpha$ and $\underline{v}^\alpha$ are respectively sub- and super-solutions of (21). We conclude by identifying the limit.

We have:

Proof of Proposition 2.2. By the definition of v_ε^A, we have $v_\varepsilon^A > -A$ and, thus, the family $(v_\varepsilon^A)_\varepsilon > 0$ is bounded from below.

To prove an upper bound we first notice that on $\mathbb{R}^d \times \{0\}$

$$v_\varepsilon^A = u_\varepsilon^0 + \varepsilon \ln(1 + e^{\frac{-A-u_\varepsilon^0}{\varepsilon}}) \quad \text{and} \quad v_\varepsilon^A = -A + \varepsilon \ln(1 + e^{\frac{A+u_\varepsilon^0}{\varepsilon}}), \tag{22}$$

and, hence,

$$v_\varepsilon^A \leq \max(u_\varepsilon^0 + \varepsilon\ln(2), -A + \varepsilon\ln(2)) \quad \text{on} \quad \mathbb{R}^d \times \{0\},$$

and, in view of (6),

$$v_\varepsilon^A \leq C_A \quad \text{on} \quad \mathbb{R}^d \times \{0\},$$

for $C_A > 0$ such that $\max(-A, u_\varepsilon^0) \leq C_A$.

Moreover, since $R \leq 0$, we have

$$v_{\varepsilon,t}^A - \varepsilon\Delta v_\varepsilon^A - |Dv_\varepsilon^A|^2 = \frac{n_\varepsilon}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})}R - \frac{\sqrt{\beta_\varepsilon n_\varepsilon}}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} \leq 0 \quad \text{in} \quad \mathbb{R}^d \times (0,\infty). \tag{23}$$

It follows from the maximum principle that

$$v_\varepsilon^A \leq C_A + \varepsilon\ln(2) \quad \text{in} \quad \mathbb{R}^d \times (0,\infty).$$

Next we show that $\underline{v}^A$ is a super-solution of (21). Since $u_m < -A$,

$$0 \leq \frac{n_\varepsilon}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} \leq 1,$$

and

$$\frac{\sqrt{\beta_\varepsilon n_\varepsilon}}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} \leq \frac{\sqrt{\beta_\varepsilon n_\varepsilon}}{2\sqrt{n_\varepsilon \exp(\frac{-A}{\varepsilon})}} = \frac{1}{2}\exp(\frac{u_m + A}{2\varepsilon}),$$

as $\varepsilon \to 0$ and uniformly on $\mathbb{R}^d \times (0,\infty)$, we have

$$\frac{\sqrt{\beta_\varepsilon n_\varepsilon}}{n_\varepsilon + \exp(\frac{-A}{\varepsilon})} \to 0. \tag{24}$$

From (15) and (23) we then deduce that, in $\mathbb{R}^d \times (0,\infty)$,

$$v^A_{\varepsilon,t} - \varepsilon \Delta v^A_\varepsilon - |Dv^A_\varepsilon|^2 \geq R - O(\varepsilon), \tag{25}$$

while by the definition of v^A_ε we also have

$$v^A_\varepsilon + A \geq 0. \tag{26}$$

Combining (25) and (26) and using the basic stability properties of the viscosity solutions (see [2]) we find that the lower semi-continuous function $\underline{v}^A$ is a viscosity super-solution of (21).

To prove that $\overline{v}^A$ is a sub-solution to (21) we assume that, for some smooth ϕ, $\overline{v}^A - \phi$ has a strict local maximum at (x_0,t_0). It follows that there exists a family, which for notational simplicity we denote again by ε, of points $(x_\varepsilon,t_\varepsilon)_{\varepsilon>0} \in \mathbb{R}^d \times (0,\infty)$ such that $v^A_\varepsilon - \phi$ has a local maximum at $(x_\varepsilon,t_\varepsilon)$, and, as $\varepsilon \to 0$, $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$ and $v^A_\varepsilon(x_\varepsilon,t_\varepsilon) \to \overline{v}^A(x_0,t_0)$ (See [2]).

We also know, still using (23) and (24), that v^A_ε solves

$$v^A_{\varepsilon,t} - \varepsilon \Delta v^A_\varepsilon - |Dv^A_\varepsilon|^2 = \big(1 - \exp(\frac{-A - v^A_\varepsilon}{\varepsilon})\big)R - O(\varepsilon).$$

It then follows that, at $(x_\varepsilon,t_\varepsilon)$,

$$\phi_t - \varepsilon \Delta \phi - |D\phi|^2 - \big(1 - \exp(\varepsilon^{-1}(-A - v^A_\varepsilon))R \leq O(\varepsilon). \tag{27}$$

Recall that $\lim_{\varepsilon\to 0} v^A_\varepsilon(x_\varepsilon,t_\varepsilon) = \overline{v}^A(x_0,t_0) \geq -A$. Hence, if $\overline{v}^A(x_0,t_0) > -A$, then

$$\lim_{\varepsilon \to 0} \exp(\varepsilon^{-1}(-A - v^A_\varepsilon(x_\varepsilon,t_\varepsilon))) = 0.$$

From this and (27) we deduce that, if $\overline{v}^A(x_0,t_0) > -A$, then, at (x_0,t_0),

$$\phi_t - |D\phi|^2 - R \leq 0,$$

and, hence, the claim.

Next we show that $\overline{v}$ and $\underline{v}$ satisfy the appropriate initial conditions. Indeed, in view of (6) and (22) we know that, as $\varepsilon \to 0$,

$$v^A_\varepsilon \to \max(-A, u^0) \quad \text{on} \quad \mathbb{R}^d \times \{0\}.$$

It also follows from a classical argument in theory of viscosity solutions ([2, 3]) that, on $\mathbb{R}^d \times \{0\}$,

$$\overline{v}^A - \max(-A,u^0) \leq 0 \quad \text{and} \quad \underline{v}^A - \max(-A,u^0) \geq 0.$$

Hence $\overline{v}^A$ ($\underline{v}^A$ resp.) satisfies the discontinuous viscosity sub-solution (super-solution resp.) initial condition corresponding to (21). (See [2], [3]).

We already know from the definition of $\overline{v}^A$ and $\underline{v}^A$ that

$$\underline{v}^A \leq \overline{v}^A,$$

while from the comparison property for (21) in the class of semi-continuous viscosity solutions (see [1, 2, 7]) we conclude from the steps above that

$$\overline{v}^A \leq \underline{v}^A \quad \text{in} \quad \mathbb{R}^d \times (0, \infty).$$

Hence $\underline{v}^A = \overline{v}^A = v^{A,1}$ is the unique continuous viscosity solution of (21) and consequently v_ε^A and $v_\varepsilon^{A,1}$ converge, as $\varepsilon \to 0$ and locally uniformly, to $v^{A,1}$.

Combining (3) and (20) we find

$$v_\varepsilon^{A,1} = u_\varepsilon^1 + \varepsilon \ln(1 + e^{\frac{-A-u_\varepsilon^1}{\varepsilon}}) \quad \text{and} \quad v_\varepsilon^{A,1} = -A + \varepsilon \ln(1 + e^{\frac{A+u_\varepsilon^1}{\varepsilon}}).$$

Moreover, from the general Hamilton-Jacobi theory [14, 4] we know that, as $\varepsilon \to 0$, $u_\varepsilon^1 \to u^1$ locally uniformly. Hence, still for $A < -u_m$, we obtain that, as $\varepsilon \to 0$,

$$v_\varepsilon^{A,1} \to \max(u^1, -A) \quad \text{in} \quad \mathbb{R}^d \times [0, \infty).$$

It also follows that the v_ε^A's converge, as $\varepsilon \to 0$, locally uniformly to $v^{A,1} = \max(u^1, -A)$. □

3 The limit for general rate

When R changes sign, the situation is much more complicated and (16) does not hold in general. In this case we are able to provide only inequalities for the limsup and liminf of the u_ε's, which we use later to characterize the limit when R is positive.

Given u_0, $\delta > 0$ and u^1 defined by (12) with $u^1 = u_0$ on $\mathbb{R}^d \times \{0\}$, next we introduce the family $(u_i^\delta[u_0], \mathcal{C}_i^\delta[u_0], \Omega_i^\delta[u_0])_{i \in \mathbb{Z}^+}$ in $C(\mathbb{R}^d \times [0,\infty)) \times (\mathbb{R}^d \times [0,\infty)) \times (\mathbb{R}^d \times [0,\infty))$ which is defined iteratively as follows:

$$u_1^\delta[u_0] = u^1, \; \mathcal{C}_1^\delta[u_0] = \mathbb{R}^d \times [0, \infty) \;\text{ and }\; \Omega_1^\delta[u_0] = \{(x,t) \in \mathbb{R}^d \times [0,\infty) : u_1^\delta[u_0](x,t) > u_m - \delta\}, \quad (28)$$

and, given $u_i^\delta[u_0], \mathcal{C}_i^\delta[u_0]$, and $\Omega_i^\delta[u_0]$, we define

$$u_{i+1}^\delta[u_0](x,t) = \sup_{\substack{(x(s),s) \in \Omega_i^\delta[u_0] \\ x(t)=x}} \int_0^t \left[-\frac{|\dot{x}(s)|^2}{4} + R(x(s)) \right] ds + u_0(x(0)), \quad (29)$$

with the sup taken over only C^1-trajectories $s \longmapsto x(s)$,

$$\mathcal{C}_{i+1}^\delta[u_0] = \{(x,t) \in \Omega_i^\delta[u_0] : \; u_{i+1}^\delta[u_0](x,t) > -\infty\}, \quad (30)$$

and

$$\Omega_{i+1}^\delta[u_0] = \{(x,t) \in \Omega_i^\delta[u_0] : \; u_{i+1}^\delta[u_0](x,t) > u_m - \delta\} \subset \mathcal{C}_{i+1}^\delta[u_0]. \quad (31)$$

It follows from (5), (28) and the theory of Optimal Control [13, 11, 1, 6] that, for all $i \in \mathbb{Z}^+$, the sets $\mathcal{C}_i^\delta[u_0]$ and Ω_1^δ are open and $u_i^\delta[u_0] \in C(\mathbb{R}^d \times [0,\infty))$.

Note that the state constraint boundary condition is hidden in the control formula. On the other hand we do not write it explicitly, because, to the best of our knowledge, there is no general theory, as in [16], for state constraint problem with time varying domains, and, in particular, in this context where we have no regularity properties for these domains.

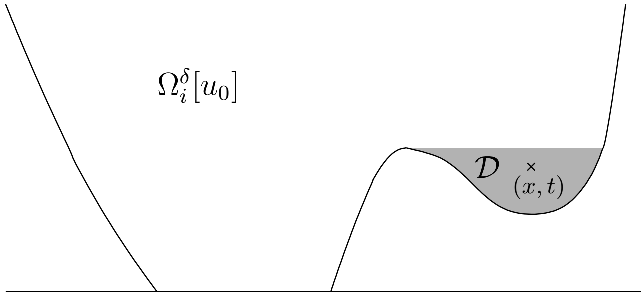

Figure 1: An example of the space-time set $\Omega_i^\delta[u_0]$. The point $(x,t) \in \Omega_i^\delta[u^0]$ cannot be connected to 0 by a trajectory $(x(s),s)$ staying within $\Omega_i^\delta[u^0]$. More generally, for the points in the hatched area, called $\mathcal{D}$, there is no admissible trajectory. We have indeed $\mathcal{C}_{i+1}^\delta[u_0] = \Omega_i^\delta[u_0] \setminus \mathcal{D}$.

We notice that in general

$$\mathcal{C}_{i+1}^\delta[u_0] \subsetneq \Omega_i^\delta[u_0].$$

The reason that $\mathcal{C}_{i+1}^\delta[u_0]$ is different from $\Omega_i^\delta[u_0]$ is that there can be points $(\bar{x},\bar{t}) \in \Omega_i^\delta[u_0]$ which can not be connected to 0 by a trajectory $(x(t),t)$ in $\Omega_i^\delta[u^0]$. (See Figure 1.)

It follows from the optimal control theory formula that, given $\mathcal{C}_{i+1}^\delta[u_0]$ from (30), $u_{i+1}^\delta[u_0]$ is the minimal viscosity solution to

$$\begin{cases} u_{i+1,t}^\delta[u_0] = |Du_{i+1}^\delta[u_0]|^2 + R \quad \text{in} \quad \mathcal{C}_{i+1}^\delta[u_0], \\ u_{i+1}^\delta[u_0] = u_0 \quad \text{in } \mathcal{C}_{i+1}^\delta[u_0] \cap (\mathbb{R}^d \times \{0\}). \end{cases} \tag{32}$$

We refer to [1, 2] for the property that $u_{i+1}^\delta[u_0]$ is a viscosity solution to (32). The minimality of $u_{i+1}^\delta[u_0]$ is proved in Appendix B.

The sequence $(u_i^\delta[u_0])_{i\in\mathbb{Z}^+}$ is non-increasing. Therefore there exists $U^\delta[u_0] \geq -\infty$, such that, as $i \to +\infty$,

$$u_i^\delta[u_0] \searrow U^\delta[u_0] \quad \text{in} \quad \mathbb{R}^d \times [0,\infty).$$

Let $U[u_0]$ to be the non-increasing limit, as $\delta \to 0$, in $\mathbb{R}^d \times [0,\infty)$

$$U^\delta[u_0] \searrow U[u_0],$$

and, for $\mu > 0$, consider the nonincreasing family of sets

$$\Omega^\delta[u_0] = \bigcap_{i\in\mathbb{Z}^+} \Omega_i^\delta[u_0] \quad \text{and} \quad \Omega[u_0-\mu] = \bigcap_{\delta>0} \Omega^\delta[u_0-\mu]. \tag{33}$$

Theorem 3.1. *Let n_ε the solution to (1), $u_\varepsilon = \varepsilon\ln(n_\varepsilon)$, and assume (5). Then, for any $\mu > 0$,*

$$\overline{u} \leq U[u_0] \quad \text{in} \quad \mathbb{R}^d \times [0,\infty) \quad \text{and} \quad U[u_0-\mu] + \mu \leq \underline{u} \quad \text{in} \quad \Omega[u_0-\mu], \tag{34}$$

We remark that, by definition, we have $u_i^\delta[u_0] = -\infty$ in $\left(\mathcal{C}_i^\delta[u_0]\right)^c$. Therefore $U^\delta[u_0] = -\infty$ in $(\Omega^\delta[u_0])^c$ and, finally, $U[u_0] = -\infty$ in $(\Omega[u_0])^c = \left(\bigcap_{i,\delta} \mathcal{C}_i^\delta[u_0]\right)^c = \left(\bigcap_\delta \Omega^\delta[u_0]\right)^c$. It follows that

$$\overline{u} = -\infty \quad \text{in} \quad (\Omega[u_0])^c.$$

Moreover, since $u_i^\delta[\cdot] \geq u_m - \delta$ in $\Omega^\delta[\cdot]$, by passing to the limit we also obtain

$$U[\cdot] \geq u_m \quad \text{in} \quad \Omega[\cdot].$$

An important question is whether, as $\mu \to 0$, $U[u_0 - \mu] \to U[u_0]$. This is wrong in general. A counterexample can be found for $u^0 = u_m$ and $R > 0$. Then $\Omega_1^\delta[u_0 - \mu]$ cannot touch $\mathbb{R}^d \times \{0\}$ and $u_i^\delta[u_0 - \mu] \equiv -\infty$. Therefore $U[u_0 - \mu] \equiv -\infty$, for any positive constant μ. On the other hand $u_i^\delta[u_0] > u_m$ and $U[u_0] = u^1$.

We continue with the

Proof of Theorem (3.1). First we show by induction that, for all $\delta > 0$ and all $i \in \mathbb{Z}^+$, $\overline{u} \leq u_i^\delta[u_0]$.

Since n_ε^1 is a super-solution to (1), by the comparison principle, we obtain $n_\varepsilon \leq n_\varepsilon^1$, and, hence, $\overline{u} \leq u_1^\delta[u_0] = u^1$.

Next assume that $\overline{u} \leq u_i^\delta[u_0]$, and, arguing by contradiction, we show, following an argument similar to that in Section 2, that $\overline{u} \leq u_{i+1}^\delta[u_0] = -\infty$ in $(\Omega_i^\delta[u_0])^c$.

To this end, suppose that, for some $(x_0, t-)) \in (\Omega_i^\delta[u_0])^c$, $\overline{u}(x_0, t_0) > -\infty$. Since $\overline{u}$ is an upper semi-continuous function, there exists a family $(\phi_\alpha)_{\alpha > o}$ of smooth functions $(\phi_\alpha)_\alpha$ such that $\overline{u} - \phi_\alpha$ attains a strict local maximum at (x_α, t_α) and, as $\alpha \to 0$,t $(x_\alpha, t_\alpha) \to (x_0, t_0)$, $\overline{u}(x_\alpha, t_\alpha) \geq \overline{u}(x_0, t_0)$, and, consequently, $\overline{u}(x_\alpha, t_\alpha) \to u(x_0, t_0)$.
It follows that there exists a family of points $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$ such that $u_\varepsilon - \phi_\alpha$ attains a local maximum at $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$, and, as $\varepsilon \to 0$, $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \to (x_\alpha, t_\alpha)$.

Moreover, in view of (4), at $(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon})$,

$$\phi_{\alpha,t} - \varepsilon\Delta\phi_\alpha - |D\phi_\alpha|^2 - R \leq -\exp((2\varepsilon)^{-1}(u_m - u_\varepsilon)).$$

Letting $\varepsilon \to 0$ yields, at (x_α, t_α),

$$\phi_{\alpha,t} - |D\phi_\alpha|^2 - R \leq \limsup_{\varepsilon\to 0}(-\exp[(2\varepsilon)^{-1}(u_m - u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}))]).$$

Since, by the definition of $\overline{u}$, we have $\limsup_{\varepsilon\to 0} u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}) \leq \overline{u}(x_\alpha, t_\alpha)$, using the assumption of the induction we find that, for α small enough, $\overline{u}(x_\alpha, t_\alpha) \leq u_i^\delta[u_0](x_\alpha, t_\alpha) \leq u_m - \delta/2$.

It follows that

$$\limsup_{\varepsilon\to 0}(-\exp[(2\varepsilon)^{-1}(u_m - u_\varepsilon(x_{\alpha,\varepsilon}, t_{\alpha,\varepsilon}))]) = -\infty,$$

and, hence, at (x_α, t_α),

$$\phi_{\alpha,t} - |D\phi_\alpha|^2 - R \leq -\infty,$$

which, of course, is not possible because ϕ_α is a smooth function.

It follows that $\overline{u} = -\infty$ in $(\Omega_i^\delta)^c$ and, in particular, $\overline{u} = -\infty$ on $\partial\Omega_i^\delta[u_0]$.

Next we show that

$$\overline{u} \leq u_{i+1}^\delta[u_0] = -\infty \quad \text{in} \quad (\mathcal{C}_{i+1}^\delta[u_0])^c.$$

To this end, let $(\bar{x}, \bar{t}) \in (\mathcal{C}_{i+1}^\delta[u_0])^c \setminus (\Omega_i^\delta[u_0])^c$. Note that the existence of such a point means that $(\bar{x}, \bar{t})$ cannot be connected to $\mathbb{R}^d \times \{0\}$ by a C^1-trajectory staying in $\Omega_i^\delta[u_0]$. Hence $(\bar{x}, \bar{t})$ belongs to a connected component $\mathcal{D}$ of $\omega_i^\delta[u_0] = \{(y, s) \in \Omega_i^\delta[u_0] : s \leq \bar{t}\}$, such that the set $\mathcal{D}$ does not touch $\mathbb{R}^d \times \{0\}$. (See Figure 1.)

Therefore $\partial_p\mathcal{D} \subset \partial\Omega_i^\delta[u_0]$, where $\partial_p\mathcal{D} = \{(y,s) \in \partial\mathcal{D} : s < \bar{t}\}$ is the parabolic boundary of $\mathcal{D}$. From the previous argument we obtain

$$\overline{u} = -\infty \qquad \text{on } \partial_p\mathcal{D}. \tag{35}$$

As in (20), for $A > 0$, we define w_ε^A by

$$n_\varepsilon + \exp\big(\frac{-A}{\varepsilon}\big) = \exp\big(\frac{w_\varepsilon^A}{\varepsilon}\big).$$

Arguing as in the previous section, we deduce that, for all $A > 0$,

$$\overline{w}^A = \max(-A, \overline{u}) \quad \text{ and } \quad \min(\overline{w}^A + A,\, \overline{w}_t^A - |D\overline{w}^A|^2 - R) \le 0,$$

and, in view of (35), that $\overline{w}^A$ solves the initial value problem

$$\begin{cases} \min(\overline{w}^A + A,\, \overline{w}_t^A - |D\overline{w}^A|^2 - R) \le 0 & \text{in} \quad \mathcal{D}, \\ \overline{w}^A = -A \quad \text{in} \quad \partial_p\mathcal{D}, \end{cases}$$

which admits, for some $C_1 > 0$,

$$\phi(x,t) = -A + C_1 t,$$

as a super-solution.

It follows from the comparison principle that, for all $A > 0$,

$$\overline{u} \le \overline{w}^A \le -A + C_1 t \quad \text{in} \quad \mathcal{D}.$$

Letting $A \to \infty$ yields $\overline{u} = -\infty$ in $\mathcal{D}$ and, consequently, $\overline{u}(\bar{x}, \bar{t}) = -\infty$. Observe that $\overline{u} = -\infty$ in $(\mathcal{C}_{i+1}^\delta[u_0])^c$ implies that $\overline{u} = -\infty$ on $\partial\mathcal{C}_{i+1}^\delta[u_0] \cap (\mathbb{R}^d \times [0,\infty))$.

Finally we show that

$$\overline{u} \le u_{i+1}^\delta[u_0] \quad \text{in} \quad \mathcal{C}_{i+1}^\delta[u_0].$$

To this end, define z_ε by

$$n_\varepsilon + \exp\big(\frac{u_{i+1}^\delta[u_0]}{\varepsilon}\big) = \exp\big(\frac{z_\varepsilon}{\varepsilon}\big),$$

and notice that

$$\overline{z} = \max(\overline{u},\, u_{i+1}^\delta[u_0]).$$

We claim that $\overline{z}$ is a sub-solution of

$$\overline{z}_t - |D\overline{z}|^2 - R \le 0 \quad \text{in} \quad \mathcal{C}_{i+1}^\delta[u_0]. \tag{36}$$

We postpone the proof of (36) and proceed by noticing that, in view of the above,

$$\overline{u} = -\infty \quad \text{on} \quad \partial\mathcal{C}_{i+1}^\delta \cap (\mathbb{R}^d \times (0, +\infty)).$$

It follows that

$$\overline{z} = u_{i+1}^\delta[u_0] \quad \text{on} \quad \partial\mathcal{C}_{i+1}^\delta[u_0],$$

and, hence,

$$\overline{z} \le u_{i+1}^\delta[u_0] \quad \text{on} \quad \partial\mathcal{C}_{i+1}^\delta[u_0].$$

Therefore, by the comparison principle for (32), we obtain

$$\overline{z} \leq u^\delta_{i+1}[u_0] \quad \text{in} \quad \mathcal{C}^\delta_{i+1}[u_0],$$

and we conclude that $\overline{u} \leq u^\delta_{i+1}[u_0]$.

It remains to prove (36). Let ϕ be a smooth test function and assume that $\overline{z} - \phi$ achieves a local maximum at $(\bar{x}, \bar{t})$ and, without loss of generality, $\overline{z}(\bar{x}, \bar{t}) = \phi(\bar{x}, \bar{t})$. In a neighborhood of $(\bar{x}, \bar{t})$ we have

$$\overline{u}(x,t) \leq \overline{z}(x,t) \leq \phi(x,t) \quad \text{and} \quad u^\delta_{i+1}[u_0](x,t) \leq \overline{z}(x,t) \leq \phi(x,t).$$

If $\overline{z}(\bar{x}, \bar{t}) = u^\delta_{i+1}[u_0](\bar{x}, \bar{t})$, then $u^\delta_{i+1}[u_0] - \phi$ achieves a local maximum at $(\bar{x}, \bar{t})$, and, since $u^\delta_{i+1}[u_0]$ is a sub-solutions of (36) in $\mathcal{C}^\delta_{i+1}[u_0]$, at $(\bar{x}, \bar{t})$ we get

$$\phi_t - |D\phi|^2 - R \leq 0.$$

If $\overline{z}(\bar{x}, \bar{t}) = \overline{u}(\bar{x}, \bar{t})$, then $\overline{u}(\bar{t}, \bar{x}) > -\infty$ and $\overline{u} - \phi$ achieves a local maximum at $(\bar{x}, \bar{t})$. Using (4) and the stability of viscosity sub-solutions we find, at $(\bar{x}, \bar{t})$,

$$\phi_t - |D\phi|^2 - R \leq 0,$$

and we conclude that $\overline{z}$ is a sub-solution of (36).

The fact that, at $(\bar{x}, \bar{t})$, $\overline{u} \leq u^\delta_i[u_0]$ yields that, for all $\delta > 0$, $\overline{u} \leq \lim_{i \to \infty} u^\delta_i[u_0] = U^\delta[u_0]$. Letting $\delta \to 0$ we obtain

$$\overline{u} \leq \lim_{\delta \to 0} U^\delta[u_0] = U[u_0] \quad \text{in} \quad \mathbb{R}^d,$$

which concludes the proof of the first part of the claim.

For the second part we need the following Lemma. Its proof is postponed to the end of this section.

Lemma 3.2. *For all $i \in \mathbb{Z}^+$ the lower semi-continuous function*

$$v^\delta_i = \max(u^\delta_i[u_0 - \mu] + 2\delta, \underline{u}), \tag{37}$$

is a super-solution of

$$\begin{cases} v^\delta_{i,t} - |Dv^\delta_i|^2 - R \geq 0 & \text{in} \quad \Omega^\delta_i[u_0 - \mu], \\ v^\delta_i = u_0 & \text{in } \{u_0 - \mu > u_m - \delta\} \cap (\mathbb{R}^d \times \{0\}). \end{cases} \tag{38}$$

Since $u^\delta_{i+1}[u_0 - \mu]$ is a minimal solution of (32) in $\mathcal{C}^\delta_{i+1}[u_0 - \mu] \subset \Omega^\delta_i[u_0 - \mu]$ with $u^\delta_{i+1}[u_0 - \mu] = u_0 - \mu$ on $\mathbb{R}^d \times \{0\}$ (see Appendix B), it follows that

$$u^\delta_{i+1}[u_0 - \mu] \leq v^\delta_i - \mu \quad \text{in} \quad \mathcal{C}^\delta_{i+1}[u_0 - \mu],$$

and, hence,

$$u^\delta_{i+1}[u_0 - \mu] + \mu \leq \max(u^\delta_i[u_0 - \mu] + 2\delta, \underline{u}) \quad \text{in} \quad \mathcal{C}^\delta_{i+1}[u_0 - \mu].$$

Letting $i \to \infty$ yields

$$U^\delta[u_0 - \mu] + \mu \leq \max(U^\delta[u_0 - \mu] + 2\delta, \underline{u}) \quad \text{in} \quad \Omega^\delta[u_0 - \mu].$$

Choosing $\mu > 2\delta$ we also get

$$U^\delta[u_0-\mu] + 2\delta < U^\delta[u_0-\mu] + \mu \quad \text{in} \quad \Omega^\delta[u_0-\mu],$$

and, therefore,

$$U^\delta[u_0-\mu] + \mu \le \underline{u} \quad \text{in} \quad \Omega^\delta[u_0-\mu].$$

Finally letting $\delta \to 0$ we obtain

$$U[u_0-\mu] + \mu = \lim_{\delta\to 0} U^\delta[u_0-\mu] + \mu \le \underline{u} \quad \text{in} \quad \Omega[u_0-\mu].$$

□

We conclude with the

Proof of Lemma 3.2. The argument relies on the property that, for concave Hamiltonians, the maximum of two super-solutions is a super-solution, which we prove in the present context of semi-continuous super-solutions in a space-time domains.

To this end, fix $i \in \mathbb{Z}^+$ and $(x,t) \in \mathcal{C}_i^\delta[u_0-\mu]$. Since $\mathcal{C}_i^\delta[u_0-\mu]$ is an open set, there exists $\rho > 0$ such that $B_\rho(x,t) \in \mathcal{C}_i^\delta[u_0-\mu]$, where $B_\rho(x,t)$ denotes the open ball of radius ρ centered at (x,t).

For $\alpha > 0$, let

$$u_i^{\delta,\alpha}(x,t) = \inf_{(y,s)\in B_\rho(x,t)} \{u_i^\delta[u_0-\mu](y,s) + (2\alpha)^{-1}(|x-y|^2 + |t-s|^2)\},$$

and

$$u_i^{\delta,\alpha,\beta} = u_i^{\delta,\alpha} * \chi_\beta,$$

where $(\chi_\beta)_\beta$ is a standard smoothing mollifier. Since $u_i^{\delta,\alpha}$ is an inf-convolution of the continuous function u_i^δ, it is locally Lipschitz continuous and semi-concave with semi-concavity constant $1/\alpha$.

It follows that $u_i^{\delta,\alpha,\beta}$ is a smooth semi-concave function with semi-concavity constant $1/\alpha$, and, moreover,

$$\liminf_{\substack{(y,s)\to(\bar y,\bar s)\\ \alpha,\,\beta\to 0}} u_i^{\delta,\alpha,\beta}(y,s) = u_i^\delta[u_0-\mu](\bar y,\bar s).$$

Using Jensen's inequality and the concavity of the Hamiltonian, we obtain, for some $K > 0$,

$$u_{i,t}^{\delta,\alpha,\beta} - \varepsilon\Delta u_i^{\delta,\alpha,\beta} - |Du_i^{\delta,\alpha,\beta}|^2 - R * \chi_\beta \ge -K\alpha - \varepsilon/\alpha \quad \text{in} \quad B_\rho(x,t); \tag{39}$$

notice that, since $u_i^{\delta,\alpha,\beta}$ is smooth, the above inequality holds in the classical sense.

To prove (38) we use the smooth approximations $v_i^{\delta,\alpha,\beta,\varepsilon}$ of v_i^δ in $B_\rho(x,t)$ given by

$$n_\varepsilon + \exp\Big(\frac{u_i^{\delta,\alpha,\beta} + 2\delta}{\varepsilon}\Big) = \exp\Big(\frac{v_i^{\delta,\alpha,\beta,\varepsilon}}{\varepsilon}\Big), \tag{40}$$

and, we show that they are almost super-solutions to (38) for α, β and ε small. Notice that in (40) we use 2δ instead of δ.

Replacing n_ε by $\exp\big(\frac{v_i^{\delta,\alpha,\beta,\varepsilon}}{\varepsilon}\big) - \exp\big(\frac{u_i^{\delta,\alpha,\beta}+2\delta}{\varepsilon}\big)$ in (1) we get

$$\begin{aligned} Rn_\varepsilon - \beta_\varepsilon\sqrt{n_\varepsilon} = {}& \big(v_{i,t}^{\delta,\alpha,\beta,\varepsilon} - \varepsilon\Delta v_i^{\delta,\alpha,\beta,\varepsilon} - |Dv_i^{\delta,\alpha,\beta,\varepsilon}|^2\big)\exp(\varepsilon^{-1}v_i^{\delta,\alpha,\beta}) \\ & - \big(u_{i,t}^{\delta,\alpha,\beta} - \varepsilon\Delta u_i^{\delta,\alpha,\beta} - |Du_i^{\delta,\alpha,\beta}|^2\big)\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta} + 2\delta)), \end{aligned}$$

and, in view of (40),

$$\begin{aligned} v_{i,t}^{\delta,\alpha,\beta,\varepsilon} - \varepsilon\Delta v_i^{\delta,\alpha,\beta,\varepsilon} - |Dv_i^{\delta,\alpha,\beta,\varepsilon}|^2 = (&u_{i,t}^{\delta,\alpha,\beta} - \varepsilon\Delta u_i^{\delta,\alpha,\beta} - |Du_i^{\delta,\alpha,\beta}|^2 \\ &- R * \chi_\beta) \exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta} + 2\delta - v_i^{\delta,\alpha,\beta,\varepsilon})) \\ &+ (R * \chi_\beta - R) \exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta} + 2\delta - v_i^{\delta,\alpha,\beta})) \\ &+ R - \beta_\varepsilon \sqrt{n_\varepsilon} \exp(-\varepsilon^{-1} v_i^{\delta,\alpha,\beta,\varepsilon}). \end{aligned}$$

Using that, in view of (40), $\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta} + 2\delta - v_i^{\delta,\alpha,\beta,\varepsilon})) \leq 1$, and (39) we find

$$\begin{aligned} v_{i,t}^{\delta,\alpha,\beta,\varepsilon} - \varepsilon\Delta v_i^{\delta,\alpha,\beta,\varepsilon} - |Dv_i^{\delta,\alpha,\beta,\varepsilon}|^2 - R \geq -K\alpha - \varepsilon/\alpha& \\ + (R * \chi_\beta - R) \exp(\varepsilon^{-1}&(u_i^{\delta,\alpha,\beta} + 2\delta - v_i^{\delta,\alpha,\beta})) \\ - \beta_\varepsilon \sqrt{n_\varepsilon} \exp(\varepsilon^{-1}&v_i^{\delta,\alpha,\beta}). \end{aligned}$$

Define

$$v_i^{\delta,\alpha,\beta}(\bar{y},\bar{s}) = \liminf_{\substack{\varepsilon\to 0 \\ (y,s)\to(\bar{y},\bar{s})}} v_i^{\delta,\alpha,\beta,\varepsilon}(y,s).$$

Letting $\varepsilon \to 0$ and using the stability of viscosity super-solutions we obtain

$$\begin{aligned} v_{i,t}^{\delta,\alpha,\beta} - |Dv_i^{\delta,\alpha,\beta}|^2 - R \geq -K\alpha& \\ + \liminf_{\substack{\varepsilon\to 0 \\ (y,s)\to(\bar{y},\bar{s})}} (R * \chi_\beta - R) &\exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta} + 2\delta - v_i^{\delta,\alpha,\beta})) - \beta_\varepsilon \sqrt{n_\varepsilon} \exp(\varepsilon^{-1} v_i^{\delta,\alpha,\beta}). \end{aligned} \tag{41}$$

Recalling that $u_i^\delta + 2\delta > u_m + \delta$ in $\Omega_i^\delta[u_0 - \mu]$, we deduce that, as $\varepsilon, \alpha, \beta \to 0$, in $B_\rho(x,t)$,

$$\beta_\varepsilon \sqrt{n_\varepsilon} \exp(-\varepsilon^{-1} v_i^{\delta,\varepsilon}) = \beta_\varepsilon \sqrt{n_\varepsilon} (n_\varepsilon + \exp(\varepsilon^{-1} u_i^{\delta,\varepsilon}))^{-1} \leq (1/2)\beta_\varepsilon \exp(-(2\varepsilon)^{-1} u_i^{\delta,\alpha,\beta}) \to 0. \tag{42}$$

Moreover, as $\varepsilon, \beta \to 0$, we also have

$$R * \chi_\beta - R \to 0 \quad \text{ and } \exp(\varepsilon^{-1}(u_i^{\delta,\alpha,\beta} + 2\delta - v_i^{\delta,\alpha,\beta})) < 1. \tag{43}$$

Finally notice that

$$v_i^\delta = \liminf_{\substack{\alpha,\beta\to 0 \\ (y,s)\to(\bar{y},\bar{s})}} v_i^{\delta,\alpha,\beta}(y,s). \tag{44}$$

Using (41), (42), (43), (44) and the stability of viscosity super-solutions we get

$$v_{i,t}^\delta - |Dv_i^\delta|^2 - R \geq 0 \quad \text{in} \quad B_\rho(x,t).$$

Since all the above hold for all $(x,t) \in \Omega_i^\delta[u_0 - \mu]$, it follows that the lower semi-continuous function v_i^δ is a super-solution of

$$v_{i,t}^\delta - |Dv_i^\delta|^2 - R \geq 0 \quad \text{in} \quad \Omega_i^\delta[u_0 - \mu], v_i^\delta = u_0 \quad \text{for} \quad \{u_0 - \mu > u_m - \delta\} \cap (\mathbb{R}^d \times \{0\}).$$

□

4 The constant rate

In this section we assume that the rate is a constant, i.e.,

$$R(x) = R \quad \text{in} \quad R^d. \tag{45}$$

We prove the following

Theorem 4.1. *Assume (45) and that, if $O = \{x \in \mathbb{R}^d : u_0(x) > u_m\}$, then $\overline{O} = \{x \in \mathbb{R}^d : u_0(x) \geq u_m\}$. Then*

$$\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = U[u_0](x,t) \quad \text{if} \quad U[u_0](x,t) \neq u_m, \tag{46}$$

with

$$\Omega[u_0] = \{(x,t) | \sup_{y \in \overline{O}} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} \geq u_m\}, \tag{47}$$

and

$$U[u_0] = \begin{cases} \sup_{y \in \overline{O}} \left\{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\right\} & \text{if} \quad (x,t) \in \Omega[u_0], \\ -\infty \quad \text{otherwise.} \end{cases} \tag{48}$$

We remark that, in particular, Theorem 4.1 shows that the limit of the u_ε's is not in general defined by (16) (otherwise $\overline{O}$ would have been replaced by $\mathbb{R}^d$). We refer to the Appendix A for an explicit example.

Proof of Theorem 4.1. When the rate R is constant after one iteration of (29), (30) and (31) we find that the set $\Omega^\delta[u_0]$ and the function $U^\delta[u_0]$, since for all $i > 1$, $j > 2$ and $\delta > 0$, we have $\Omega_i^\delta[u_0] = \mathcal{C}_j^\delta[u_0] = \Omega^\delta[u_0]$ and $u_i^\delta[u_0] = U^\delta[u_0]$.

In fact, since every optimal trajectory in $\mathcal{C}_2^\delta$ is a straight line connecting a point in Ω_2^δ to a point in $I^\delta = \{x \in \mathbb{R}^d : u_0(x) > u_m - \delta\}$, it is included in Ω_2^δ. This is because

$$\phi(x,t) = -\frac{|x-c|^2}{4t} + Rt + u_0(c)$$

is concave in (x,t) and, therefore, all the optimal trajectories of the points in Ω_2^δ are included in Ω_2^δ. It follows that $\Omega_2^\delta = \mathcal{C}_3^\delta$, $u_2^\delta = u_3^\delta$ and consequently $\Omega_2^\delta = \Omega_3^\delta$. By iteration we obtain, for all $i > 2$, $\Omega_2^\delta = \Omega_i^\delta = \Omega^\delta$ and $u_2^\delta = u_i^\delta = U^\delta$.

Using (31) and (29) we see that, for all $i \geq 2$,

$$\Omega^\delta[u_0] = \Omega_i^\delta[u_0] = \{(x,t) : \sup_{y \in I^\delta} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} > u_m - \delta\}, \tag{49}$$

and

$$U^\delta[u_0] = u_i^\delta[u_0](x,t) = \begin{cases} \sup_{y \in I^\delta} \left\{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\right\} & \text{if} \quad (x,t) \in \Omega^\delta[u_0], \\ -\infty \quad \text{otherwise.} \end{cases} \tag{50}$$

It is easy to verify that (32) holds, since, for all $i > 2$ and $\delta > 0$,

$$u_{i,t}^\delta[u_0] - |Du_i^\delta[u_0]|^2 - R = 0 \quad \text{in} \quad \Omega_i^\delta[u_0] = \mathcal{C}_{i+1}^\delta[u_0].$$

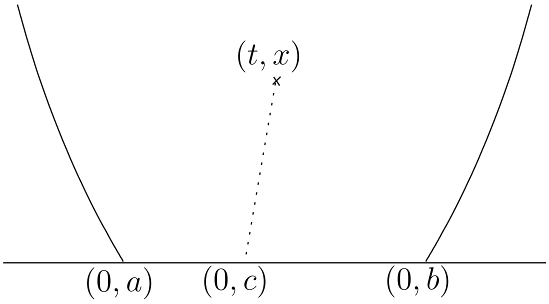

Figure 2: The case with $R(x) = R$ a positive constant, $\{x \in \mathbb{R} | \, u_0(x) > u_m\} = (a,b)$ and $u_0(\cdot) \geq u_m$ on an interval $[a,b]$. The set Ω is a union of hyperbolas. The optimal trajectories are straight lines and we have $U(x,t) = -\frac{|x-c|^2}{4t} + Rt + u_0(c)$, where c is a point where the maximum in (48) is attained.

Letting $\delta \to 0$ in (49) and (50) we obtain (47)–(48). (See Figure 2.)

We also have

$$\Omega[u_0 - \mu] = \bigcap_{\delta>0} \Omega^\delta[u_0 - \mu] = \{(x,t) : \sup_{y \in J^\mu} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} \geq u_m + \mu\},$$

with

$$J^\mu = \{(x,t) : \, u_0 \geq u_m + \mu\}.$$

It follows that

$$\cup_{\mu>0}\Omega[u_0 - \mu] = \{(x,t) : \sup_{y \in O} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} > u_m\}, \tag{51}$$

and

$$\lim_{\mu \to 0^+} U[u_0 - \mu] = \begin{cases} \sup_{y \in O} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} & \text{for} \quad (x,t) \in \cup_{\mu>0}\Omega[u_0 - \mu], \\ -\infty \quad \text{otherwise.} \end{cases} \tag{52}$$

We also notice that

$$\sup_{y \in O} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\} = \sup_{y \in \bar{O}} \{-\frac{|x-y|^2}{4t} + Rt + u_0(y)\}. \tag{53}$$

Comparing (47), (48) with (51), (52) and using (53) we deduce that

$$\lim_{\mu \to 0} U[u_0 - \mu](x,t) = U[u_0](x,t) \quad \text{for} \quad U[u_0](x,t) \neq u_m,$$

and, consequently, that

$$\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = U[u_0](x,t) \quad \text{if} \quad U[u_0](x,t) \neq u_m.$$

□

5 Strictly positive rate

In this section we study the limiting behavior of the u_ε's when

$$R \geq a > 0 \quad \text{in} \quad \mathbb{R}^d, \tag{54}$$

and we show that the limit is not, in general, defined by (16).

For this we need to assume that, for sufficiently small $\mu > \delta > 0$, there exists $\rho_{\delta,\mu} > 0$ such that

$$\lim_{\mu\to 0}\lim_{\delta\to 0}\rho_{\delta,\mu} = 0 \quad \text{and, if } u_0(y) > u_m - \delta, \quad \text{then} \quad \sup_{|y-z|\leq\rho_{\delta,\mu}} u_0(z) > u_m - \delta + \mu. \tag{55}$$

Notice that it is important $\rho_{\delta,\mu}$ is chosen independently of y. If $u_0 \in C^1$, (55) implies u_m is never a local maximum of u.

We have:

Theorem 5.1. *Assume (54) and (55). Then*

$$\lim_{\varepsilon\to 0} u_\varepsilon = U[u_0] \quad \textit{in} \quad \cup_{\mu>0}\,\Omega[u_0-\mu]. \tag{56}$$

Recall that we already know that $\lim_{\varepsilon\to 0} u_\varepsilon = -\infty$ in $\Omega[u_0]^c$. When R is negative the claim of Theorem (5.1) together with (56) are equivalent to (16).

Proof of Theorem (5.1). For $h > \bar{h} = \frac{\mu}{2a} + \frac{1}{2}\sqrt{\frac{\mu^2}{a^2} + \frac{\rho^2_{\delta,\mu}}{a}}$, $(x,t) \in \mathbb{R}^d \times [0,\infty)$, $i \geq 1$ and $\mu, \delta > 0$, we have

$$u_i^\delta[u_0](x,t) \leq u_i^\delta[u_0-\mu](x,t+h). \tag{57}$$

We postpone the proof of this inequality to Appendix C and we continue with the ongoing one.

Letting $i \to +\infty$ and $\delta, \mu \to 0$ we obtain, for all $h > 0$ and $t > 0$,

$$U[u_0](\cdot,\cdot) \leq \lim_{\mu\to 0^+} U[u_0-\mu](\cdot,\cdot+h), \tag{58}$$

and, hence,

$$U[u_0](x,t) \leq \lim_{\mu\to 0^+} U[u_0-\mu](x,t+h) \leq \underline{u}(x,t+h) \leq \overline{u}(x,t+h) \quad \text{for all} \quad (x,t) \in \cup_{\mu>0}\Omega[u_0-\mu],$$

and, finally,

$$U[u_0](x,t) \leq \liminf_{h\to 0^+} \underline{u}(x,t+h) \leq \limsup_{h\to 0^+} \overline{u}(x,t+h) \quad \text{for all} \quad (x,t) \in \cup_{\mu>0}\Omega[u_0-\mu].$$

The definitions of $\underline{u}$ and $\overline{u}$ also imply that

$$\liminf_{h\to 0^+} \underline{u}(x,t+h) = \underline{u}(x,t) \quad \text{and} \quad \limsup_{h\to 0^+} \overline{u}(x,t+h) = \overline{u}(x,t).$$

Combining all the above we obtain

$$U[u_0] \leq \underline{u} \leq \overline{u} \quad \text{in} \quad \cup_{\mu>0}\,\Omega[u_0-\mu].$$

Using this last inequality and (34) yields

$$\underline{u} = \overline{u} = U[u_0] \quad \text{in} \quad \cup_{\mu>0}\,\Omega[u_0-\mu],$$

and, hence,

$$\lim_{\varepsilon\to 0} u_\varepsilon = U[u_0] \quad \text{in} \quad \cup_{\mu>0}\,\Omega[u_0-\mu].$$

□

6 Conclusions

We showed that the local uniform limit, as $\varepsilon \to 0$, for the parabolic problem (1) with finite time extinction is naturally set in the exponential regime (3) and that the formal limit is apparently the variant (7) of the standard eikonal equation. The new ingredient is an obstacle that depends on the solution itself.

The variational inequality admits many solutions (see Appendix A) and the difficulty is to select the correct additional information. This is easy when the rate R is negative, as shown in Section 2. Since, in this case it is enough to enforce the Dirichlet boundary condition on the boundary of the unknown open set Ω where the liminf of u_ε's is finite. This is due to the fact that, for concave Hamiltonians, the supremum of two supersolutions is still a supersolution.

When the rate R is positive we do not have easy super-solutions at hand, and the answer is more elaborate. It requires an iterative procedure which allows us to identify again the limit of the u_ε's. The key ingredients are boundary conditions for (7) that involve state constraints and which are seen through a control problem.

If the R changes sign we can only bound from above and below the limsup and liminf of the u_ε's by upper and lower solutions, $\bar{u}$ and $\underline{u}$ (Section 3).

In terms of the biological motivation, our results qualitatively mean that the specific form of the survival threshold (a square root here) is irrelevant for the asymptotic problem. It also shows that the exponential shape is deeply influenced by the survival threshold except when R is nonpositive.

We conjecture that these upper and lower solutions are in fact equal and the correct setting (implying uniqueness) is to find a pair (u, Ω) for which we can impose both Dirichlet and state constraints boundary conditions. Both establishing directly these boundary conditions for the semi-limits of the u_ε's as well as developing a theory of state constraints boundary conditions for time varying, non-smooth domains are very challenging mathematical issues.

A Non-uniqueness

To explain the difficulty for (7), we present here counter-examples for uniqueness and elaborate further conditions. Recall that the problem is to find pairs (u, Ω) such that u is a viscosity solutions to (7).

A first source for non-uniqueness is the value of u on $\partial\Omega$. Indeed assume that R and u_0 are such that there exists a unique viscosity solution u^1 of (12) or, more generally, with u^1 defined in (10) and (11). For all $\eta \geq u_m$, we introduce the pair

$$\Omega_\eta = \{(x,t) : \; u^1(x,t) \geq \eta\} \quad \text{and} \quad w_\eta(x,t) = \begin{cases} u^1(x,t) & \text{if} \quad (x,t) \in \Omega_\eta, \\ -\infty & \text{otherwise.} \end{cases}$$

It can be easily verified that (w_η, Ω_η) is a viscosity solution of (7). In order to avoid this artefact, one can add the Dirchlet boundary condition (8) which appeared throughout our constructions. However in the next example we see that this Dirichlet condition is not enough to obtain uniqueness. In fact a state constraint boundary condition is hidden behind the property $u^1 = -\infty$ in the complement of $\overline{\Omega}_\eta$ and we do not take it into account here.

Example Let

$$R(x) = 1 \quad \text{and} \quad u_0(x) = -x^2.$$

A simple computation shows that in this case the u^1 defined in (12) is given by

$$u^1(x,t) = t - \frac{x^2}{1+4t}.$$

Therefore the first truncation of u^1, namely the pair $(\widetilde{u}, \widetilde{\Omega})$

$$\widetilde{u}(x,t) = \begin{cases} t - \frac{x^2}{1+4t} & \text{for} \quad t - \frac{x^2}{1+4t} \geq u_m, \\ -\infty & \text{otherwise}, \end{cases}$$

and

$$\widetilde{\Omega} = \{(x,t) : \widetilde{u}(x,t) > -\infty\},$$

is a viscosity solution of (7). As a matter of fact this is the maximal sub-solution to (7), (8) but it does not satisfy the state constraint boundary condition. To see this choose $u_m = -0.04$. The point $(1,2)$ is included in $\widetilde{\Omega}$ since $\widetilde{u}(1,2) = 0.2 > -004$. The optimal trajectory associated to this point, giving the value $\widetilde{u}(1,2) = 0.2$, is the straight line that connects $(0,0.4)$ to $(1,2)$. But we have $u_0(0.4) = -0.16 < -0.04$. So the point $(0,0.4)$ is not included in $\widetilde{\Omega}$. Therefore a part of the optimal trajectory of the point $(1,2)$ is not included in $\widetilde{\Omega}$. Hence $\widetilde{u}$ does not satisfy the state constraint condition.

Following the arguments in Section 4 we can find a viscosity solution to (7) and (8). Indeed using (48) it is possible to compute explicitly the function $U[u_0] = \lim_{\delta \to 0} U^\delta[u_0] = \lim_{\delta \to 0} u_2^\delta[u_0]$ to find

$$\breve{u}(x,t) = \begin{cases} t - \frac{x^2}{1+4t} & \text{if} \quad -\frac{x^2}{(1+4t)^2} \geq u_m,\ t - \frac{x^2}{1+4t} \geq u_m, \\ t - \frac{(x-\sqrt{-u_m})^2}{4t} + u_m & \text{if} \quad x > 0,\ -\frac{x^2}{(1+4t)^2} \leq u_m,\ t \geq \frac{(x-\sqrt{-u_m})^2}{4t}, \\ t - \frac{(x+\sqrt{-u_m})^2}{4t} + u_m & \text{if} \quad x < 0,\ -\frac{x^2}{(1+4t)^2} \leq u_m,\ t \geq \frac{(x+\sqrt{-u_m})^2}{4t}, \\ -\infty & \text{otherwise}, \end{cases}$$

with

$$\breve{\Omega} = \{(x,t) : \breve{u}(x,t) > -\infty\}.$$

From Theorem 4.1 we know that $\breve{u}$ is indeed the pointwise limit of the u_ε's outside the exceptional set $\{(x,t) : \breve{u}(x,t) = u_m\}$.

However, in general $\widetilde{u} \neq \breve{u}$. Consider, for instance, the value $u_m = -0.04$. Then

$$\widetilde{u}(1,2) = 0.2, \quad \breve{u}(1,2) = 0.15, \quad \widetilde{u}(1,2.21) = 0.02, \quad \breve{u}(1,2.21) = -\infty.$$

and, consequently,

$$\breve{\Omega} \subsetneq \widetilde{\Omega}.$$

According to Section 4, the state constraint boundary condition is satisfied for $\breve{u}$, which motivates our conjecture in Section 6.

B $u_i^\delta[u_0]$ is a minimal solution of (32) in $\mathcal{C}_i^\delta[u_0]$

Here we prove that $u_i^\delta[u_0]$ is a minimal solution of (32) in $\mathcal{C}_i^\delta[u_0]$ by showing that, for any super-solution w of (32) in $\mathcal{C}_i^\delta[u_0]$, we have

$$u_i^\delta[u_0] \le w \quad \text{in} \quad \mathcal{C}_i^\delta[u_0]. \tag{59}$$

To this end we assume that $\gamma : [0,t] \to \Omega_{i-1}^\delta[u_0]$ is a C^1-trajectory, with $\gamma(t) = x \in \mathcal{C}_i^\delta[u_0]$. Since $\mathcal{C}_i^\delta[u_0]$ is the set of points that can be connected by a C^1-trajectory in $\Omega_{i-1}^\delta[u_0]$ to some point in $\mathbb{R}^d$ at time $t = 0$, it follows that γ is included in $\mathcal{C}_i^\delta[u_0]$.

Let w be a super-solution of (32) in $\mathcal{C}_i^\delta[u_0]$ and define, for $s \in [0,t]$, $\varphi(s) = w(\gamma(s), s)$. It is clear that φ is lower semi-continuous.

We claim that φ is a viscosity super-solution of

$$\varphi' \ge -\frac{|\dot\gamma|^2}{4} + R(\gamma) \qquad \text{in} \qquad (0,t). \tag{60}$$

We postpone the proof of this claim to the end of the present paragraph and we proceed noticing that the function

$$\psi(t) = \int_0^t \big(-\frac{|\dot\gamma(s)|^2}{4} + R(\gamma(s))\big) ds + w(\gamma(0), 0),$$

is a subsolution of (60). Using the standard comparison principle we then obtain

$$w(x,t) = \varphi(t) \ge \int_0^t \big(-\frac{|\dot\gamma(s)|^2}{4} + R(\gamma(s))\big) ds + u_0(\gamma(0)),$$

and, since this is true for any C^1-trajectory γ, (59) follows.

It remains to prove (60). Let $\phi \in C^1((0,t))$ be a test function, assume that $\bar t$ is a strict minimum point of $\varphi - \phi$ and consider the function

$$F_\mu(y,t) = w(y,t) - \phi(t) + \frac{|y - \gamma(t)|^2}{\mu^2} + (t - \bar t)^2,$$

which attains a local minimum at a point (y_μ, t_μ) such that, as $\mu \to 0$,

$$(t_\mu - \bar t) \to 0 \quad \text{and} \quad \frac{|y_\mu - \gamma(t_\mu)|^2}{\mu^2} \to 0. \tag{61}$$

Since w is a super-solution we have

$$\phi'(t_\mu) + \frac{2\big(\gamma(t_\mu) - y_\mu\big)}{\mu^2} \cdot \dot\gamma(t_\mu) + 2(t_\mu - \bar t) \ge \Big|\frac{2(y_\mu - \gamma(t_\mu))}{\mu^2}\Big|^2 + R(y_\mu).$$

Using the latter and the elementary inequality

$$q \cdot \dot\gamma(t) - q^2 \le \frac{|\dot\gamma(t_\mu)|^2}{4},$$

we obtain

$$\phi'(t_\mu) + 2(t_\mu - \bar t) \ge -\frac{|\dot\gamma(t_\mu)|^2}{4} + R(y_\mu),$$

and, after letting $\mu \to 0$, we conclude using (61).

C Proof of (57)

We prove by induction on i that, for all $h > \bar{h} = \frac{\mu}{2a} + \frac{1}{2}\sqrt{\frac{\mu^2}{a^2} + \frac{\rho^2_{\delta,\mu}}{a}}$, $i > 1$, $\delta > 0$, and $(x,t) \in \mathbb{R}^d \times [0,\infty)$,

$$u_i^\delta[u_0](x,t) \leq u_i^\delta[u_0 - \mu](x, t+h).$$

Recall that $u_1^\delta[u_0] = u^1[u_0]$ and $u_1^\delta[u_0 - \mu] = u^1[u_0 - \mu] = u^1[u_0] - \mu$, where $u^1[u_0]$ solves

$$\begin{cases} u_t^1[u_0] = |Du^1[u_0]|^2 + R & \text{in} \quad \mathbb{R}^d \times (0, +\infty), \\ u^1[u_0] = u_0 & \text{on} \quad \mathbb{R}^d \cap \{0\}. \end{cases} \tag{62}$$

From (62) and $R \geq a$ we find

$$u^1[u_0](\cdot,t) + ah - \mu \leq u^1[u_0](\cdot, t+h) - \mu = u^1[u_0 - \mu](\cdot, t+h).$$

Therefore, for all $h > \bar{h} \geq \mu/a$, we have

$$u^1[u_0](\cdot,t) \leq u^1[u_0 - \mu](\cdot, t+h),$$

and, consequently,

$$u_1^\delta[u_0](\cdot,t) \leq u_1^\delta[u_0 - \mu](\cdot, t+h).$$

Assume next that, for all $h > \bar{h}$ and $t > 0$,

$$u_i^\delta[u_0](\cdot,t) \leq u_i^\delta[u_0 - \mu](\cdot, t+h).$$

It follows that, for all $h > \bar{h}$,

$$\Omega_i^\delta[u_0] + he_t \subset \Omega_i^\delta[u_0 - \mu], \tag{63}$$

where e_t is the unit vector in the direction of time axis.

Assume that $(x,t) \in \mathcal{C}_{i+1}^\delta[u_0] \subset \Omega_i^\delta[u_0]$ and let γ be a C^1-trajectory in $\Omega_i^\delta[u_0]$ connecting (x,t) to a point $(y,0)$ with $u_0(y) > u_m - \delta$. It follows from (55) that there exists $z \in \mathbb{R}^d$ such that $|z - y| < \rho_{\delta,\mu}$ and $u_0(z) > u_m - \delta + \mu$. Without loss of generality we can take $u_0(z) \geq u_0(y)$.

The claim is that the trajectory $\widetilde{\gamma} : [0, t+h] \to \mathbb{R}^d$ defined by

$$\widetilde{\gamma}(s) = \begin{cases} h^{-1}s(y - z) + z & \text{if} \quad 0 \leq s \leq h, \\ \gamma(s - h) & \text{for} \quad h < s \leq t + h, \end{cases} \tag{64}$$

is included in $\Omega_i^\delta[u_0 - \mu]$.

Indeed notice that, for all $h > \bar{h}$,

$$-\frac{|y - z|^2}{4h} + ah \geq \mu \geq 0.$$

Consequently, it follows from $R(x) \geq a$ and (49) that the straight line connecting (y,h) to $(z,0)$ is included in $\Omega^\delta[u_0 - \mu] = \cap_j \Omega_j^\delta[u_0 - \mu]$, and, in particular, it is included in $\Omega_i^\delta[u_0 - \mu]$. Therefore, for all $0 \leq s \leq h$, the point $(\widetilde{\gamma}(s), s)$ is included in $\Omega_i^\delta[u_0 - \mu]$.

Moreover using (63) we find that $\gamma + he_t \in \Omega_i^\delta[u_0 - \mu]$. Hence, for all $h < s$, $(\widetilde{\gamma}(s), s) \in \Omega_i^\delta[u_0 - \mu]$. Thus we conclude that $\widetilde{\gamma}$ is included in $\Omega_i^\delta[u_0 - \mu]$.

Next write

$$\int_0^{t+h} \left(-\frac{|\dot{\widetilde{\gamma}}(s)|^2}{4} + R(\widetilde{\gamma}(s))\right)ds + u_0(z) - \mu = \int_0^t \left(-\frac{|\dot{\gamma}(s)|^2}{4} + R(\gamma(s))\right)ds + \int_0^h \left(-\frac{|\dot{\widetilde{\gamma}}(s)|^2}{4} + R(\widetilde{\gamma}(s))\right)ds + u_0(z) - \mu. \tag{65}$$

It follows that

$$\int_0^h \left(-\frac{|\dot{\widetilde{\gamma}}(s)|^2}{4} + R(\widetilde{\gamma}(s))\right)ds + u_0(z) - \mu \geq u_0(y). \tag{66}$$

If this is true, then using (29), (65) and (66) we obtain, for all $h > \bar{h}$ and $t > 0$,

$$u_{i+1}^\delta[u_0](\cdot, t) \leq u_{i+1}^\delta[u_0 - \mu](\cdot, t + h),$$

and we deduce (57).

It remains to prove (66). Since $R \geq a$, in view of (64), we have

$$\int_0^h \left(-\frac{|\dot{\widetilde{\gamma}}(s)|^2}{4} + R(\widetilde{\gamma}(s))\right)ds + u_0(z) - \mu \geq -\frac{|y - z|^2}{4h} + ah + u_0(z) - \mu,$$

$u_0(z) \geq u_0(y)$ and, for all $h > \bar{h}$,

$$-\frac{|y - z|^2}{4h} + ah \geq \mu.$$

References

[1] M. Bardi and I. Capuzzo-Dolcetta. *Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1997. With appendices by Maurizio Falcone and Pierpaolo Soravia.

[2] G. Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*, volume 17 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1994.

[3] G. Barles, L. C. Evans, and P. E. Souganidis. Wavefront propagation for reaction-diffusion systems of PDE. *Duke Math. J.*, 61(3):835–858, 1990.

[4] G. Barles, S. Mirrahimi, and B. Perthame. Concentration in Lotka-Volterra parabolic or integral equations: a general convergence result. *Methods Appl. Anal.*, 16(3):321–340, 2009.

[5] G. Barles and B. Perthame. Concentrations and constrained Hamilton-Jacobi equations arising in adaptive dynamics. *Contemp. Math.*, 439:57–68, 2007.

[6] P. Cardaliaguet. A note on the regularity of solutions of Hamilton-Jacobi equations with superlinear growth in the gradient variable. *ESAIM: COCV*, 15(2):367–376, 2009.

[7] M. G. Crandall, H. Ishii, and P.-L. Lions. User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.

[8] M. G. Crandall and P.-L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, (1):1–42, 1983.

[9] L. C. Evans and B. K. Knerr. Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities. *Illinois J. Math.*, 23(1):153–166, 1979.

[10] L. C. Evans and P. E. Souganidis. A PDE approach to geometric optics for certain semilinear parabolic equations. *Indiana Univ. Math. J.*, 38(1):141–172, 1989.

[11] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability*. Springer, New York, second edition, 2006.

[12] M. Gauduchon and B. Perthame. Survival thresholds and mortality rates in adaptive dynamics: conciliating deterministic and stochastic simulations. *Mathematical Medicine and Biology ; doi: 10.1093/imammb/dqp018*, 2009.

[13] P. L. Lions. *Generalized solutions of Hamilton-Jacobi equations*, volume 69 of *Research Notes in Mathematics*. Pitman Advanced Publishing Program, Boston, 1982.

[14] B. Perthame and G. Barles. Dirac concentrations in Lotka-Volterra parabolic PDEs. *Indiana Univ. Math. J.*, 57(7):3275–3301, 2008.

[15] P. Quittner and P. Souplet. *Superlinear parabolic problems. Blow-up, global existence and steady state.* Birkhäuser Advanced texts, 2007.

[16] H. M. Soner. Optimal control with state-space constraint i. *SIAM J. Control Optim.*, 24(3):552–561, 1986.

[17] P. E. Souganidis. Front propagation: theory and applications. In *Viscosity solutions and applications (Montecatini Terme, 1995)*, volume 1660 of *Lecture Notes in Math.*, pages 186–242. Springer, Berlin, 1997.