On Transversals to Tetrahedra

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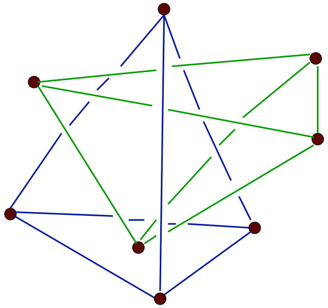
Let us consider 8 points in $\mathbb{R}^3$ general position.



Question : Is there a transversal line to all tetrahedra ?

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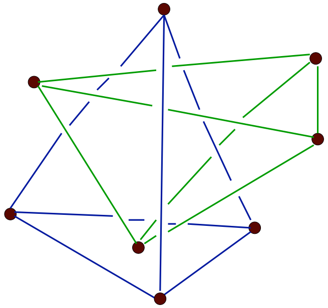
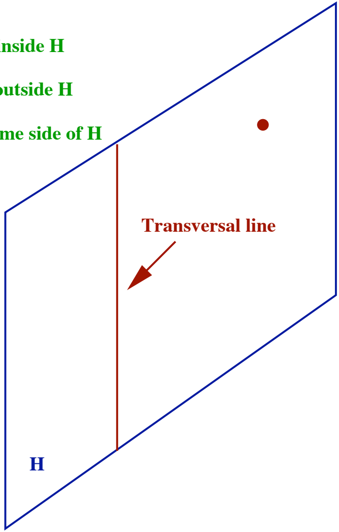


Question : Is there a transversal line to all tetrahedra ?

NEVER

- There are at most 3 points inside H
- There are at least 5 points outside H
- There are 3 points in the same side of H



Question : Let A be a set of 6 points in $\mathbb{R}^3$ in general position. Is there a transversal line to all tetrahedra of A?

ALWAYS

Let $x \in A$ and let T_1 be the set of tetrahedra containing x and let T_2 be the set of tetrahedra not containing x.

T_2 has the 4-Helly property, and therefore, there exists a point y in the intersection of all tetrahedra in T_2.

So, the line passing through x and y gives the desired transversal.

Question : Let A be a set of 7 points in $\mathbb{R}^3$ in general position. Is there a transversal line to all tetrahedra of A?

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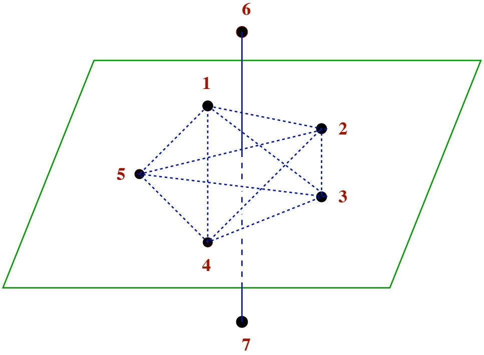
Sometimes YES



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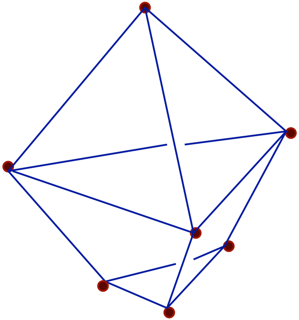

Definitions

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

$m(k, d, \lambda) \stackrel{\text{def}}{=}$ the maximum positive integer n such that every set of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hull of all k-set have a transversal $(d - \lambda)$-plane.

$M(k, d, \lambda) \stackrel{\text{def}}{=}$ the minimum positive integer n such that for every set of n points in general position in $\mathbb{R}^d$ the convex hull of the k-sets does not have a transversal $(d - \lambda)$-plane.

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- $m(4,3,2) = 6$ and $M(4,3,2) = 8$.
- $m(k,d,\lambda) < M(k,d,\lambda)$.

Theorem (Arocha, Bracho, Montejano, R.A., 2011)

$$M(k,d,\lambda) = \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$

Theorem (Arocha, Bracho, Montejano, R.A., 2011)

$$m(k,d,\lambda) \geq d - \lambda + k + \left\lceil \frac{k}{\lambda} \right\rceil - 1.$$

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Theorem (Rado's central point theorem, 1947) If X is a bounded measurable set in $\mathbb{R}^d$ then, there exists a center point $x \in \mathbb{R}^d$ such that for each half-space H that contains x

$$measure(H \cap X) \geq \; measure(X/(d+1)).$$

Discrete version of the central point theorem If X be a finite set of n points in $\mathbb{R}^d$ then, there exists a center point $x \in \mathbb{R}^d$ such that for any closed half-space H containing x

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Our lower bound on $m(k, d, \lambda)$ led to the following generalisation of the discrete version of Rado's central point theorem.

Theorem 1 (Arocha, Bracho, Montejano, R.A., 2011) Let X be a finite set of n points in $\mathbb{R}^d$. Then, there is a $(d-\lambda)$-plane L such that any closed half-space H through L contains at least $\lfloor \frac{n-d+2\lambda}{\lambda+1} \rfloor + d - \lambda$ points.

Corollary (Arocha, Bracho, Montejano, R.A., 2011)
Let $A_i \subset \mathbb{R}^d$ for $i = 1, \ldots, d-\lambda+1$. Then, there is a $(d-\lambda)$-plane L such that any closed half-space H through L contains at least $\left\lfloor \frac{|A_i|+\lambda}{\lambda+1} \right\rfloor$ points of A_i.

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Conjecture 1 $m(k,d,\lambda) = d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1$.

Remark : Theorem 1 is sharp if Conjecture 1 is true.

Theorem (Arocha, Bracho, Montejano, R.A., 2011)
Conjecture 1 is true if either

a) $d = \lambda$ or
b) $\lambda = 1$ (equivalent to Lovász' result on a Kneser problem) or
c) $k \leq \lambda$ or
d) $\lambda = k - 1$ or
e) $k = 2, 3$.

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Question : Is Conjecture 1 true for tetrahedra, that is, for $k = 4$?

Conjecture 1 is equivalent to the following conjecture (by setting $d = \alpha + \lambda$)

Conjecture 2 There is a set of $\alpha + k + \lceil \frac{k}{\lambda} \rceil$ points in $\mathbb{R}^{\alpha+\lambda}$ such that the convex hulls of the k-sets do not admit a transversal α-plane.

Since Conjecture 1 is true when $k - 1 \leq \lambda$ and $\lambda = 1$ then the validity of Conjecture 1 in the case when $k = 4$ follows by showing Conjecture 2 in the case when $\lambda = 2$.

Question : Is there a set of $\alpha + 6$ points in $\mathbb{R}^{\alpha+2}$ such that the convex hull of the 4-sets (the tetrahedra) do not admit a transversal α-plane ?

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Lemma 1 Let $R \subset \mathbb{R}^{\alpha+2}$ be a set of $\alpha+6$ points in general position and let H be a transversal α-plane to the tetrahedra of R. Then, either

(a) H contains $\alpha+1$ points of R or

(b) H contains only α points of R and the six points of R which are not in H can be partitioned in three pairs with the property that the convex hull of each pair intersect transversely H.

Definitions Let $R \subset \mathbb{R}^{\alpha+1}$ be a set of $\alpha + 4$ points in general position and let $\phi \in \mathbb{R}^{\alpha+1}$.

A $(\alpha + 1)$-set $\{x_1, \ldots, x_{\alpha+1}\} \subset R$ is even if the hyperplane generated by $\{x_1, \ldots, x_{\alpha+1}\}$ does not leave exactly three points of $R \cup \{\phi\}$ in some open half-space determined by it.

(R, ϕ) is α-even if every α-set of R is contained in an even $(\alpha + 1)$-set of R.

Lemma 2 Let R be a set of $\alpha+4$ points of $\mathbb{R}^{\alpha+1}$ and let $\phi \in \mathbb{R}^{\alpha+1}$ be such that $R \cup \{\phi\}$ is a set of points in general position in $\mathbb{R}^{\alpha+1}$. Let $\psi, \varphi \in \mathbb{R}^{\alpha+2} - \mathbb{R}^{\alpha+1}$ be such that $\varphi \in (\phi, \psi)$.

If the α-plane $H \subset \mathbb{R}^{\alpha+2}$ through α points of R and φ is a transversal α-plane to the tetrahedra of $R \cup \{\varphi, \psi\}$ then (R, ϕ) is not α-even.

Lemma 3 Let R be a set of $\alpha+4$ points in $\mathbb{R}^{\alpha+1}$ and let $\phi \in \mathbb{R}^{\alpha+1}$ such that the points of $R \cup \{\phi\}$ are in general position and satisfying :

- the set $R \cup \{\phi\}$ has $\alpha+5$ points in $\mathbb{R}^{\alpha+1}$ without a transversal $(\alpha-1)$-plane to the tetrahedra and
- (R, ϕ) is α-even,

Let $\psi \in \mathbb{R}^{\alpha+2} - \mathbb{R}^{\alpha+1}$ be any point and let $\varphi \in \mathbb{R}^{\alpha+2} - \mathbb{R}^{\alpha+1}$ be a point such that $\varphi \in (\psi, \phi)$.

Then, the set of points $R \cup \{\varphi, \psi\}$ has $\alpha+6$ points in $\mathbb{R}^{\alpha+2}$ without a transversal α-plane to the tetrahedra.

Theorem (Arocha, Bracho, Montejano, R.A., 2012)
Conjecture 1 is true for $k = 4$ and $d = 4$.

Theorem (Arocha, Bracho, Montejano, R.A., 2012)
There is a function $f(4,5,\lambda)$ such that

$$m(4,5,\lambda) < f(4,5,\lambda) < M(4,5,\lambda).$$

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