Borsuk and Vázsonyi problems through Reuleaux polyhedra

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Abstract

The Borsuk conjecture and the Vázsonyi problem are two attractive and famous questions in discrete and combinatorial geometry, both based on the notion of diameter of a bounded sets. In this paper, we present an equivalence between the critical sets with Borsuk number 4 in $\mathbb{R}^3$ and the minimal structures for the Vázsonyi problem by using the well-known Reuleaux polyhedra. The latter lead to a full characterization of all finite sets in $\mathbb{R}^3$ with Borsuk number 4.

The proof of such equivalence needs various ingredients, in particular, we proved a conjecture dealing with *strongly critical configuration* for the Vázsonyi problem and showed that the diameter graph arising from involutive polyhedra is vertex (and edge) 4-critical.

1 Introduction

The Borsuk partition and *The frequent large distance problems* are two attractive and well-known questions in discrete and combinatorial geometry, both based on the notion of *diameter* of bounded sets. The *diameter* of a bounded set $S \subset \mathbb{R}^d$ is define as $\text{diam}(S) := \sup_{x,y \in S} ||x - y||$. If S is a finite set of points, the diameter would be the maximum euclidean distance between any two points of S. In this paper we put forward an equivalent of these problems by considering their finite *strongly critical* configurations.

In 1933, Borsuk [5] proposed the following question (sometimes known as the *Borsuk conjecture:*)

> Does every set $S \subset \mathbb{R}^d$ with finite diameter $\text{Diam}(S)$ is the union of at most $d+1$ sets of diameter less than $\text{Diam}(S)$?

It is known to be true for $d = 2$ (see [5]) and for $d = 3$ (see [25], [7] and [9] for a simpler proof).

During fifty years, Borsuk's conjecture was believed to be true until 1993 when Kahn and Kalai [14] proved to be false for $d = 1325$ and for each $d > 2014$. Nowadays, there are known counterexamples in dimensions 64 and higher [13] but the problem still open for $4 \leq d \leq 63$. We refer the reader to [26] for a survey on the Borsuk conjecture.

Recall that the *Borsuk number* of $S \subset \mathbb{R}^d$, denoted by $a(S)$, is the smallest number of subsets that S can be partitioned, such that each of which has smaller diameter than S. Also, recall that the *diameter graph* Diam_V of finite $V \subset \mathbb{R}^3$ is the graph with set of vertices V and two vertices are joined by an edge if their distance is a diameter. These are helpful definitions in order to deal with the Borsuk problem for a finite set of points V, since in this case the equality $\chi(\text{Diam}_V) = a(V)$ holds, where $\chi(G)$ denote the chromatic number of the graph G.

Boltyanski proved that in a two dimensional Banach space, every bounded set is not the union of two sets with smaller diameter if and only if it has a unique *completion* to a body of constant width ([3] for the original proof in Russian or [4, pp-245] for English).

By using above definitions, Boltyanski characterized all the sets in $\mathbb{R}^2$ having Borsuk number 3 (that is, attaining the maximum). Unfortunately, the same argument does not work in $\mathbb{R}^3$ for the sets with Borsuk number 4. For instance, four points in tetrahedral position has Borsuk number 4 but its completition to a body of constant width is not unique (see [18], [23], [17, pp-358]).

In the same spirit, it turned out to be a challenging problem to characterize all the sets in $\mathbb{R}^3$ with Borsuk number 4. In [12], Hujter and Lángi give all the configurations of these sets up to 7 points and mentioned, we cite:

> "A complete characterization of the Borsuk number of finite sets in $\mathbb{R}^3$, even of those with $a(S) = 4$, looks hopeless."

Our main result gives a complete characterization of finite subsets in $\mathbb{R}^3$ with Borsuk number 4. We do so by using some recent tools/results about involutive polyhedra and by characterizing the *critical* Borsuk configurations, that is, the finite sets not having subsets with the same Borsuk number.

Our approach is closely related to the *frequent large distance problem*:

> Given $0 < d < n$, what is the maximum number of diameters over all the sets of n points in $\mathbb{R}^d$?

We denote by $e(d, n)$ such maximum number of diameters. This is one of the oldest problems in discrete and combinatorial geometry. It was first proposed in 1934 by Hopf and Pannwitz [11] in the plane and then generalized to all dimensions.

Given a finite set $V \subset \mathbb{R}^d$, we let $e(V)$ be the number of diameters in V (we keep the same notation introduced in [15]). We say that V is an *extremal configuration* for the frequent large distance problem if $e(V) = e(d, |V|)$.

It is well known that $e(2, n) = n$ and how all the extremal configurations look like (see [24, pp 213-214], [16]). For $d = 3$, the problem is better known as *the Vázsonyi problem* in honor to Vázsonyi, who conjectured that $e(3, n) = 2n - 2$. Grünbaum [8], Heppes [10] and Straszewicz [29] proved independently to be true and Kupitz, Martini and Perles [15] characterize all the extremal configurations.

We say that V is a *critical configuration* for the Vázsonyi problem if V is an extremal configuration and any point of V is adjacent to at least 3 diameters. We also say that V is *strongly critical* if V does not have an extremal configuration subset. By using the characterization of the extremal configurations, we have that being strongly critical implies to be critical, however the opposite direction is not true.

The existence of a set of 8 points that is critical but not strongly critical was claimed in [15] and intended to be given in a future work, however, as far as we are aware it was never published. By using bodies of constant width, we were able to construct an explicitly critical configuration of 8 points in $\mathbb{R}^3$ that is not strongly critical (see end of Section 4.1).

Our approach led us to investigate the *ball polyhedra*. In [15], it was proved that the 1-skeleton of ball polytopes arising from extremal set of points in $\mathbb{R}^3$ are 2-connected planar graphs. In the same paper, the authors also proposed the following

Conjecture 1. *[15] An extremal set $V \subset \mathbb{R}^3$ has a polytopal ball polytope $\mathcal{B}(V)$ (i.e. the 1-skeleton of $\mathcal{B}(V)$ can be realized as the 1-skeleton of a 3-polytope) if and only if V is strongly critical.*

We are able to prove this characterization (Lemma 2). Furthermore, the latter yields to a nice equivalence between strongly critical configurations for the Vázsonyi problem and the *Reuleaux polyhedra* (Theorem 6).

This relationship, combined with a result about the 4-critically of the *diagonal* graph arising from *involutive polyhedra* (Lemma 1), led us to a full characterization of all finite sets in $\mathbb{R}^3$ with Borsuk number 4.

Theorem 1. *Let $V \subset \mathbb{R}^3$ be a finite set with finite diameter and $| V |= n \geq 4$. The following statements are equivalent*

i. V has a subset that is an extremal configuration for the Vázsonyi problem.

ii. V has Borsuk number 4.

iii. There is a $V_1 \subset V$ such that $\mathcal{B}(V_1)$ is a Reuleaux polyhedron.

The organization of the paper is the following. In the next section we present a number of results and notions needed for the rest of the paper. In particular, we discuss some background on both the *ball polyhedra* and the *Reuleaux polyhedra* as well as their properties. In Section 3, we prove a key lemma on the chromatic number of the *diagonal* graph of involutive polyhedra. This is not only interesting for its own sake, but it is a crucial brick for our contributions. Section 4.1 is mainly devoted to prove our main results. We finally end with some concluding remarks.

2 Preliminaries

We review some results and notions on Ball polytopes and Reuleaux polyhedra needed throughout the paper. We refer the reader to [17, pp 132-141] for further details. We also discuss some needed background on involutive polyhedra.

2.1 Ball-polyhedra

Given a finite subset V of $\mathbb{R}^3$, the ball set of V is define as $\mathcal{B}(V) = \{y \in \mathbb{R}^3 : \forall x \in V, \|x - y\| \leq 1\}$. If the radii of the *circumball* of V, denoted by $\mathrm{cr}(V)$, is less than 1, then $\mathcal{B}(V)$ is called the *ball polyhedron* associated with V. A point $v \in V$ is *essential* if $\mathcal{B}(V) \subsetneq \mathcal{B}(V \backslash \{v\})$, otherwise it is *inessential*. The subset of essential points will be denoted as $\mathrm{ess}(V)$. A finite set $V \subset \mathbb{R}^3$ satisfying $\mathrm{cr}(V) < 1$ and $V = \mathrm{ess}(V)$ is *tight*.

The following four theorems are due to Martini, Kupitz and Perles [15].

Theorem 2. *[15] Assume that $V \subset \mathbb{R}^3$ is finite and $\mathrm{diam}\, V = 1$. Then*

1. $\mathrm{cr}(V) < 1$

2. *If a point $v \in V$ is incident with (at least) two diameters of V, then $v \in \mathrm{ess}(V)$.*

3. *If V is extremal for the Vázsonyi problem, then V is tight.*

Definition 1. *Facial structure of a ball polyhedron $\mathcal{B}(V)$.*

1. *For a point $p \in V$ the set $F_p := \{x \in \mathcal{B}(V) : ||x - p|| = 1\}$ is a* facet *of $\mathcal{B}(V)$.*

2. *A boundary point z of $\mathcal{B}(V)$ is a* vertex *of $\mathcal{B}(V)$ if either z belongs to three or more distinct facets of $\mathcal{B}(V)$, in which case z is a* principal vertex*, or $z \in V \cap \mathcal{B}(V)$ and z belongs to exactly two facets of $\mathcal{B}(V)$, in which case z is called a* dangling vertex*. Denote by $\mathrm{vert}\, \mathcal{B}(V)$ the set of vertices of $\mathcal{B}(V)$. In other words, $z \in \mathrm{vert}\, \mathcal{B}(V)$ if and only if $z \in \mathcal{B}(V)$ and $\|z - p\| = 1$ holds for at least three points $p \in V$, or if $z \in V \cap \mathcal{B}(V)$ and $\|z - p\| = 1$ holds for exactly two points $p \in V$.*

3. *An* edge *of $\mathcal{B}(V)$ is the closure of a connected component of $(F_p \cap F_q) \backslash (\mathrm{vert}\, \mathcal{B}(V))$, where $\{p, q\}$ ranges*

over all pairs of distinct points of V.

4. *The set of faces of $\mathcal{B}(V)$, including facets, edges, vertices and improper faces $\mathcal{B}(V)$ and $\emptyset$, is the* spherical face complex *of $\mathcal{B}(V)$ denoted by $\mathcal{SF}(\mathcal{B}(V))$. In particular, the 1-skeleton of $\mathcal{SF}(\mathcal{B}(V))$ is the set of vertices and edges of $\mathcal{B}(V)$ viewed as a graph.*

Theorem 3. *[15] Given a tight finite set $V \subset \mathbb{R}^3$ and $|V| \geq 3$, the 1-skeleton of $\mathcal{SF}(\mathcal{B}(V))$ is planar and 2-connected.*

The following result was called the *extended GHS Theorem* in [15] after Grümbaum, Heppes and Straszewicz who gave the proofs for the Vázsonyi problem independently.

Theorem 4. *[15]* ***(GHS)*** *Let $V \subset \mathbb{R}^3$ be finite with $\mid V \mid = n \geq 4$ and $\operatorname{diam} V = 1$. The following three statements are equivalent*

1. *V is extremal for the Vázsonyi problem, i.e., $e(V) = e(3, n)$.*

2. *$e(V) = 2n - 2$.*

3. *V is tight and $V = \operatorname{vert} \mathcal{B}(V)$.*

An *involutory self-duality* of $\mathcal{SF}(\mathcal{B}(V))$ is an order reversing map $\varphi : \mathcal{SF}(\mathcal{B}(V)) \to \mathcal{SF}(\mathcal{B}(V))$ of order two ($\varphi^2 = Id$) and that sends every vertex $v \in \mathcal{SF}(\mathcal{B}(V))$ to its corresponding *dual face* $F_v \in \mathcal{SF}(\mathcal{B}(V))$. This involution can be naturally extended to the edges as follows: for every edge $ab \in \mathcal{SF}(\mathcal{B}(V))$, $\varphi(ab) = \varphi(a)\varphi(b)$ is the edge induced by the intersection of F_a and F_b.

Theorem 5. *[15] Let V be an extremal Vázsonyi configuration in $\mathbb{R}^3$. Then, there is always an unique edge-extension involution $\varphi : \mathcal{SF}(\mathcal{B}(V)) \to \mathcal{SF}(\mathcal{B}(V))$ without* fixed point*, that is $v \notin \varphi(v)$ for all $v \in V$.*

We will refer to this involution as *the canonical involution*.

A ball polyhedra $\mathcal{B}(V)$ is called *standard* if $\mathcal{SF}(\mathcal{B}(V))$ is a *polytopal* lattice (that is, $\mathcal{SF}(\mathcal{B}(V))$ can be realized as the face lattice of a 3-polytope). Numerous papers have focus their attention in studying this kind of ball polytopes. For instance, it is known that Q is a standard ball polytope if and only if either for any supporting sphere $\mathbb{S}(p, r)$ of Q, the intersection $Q \cap \mathbb{S}(p, r)$ is homeomorphic to a closed Euclidean ball of some dimension [15, Remark 9.1] or the intersection of two faces is either empty, a vertex or an edge [19] (see also [2, 23]).

In [15], it was mentioned that not all the extremal configurations for the Vázsonyi problem induce a standard ball polytope. The example that we will present in Section 4.3 is a critical configuration for the Vázsonyi problem, but it turns out not to be a standard ball polytope.

2.2 Reuleaux polyhedra

A standard ball polyhedron $\mathcal{B}(V)$ satisfying $V = \operatorname{vert} \mathcal{B}(V)$ is called a *Reuleaux* polyhedron, and denoted by $\mathcal{R}(V)$. Reuleaux polyhedra enjoy several attractive properties. For instance, they are "frames" of bodies of *constant width* in $\mathbb{R}^3$; see for example, the *Meissner* polyhedra constructed in [23] or the *Pea* bodies built in [1].

It is known that the set of vertices of a Reuleaux polyhedron V form an extremal configuration for the Vázsonyi problem. Furthermore, by using the density of the Reuleaux polyhedra in the set of bodies of constant width (investigated in [27]), it was showed in [12], that the vertex set of a Reuleaux polytope has Borsuk number 4. This fact can also be deduced from [19, Theorem 3] where the chromatic number for the diameter graph of V was shown to be equal 4.

A graph G is called *polyhedron* if it is a simple, 3-connected, planar graph. The name comes after Steinitz' characterization [28] stating that G is a polyhedron if and only if it is the 1-skeleton of a convex 3-polytope. Since the Reuleaux polyhedra are standard ball polytopes, then they have polytopal structure and hence their 1-skeleton is a polyhedron.

2.3 Involutive graphs

Let G be a self-dual graph and let G^* be its dual. A map $\tau : V(G) \to V(G^*)$ is called an *involution* if it satisfies the following:

1) $v \notin \tau(v)$ for every $v \in V$ and

2) $u \in \tau(v) \iff v \in \tau(u)$

A self-dual polyhedron G admitting an involution is called an *involutive* polyhedron (see [19]). Note that $\tau(v)$ can be thought as a face of G (called *dual* face of v, and denoted by F_v). It is easy to verify that for any edge $ab \in E$, there is an other edge $xy \in E$ such that $\tau(a) \cap \tau(b) = xy$ and $\tau(x) \cap \tau(y) = ab$. We will write $\tau(a, b) = xy$ and call them *dual edges*. Since the vertices of a Reuleaux polyhedron are in extremal configuration for the Vázsonyi problem, an involutive map exists and it is actually the canonical involution defined above in Theorem 5. Hence, the 1-skeleton of a Reuleaux polyhedron is an involutive polyhedron.

Let $G = (V, E)$ be an involutive polyhedron and let $a, x \in V$. We say that $[a, x]$ is a *diagonal* of G if $x \in \tau(a)$. We define the the *diagonal* graph Diag_G arising from G, the graph where the set of vertices is V and set of edges consisting of the set of all the diagonals of G. We notice that our diagonal graph correspond to the diameter graph used in [19]. We rather prefer to use the term diagonal to insist that it arises from the involutive map of the abstract graph. In [19], the authors studied involutive graphs from a more geometric point of view (in connection with *metric mappings* and *metric embeddings*) and thus the term diameter seems more appropriate.

In [19], it was stated the following

Conjecture 2. *[19] Every involutive polyhedron $G = (V, E)$ is isomorphic to the 1-skeleton of a Reuleaux polyhedron $\mathcal{R}(S)$ for some set of points S.*

If this conjecture were true then we would have that Diag_G is isomorphic to Diam_S. Indeed, in such a case, there is a bijection $f : V \to S$ such that $[x, y]$ is a diagonal in G if and only if the distance between $f(x)$ and $f(y)$ (vertices in the realization of $\mathcal{R}(S)$) is equal to Diam_S. Conjecture 2 will be discussed further in the last section.

By Withney's work [31], it is known that any polyhedron G can be drawn in the plane or in the 2-sphere (in this case, G is said to be a *map*, that is, a graph cellularly embedded in $\mathbb{S}^2$) essentially in a unique way. Montejano, Ramírez and Rasskin [22] proved that any involutive polyhedra is *antipodally self-dual*, that is, there are maps $\hat{G}$ and $\hat{G}^*$ of G and its dual respectively (simultaneously embedded in $\mathbb{S}^2$) such that $\hat{G} = -\hat{G}^*$.

Let $I(G)$ be the *incidence* graph of the planar graph G. We recall that the vertices of $I(G)$ is given by $V(G) \cup V(G^*)$ and $\{v, w\}$ is an edge of $I(G)$ if $v \in V(G), w \in V(G^*)$ and $v \in F_w$ where F_w is the face in G corresponding to w. By a *symmetric cycle* C of a planar graph G, we mean that there is an automorphism $\sigma(G)$ such that $\sigma(C) = C$ and $\sigma(int(C)) = ext(C)$, that is, the induced graph in the *interior* of C is isomorphic to the induced graph in the *exterior* of C.

In [22, Lemma 1], it was proved that if G is an antipodally self-dual map then $I(G)$ is *antipodally symmetric*, that is, there is a map $\hat{G}$ of G such that $-\hat{G} = \hat{G}$. Furthermore, in [22, Theorem 1] it was proved that if G is an antipodally self-dual map then there is a *symmetric cycle* C_I with $2r$ vertices in $I(G)$, with r odd. We shall denote by Embed($I(G)$) such embedding with C_I placed along the equator of $\mathbb{S}^2$.

The notion of symmetric cycle in maps has already been used in other contexts, for instance, to study knot theory problems [[21], [20]].

3 The Key Lemma

This section is devoted to prove the following lemma that plays a central role throughout this paper.

Lemma 1. *Let G be an involutive polyhedron. Then,* Diag_G *is 4-critical, that is, it is vertex 4-chromatic and the removal of any vertex decreases its chromatic number.*

In order to prove the above lemma, we first establish a number of important properties needed as basic bricks for its proof.

Let G be an involutive graph. We shall consider the above mentioned antipodal embedding Embed($I(G)$) in $\mathbb{S}^2$ where the symmetric cycle C_I is minimal, that is, with a minimal number of edges. We suppose that $int(G)$ and $ext(G)$ are drawn in the Northern and the Southern hemispheres (denoted by $\mathbb{S}^2_{\text{N}}$ and $\mathbb{S}^2_{\text{S}}$) respectively.

[P1] We suppose that $|C_I| = 2r$ where r is an odd integer. We label the black (resp. white) vertices of C_I with $v_0, \ldots, v_{r-1}$ (resp with $v_0^*, \ldots, v_{r-1}^*$) clockwise around the equator. Since vertex v_i is antipodally embedded to v_i^* then C_I is cyclically labeled as follows $\{v_0, v^*_{\frac{r+1}{2}}, v_1, \ldots, v_{\frac{r-1}{2}}, v_0^*, v_{\frac{r+1}{2}}, v_1^* \ldots, v^*_{\frac{r-1}{2}}\}$, see Figure 1(a)

[P2] We claim that any v_i is adjacent to exactly two vertices of C_I in Diag_G. We may show this for v_0 (the argument is the same for any v_i). We clearly have that $v_{\frac{r-1}{2}}$ and $v_{\frac{r+1}{2}}$ are adjacent to v_0 since they both are vertices of the dual face F_{v_0} represented by v_0^*. Now, suppose that there is another v_j, $j \neq \frac{r-1}{2}, \frac{r+1}{2}$ adjacent to v_0. The latter means that v_j is also in the face F_{v_0} and therefore there must also exists an edge joining v_j and v_0^* in G_I, see Figure 1(b).

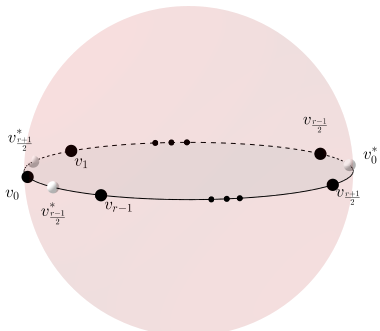
Since $I(G)$ is antipodally symmetric then, there is also an edge joining v_j^* and v_0. We way construct the cycle $C'_I = v_0, v_j^*, [v_j^*; v_0^*], v_0^*, v_j, [v_j; v_0]$ where $[a; b]$ denotes the path along the equator joining a and b without intersecting any other previous vertex in C'_I. By the antipodality of $I(G)$, we have that C'_I induce a symmetric cycle of $I(G)$ with $|C'_I| < |C_I|$, which is a contradiction to the minimality of C_I, see Figure 1(b).

[P3] By [P2], the degree of each vertex v_i of C_I in Diag_G is equals two. In other words, v_i form two diagonals with the two vertices adjacent to v_i^* in C_I. Since r is odd then the set of all these couple of diagonals form a cycle C_D in Diag_G. C_D is a star with r vertices in C_I. For commodity, we preserve the same vertex labels of C_I , given by the order of appearance around the equator for C_D, see Figure 1(c)

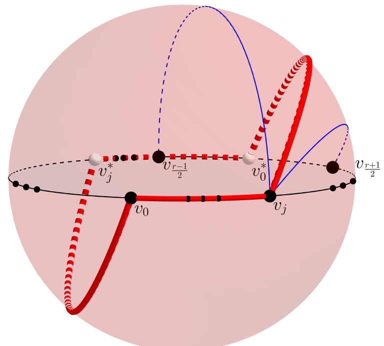
[P4] We claim that there is not face of G containing two non-consecutive vertices of C_D (recall that consecutive is with respect to the order of appearance around the equator and not in the order of appearance while traveling through C_D). We proceed by contradiction, suppose that there is a face F_w containing two non-consecutive vertices, say v_0 and v_j. We thus have that the vertex w^*, representing the dual face F_w in $I(G)$, must be adjacent to both v_0 and v_j. By antipodality, we also have that w is adjacent to both v_0^* and v_j^*. We may thus construct a symmetric cycle $C' = [v_0; v_j^*], v_j^*, w, v_0^*, [v_0^*; v_j], v_j, w^*, v_0$ with $|C'| < |C_I|$, which is a contradiction to the minimality of C_I, see Figure 2(a).

[P5] Notice that a face F of G can never contain four or more vertices of C_D, otherwise F would have at least two non-consecutive vertices of C_D which, by [P4], is impossible.

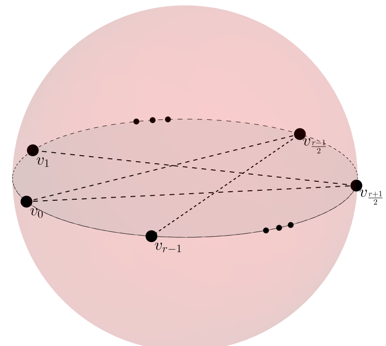
There might exist a face F containing exactly three consecutive vertices of C_D, in this case, G is actually



(a) Cycle C_I with $r = 7$.

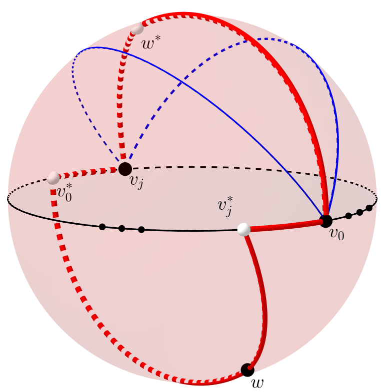


(b) Red edges inducing the new shorter symmetric cycle C_I. Blue arcs are edges in G.

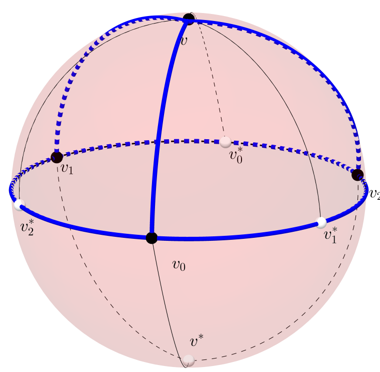


(c) Cycle C_D.

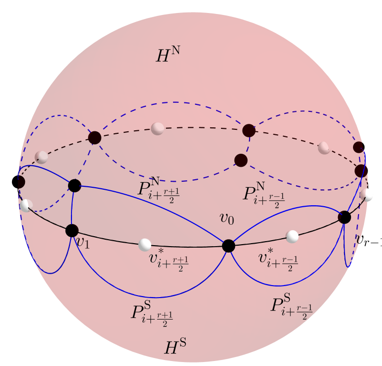
Figure 1: Edges of $I(G)$ in black and edges of G in blue



(a) Red edges inducing the new shorter symetric cycle C_I'.



(b) The Tetrahedron.



(c) $H^{\mathrm{N}}, H^{\mathrm{S}}, P_i^N$ and P_i^S

Figure 2: Edges of $I(G)$ in black and edges of G in blue

the tetrahedron. Indeed, since the vertices are consecutive then C_D consist of three vertices and thus the drawing of $I(G)$ consist of six vertices in the equator (three black and three white appearing alternating) with one black vertex say in $int(G)$ joined to the three white vertices in the equator and one white vertex in $ext(G)$ (representing the face F) joined to the three black vertices in the equator. We thus have that G consist of 4 black vertices forming a tetrahedron, see Figure 2(b)

[P6] Let us consider the embedding of G in $\mathbb{S}^2$, say Embed(G), induced by the embedding of $I(G)$. By the symmetry of C_I then the only faces in Embed(G) lying in $\mathbb{S}^2_{\text{N}}$ and $\mathbb{S}^2_{\text{S}}$ at the same time are the faces corresponding to each blue vertex in C_I. Any other face completely lies in either of the hemispheres, see Figure 2(c).

[P7] Recall that F_{v_i} is the dual face of v_i represented by vertice v_i^*. We define P_i^{N} (resp. P_i^{S}) as the path going from $v_{i+\frac{r-1}{2}}$ to $v_{i+\frac{r+1}{2}}$ for each $i = 0, \ldots, \frac{r+1}{2}$ (sum mod r) through the vertices of F_{v_i} appearing in $\mathbb{S}^2_{\text{N}}$ (resp. in $\mathbb{S}^2_{\text{S}}$).

We also let H^{N} (resp. H^{S}) be the union of all P_i^{N} (resp. all P_i^{S}), see Figure 2(c).

[P8] Since G is a polyhedra (and thus simple) then any pair of faces share at most one edge. Therefore, we may have repeated consecutive edges in H^{N} (or H^{S}) if F_{v_i} and $F_{v_{i+1}}$ share an edge, see Figure 2(c).

[P9] Notice that H^{N} (resp. H^{S}) induce to a path of G separating all the faces completely contained in $\mathbb{S}^2_{\text{N}}$ (resp. in $\mathbb{S}^2_{\text{S}}$) from the rest of faces, see Figure 2(c).

We may now prove Lemma 1.

Proof of Lemma 1.

By [19, Theorem 3], $\chi(\text{Diag}_G) = 4$. We shall show that $\chi(\text{Diag}_G \setminus \{v\}) = 3$ for any $v \in V(\text{Diag}_G)$. To this end, for each $v \in V(G)$ we will show that there always exists a map $c : V(G) \to \{0, 1, 2, 3\}$ from the vertices of G to colors 0,1,2 and 3 inducing a proper coloring with $c(v) = 3$ and $c(v) \neq c(u)$ for all $u \neq v$.

We have that either v is a vertex in $V(C_D)$ or it lies in an hemisphere. Let us see each of these two cases.

Case 1) Let $v \in V(C_D)$. W.l.o.g., we may take $v = v_0$ (in the labeling of C_D). We have that the dual face F_{v_0} contains at least three vertices, say $v_{\frac{r-1}{2}}, v_{\frac{r+1}{2}}$ (see **[P7]**) and u. W.l.o.g., we may assume that u lies in $\mathbb{S}^2_{\text{N}}$.

Let us notice that, by definition of the paths P_i^{N} (see **[P7]**), $u \in P^{\text{N}}_{\frac{r-1}{2}}$. We will use this fact later on in the Subcase 1.2 below.

Let $A[v_0, v_{\frac{r+1}{2}}]$ (resp. $A[v_0, v_{\frac{r-1}{2}}]$) be the vertices in the arc of the equator between v_0 and $v_{\frac{r+1}{2}}$ not containing $v_{\frac{r-1}{2}}$ (resp. the arc between v_0 and $v_{\frac{r-1}{2}}$ not containing $v_{\frac{r+1}{2}}$), see Figure 3.

We color the vertices of G as follows.

- $c(v = v_0) = 3$,
- $c(x) = 2$ if $x \in A[v_0, v_{\frac{r+1}{2}}] \setminus \{v_0\}$,
- $c(x) = 1$ if $x \in A[v_0, v_{\frac{r-1}{2}}] \setminus \{v_0\}$,
- $c(x) = 0$ if x lies in $\mathbb{S}^2_{\text{N}}$,

see Figure 3

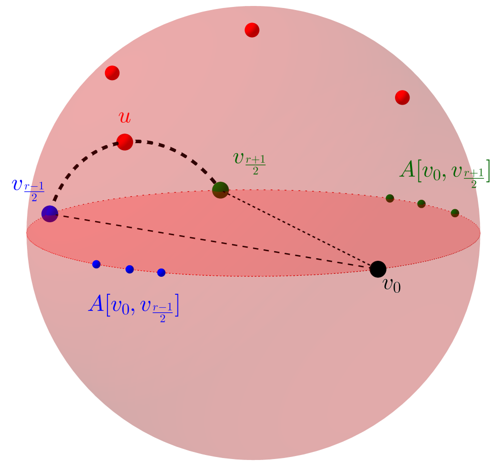


Figure 3: The blue vertices are color 1, the green ones are color 2 and the red ones are color 0.

We first notice that the vertices of an edge in C_D have different colors (and thus colored properly). Moreover, since there is not edge of Diag_G between two vertices in $\mathbb{S}^2_{\text{N}}$ (all the neighbors of the vertices in $\mathbb{S}^2_{\text{N}}$ in Diag_G lie in $\mathbb{S}^2_{\text{S}}$ our coloring works so far.

We finally need to color each vertex lying in $\mathbb{S}^2_{\text{S}}$. Let w be a vertex of G in $\mathbb{S}^2_{\text{S}}$ and let F_w its dual face lying in $\mathbb{S}^2_{\text{N}}$. We claim that at most two out of the three colors 0,1 and 2 could be used for the vertices in F_w. If this is the case, we may then color vertex w with a color different from 0,1 and 2. We prove the claim by contradiction. Let us suppose that the three colors 0,1 and 2 are used in the vertices of F_w. If colors 1 and 2 are used then F_w must have two vertices of C_D. By **[P4]**, these vertices cannot be non-consecutive, and therefore the only choice for these vertices to be in F_w are $v_{i+\frac{r-1}{2}}$ and $v_{i+\frac{r+1}{2}}$.

We have two subcases.

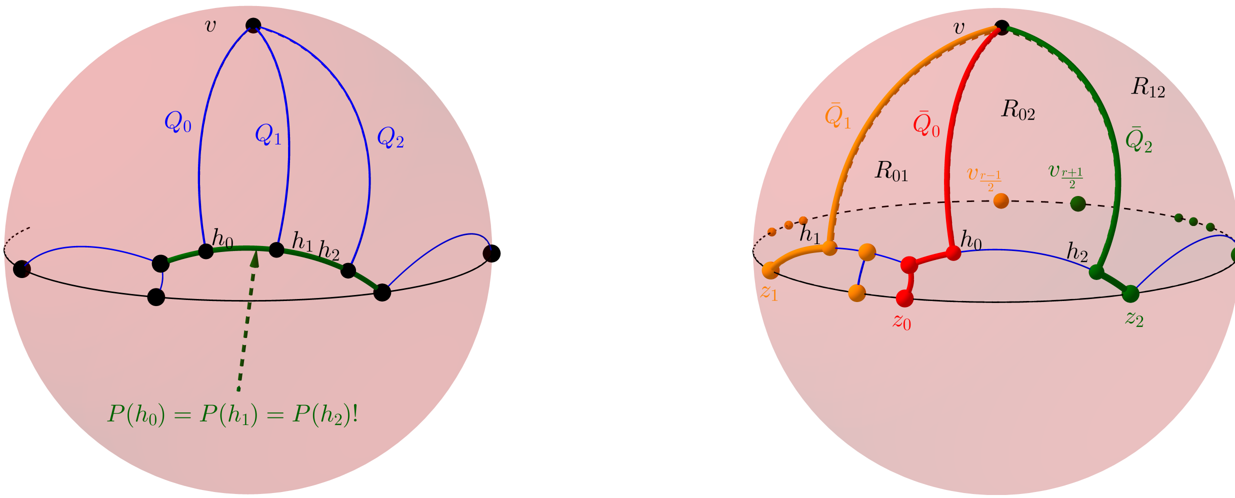
Subcase 1.1) We suppose that $u \in F_w$. We claim that $F_w = F_{v_0}$. Indeed, F_w and F_{v_0} have three common vertices and since any two faces share at most one edge (since G is 3-connected) then the only way for this to happen is if $F_w = F_{v_0}$. However, the latter implies that $w = v_0$, contradicting the fact that w is in $\mathbb{S}^2_{\text{S}}$.

Subcase 1.2) We suppose that $u \notin F_w$. Since both faces F_{v_0} and F_w passe through $v_{\frac{r-1}{2}}$ and $v_{\frac{r+1}{2}}$ then F_w must contains F_{v_0}, in particular, F_w contains $P^{\text{N}}_{\frac{r-1}{2}}$, see Figure 3. As noticed above, $u \in P^{\text{N}}_{\frac{r-1}{2}}$. We clearly have that any path connecting u with any other vertex in the exterior of F_w must going through either $v_{\frac{r-1}{2}}$ or $v_{\frac{r+1}{2}}$, implying that these are cut vertices, contradicting the 3-connectivity of G.

Case 2) Let v be a vertex lying in $\mathbb{S}^2_{\text{N}}$ (the case when v lies in $\mathbb{S}^2_{\text{S}}$ is analogous). We will first construct three vertex-disjoint paths joining v with three different vertices of C_D.

Let w be a vertex in $\mathbb{S}^2_{\text{S}}$ (this vertex exists, otherwise G would be the tetrahedron which is clearly 4-critical). Since G is 3-connected then, by Menger's theorem, there exist three vertex-disjoint paths Q_0, Q_1 and Q_2 joining u to w. We clearly have that each of these paths must intersect H^{N}. Let h_i be the first vertex of H^{N} hit by Q_i for each $i = 0, 1, 2$. Suppose that h_i is in one of the $P^{\text{N}}_{v_i}$, we denoted it by $P(h_i)$ for short. We observe that there are two ways to reach C_D from h_i : either by following the vertices of $P(h_i)$ appearing to the right of h_i (denoted by R_i) or by following the vertices of $P(h_i)$ appearing to the left of h_i (denoted by L_i). Notice that R_i or L_i maybe consist of only the vertex h_i, which is already a vertex in C_D.

Here are the desired paths:



(a) Q_1 would intersect either Q_0 or Q_2.

(b) Coloration by $\bar{Q}_0, \bar{Q}_1$ and $\bar{Q}_2$.

Figure 4: Division of the north hemisphere by the paths $\bar{Q}_0, \bar{Q}_1$ and $\bar{Q}_2$.

- $\bar{Q}_0 := Q_0[v, h_0] \cup T_0$, where T_0 is either R_0 or L_0.
- $\bar{Q}_1 := Q_1[v, h_1] \cup T_1$, where T_1 is either R_1 or L_1. Notice that if $P(h_0) = P(h_1)$ then we can always take T_1 as the side not used in T_0.
- $\bar{Q}_2 := Q_2[v, h_2] \cup T_2$, where T_2 is either R_2 or L_2. Notice that if $P(h_0) = P(h_1)$ then $P(h_2) \neq P(h_0), P(h_1)$ otherwise there will be two Q_i's with a common vertex (which is not possible since they are vertex-disjoint), see Figure 4(a)

Suppose that the vertices v_i's are placed in a r-regular polygon all on the equator. Let z_i be the common vertex of $\bar{Q}_i$ and C_D. Draw a line ℓ going through of the z_0 perpendicular to the opposite side in the regular polygon. We may suppose that we have the situation in which z_1 and z_2 are in opposite sides of ℓ. Otherwise, if both z_1 and z_2 are on the same side of ℓ then either z_1 is between z_0 and z_2 or z_2 is between z_0 and z_1. If z_1 is between z_0 and z_2 then we clearly have that the line ℓ' going through z_1 perpendicular to the opposite side in the regular polygon will leave z_0 and z_2 in different sides (similarly if z_2 were the middle vertex).

W.l.o.g., we may assume that $z_0 = v_0$. Let $A[v_0, v_{\frac{r+1}{2}}]$ (resp. $A[v_0, v_{\frac{r-1}{2}}]$) be the vertices in the arc of the equator between v_0 and $v_{\frac{r+1}{2}}$ containing $z_1 = z_x$ (resp. between v_0 to $v_{\frac{r-1}{2}}$ containing $z_2 = z_y$), see Figure 4(b).

We begin coloring some vertices lying in C_D and $\mathbb{S}^2_{\mathrm{N}}$ as follows:

- $c(v) = 3$,
- $c(v_0) = 0$,
- $c(x) = 1$ for all vertex $x \in A[v_0, v_{\frac{r+1}{2}}] \setminus \{v_0\}$,
- $c(x) = 2$ for all vertex $x \in A[v_0, v_{\frac{r-1}{2}}] \setminus \{v_0\}$,
- $c(x) = 0$ for all vertex $x \in \bar{Q}_0 \setminus \{v\}$,
- $c(x) = 1$ for all vertex $x \in \bar{Q}_1 \setminus \{v\}$ and

- $c(x) = 2$ for all vertex $x \in \bar{Q}_2 \setminus \{v\}$, see Figure 4(b)

Let us verify that this partial coloring is fine so far. We first remark that any vertex in $A[v_0, v_{\frac{r+1}{2}}] \setminus \{v_0\}$ (with color 1) is well colored since its neighbors are two opposite vertices lying in $A[v_0, v_{\frac{r-1}{2}}]$ having color 2 (similarly, for the vertices in $A[v_0, v_{\frac{r-1}{2}}] \setminus \{v_0\}$).

Let us check that the vertices in $\bar{Q}_i = Q_i[v, h_i] \cup T_i$ are all well colored. We notice that there is not problem with the colors of vertices in $Q_i[v, h_i]$ since all their neighbors (in Diag_G) are vertices in $\mathbb{S}^2_{\text{S}}$ (which are not colored yet). Let us now check the vertices of T_i. We will do so for T_0 (analogue arguments can be used to check that the vertices in both T_1 and T_2 are also properly colored).

We have that the vertices of $T_0 = [h_0, \ldots, v_0]$ (colored with color 0 since they are contained in $\bar{Q}_0$) is a subset of $P_{v_{\frac{r-1}{2}}}$ which, in turn, as pointed out in **[P7]**, is a subset of the dual face $F_{v_{\frac{r-1}{2}}}$. Therefore, the neighbor of each vertex of T_0 is $v_{\frac{r-1}{2}}$ that is colored with color 2. It may happen (see **[P8]**) that $F_{v_{\frac{r-1}{2}}}$ share an edge with face $F_{v_{\frac{r+1}{2}}}$, in such a case, the last two vertices in T_0 belong to these both faces and therefore they have both $v_{\frac{r-1}{2}}$ and $v_{\frac{r-1}{2}}$ as neighbors, but this is not a problem since $v_{\frac{r-1}{2}}$ is colored with color 1 (and vertices in T_0 are colored with 0.

In order to complete the coloring (the rest of vertices in $\mathbb{S}^2_{\text{N}}$ and the vertices in $\mathbb{S}^2_{\text{S}}$, we need to partition $\mathbb{S}^2_{\text{N}}$ into 3 regions as follows:

- $R_{0,1} :=$ boarded by $\bar{Q}_0, \bar{Q}_1$ and the arc of the equator between v_0 and z_y,
- $R_{1,2} :=$ boarded by $\bar{Q}_1, \bar{Q}_2$ and the arc of the equator between z_y and z_x and
- $R_{0,2} :=$ boarded by $\bar{Q}_0, \bar{Q}_2$ and the arc of the equator between z_x and v_0, see Figure 4(b)

The goal of such a partition is to divide the set of faces lying in $\mathbb{S}^2_{\text{N}}$ into three parts (each partitioned into faces). With this, we may color the vertices of each region using only two colors and so any face lying in that region would have either two colors (or three if the face contains the vertex v). But this face is the dual face of a vertex u lying in the equator or $\mathbb{S}^2_{\text{S}}$. We would then have a color left (other than color 3) to be used to color u.

We shall color the vertices lying in the interior of region $R_{0,1}$ (similarly for the other two regions). Let u be a vertex in the interior of $R_{0,1}$. Then,

$$c(u) = \begin{cases} c(v_i) & \text{if } u \in P_i \text{ for some } i, \\ 0 \text{ or } 1 & \text{otherwise.} \end{cases}$$

If $u \in P_i$ then u and v_i are both in the same dual face F_{v_j} and thus both neighbors of v_j. Therefore, if the color given to u is the same as the one given to v_i then it would clearly be well colored with respect to v_j (that is already colored).

If $u \notin P_i$ then u would belong to a face F lying within region $R_{0,1}$ with vertices colored with colors 0 or 1 (or 3 if the F touches vertex v). Since F is the dual face F_w for some vertex w lying in $\mathbb{S}^2_{\text{S}}$ then it would be enough to color $c(w) = 2$.

On this way, we can always find a proper 4-coloring (with colors 0,1,2 and 3) in which v is the only vertex having color 3, as desired. □

4 Main Results

In this section, we prove our main contributions. We first show the validity of Conjecture 1 (see Theorem 6), which lead us to the proof our main result. Then towards the end of this section, we present a special configuration of points that is critical but not strongly critical for the Vázsonyi problem.

4.1 Reuleaux polyhedra in the Vázsonyi problem

In order to show Conjecture 1, we need the following

Lemma 2. *Let $V \subset \mathbb{R}^3$ be an extremal configuration for the Vázsonyi problem. Then, the 1-skeleton of $\mathcal{B}(V)$ is simple and 3-connected if and only if V is strongly critical.*

Proof. Let us denote by G the 1-skeleton of $\mathcal{SF}(\mathcal{B}(V))$.

(*Necessity*) Suppose that V is strongly critical, that is, V does not have an extremal proper subset. Since in particular V is an extremal configuration, by Theorem 5, G admits a canonical involution, say φ. Furthermore, by the (GHS) Theorem 4 V is tight and by Theorem 3, G is a 2-connected planar graph.

By the canonical involution, Diam_V can not have vertices with degree less than three (by the strongly critical), then all the faces of G most have at least three vertices. Therefore G is simple.

We shall prove that G is 3-connected by contradiction. Suppose then that G admits a 2-cutting set, say $\{x, y\}$. Let $A_1, \ldots, A_k$, $k \geq 2$, be the connected components of $G \setminus \{x, y\}$.

Remark 1. *Let $F_x = \varphi(x)$ and $F_y = \varphi(y)$ be the dual faces of x and y respectively and let $B_i = \varphi(V(A_i))$ for each $1 \leq i \leq k$.*

(a) B_i is the union of faces where, by Theorem 5, two faces F_u and F_v share an edge if and only if u and v are joined by an edge in A_i. We have thus that B_i is a planar graph with more than tree vertices (otherwise, A_i would consist of a dangling vertex which is not possible since V is strongly critical).

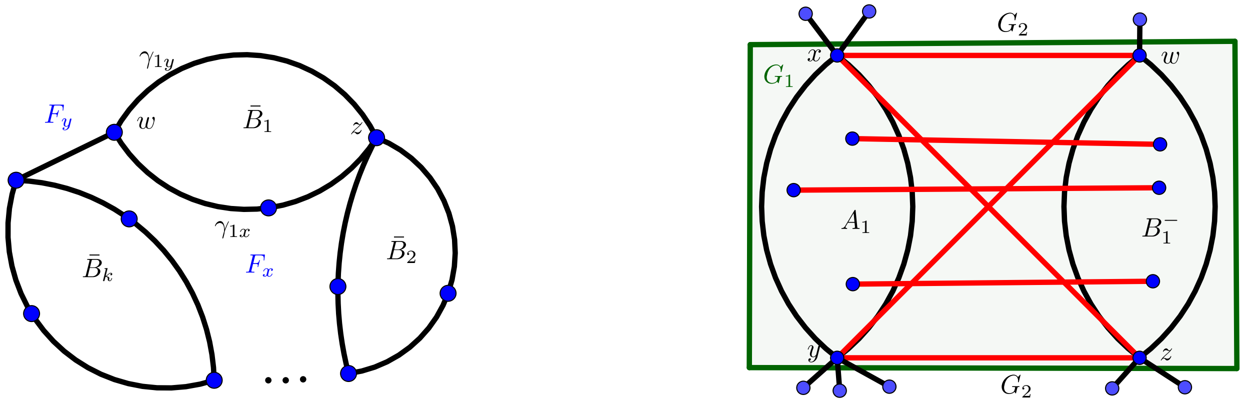
(b) B_i is connected for each $1 \leq i \leq k$. Indeed, Let $p, q \in V(B_i)$ we show that there is a path $\gamma_{p,q}$ joining p and q. Suppose that $p \in F_r$ and $q \in F_s$ were F_r and F_s are some faces in B_i, $r, s \in A_i$. Since A_i is connected then there exists a path $\gamma[r, s]$ between the vertices r and s. Assume first that $\gamma[r, s]$ consists of one edge. Since F_r and F_s share one edge then can construct a path from p to q by a proper sequence of vertices in F_r and F_s. We can clearly proceed by induction if the length of $\gamma[r, s]$ is greater or equal to 2.

Let H be a planar graph. We call the border*, denoted by ∂H, the cycle of the exterior face of H.*

(c) Since the canonical involution preserve adjacencies then, we have that every edge in ∂B_i, $1 \leq i \leq k$ is an edge of either F_x or F_y. Therefore,

$$\bigcup_{i=1}^{k} \partial B_i \subset (F_x \cup F_y).$$

Since F_x and F_y are cycles then ∂B_i can be thought as the union of two paths $\gamma_i^x \subset F_x$ and $\gamma_i^y \subset F_y$ and both paths having the same ends. In fact, ∂B_i can be viewed as a "digon" (a graph with two vertices, say w and z, connected by two edges) with some possible extra vertices in each edge and where w and z are the common ends of the paths. Furthermore, F_x (resp. F_y) is the union of the k paths γ_i^x (resp. γ_i^y), and possibly an extra edge shared by F_x and F_y, in particular, $V(F_x) \cap V(F_y) \neq \emptyset$ (see Figure 5(a)).



(a) The boundaries of F_x, F_y and $B_i's$

(b) A_1, B_1^-, G_1 and G_2.

Figure 5

We focus our attention to A_1 and B_1. Let $A_1^+ = A_1 \cup \{x, y\}$ and $B_1^- = B_1 \setminus \{w, z\}$. Let w and z be the shared ends of paths γ_1^x and γ_1^y. Then, we clearly have that any vertex in B_1^- cannot be connected by a path to any other vertex in B_i, $i \neq 1$, in other words, $\{w, z\}$ is also a 2-cutting set of G.

We observe that B_1^- has the same "shape" as A_1, that is, B_1^- is connected and the vertices w and z play the same role as the vertices x and y for A_1. Since φ is involutive and $\varphi(A_1) = B_1$ then $\varphi(B_1^-) = A_1^+$. We thus have that A_1^+ has the same shape as B_1, that is, it is a digon with x and y the common ends of the corresponding paths (see Figure 5(b)).

We notice that x, y, w, z are four different vertices. Moreover, we claim that $x, y \notin V(B_1)$ and $w, z \notin V(A_1)$. Indeed, suppose, for an instance, that $w \in V(A)$ (the other cases are similar). Then, $F_w \subset B_1$ which implies that $F_z \subset B_1$ as well. Now, since $V(F_w) \cap V(F_z) \neq \emptyset$ then $B_1 \subset A_1$. Now, since $w, z \in V(F_x) \cap V(F_y)$ then $\{x, y\} = \{w, z\}$ (both are 2-cutting sets) which is a contradiction since x, y, w, z are all different vertices.

We now claim that $V(A_1^+) \cap V(B_1) = \emptyset$. We proceed by contradiction, suppose that there is $v \in V(A_1^+) \cap V(B_1)$, then there is path $\gamma[v, x]$ (completely contained in A_1) joining v to x. Since $w, z \notin V(A_1) \subset V(A_1^+)$ then $\gamma[v, x]$ contains neither w nor z. Since B_1^- is connected then any path starting in a vertex in B_1^- not using either z or w (like the path $\gamma[v, x]$) must contain only vertices in B_1. The latter implies that $x \in B_1$, which is a contradiction.

We shall now count the number of diameters induced by $V(G)$. Let G_1 be the subgraph generated by $V(A_1^+ \cup B_1)$ and let G_2 be the subgraph generated by $V(G) \setminus V(A_1 \cup B_1^-)$ (see Figure 5(b)). Since $V(A_1^+) \cap V(B_1) = \emptyset$ then $\{x, y, w, z\} = V(G_1 \cap G_2)$, so

$$|V(G)| = |V(G_1)| + |V(G_2)| - 4.$$

We have that $|E(\mathrm{Diam}_G)| = |E(\mathrm{Diam}_{G_1})| + |E(\mathrm{Diam}_{G_2})| - r$ where r denotes the number of diameters having ends in $\{x, y, w, z\}$. Notice that $r \geq 4$ because xz, xw, yz, yw are diameters.

Since V is an extremal configuration, then

$$|E(\mathrm{Diam}_{G_1})| + |E(\mathrm{Diam}_{G_2})| - r = |E(\mathrm{Diam}_G)| = 2|V(G)| - 2 = 2(|V(G_1)| + 2|V(G_2)| - 4) - 2,$$

and thus,

$$|E(\mathrm{Diam}_{G_1})| + |E(\mathrm{Diam}_{G_2})| = 2(|V(G_1)| + 2|V(G_2)| - 10 + r. \tag{1}$$

Since V is strongly critical then

$$|E(\mathrm{Diam}_{G_1})| \leq 2|V(G_1)| - 3 \text{ and } |E(\mathrm{Diam}_{G_2})| \leq 2|V(G_2)| - 3, \tag{2}$$

and thus, by adding these inequalities, we obtain

$$|E(\mathrm{Diam}_{G_1})| + |E(\mathrm{Diam}_{G_2})| \leq 2|V(G_1)| + 2|V(G_2)| - 6. \tag{3}$$

By combining (1) with (3), we have that $r = 4$, that is,

$$|E(\mathrm{Diam}_{G_1})| + |E(\mathrm{Diam}_{G_2})| = 2(|V(G_1)| + 2|V(G_2)| - 6 \tag{4}$$

and so xy, xw, yz, yw are the only diameters of Diam_G on the set $\{x, y, z, w\}$. Furthermore, by combining (2) with (4), we obtain that $|E(\mathrm{Diam}_{G_1})| = 2|V(G_1)| - 3$ and $|E(\mathrm{Diam}_{G_2})| = 2|V(G_2)| - 3$ are both odd integers.

We shall show that $|E(\mathrm{Diam}_{G_1})|$ is also an even integer, leading to the desired contradiction. To this end, we first count the edges in $E(G_1)$ not having both ends in $\{x, y, w, z\}$, we denote by $\tilde{E}(G_1)$ such a set of edges. We know that, by construction, the dual edge of an edge adjacent to a vertex $a \in V(A_1)$ is an edge in B_1 and, symmetrically, the dual edge of an edge adjacent to a vertex $b \in V(B_1^-)$ is an edge in A_1^+ In other words, any edge in $\tilde{E}(G_1)$ will have its duals in $\tilde{E}(G_1)$. Then, the number of edges in $\tilde{E}(G_1)$ is even.

Now, we clearly have that

$$\sum_{v \in A} \delta(v) + \sum_{v \in B} \delta(v) + \sum_{v \in \{x,y\}} \delta(v)\big|_A + \sum_{v \in \{w,z\}} \delta(v)\big|_B = 2|\tilde{E}(G_1)| \tag{5}$$

where $\delta(v)$ denotes the degree in the graph G of a vertex v and $\delta(v)\big|_S$ the degree of vertex v with endpoints only on set S.

We observe that, by duality, the degree of each vertex $v \in A_1$ is the same as the number of vertices of its dual face and thus the number of diameters adjacent to v. Then, the diameters with one end in A_1 is $\sum_{v \in A_1} \delta(v)$. By the same argument, $\sum_{v \in B_1^-} \delta(v)$ gives the diameters with one end in B_1^-.

Finally, $\sum_{v \in \{x,y\}} \delta_{A_1}(v)$ is the number of diameters with one end in $\{x, y\}$ and the other end in B_1, which is, in fact, a vertex in ∂B_1. Similarly, $\sum_{v \in \{x,y\}} \delta_{B_1}(v)$ is the number of diameters with one end in $\{z, w\}$ and the other in A_1^+, which is in fact ∂A_1^+.

We have that the left-hand side of equality (5) is equals to $2|E(\mathrm{Diam}_{G_1})|$. Therefore, $2|E(\mathrm{Diam}_{G_1})| = 2|\tilde{E}(G_1)|$ implying that $|E(\mathrm{Diam}_{G_1})| = |\tilde{E}(G_1)|$ and, since $|\tilde{E}(G_1)|$ is even (as remarked above) then $|E(\mathrm{Diam}_{G_1})|$ is also even, as claimed above. Therefore, G cannot have 2-cutting set and so G is 3-connected.

(*Sufficiency*) Suppose that the graph G is 3-connected and simple. Since V is an extremal configuration then, by the (GHS) Theorem 4, V is tight and thus, by Theorem 3, is a planar graph. Hence, G is a polyhedron. Moreover, by Theorem 5, G admits a canonical involution, and thus G is an involutive polyhedron. Therefore, by Lemma 1, Diag_G is 4-critical.

We proceed by contradiction. Let us suppose that V is not strongly critical for the Vázsonyi problem. Then, there is a strongly critical subset $V_1 \subset V$ implying, by the necessity condition, that the 1-skeleton of $\mathcal{B}(V_1)$, say G_1, is planar, simple and 3-connected. By the same arguments as above, the latter implies that

G_1 is an involutive polyhedron, and again by Lemma 1, Diag_{G_1} is 4-critical, contradicting that Diag_G is 4-critical. □

The following result, in terms of Reuleaux polyhedra, implies Conjecture 1.

Theorem 6. *Let $V \subset \mathbb{R}^3$ be an extremal set. Then, $\mathcal{B}(V)$ is a Reuleaux polyhedron if and only if V is strongly critical.*

Proof. Suppose that V is strongly critical. Then, by Lemma 2, the 1-skeleton of $\mathcal{B}(V)$ is simple and 3-connected and by Theorem 3, is a planar graph. Therefore, by Steinitz's characterization, $\mathcal{B}(V)$ is an standard ball polytope. Moreover, since V is an extremal configuration then, by Theorem 4, $\text{vert}(\mathcal{B}(V)) = V$ implying thus that $\mathcal{B}(V)$ is a Reuleaux polyhedron.

Suppose now that $\mathcal{B}(V)$ is a Reuleaux polyhedron. Then, $\mathcal{B}(V)$ is a standard ball polytope. Since the 1-skeleton of $\mathcal{B}(V)$ has a polytopal structure then, again by Steinitz's characterization, it is simple and 3-connected, therefore by Lemma 2, V is strongly critical. □

4.2 Proof ot Theorem 1

We prove our main contribution by analyzing the *minimal* structures for the Borsuk and Vázsonyi problem in $\mathbb{R}^3$, which astonishingly are the Reuleaux polyhedra in both cases.

Theorem 7. *Let $V \subset \mathbb{R}^3$ be a finite set of points with $\mid V \mid= n \geq 4$. The following three statements are equivalent:*

(i) V is strongly critical for the Vázsonyi problem.

(ii) Diam_V is 4-critical.

(iii) $\mathcal{B}(V)$ is a Reuleaux polyhedron.

Proof. The equivalence $(i) \iff (iii)$. follows by Theorem 6, and $(iii) \implies (ii)$ is a consequence of Lemma 1. We shall prove that $(ii) \implies (i)$
Since Diam_V is 4-critical then each $v \in V$ has degree at least 3 in Diam_V, thus by Theorem 2 V is tight and then we have $V \subset \text{vert}\,\mathcal{B}(V)$. We consider two cases.

Case 1) If $V = \text{vert}\,\mathcal{B}(V)$, by Theorem 4, V is extremal for the Vázsonyi problem. Suppose that V is not strongly critical for the Vázsonyi problem, then there is a proper subset V_1 of V, which is strongly critical for the Vázsonyi problem. The latter implies that Diam_{V_1} is 4-critical (since $(i). \implies (ii)$), contradicting that Diam_V is 4-critical.

Case 2) If $V \subsetneq \text{vert}\,\mathcal{B}(V)$, by Theorem 4, $e(V) < 2n-2$. Let $m_0 = (2n-2) - e(V)$. We may assume that V does not have an extremal subset for the Vázsonyi problem, otherwise it would lead a contradiction as in Case 1.
Let $v \in \text{vert}\,\mathcal{B}(V)\backslash V$, then v has to be adjacent to at least 3 diameters (Definition 1), so we can define a new subset $V_1 = V \bigcup \{v\}$ in $\mathbb{R}^3$, having at least 3 more diameters than V, and the difference to be Vázsonyi would be $m_1 := (2(n+1)-2) - e(V_1) < m_0$. We may repeat this procedure at most m_0 times in order to obtain a set V_r, with $r \leq m_0$, which is extremal for the Vázsonyi problem. Since V_r has no extremal subset for the Vázsonyi problem, then V_r would be strongly critical for the Vázsonyi problem and so by $(i) \implies (ii)$, Diam_{V_r} would be 4-critical, which is a contradiction.

Therefore, (i), (ii) and (iii) are equivalent. □

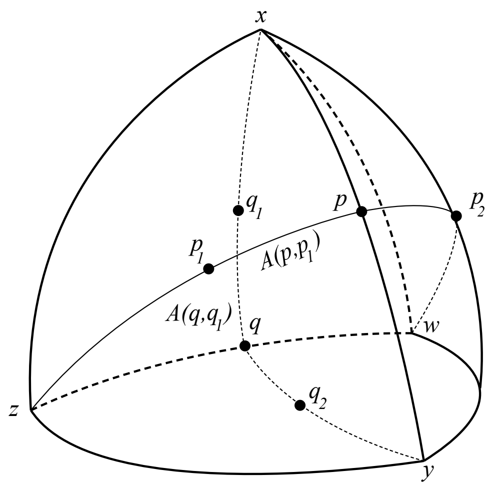


Figure 6: Reuleaux Tetrahedron

We clearly have that Theorem 1 is a straightforward consequence of Theorem 7.

4.3 Special configuration of points

Let us consider the following configuration of 8 points in $\mathbb{R}^3$. Four points, say of w, x, y and z, are the vertices of a regular tetrahedron with edges of length 1.

We shall add other appropriate four points, say a, b, c and d (this is the tricky part of the construction). Let $S(c)$ be the sphere of center c and radius 1. The four desired points will be lie at the Reuleaux tetrahedron formed by $S(x) \cap S(y) \cap S(w) \cap S(z)$ as follows. Let p (resp. q) be the mid-point of the circular-arc edge between x and y (resp. circular-arc edge between z and w). Let p_1 (resp. q_1) be the mid-point of the circular-arc in $S(w)$ joining p to z (resp. in $S(y)$ joining q to x).

Notice that p_1 (resp. q_1) is the centroid of the spherical triangle with vertices z, y and x (resp. spherical triangle with vertices x, w and z). It is known [17] that $\|p, q\| = \left(\sqrt{3} - \frac{\sqrt{2}}{2}\right) \approx 1.0249$, see Figure 6. Observe next, that points p_1 and q_1 are boundary points of both the Reuleaux tetrahedron and the Meissner's bodies of constant width. Moreover, in such bodies, the segment $[p_1, q_1]$ is not a diameter (see [17, pp 171-173]), then the distance between the two centroids is strictly less than one, i.e. $\|p_1, q_1\| < 1$.

Let $A(p, p_1)$ (resp. $A(q, q_1)$) be the circular-arc in $S(w)$ joining p to p_1 (resp. the circular-arc in $S(y)$ joining q to q_1). Let

$$\begin{array}{rccc} \alpha_1 : & [0,1] & \longrightarrow & A(p,p_1) \\ & t & \mapsto & \alpha_1(t) \end{array} \quad \text{and} \quad \begin{array}{rccc} \beta_1 : & [0,1] & \longrightarrow & A(q,q_1) \\ & t & \mapsto & \beta_1(t) \end{array}$$

where $\alpha_1(0) = p, \alpha_1(1) = p_1, \beta_1(0) = q$ and $\beta_1(1) = q_1$.

Finally, let

$$\begin{array}{rccc} \gamma_1 : & [0,1] & \longrightarrow & \mathbb{R} \\ & t & \mapsto & \gamma_1(t) = \|\alpha_1(t), \beta_1(t)\| \end{array}$$

We have that $\gamma_1(t)$ is a continuous function in $[0, 1]$. Moreover, since $\gamma_1(0) = \|\alpha_1(0), \beta_1(0)\| = \|p, q\| > 1$ and $\gamma_1(1) = \|\alpha_1(1), \beta_1(1)\| = \|p_1, q_1\| < 1$ then, by the Mean Value Theorem, there is $t_1 \in [0, 1]$ such that $\gamma_1(t_1) = 1$.

We set $a = \alpha_1(t_1)$ and $c = \beta_1(t_1)$. Let us now use the symmetry of the Reuleaux tetrahedron in order to obtain points b and d. For, we have the following

Remark 2. *Let q_2 (resp. p_2) be the centroid of the spherical triangle with vertices z, y and w (resp. with vertices x, y and w). Let $A(p, p_2)$ (resp. $A(q, q_2)$) be the circular-arc in $S(z)$ joining p to p_2 (resp. in $S(x)$ joining q to q_2). Let α_2 (resp. β_2) be defined similarly as α_1 (resp. as β_1) having as codomain $A(p, p_2)$ (resp. $A(q, q_2)$) instead of $A(p, p_1)$ (resp. $A(q, q_1)$).*

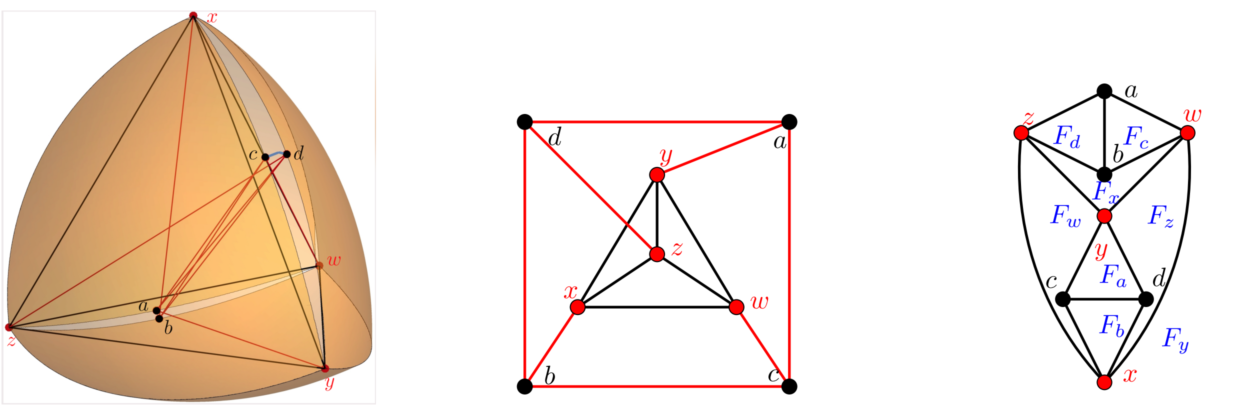
(1) Let γ_2 be defined similarly as γ_1 but taking β_2 instead of β_1. By the same argument as above, there is $t_2 \in [0, 1]$ such that $\gamma_2(t_2) = 1$. By the symmetry with respect to the circular-arc edge between z and w, we have that $t_1 = t_2$. We set $b = \alpha_2(t_2)$.

(2) Let γ_3 be defined similarly as γ_1 but taking α_2 instead of α_1. By the same argument as above, there is $t_3 \in [0, 1]$ such that $\gamma_3(t_3) = 1$. By the symmetry with respect to the circular-arc edge between x and y, we have that $t_1 = t_3$. We set $d = \beta_3(t_3)$.

Moreover,

(3) Let γ_4 be defined similarly as γ_1 but taking α_2 instead of α_1 and β_2 instead of β_1. By the same argument as above, there is $t_4 \in [0, 1]$ such that $\gamma_4(t_4) = 1$. By the symmetry with respect to the circular-arc edge between x and y, we have that $t_1 = t_4$.

Since the original tetrahedron is regular (and each edge is of length one) then the six couples of points formed by $\{w, x, y, z\}$ are at distance one. Moreover, by construction, $\|a, c\| = \|a, d\| = \|b, c\| = \|b, d\| = 1$. Furthermore, $\|c, w\| = \|d, z\| = \|a, y\| = \|b, x\| = 1$ since $c \in S(w), d \in S(z), a \in S(y)$ and $b \in S(x)$. It can be checked that the distance of any other couple of points in $\{a, b, c, d, w, x, y, z\}$ is less than one. The diameter graph is illustrate in Figure 7 (b).



(a) The embedding

(b) The diameter graph.

(c) 1-skeleton.

Figure 7: Critical configuration of 8 points that is not strongly critical for the Vázsonyi problem.

The above configuration of 8 points is an extremal Vázsonyi configuration since it contains $(2 \times 8) - 2 = 14$ diameters. Moreover, it is critical since all points are adjacent to at least 3 diameters and there is not dangling edge (see Figure 7 center). However, it is not strongly critical since it contains the tetrahedron as an extremal subset. Moreover, this configuration is an extremal Vázsonyi configuration but its ball set is not polytopal since it is not 3-connected, for instance $\{z, w\}$ is a 2-cutting set of its 1-skeleton (see Figure 7 (c)). The 1-skeleton is indeed planar but just 2-connected.

We computed explicitly the coordinates of the points of such configuration. In order to simplify the calculations, we set the diameter equal to $\sqrt{3}$ and the coordinates for a, b, c, d are approximated with an error of order of 10^{-4}.

$$\begin{aligned}
x &= (0, 0, \sqrt{2})\\
y &= (1, 0, 0)\\
w &= (\cos(2\pi/3), \sin(2\pi/3), 0)\\
z &= (\cos(4\pi/3), \sin(4\pi/3), 0)\\
a &= (-0.72849, 0, -0.11106)\\
b &= (-0.68087, 0, -0.1784)\\
c &= (0.7095, -0.03157, 0.85524)\\
d &= (0.7095, 0.03157, 0.85524)
\end{aligned}$$

5 Concluding remarks

In this section, we point out some interesting observations and posibilities for future work concerning realizations of Reuleaux polyhedra.

In [19], the authors proved (computationally) the validity of Conjecture 2 up to 14 vertices. They do so by finding first all involutive graphs up to 14 vertices and then constructing explicitly the corresponding desired embedding in each case. We observe that this list of involutive graphs combined with Theorem 7 may allow to construct sets up to 14 points in $\mathbb{R}^3$ having Borsuk number 4 (extending the examples given in [12, Lemma 3] with at most 7 points).

In order to find the above list of involutive graphs, the authors generated all 3-connected planar graphs and then they searched for the existence of an involutive map in each case. We propose an alternative (more direct) way to find all involutive graphs by using the classification of the family of involutive polyhedra given by Bracho *et al.* [6, Theorem 6]. They showed that if P is an involutive polyhedron then there is always an edge $e \in E(P)$ such that $P/\{e\} \setminus \{\tau(e)\}$ is also an involutive polyhedron where τ is the involution and $G \setminus \{f\}$ (resp. denote $G/\{f\}$) denotes the deletion (resp. contraction) of edge f in G. The latter implies that any involutive polyhedra can be reduced to a wheel (with an odd number of vertices in the main cycle) by a finite sequence of *delete-contraction* operation (applied simultaneously each time).

As Tutte [30] remarked, the inverse of the delete-contraction operation correspond to diagonalize faces of the graph and its dual simultaneously. The latter can be settled as an *add-expansion* operation in P as follows.

Let $v \in V(P)$ with degree at least 4. Let F_v be the dual face of v. Notice that v is a vertex of the dual face F_w for any vertex $w \in F_v$.

- **Split** the vertices F_v into two paths P_1 and P_2 with at least 3 vertices each (which is possible since the F_v contains at least 4 vertices) with P_1 and P_2 having only x and y as common vertices. **Add** an edge joining x and y. Let F_1 and F_2 be the faces formed by $P_1 \cup xy$ and $P_2 \cup xy$ respectively.

- **Expand** v into two vertices v_1 and v_2, that is, delete v and add vertices v_1 and v_2 joined by an edge. Also, for $i = 1, 2$, add an edge joining v_i to a neighbor w of v such that $\tau(vw)$ (the dual edge of vw) is an edge in P_i.

We invite the reader to check that this procedure is the inverse operation of the delete-contraction operation. Let us verify that the resulting graph G' is also an involutive polyhedron. We clearly have that G' is a simple, 3-connected, planar graph. Moreover, the involution τ' of G' is given by

$$\tau'(w) = \begin{cases} F_1 & \text{if } w = v_1, \\ F_2 & \text{if } w = v_2, \\ \tau(x) \text{ with } v \text{ replaced by the edge } v_1v_2 & \text{if } w = x, \\ \tau(y) \text{ with } v \text{ replaced by the edge } v_1v_2 & \text{if } w = y, \\ \tau(w) & \text{otherwise.} \end{cases}$$

We thus have that any involutive polyhedron can be obtained from an odd wheel by a finite sequence of add-expansion operation. We observe that the latter would lead to a method to construct Reuleaux polyhedra if Conjecture 2 were true. Moreover, by Theorem 7, the former would give infinite families of strongly critical Borsuk configurations as well as strongly critical Vázsonyi configurations.

Also by the above, we can deduce that Lemma 1 gives infinitely many 4-critical graphs that can be actually constructed systematically. It turns out that, this infinite family also satisfy the following property that graph theorists might find of some interest.

Proposition 1. *Let G be an involutive polyhedron. Then,* Diag_G *is edge 4-critical, that is, it is vertex 4-chromatic and the removal of any edge decreases its chromatic number.*

Proof. We know, by Lemma 1, that Diag_G is vertex 4-critical. Then, $\chi(\mathrm{Diag}_G) = 4$ and $\chi(\mathrm{Diag}_G \setminus \{v\}) < 4$ for every $v \in V(\mathrm{Diag}_G)$. Let $e := xy \in E(\mathrm{Diag}_G)$ with $x, y \in V(\mathrm{Diag}_G)$. We shall show that $\chi(\mathrm{Diag}_G \setminus \{e\}) < 4$.

Since G is a polyhedron, then $\delta_{\mathrm{Diag}_G}(x) \geq 3$ for all $v \in V(\mathrm{Diag}_G)$. We have two cases

Case 1: $\delta_{\mathrm{Diag}_G}(x) = 3$. Set $F_x := (y, w_0, w_1)$ and assume the color of x is $c(x) = 0$. By Lemma 1 we know that there is a 3 coloring of $\mathrm{Diag}_G \setminus \{x\}$ with colors $\{1, 2, 3\}$. Suppose $c(y) = 1$. If $c(w_0), c(w_1) \neq 1$ then we may re-color x with color $c(x) = c(y) = 1$ and obtain a proper 3-coloring of $\mathrm{Diag}_G \setminus \{e\}$. If say $c(w_0) = 1$ then we may re-color x with color $c(x) = j \in \{2, 3\} \setminus c(w_1)$ which yields a proper coloring of $\mathrm{Diag}_G \setminus \{e\}$.

Case 2: $\delta_{\mathrm{Diag}_G}(x) \geq 4$. In this case, we can apply an add-expansion operation. We do so by expanding x into v_1 and v_2 in G with $P_1 = (w_n, y, w_0)$ and $P_2 = (w_0, \ldots, w_n)$ (see the above notation). By the above discussion, the new graph G' is an involutive polyhedron.

By construction, we have that $\mathrm{Diag}_G \setminus \{e\} \subset \mathrm{Diag}_{G'}$. Furthermore, we can obtain $\mathrm{Diag}_{G'}$ from $\mathrm{Diag}_G \setminus \{e\}$ by adding a new vertex z and the edges zw_n, zy and zw_0 (in the above notation, we are taking $v_1 = z$ and $v_2 = x$).

We thus have that $\mathrm{Diag}_{G'} \setminus \{z\} = \mathrm{Diag}_G \setminus \{e\}$. Since G' is also an involutive polyhedron we know that $\chi(\mathrm{Diag}_{G'}) = 4$, and by Lemma 1, $\chi(\mathrm{Diag}_{G'} \setminus \{z\}) = 3$, then $\chi(\mathrm{Diag}_G \setminus \{e\}) = 3$. Therefore, Diag_G is edge 4-critical.

□

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References

[1] Isaac Arelio, Luis Montejano, and Déborah Oliveros. Peabodies of constant width. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 64:367–385, 2023.

[2] Károly Bezdek, Zsolt Lángi, Márton Naszódi, and Peter Papez. Ball-polyhedra. *Discrete & Computational Geometry*, 38:201–230, 2007.

[3] V.G. Boltyanski and P.S. Soltan. Borsuk's problem. *Mat. Zametki*, 22:621–631, 1977.

[4] Vladimir Boltyanski, Horst Martini, and Petru S Soltan. *Excursions into combinatorial geometry.* Springer Science & Business Media, 2012.

[5] Karol Borsuk. Drei sätze über die n-dimensionale euklidische sphäre. *Fundamenta Mathematicae*, 20(1):177–190, 1933.

[6] Javier Bracho, Luis Montejano, Eric Pauli Pérez, and Jorge Luis Ramírez Alfonsín. Strongly involutive self-dual polyhedra. *Ars Mathematica Conteporanea*, 20(1):143–149, 2021.

[7] H.G. Eggleston. Covering a three-dimensional set with sets of smaller diameter. *Journal of the London Mathematical Society*, 1(1):11–24, 1955.

[8] Branko Grünbaum. A proof of Vázsonyi's conjecture. *Bull. Res. Council Israel, Sect. A*, 6:77–78, 1956.

[9] Branko Grünbaum. A simple proof of Borsuk's conjecture in three dimensions. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 53, pages 776–778. Cambridge University Press, 1957.

[10] Aladár Heppes. Beweis einer vermutung von a. Vázsonyi. *Acta Mathematica Hungarica*, 7(3-4):463–466, 1956.

[11] Heinz Hopf and Erika Pannwitz. Aufgabe nr. 167. *Jahresbericht Deutsch. Math.-Verein*, 43:114, 1934.

[12] Mihály Hujter and Zsolt Lángi. On the multiple Borsuk numbers of sets. *Israel Journal of Mathematics*, 199(1):219–239, 2014.

[13] Thomas Jenrich and Andries E Brouwer. A 64-dimensional counterexample to Borsuk's conjecture. *The Electronic Journal of Combinatorics*, pages 4–29, 2014.

[14] Jeff Kahn and Gil Kalai. A counterexample to Borsuk's conjecture. *Bulletin of the American Mathematical Society*, 29(1):60–62, 1993.

[15] Y Kupitz, Horst Martini, and M. Perles. Ball polytopes and the Vázsonyi problem. *Acta Mathematica Hungarica*, 126(1-2):99–163, 2010.

[16] Yakov Shimeon Kupitz. *Extremal problems in combinatorial geometry*, volume 53. Matematisk institut, Aarhus universitet, 1979.

[17] Horst Martini, Luis Montejano, and Déborah Oliveros. *Bodies of constant width.* Springer, 2019.

[18] Ernst Meissner. Über die durch reguläre Polyeder nicht stützbaren Körper. *Vierteljahresschr. Naturfor. Ges. Zürich*, 63:544–551, 1918.

[19] Luis Montejano, Eric Pauli, Miguel Raggi, and Edgardo Roldán-Pensado. The graphs behind Reuleaux polyhedra. *Discrete & Computational Geometry*, 64(3):1013–1022, 2020.

[20] Luis Montejano, Jorge L Ramírez Alfonsín, and Iván Rasskin. Self-dual maps III: projective links, 2022. `arXiv:2210.04053`.

[21] Luis Montejano, Jorge L Ramírez Alfonsín, and Iván Rasskin. Self-dual maps II: links and symmetry. *SIAM Journal on Discrete Mathematics*, 37(1):191–220, 2023.

[22] Luis Montejano, Jorge Luis Ramírez Alfonsín, and Iván Rasskin. Self-dual maps I: antipodality. *SIAM Journal on Discrete Mathematics*, 36(3):1551–1566, 2022.

[23] Luis Montejano and Edgardo Roldán-Pensado. Meissner polyhedra. *Acta Mathematica Hungarica*, 151(2):482–494, 2017.

[24] János Pach and Pankaj K Agarwal. *Combinatorial Geometry*. John Wiley & Sons, 2011.

[25] Julian Perkal. Sur la subdivision des ensembles en parties de diamètre inférieur. In *Colloq. Math*, volume 1, page 45, 1947.

[26] Andreĭ M Raĭgorodskiĭ. The Borsuk partition problem: the seventieth anniversary. *The Mathematical Intelligencer*, 26(3):4–12, 2004.

[27] G.T. Sallee. Reuleaux polytopes. *Mathematika*, 17(2):315–323, 1970.

[28] Ernst Steinitz. Polyeder und raumeinteilungen. *Encyk der Math Wiss*, 12:38–43, 1922.

[29] Stefan Straszewicz. Sur un probleme géométrique de P. Erdös. *Bull. Acad. Polon. Sci. Cl. III*, 5:39–40, 1957.

[30] William Thomas Tutte. A theory of 3-connected graphs. *Indag. Math*, 23(441-455):8, 1961.

[31] Hassler Whitney. 2-isomorphic graphs. *American Journal of Mathematics*, 55(1):245–254, 1933.