Ehrhart theory II : further results

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Euler formula

Let P be a d-polytope. We recall that Euler's formula for P is

$$\sum_{k=0}^{d}(-1)^k n_k(P) = 1$$

where $n_k(P)$ is the number of k-faces of P.

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When P is simple (that is, each vertex of P is of degree d) the Dehn-Sommerville's relations are

$$\sum_{j=0}^{k}(-1)^j \binom{d-j}{d-k} n_j(P) = n_k(P), \qquad k = 0, \ldots, d.$$

Proof of Euler's formula via Ehrhart

We count integer points in tP according to the (relative) interior points

$$L_P(t) = \sum_{F \subseteq P} L_{F^\circ}(t) = \sum_{F \subseteq P} (-1)^{dim(F)} L_F(-t)$$

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Now, the constant term of $L_F(t)$ is 1 for every face F. Hence

$$1 = \sum_{F \subseteq P} (-1)^{dim(F)} = \sum_{j=0}^{d} (-1)^j n_j(P).$$

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Let F be a k-face of P and let us count integers points of F according to the relative interior faces of F

$$L_F(t) = \sum_{G \subseteq F} L_{G^\circ}(t)$$

and by the reciprocity law

$$L_F(t) = \sum_{G \subseteq F} (-1)^{dim(G)} L_G(-t) = \sum_{j=0}^{k} (-1)^j \sum_{G \subseteq F, dim(G)=j} L_G(-t).$$

Dehn-Summerville generalisation

Obtaining

$$\begin{aligned}
F_k(t) &= \sum_{F\subseteq P, dim(F)=k} L_F(t) \\
&= \sum_{F\subseteq P, dim(F)=k} \sum_{j=0}^{k} (-1)^j \sum_{G\subseteq F, dim(G)=j} L_G(-t) \\
&= \sum_{j=0}^{k} (-1)^j \sum_{F\subseteq P, dim(F)=k} \sum_{G\subseteq F, dim(G)=j} L_G(-t) \\
&= \sum_{j=0}^{k} (-1)^j \sum_{G\subseteq F, dim(G)=j} n_k(P/G) L_G(-t)
\end{aligned}$$

where $n_k(P/G)$ denotes the number of k-faces in P containing a given j-face of G of P.

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Moreover, for $k = d$ we have

$$L_P(-t) = F_d(-t) = \sum_{j=0}^{d} (-1)^j F_j(t) = (-1)^d \sum_{j=0}^{d} (-1)^{d-j} F_j(t)$$

inclusion-exclusion formula for the number of integer points in the interior of tP.

Applications to coefficients

Consider $F_k(t)$ when $k = d$

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The last term of this sum is

$$(-1)^d F_d(-t) = (-1)^d L_P(-t) = L_{P^\circ}(t)$$

obtaining

$$L_P(t) - L_{P^\circ}(t) = \sum_{j=0}^{d-1} (-1)^j F_j(-t)$$

the number of integer points on the boundary of tP.

If we let $L_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_0$ then

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and then

$$L_P(t) - L_{P^\circ}(t) = 2c_{d-1} t^{d-1} + 2c_{d-3} t^{d-3} + \cdots$$

the sum ends with $2c_0$ if d is odd and $2c_1 t$ if d is even.

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the sum ends with $2c_0$ if d is odd and $2c_1 t$ if d is even. Obtaining

$$c_{d-1} t^{d-1} + c_{d-3} t^{d-3} + \cdots = \frac{1}{2} \sum_{j=0}^{d-1} (-1)^j F_j(-t)$$

Brion's formula

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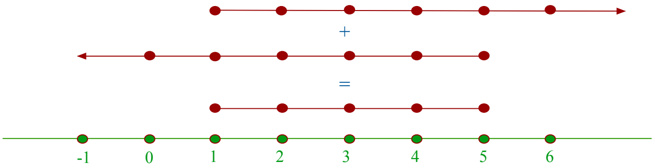
By adding these two equalities we obtain a kind of miracle

$$\frac{x}{1-x} + \frac{x^5}{1-x^{-1}} = \frac{x}{1-x} + \frac{x^6}{x-1} = \frac{x-x^6}{1-x} = x + x^2 + x^3 + x^4 + x^5.$$

We can thought this as a function enumerating the positive integers in the finite segment $[1, 5]$



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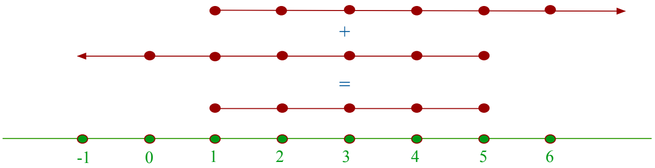
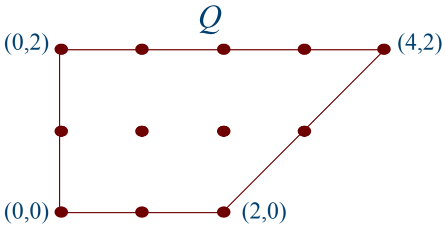


Consider the polygon Q with vertices $(0, 0, (2, 0), (0, 2)$ et $(4, 2)$



The two edges incident to the origine generate the nonnegative quadrant admitting thus the generating function

$$\sum_{m,n\geq 0} x^m y^n = \sum_{m\geq 0} x^m \sum_{n\geq 0} y^n = \frac{1}{1-x}\frac{1}{1-y} = \frac{1}{(1-x)(1-y)}.$$

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The two edges incident to vertex $(0,2)$ generate de cone $(0,2)+\mathbb{R}_{\geq 0}(0,-2)+\mathbb{R}_{\geq 0}(4,0)$ admitting thus the generating function

$$\sum_{m\geq 0, n\geq 2} x^m y^n = \sum_{m\geq 0} x^m \sum_{n\leq 2} y^n = \frac{1}{1-x}\frac{y^2}{1-y^{-1}}.$$

The two edges incident to vertex $(4,2)$ generate de cone $(4,2)+\mathbb{R}_{\geq 0}(-4,0)+\mathbb{R}_{\geq 0}(-2,-2)$ admitting thus the generating function

$$\frac{x^4y^2}{(1-x^{-1})(1-x^{-1}y^{-1})}.$$

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$$\frac{x^2}{(1-xy)(1-x^{-1})}.$$

By adding these functions we obtain

$$\frac{1}{(1-x)(1-y)}+\frac{y^2}{1-y^{-1}}+\frac{x^4y^2}{(1-x^{-1})(1-x^{-1}y^{-1})}+\frac{x^2}{(1-xy)(1-x^{-1})}$$

$$=y^2+xy^2+x^2y^2+x^3y^2+x^4y^2+y+xy+x^2y+x^3y+1+x+x^2.$$

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We know that the generating function

$$\sigma_{C_u}(x) = \sum_{m \in (C_u \cap \mathbb{Z}^d)} \mathbf{x}^{\mathbf{m}}$$

is a rational function.

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We donc have that $\sigma_P(1, \ldots, 1)$ counts the number of integer points in P.

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Example ... continuation.

$$\sigma_Q(1, \ldots, 1) = \sum_{m \in (Q \cap \mathbb{Z}^2)} \mathbf{1^m} = 1+1+1+1+1+1+1+1+1+1+1+1 = 12.$$

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Barvinok's algorithm

In 1993 Barvinok found an algorithme to count integer points in polyhedra.

When the dimension is fixed the algorithm can count the number of integer points in a polytope in polynomial time on the size of the input.

It computes

$$\sum_{m \in (C_u \cap \mathbb{Z}^d)} \mathbf{x}^{\mathbf{m}}$$

where $\mathbf{x}^{\mathbf{m}}$ for $x_1^{m_1} x_2^{m_2} \cdots x_d^{m_d}$.

Quasi-polynomial

A periodic rational number $c(n)$ is a function $c : \mathbb{Z} \to \mathbb{Q}$ with a period q such that $c(n) = c(n')$ when $n \equiv n' \pmod{q}$.

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Example Consider $c(n) = [5/2, 1/3, 1, 1/4]_n$.
$c(n)$ is a periodic number of period 4.

n	$c(n)$
$0 \pmod 4$	$5/2$
$1 \pmod 4$	$1/3$
$2 \pmod 4$	1
$3 \pmod 4$	$1/4$

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A quasi-polinomial f of degree d is a function

$$f(n) = c_d(n)n^d + \cdots + c_1(n)n + c_0$$

where $c_i(n)$ is a periodic number. The period q of f is the least commun multiple of the periods of its coefficients.

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The list of polynomials that f represents are

n	$f_{n \pmod 6}(n)$
$0 \pmod 6$	$5n^3 + 1/2n^2 + n + 3/7$
$1 \pmod 6$	$5n^3 + 2n^2 + 1/2n + 3$
$2 \pmod 6$	$5n^3 + 1/3n^2 + n + 3/7$
$3 \pmod 6$	$5n^3 + 1/2n^2 + 1/2n + 3$
$4 \pmod 6$	$5n^3 + 2n^2 + n + 3/7$
$5 \pmod 6$	$5n^3 + 1/3n^2 + 1/2n + 3$

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Example Consider P given by the interval $[0, n/3]$. The denominator of P is 3 and its quasi-polynomial of the form

$$L_P(t) = \alpha n + [\beta_1, \beta_2, \beta_3]_n$$

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In order to determine $\alpha, \beta_1, \beta_2, \beta_3$ we use Lagrange interpolation.

Compute the number of integer points of nP for some values of n

n	$L_P(n)$
0	1
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2	1
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$$1 = L_P(2) = 1/3(2) + \beta_3 \text{ impliying that } \beta_3 = 1/3$$

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We get

$$L_P(n) = 1/3n + [1, 2/3, 1/3]_n.$$

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Example Consider the pyramide P with vertices $(0,0,0),(1,0,0),(0,1,0),(1,1,0)$ and $(1/2,0,1/2)$. In this case, the denominator of P is 2 however $L_P(n) = \binom{n+3}{3}$ is of period 1.

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Theorem The quasi-polynomial of a rational 1-polytope is always of full period.

Cyclic polytope

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Theorem

$$L_{C_d}(k) = \sum_{i=0}^{d} f_i k^i$$

where $f_i = vol(C_i(t_1, \ldots, t_n))$.

Semimagic matrix

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Let $H_n(t)$ be the total number of semimagic matrices of order n and line sum t.

Semimagic matrix

Consider the polytope

$$B_n := \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} : x_{jk} \geq 0, \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

consisting of nonnegative real matrices in which all rows and columns sum to 1. B_n is called Birkhoff-von Neumann polytope.

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consisting of nonnegative real matrices in which all rows and columns sum to 1. B_n is called Birkhoff-von Neumann polytope. $H_n(t)$ enumerates precisely the integer points in tB_n, that is,

$$H_n(t) = \#(tB_n \cap \mathbb{Z}^{n^2}) = L_{B_n}(t).$$

Semimagic matrix

Consider the polytope

$$B_n := \left\{ \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} : x_{jk} \geq 0, \begin{array}{l} \sum_j x_{jk} = 1 \text{ for all } 1 \leq k \leq n \\ \sum_k x_{jk} = 1 \text{ for all } 1 \leq j \leq n \end{array} \right\}$$

consisting of nonnegative real matrices in which all rows and columns sum to 1. B_n is called Birkhoff-von Neumann polytope. $H_n(t)$ enumerates precisely the integer points in tB_n, that is,

$$H_n(t) = \#(tB_n \cap \mathbb{Z}^{n^2}) = L_{B_n}(t).$$

A permutation matrix is a square matrix with 0,1 entries with exactly one 1 in each row and each column.

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Frobenius coin-exchange problem

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Consider the restricted partition function

$$p_A(n) := \#\{(m_1, \ldots, m_d) \in \mathbb{Z}^d : m_j \geq 0, m_1 a_1 + \cdots + m_d a_d = n\}$$

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$g(a_1, \ldots, a_n)$ is the largest positive integer n for which $p_A(n) = 0$.

There is a nice geometric interpretation of $p_A(n)$. Let

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The function $p_A(n)$ counts precisely those integer points in $\mathbb{Z}^d$ that lie in the nP (that is, we replace $x_1a_1 + \cdots + x_da_d = 1$ by $x_1a_1 + \cdots + x_da_d = n$).