Ehrhart theory IV : matroid polytope

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Lemma All the bases of a matroid have the same cardinality r.

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The family $\mathcal{B}$ verifies the following conditions :

$(B1)$ $\mathcal{B} \neq \emptyset$,

$(B2)$ (**exchange propety**) $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

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- Let $U_{r,n} = \binom{[n]}{r}$ (i.e., the family of all r-sets of $\{1, \ldots, n\}$). $U_{r,n}$ is a matroid (called the uniform matroid of rank r on n elements).

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• Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. Let $\mathcal{B}$ be the set of all maximal forest in G.
Then, $M(G) = (\mathcal{B}, E)$ is a matroid with $r(M(G)) = n - c$ where c is the number of connected components of G.
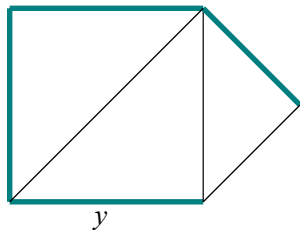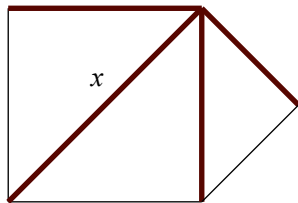
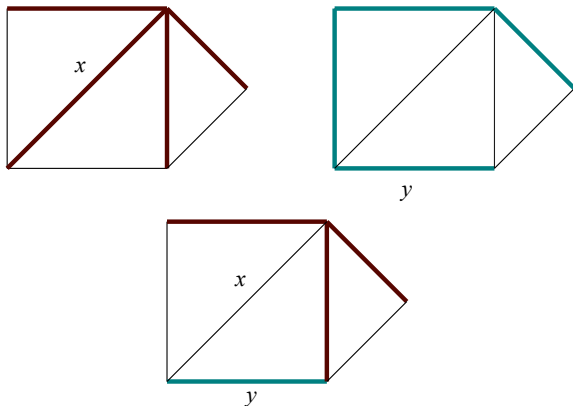# Bases
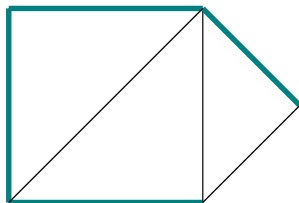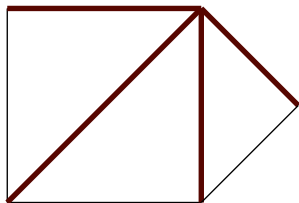
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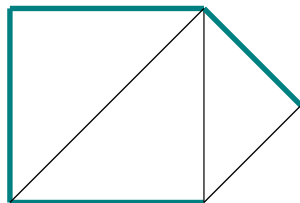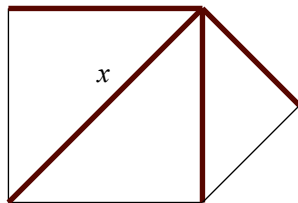
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Matroid base polytope

Let $M = (\mathcal{B}, E)$ with $|E| = n$. For each base $B \in \mathcal{B}$, the incident vector $e_B \in \mathbb{R}^E$ is defined by

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Let Δ_E be the simplexe in $\mathbb{R}^E$, i.e.,

$$\Delta_E = conv(e_i : i \in E) = \{x \in \mathbb{R}^E : \sum_{i \in E} x_i = 1, \ x_i \geq 0 \text{ for all } i \in E\}$$

Matroid base polytope

Theorem (Gel'fand, Goresky, MacPherson, Serganova 1987)
Let $P \subseteq \mathbb{R}^E$ be a convex polytope. Then, P is the base polytope of a matroid $M = (\mathcal{B}, E)$ if and only if
- $P_M \subseteq r\Delta_E$ where $r = r(M)$ (implying that $dim(P) \leq n - 1$
- the vertices of P belong to $\{0, 1\}^E$ and
- each edge of P_M is a translation of $conv(e_i, e_j)$ for all $i, j \in E, i \neq j$.

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$$P_{U_{2,3}} = conv\{(1,1,0),(1,0,1),(0,1,1)\}$$

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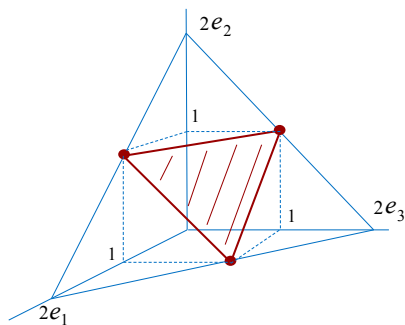
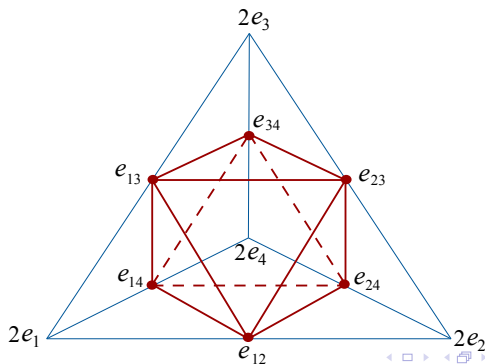# Example

$$P_{U_{2,4}} = conv \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4$$

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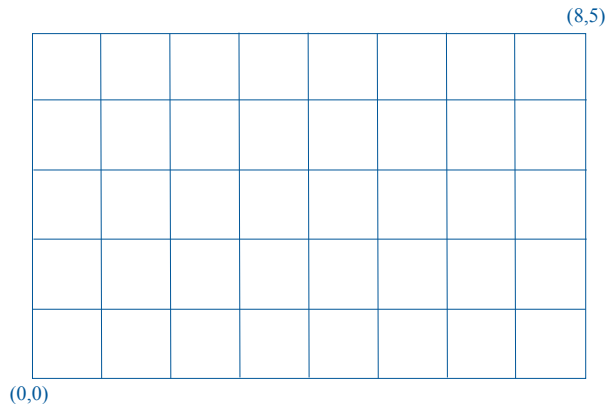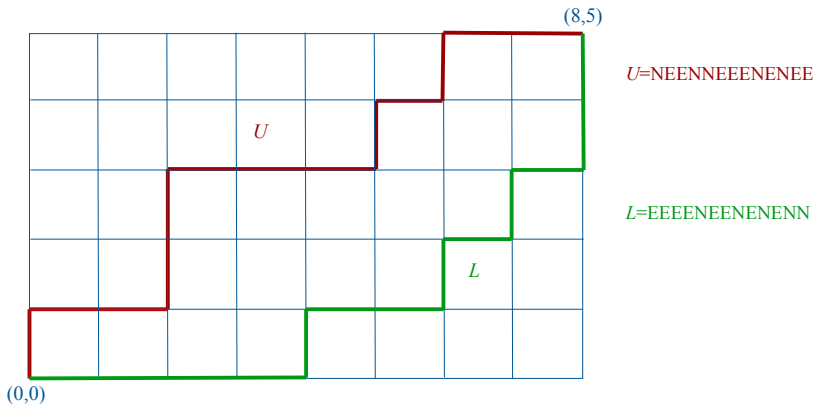
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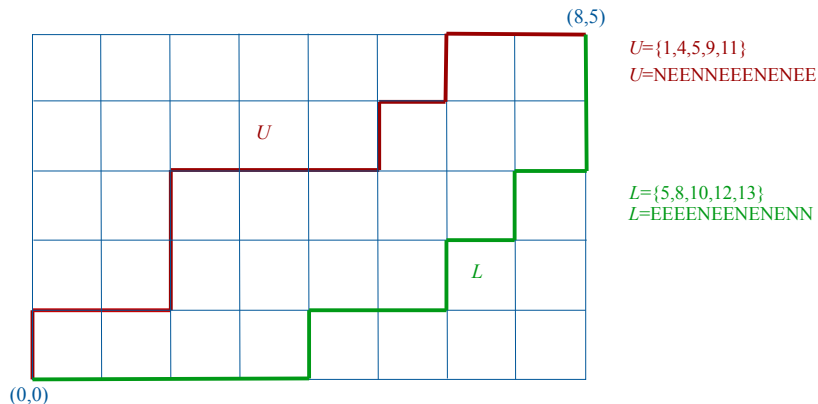
Theorem $dim(P_M) = n - c(M)$ where $n = |E|$.
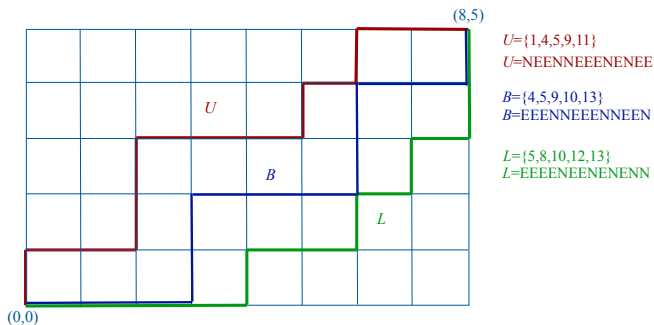
Lattice path matroid

(8,5)

(0,0)

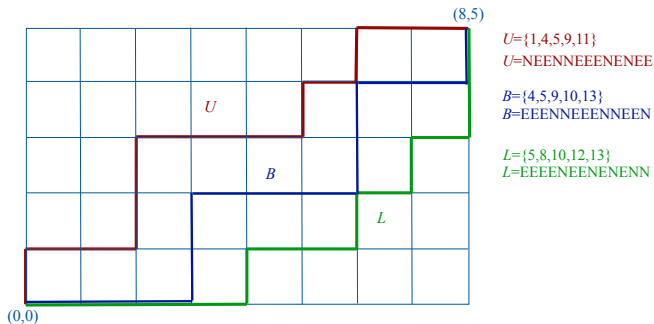# Lattice path matroid

Lattice path matroid

U={1,4,5,9,11}
U=NEENNEEENENEE

L={5,8,10,12,13}
L=EEEENEENENENN

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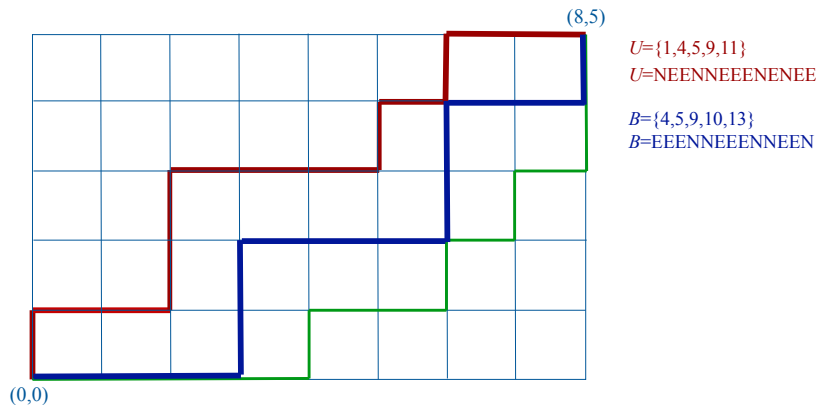
$B=\{4,5,9,10,13\}$
B=EEENNEEENNEEN

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$M[U, L]$ lattice path matroid (LPM) of rank r (# rows) on $r + m$ (# rows + # columns) elements.
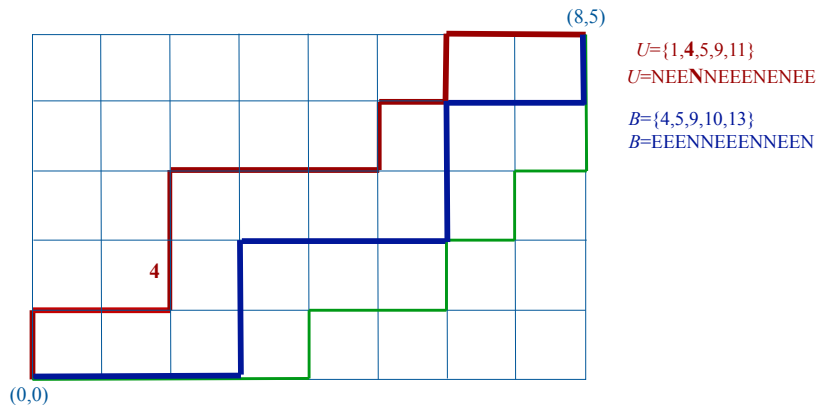
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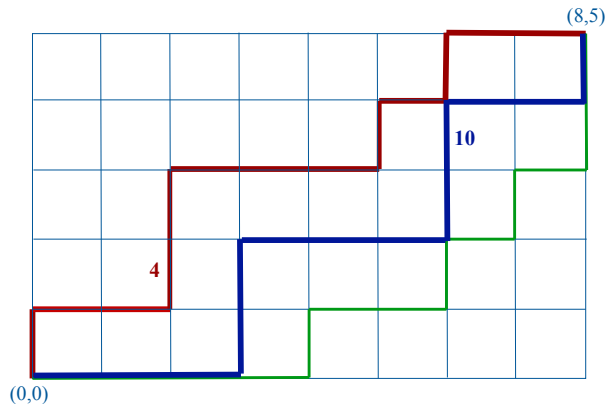U=NEENNEEENENEE

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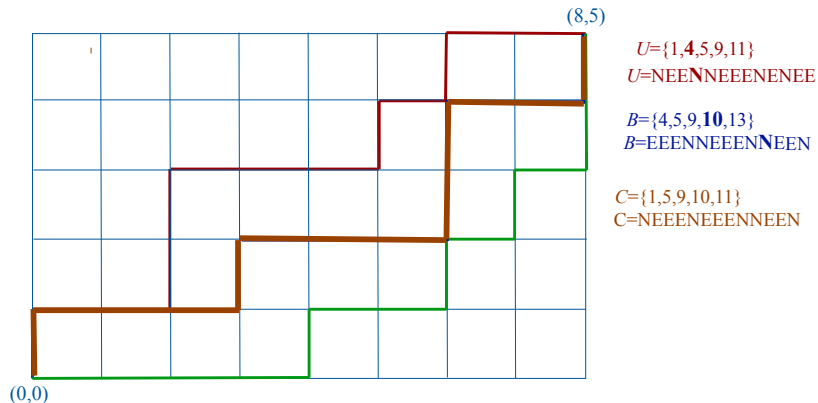
Generalized lattice path
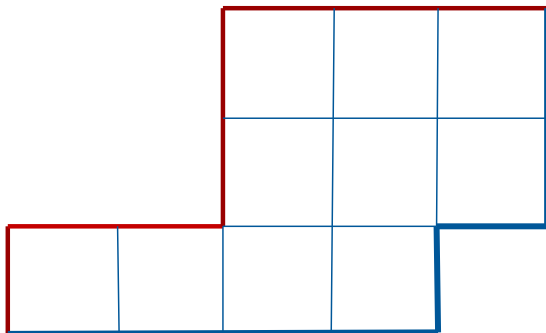
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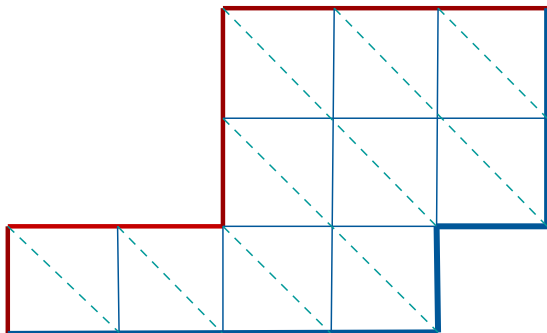
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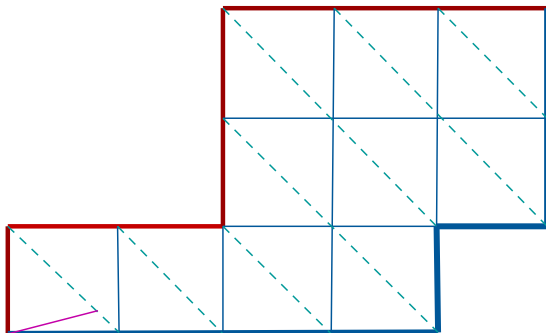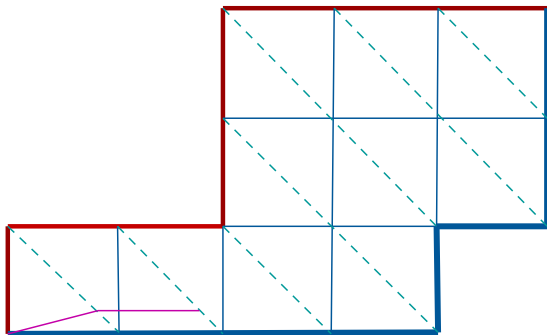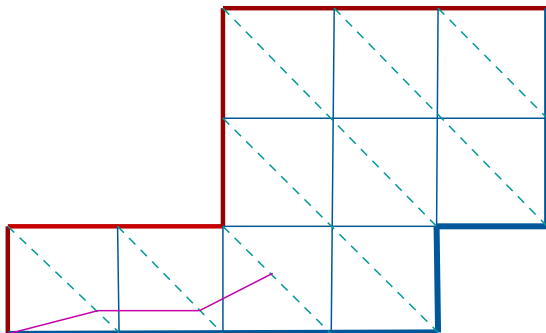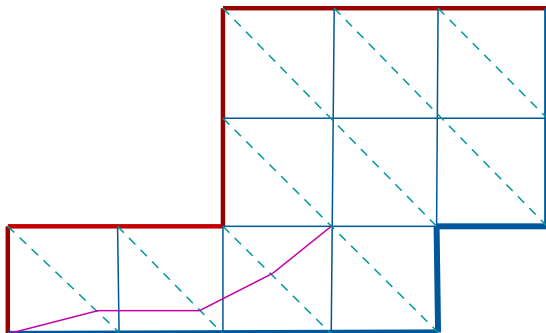
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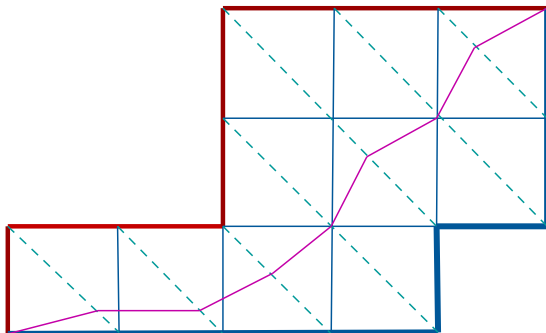
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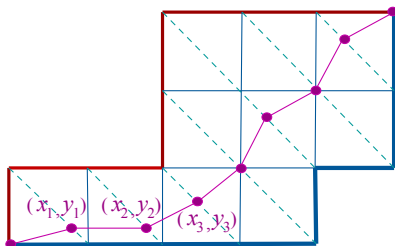
Let $st(P) = (p_1, \ldots, p_{r+m})$ where $p_{i+1} = y_{i+1} - y_i$ for each i.
We call $st(P)$ step vector of P.

Characterizing step vectors

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M[U, L]$ be a LPM of rank r on $r + m$ elements.
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Let $\mathcal{C}_M$ be the family of step vectors of all generalized lattice path in $M[U, L]$. Then,

$$\mathcal{C}_M = \left\{ p \in \mathbb{R}^{r+m} \mid 0 \le p_i \le 1, \sum_{j=1}^{i} l_j \le \sum_{j=1}^{i} p_j \le \sum_{j=1}^{i} u_j \; \forall i \right\}$$

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- $0 \leq y_{i+1} - y_i \leq 1$
- Any generalized path stay between U and L.

Points in LPM polytope

Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $M = M[U, L]$ be a LPM of rank r on $r + m$ elements and let P_M be the matroid polytope. Then, $P_M = \mathcal{C}_M$.

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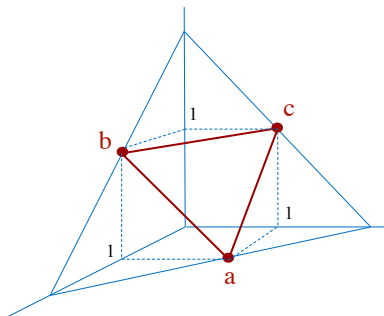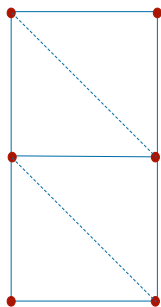$P_M \supseteq \mathcal{C}_M$ by induction on the number of elements.

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$$kP_M \cap \mathbb{Z}^{r+m} = \mathcal{C}_M^k$$

Integer points in LPM polytopes
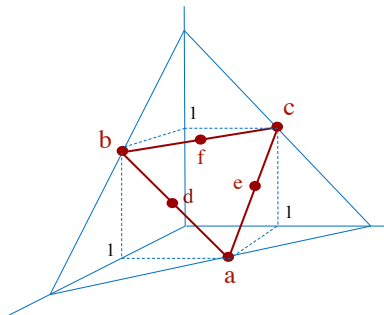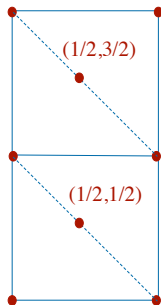
Example : Consider $P_{U_{2,3}}$

a=(1,1,0)
b=(1,0,1)
c=(0,1,1)

Example : Construct paths in $\mathcal{C}^2_{U_{2,3}}$

a=(1,1,0)   d=(1,1/2,1/2)
b=(1,0,1)   e=(1/2,1,1/2)
c=(0,1,1)   f=(1/2,1/2,1)

Example : $2P_{U_{2,3}}$
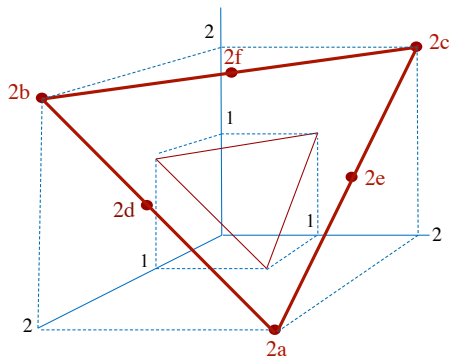
2a=(2,2,0)   2d=(2,1,1)
2b=(2,0,2)   2e=(1,2,1)
2c=(0,2,2)   2f=(1,1,2)

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$\mathcal{C}^k_{U_{2,3}}$

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$\mathcal{C}^k_{U_{2,3}} = \frac{1}{2}(k+1)(k+2)$
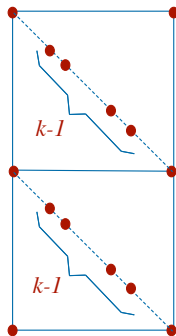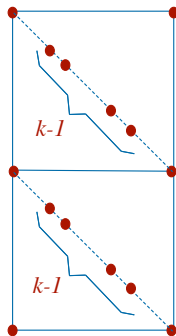
Let us consider $kP_{U_{2,3}}$

$$\mathcal{C}^k_{U_{2,3}} = \tfrac{1}{2}(k+1)(k+2) = \tfrac{1}{2}k^2 + \tfrac{3}{2}k + 1$$

Let us consider $kP_{U_{2,3}}$
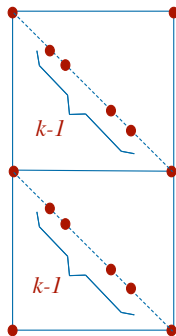
$\mathcal{C}_{U_{2,3}}^{k} = \frac{1}{2}(k+1)(k+2) = \frac{1}{2}k^2 + \frac{3}{2}k + 1 = kP_{U_{2,3}} \cap \mathbb{Z}^3$

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Integer points in LPM polytopes

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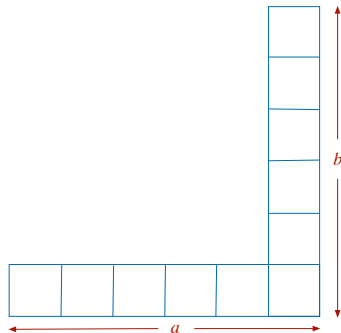
$$L_{S(a,b)}(k) = \sum_{i=0}^{k} \binom{a + k - 1 - i}{a - 1} \binom{b + k - 1 - i}{b - 1}$$

Distributive polytopes

A polytope $P \subseteq \mathbb{R}^n$ is called distributive if for all $x, y \in P$ also their componentwise maximum and minimum $\max(x, y)$ and $\min(x, y)$ are in P.

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Example : A distributive polytope in $\mathbb{R}^2$.
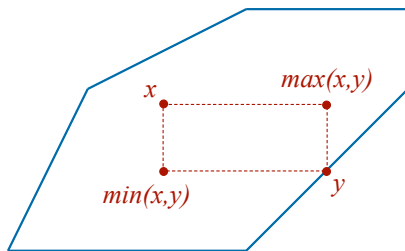
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Let $M = M[U, L]$ be a connected rank r LPM on $r + m$ elements. Then, there exists a bijective affine transformation taking $P_M \subset \mathbb{R}^{r+m}$ into a full-dimensional distributive integer polytope $Q_M \subset \mathbb{R}^{r+m-1}$ such that $L_{P_M}(t) = L_{Q_M}(t)$.

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Proof (idea). Recall that $st(L) = (l_1, \ldots, l_{r+m})$. Check that

$$\pi: \begin{array}{ccc} P_M \subset \mathbb{R}^{r+m} & \longrightarrow & \mathbb{R}^{r+m-1} \\ p = (p_1, \ldots, p_{r+m}) & \mapsto & (p_1 - l_1, \ldots, \sum_{j=1}^{r+m-1}(p_j - l_j)) \end{array}$$

is a suitable transformation.

Example

$P_{U_{2,3}}$
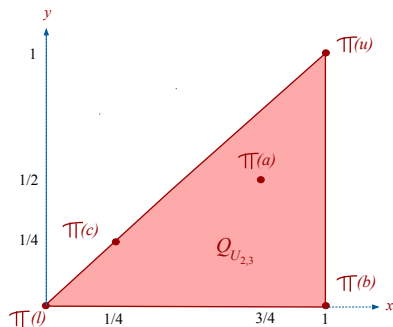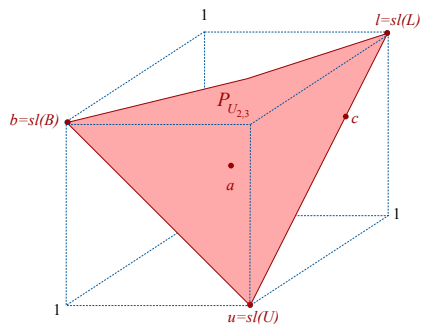
We have $\pi(a) = (\frac{3}{4}, \frac{1}{2})$, $\pi(b) = (1, 0)$ and $\pi(c) = (\frac{1}{4}, \frac{1}{4})$.

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Order polytopes

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The order polytope $\mathcal{O}(X)$ of X is defined as the set of those $x \in \mathbb{R}^n$ such that

$$0 \leq x_i \leq 1, \text{ for all } i \in X \text{ and}$$
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Remark. $\mathcal{O}(X)$ is a bounded convex polytope

Snake polytopes

Snake $S(a_1, a_2, a_3, a_4)$
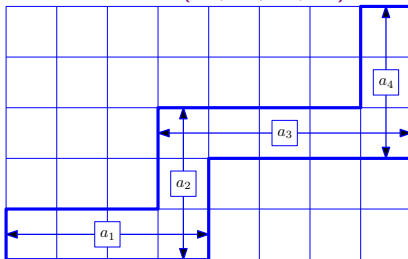
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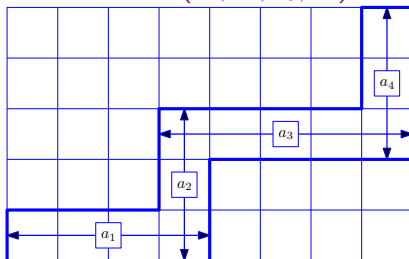
Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $a_1, \ldots, a_k \geq 2$ be integers. Then, a connected LPM M is the snake $S(a_1, \ldots a_k)$ if and only if Q_M is the order polytope of the zig-zag chain poset on $a_1, \ldots, a_k$.

Ehrhart series

The standard d-simplex

$$\begin{aligned} \Delta \quad &= \{\mathbf{x} \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq 1\} \\ &= conv\{(0,\ldots,0),(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\} \end{aligned}$$

$$L_\Delta(t) = \binom{t+d}{d}$$

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This motivate to define the Ehrhart serie of the lattice polytope P as

$$
Ehr_P(z) := 1 + \sum_{t \geq 1} L_P(t) z^t
$$

Ehrhart series

Ehrhart's theorem (Equivalent) For any lattice polytope P of dimension d the Ehrhart serie $Ehr_P(z)$ is a rational function of the form

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- Theorem (Stanley 1980) $h_0^*, \ldots, h_d^*$ are nonnegative integers

Conjecture (De Loera, Haws, Köppe, 2009) The h^*-vector of base matroid polytopes are unimodal, i.e.,

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Theorem (Knauer, Martinez-Sandoval, R.A., 2017)
Let $a, b \geq 2$ be integers. The h^*-vectors of the snake polytopes $P_{S(a,\ldots,a)}$ and $P_{S(a,b)}$ are unimodal.