On the Möbius function of semigroup posets

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Joint work : J.Chappelon, I. García Marco, L.P. Montejano.

Basics on posets

Let $(\mathcal{P}, \leq)$ be a **locally finite poset**, i.e,

- the set $\mathcal{P}$ is partially ordered by $\leq$, and
- for every $a, b \in \mathcal{P}$ the set $\{c \in \mathcal{P} \,|\, a \leq c \leq b\}$ is finite.

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A **chain** of length $l \geq 0$ between $a, b \in \mathcal{P}$ is

$$\{a = a_0 < a_1 < \cdots < a_l = b\} \subset \mathcal{P}.$$

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The **Möbius function** $\mu_{\mathcal{P}}$ is the function

$$\mu_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \longrightarrow \mathbb{Z}$$

$$\mu_{\mathcal{P}}(a, b) = \sum_{l \geq 0} (-1)^l c_l(a, b).$$

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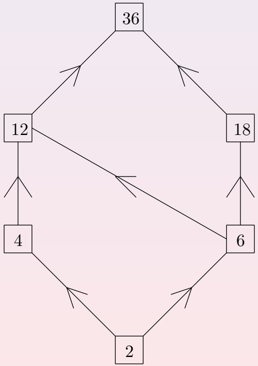
- length 3 $\left\{ \begin{array}{c} \{2,4,12,36\} \\ \{2,6,12,26\} \\ \{2,6,18,36\} \end{array} \right.$



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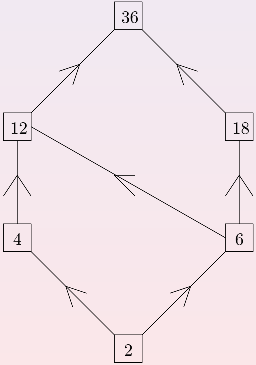

Thus,

$\mu_{\mathbb{N}}(2, 36) = -c_1(2, 36) + c_2(2, 36) - c_3(2, 36) = 1 - 4 + 3 = 0.$

Möbius classical arithmetic function

Given $n \in \mathbb{N}$ the *Möbius arithmetic function* $\mu(n)$ is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with } p_i \text{ distinct primes}, \\ 0 & \text{otherwise (i.e., } n \text{ admits at least one square} \\ & \text{factor bigger than one).} \end{cases}$$

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The inverse of the ζ Riemann function, $s \in \mathbb{C}, Re(s) > 0$

$$\zeta^{-1}(s) = \left(\sum_{n=1}^{+\infty} \frac{1}{n^s} \right)^{-1} = \prod_{p-primes} (1 - p^{-s}) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}$$

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Example: $\mu_{\mathbb{N}}(2,36) = 0$ because $36/2 = 18 = 2 \cdot 3^2$

Möbius inversion formula

Theorem (Rota)

Let $(\mathcal{P}, \leq)$ be a poset, let p be an element of $\mathcal{P}$ and consider $f : \mathcal{P} \to \mathbb{R}$ a function such that $f(x) = 0$ for all $x \not\geq p$. Suppose that

$$g(x) = \sum_{y \leq x} f(y) \text{ for all } x \in \mathcal{P}.$$

Then,

$$f(x) = \sum_{y \leq x} g(y)\, \mu_{\mathcal{P}}(y, x) \text{ for all } x \in \mathcal{P}.$$

Compute the *Euler function* $\phi(n)$ (the number of integers smaller or equal to n and coprime with n)

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Let D be a finite set and consider the **poset of multisets over** D **ordered by inclusion** $\mathcal{P}$. Then, for all A, B multisets over D we have that

$$\mu_{\mathcal{P}}(A, B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subset B \text{ and } B \setminus A \text{ is a set,} \\ 0 & \text{otherwise.} \end{cases}$$

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An immediate consequence is the classical **inclusion-exclusion** counting formula !!

Semigroup poset

Let $\mathcal{S} := \langle a_1, \ldots, a_n \rangle$ denote the **subsemigroup** of $\mathbb{Z}^m$ generated by $a_1, \ldots, a_n \in \mathbb{N}^m$, i.e.,

$$\mathcal{S} := \langle a_1, \ldots, a_n \rangle = \{x_1 a_1 + \cdots + x_n a_n \mid x_1, \ldots, x_n \in \mathbb{N}\}.$$

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The semigroup $\mathcal{S}$ induces a binary relation $\leq_{\mathcal{S}}$ on $\mathbb{Z}^m$ given by

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It turns out that $\leq_{\mathcal{S}}$ is an **order** iff $\mathcal{S}$ is **pointed** (i.e., $\mathcal{S} \cap -\mathcal{S} = \{0\}$). Moreover, whenever $\mathcal{S}$ is pointed the poset $(\mathbb{Z}^m, \leq_{\mathcal{S}})$ is locally finite.

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It is easy to check that $\mu_{\mathcal{S}}(x, y) = \mu_{\mathcal{S}}(0, y - x)$, hence we shall only consider the reduced Möbius function $\mu_{\mathcal{S}} : \mathbb{Z}^m \longrightarrow \mathbb{Z}$ defined by

$$\mu_{\mathcal{S}}(x) := \mu_{\mathcal{S}}(0, x) \quad \text{for all } x \in \mathbb{Z}^m.$$

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Proposition (Key)

If $\mathcal{S}$ is a pointed semigroup, $x \in \mathbb{Z}^m$, then

$$\sum_{b \in \mathcal{S}} \mu_{\mathcal{S}}(x - b) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Known results about $\mu_{\mathcal{S}}$

1. *Deddens* (1979).

   For $\mathcal{S} = \langle a, b \rangle \subset \mathbb{N}$ where $a, b \in \mathbb{Z}^+$ are relatively prime:

$$\mu_{\mathcal{S}}(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \equiv 0 \text{ or } a+b \pmod{ab}, \\ -1 & \text{if } x \geq 0 \text{ and } x \equiv a \text{ or } b \pmod{ab}, \\ 0 & \text{otherwise.} \end{cases}$$

2. *Chappelon and R.A.* (2013).
   - A **recursive formula** for $\mu_{\mathcal{S}}$ when $\mathcal{S} = \langle a, a+d, \ldots, a+kd \rangle \subset \mathbb{N}$ for some $a, k, d \in \mathbb{Z}^+$, and
   - a **semi-explicit formula** for $\mathcal{S} = \langle 2q, 2q+d, 2q+2d \rangle \subset \mathbb{N}$ where $q, d \in \mathbb{Z}^+$ and $\gcd\{2q, 2q+d, 2q+2d\} = 1$.

Goals

1. Provide **general tools** to study $\mu_{\mathcal{S}}$ for every semigroup $\mathcal{S} \subset \mathbb{Z}^m$.

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2. Provide **explicit formulas** for certain families of semigroups $\mathcal{S} \subset \mathbb{Z}^m$.

Multigraded Hilbert series

- Let k be a field. A semigroup $\mathcal{S} = \langle a_1, \ldots, a_n \rangle \subset \mathbb{N}^m$ induces a **grading** in the **ring of polynomials** $k[x_1, \ldots, x_n]$ by assigning $\deg_{\mathcal{S}}(x_i) := a_i$ for all $i \in \{1, \ldots, n\}$.

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- For all $b \in \mathbb{N}^m$, we denote by $k[x_1, \ldots, x_n]_b$ the k-vector space formed by all polynomials $\mathcal{S}$-homogeneous of $\mathcal{S}$-degree b.

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- For all $b \in \mathbb{N}^m$, we denote by $k[x_1, \dots, x_n]_b$ the k-vector space formed by all polynomials $\mathcal{S}$-homogeneous of $\mathcal{S}$-degree b.
- Consider $I \subset k[\mathbf{x}]$ an ideal generated by $\mathcal{S}$-homogeneous polynomials. For all $b \in \mathbb{N}^m$ we denote by I_b the k-vector space formed by the $\mathcal{S}$-homogeneous polynomials of I of $\mathcal{S}$-degree b.

The **(multigraded) Hilbert function** of $M := k[x_1, \ldots, x_n]/I$ is

$$HF_M : \mathbb{N}^m \longrightarrow \mathbb{N},$$

where $HF_M(b) := \dim_k(k[x_1, \ldots, x_n]_b) - \dim_k(I_b)$ for all $b \in \mathbb{N}^m$.

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We define the **(multivariate) Hilbert series** of M as the formal power series in $\mathbb{Z}[[t_1, \ldots, t_m]]$:

$$\mathcal{H}_M(\mathbf{t}) := \sum_{b \in \mathbb{N}^m} HF_M(b)\ \mathbf{t}^b$$

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Theorem

$$\mathcal{H}_M(\mathbf{t}) = \frac{\mathbf{t}^\alpha h(\mathbf{t})}{(1 - \mathbf{t}^{a_1}) \cdots (1 - \mathbf{t}^{a_n})},$$

where $\alpha \in \mathbb{Z}^m$ and $h(\mathbf{t}) \in \mathbb{Z}[t_1, \ldots, t_m]$.

We denote by $I_{\mathcal{S}}$ the **toric ideal** of $\mathcal{S}$, i.e., the kernel of the homomorphism of k-algebras

$$\varphi : k[x_1, \ldots, x_n] \longrightarrow k[t_1, \ldots, t_m]$$

induced by $\varphi(x_i) = \mathbf{t}^{a_i}$ for all $i \in \{1, \ldots, n\}$.

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$$\mathcal{H}_{k[x_1,\ldots,x_n]/I_{\mathcal{S}}}(\mathbf{t}) = \sum_{b \in \mathcal{S}} \mathbf{t}^b.$$

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From now on, we denote $\mathcal{H}_{\mathcal{S}}(\mathbf{t}) := \mathcal{H}_{k[x_1,\ldots,x_n]/I_{\mathcal{S}}}(\mathbf{t})$ and we call it the **Hilbert series of** $\mathcal{S}$.

Möbius function via Hilbert series

Example: For $\mathcal{S} = \langle 2, 3 \rangle \subset \mathbb{N}$, we have that $\mathcal{S} = \{0, 2, 3, 4, 5, \ldots\}$

$$\mathcal{H}_{\mathcal{S}}(t) = 1 + t^2 + t^3 + t^4 + t^5 + \cdots$$

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Then, $(1 - t^2)\,\mathcal{H}_{\mathcal{S}}(t) = 1 + t^3$ and $\mathcal{H}_{\mathcal{S}}(t) = \dfrac{1 + t^3}{1 - t^2}$.

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Theorem (1 (Chappelon, García-Marco, Montejano, R.A. 2014))

Let $a_1,\ldots,a_k$ nonzero vectors of $\mathbb{Z}$ and denote

$$(1-\mathbf{t}^{a_1})\cdots(1-\mathbf{t}^{a_n})\mathcal{H}_\mathcal{S}(\mathbf{t}) = \sum_{b\in\mathbb{Z}^m} f_b \mathbf{t}^b.$$

Then,

$\sum_{b\in\mathbb{Z}^m} f_b\,\mu(x-b) = 0$ *for all* $x \notin \{\sum_{i\in A} a_i \,|\, A \subset \{1,\ldots,n\}\}$.

Example: $\mathcal{S} = \langle 2, 3 \rangle$

We know that,

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It is evident that $\mu_{\mathcal{S}}(0) = 1$. A direct computation yields $\mu_{\mathcal{S}}(2) = -1$.

Hence

$$\mu_{\mathcal{S}}(x) = \begin{cases} 1 & \textit{if } x \equiv 0 \textit{ or } 5 \ (\textit{mod } 6), \\ -1 & \textit{if } x \equiv 2 \textit{ or } 3 \ (\textit{mod } 6), \\ 0 & \textit{otherwise}. \end{cases}$$

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Semigroup $\mathbb{N}^m$

Let $\mathcal{S} = \langle e_1, \ldots, e_m \rangle$ where $\{e_1, \ldots, e_m\}$ denote the canonical basis of $\mathbb{N}^m$. Then,

$$\mu_{\mathcal{S}}(x) = \begin{cases} (-1)^{|A|} & \text{if } x = \sum_{i \in A} e_i \text{ for some } A \subset \{1, \ldots, m\} \\ 0 & \text{otherwise} \end{cases}$$

Proof #1.

We observe that $\mathbb{N}^m = \mathcal{S}$ and thus

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \sum_{(b_1, \ldots, b_m) \in \mathbb{N}^m} t_1^{b_1} \cdots t_m^{b_m} = \frac{1}{(1-t_1)\cdots(1-t_m)}.$$

By Theorem (1) we have that $\mu_{\mathbb{N}^m}(x) = 0$ for all $x \notin \{\sum_{i \in A} e_i \mid A \subset \{1, \ldots, m\}\}$.

A direct computation yields $\mu_{\mathbb{N}^m}(\sum_{i \in A} e_i) = (-1)^{|A|}$ for every $A \subset \{1, \ldots, m\}$. □

Möbius function via Hilbert series

We consider $\mathcal{G}_{\mathcal{S}}$ the **generating function of the Möbius function,** which is

$$\mathcal{G}_{\mathcal{S}}(\mathbf{t}) := \sum_{b \in \mathbb{N}^m} \mu_{\mathcal{S}}(b)\, \mathbf{t}^b.$$

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Theorem (2 (Chappelon, García-Marco, Montejano, R.A. 2014))

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t})\, \mathcal{G}_{\mathcal{S}}(\mathbf{t}) = 1.$$

Again semigroup $\mathcal{S} = \mathbb{N}^m$

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Proof #2.

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \sum_{(b_1,\ldots,b_m)\in\mathbb{N}^m} t_1^{b_1}\cdots t_m^{b_m} = \frac{1}{(1-t_1)\cdots(1-t_m)}.$$

By Theorem (2) we have that

$$\mathcal{G}_{\mathcal{S}}(\mathbf{t}) = (1-t_1)\cdots(1-t_m) = \sum_{A\subset\{1,\ldots,m\}} (-1)^{|A|}\,\mathbf{t}^{\sum_{i\in A} e_i}.$$

□

Semigroups with unique Betti element

A semigroup $\mathcal{S} \subset \mathbb{N}^m$ is said to be a **semigroup with a unique Betti element** $b \in \mathbb{N}^m$ if $I_{\mathcal{S}}$ is generated by polynomials of $\mathcal{S}$-degree b.

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We denote $d := \dim(\mathbb{Q}\{a_1, \ldots, a_n\})$. In this setting we have the following result.

Theorem (Chappelon, García-Marco, Montejano, R.A. 2014)

For $\mathcal{S} = \langle a_1, \ldots, a_m \rangle \subset \mathbb{N}^m$ with a unique Betti element b

$$\mu_{\mathcal{S}}(x) = \sum_{j=1}^{t} (-1)^{|A_j|} \binom{k_j + n - d - 1}{k_j},$$

if $x = \sum_{i \in A_1} a_i + k_1\, b = \cdots = \sum_{i \in A_t} a_i + k_t\, b$.

Numerical semigroups with unique Betti element

When $\mathcal{S} = \langle a_1, \ldots, a_n \rangle \subset \mathbb{N}$ is a **semigroup with a unique Betti element** and $\gcd\{a_1, \ldots, a_n\} = 1$, it is known that there exist pairwise relatively prime different integers $b_1, \ldots, b_n \geq 2$ such that $a_i := \prod_{j \neq i} b_j$ for all $i \in \{1, \ldots, n\}$.

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In this setting we can refine the previous Theorem to obtain the following result.

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In this setting we can refine the previous Theorem to obtain the following result.

Corollary (Chappelon, García-Marco, Montejano, R.A. 2014)

Set $b := \prod_{i=1}^{n} b_i$, *then*

$$\mu_{\mathcal{S}}(x) = \begin{cases} (-1)^{|A|} \binom{k+n-2}{k} & \text{if } x = \sum_{i \in A} a_i + k\, b \\ & \text{for some } A \subset \{1, \ldots, n\} \\ 0 & \text{otherwise} \end{cases}$$

3-generated numerical semigroups complete intersection

Whenever $\mathcal{S} := \langle a_1, a_2, a_3 \rangle \subset \mathbb{N}$ with $\gcd\{a_1, a_2, a_3\} = 1$, we say that $\mathcal{S}$ is a **complete intersection** if there exists two $\mathcal{S}$-homogeneous polynomials f_1, f_2 such that $I_{\mathcal{S}} = (f_1, f_2)$.

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Theorem (Herzog (1970))

$\mathcal{S}$ is a complete intersection $\Longleftrightarrow$ $\gcd\{a_i, a_j\} a_k \in \langle a_i, a_j \rangle$, *where* $\{i, j, k\} = \{1, 2, 3\}$.

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We aim at presenting a formula for $\mathcal{S} = \langle a_1, a_2, a_3 \rangle \subset \mathbb{N}$ complete intersection and $\gcd\{a_1, a_2, a_3\} = 1$, so we assume that $d a_1 \in \langle a_2, a_3 \rangle$, where $d := \gcd\{a_2, a_3\}$.

3-generated numerical semigroups complete intersection

For every $x \in \mathbb{Z}$ and every $B = (b_1, \ldots, b_k) \subset (\mathbb{Z}^+)^k$, the **Sylvester denumerant** $d_B(x)$ is the number of non-negative integer solutions $(x_1, \ldots, x_k) \in \mathbb{N}^k$ to the equation $x = \sum_{i=1}^k x_i b_i$.

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For every $x \in \mathbb{Z}$ we denote by $\alpha(x)$ the only integer such that $0 \leq \alpha(x) \leq d - 1$ and $\alpha(x)\, a_1 \equiv x \pmod d$.

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For $\mathcal{S} = \langle a_1, a_2, a_3 \rangle$ complete intersection and $\gcd\{a_1, a_2, a_3\} = 1$, we have the following result.

Theorem (Chappelon, García-Marco, Montejano, R.A. 2014)

$\mu_{\mathcal{S}}(x) = 0$ *if* $\alpha(x) \geq 2$, *or*
$\mu_{\mathcal{S}}(x) = (-1)^{\alpha} \left(d_B(x') - d_B(x' - a_2) - d_B(x' - a_3) + d_B(x' - a_2 - a_3) \right)$
otherwise, where $x' := x - \alpha(x)\, a_1$ *and* $B := (da_1, a_2\, a_3/d)$.

How general are semigroup posets?

Let $D = \{d_1, \ldots, d_m\}$ be a finite set and let us consider $(\mathcal{P}, \subset)$, the **poset of all multisets of** D **ordered by inclusion**.

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For the semigroup $\mathcal{S} := \mathbb{N}^m$, we consider the map

$$\begin{array}{rcl} \psi : \ (\mathcal{P}, \subset) & \to & (\mathbb{N}^m, \leq_{\mathbb{N}^m}) \\ A & \mapsto & (m_A(d_1), \ldots, m_A(d_m)), \end{array}$$

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where $m_A(d_i)$ denotes the number of times that d_i belongs to A.

It is easy to check that ψ is an **order isomorphism** (an order preserving and order reflecting bijection).

How general are semigroup posets?

Hence,

$$\mu_{\mathcal{P}}(A, B) = \mu_{\mathbb{N}^m}(\psi(A), \psi(B)),$$

and we can recover the formula for $\mu_{\mathcal{P}}$ by means of $\mu_{\mathbb{N}^m}$.

$$\mu_{\mathcal{P}}(A, B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subset B \text{ and } B \setminus A \text{ is a set} \\ 0 & \text{otherwise} \end{cases}$$

Take $p_1, \ldots, p_m$ the m first prime numbers, and consider

$$\mathbb{N}_m := \{p_1^{\alpha_1} \cdots p_m^{\alpha_m} \mid \alpha_1, \ldots, \alpha_m \in \mathbb{N}\} \subset \mathbb{N}.$$

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Hence,

$$\mu_{\mathbb{N}_m}(a, b) = \mu_{\mathbb{N}^m}(\psi(a), \psi(b)),$$

and we can recover the formula for $\mu_{\mathbb{N}_m}$ by means of $\mu_{\mathbb{N}^m}$.

$$\mu_{\mathbb{N}_m}(a, b) = \begin{cases} (-1)^r & \text{if } b/a \text{ is a product of } r \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$