Apollonian packings, polytopes and Descartes-type identities

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Joint work with I. Rasskin

“Celebrating the 70th birthday of Luis Montejano”

Mexico, October 15th, 2021

Apollonius of Perga

260 BC. - 190 BC. (70 years)

- Known as "the great geometer"
- His famous book *Conics* introduced the terms of hyperbola y ellipse.

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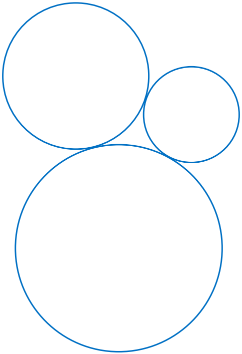

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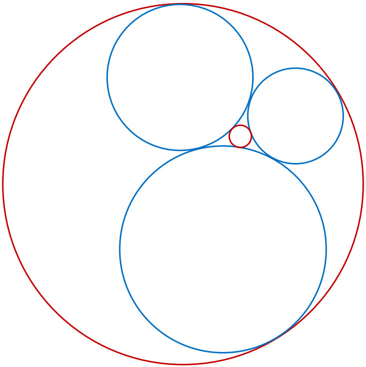

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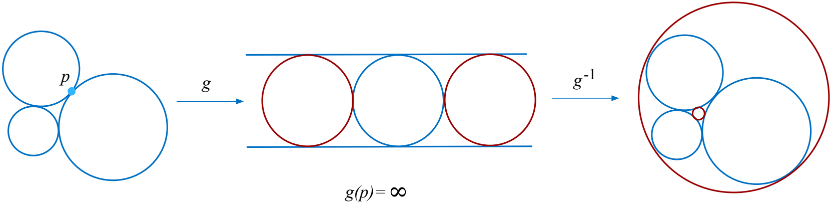
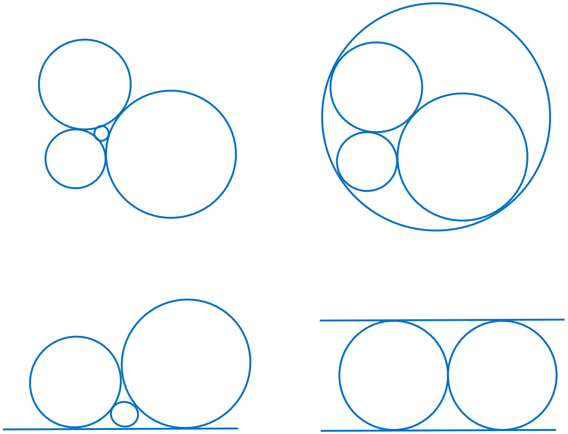
Proof (idea) : We take g a Möbius transformation



Differents representations

Possibles representations of 4 circles pairwise tangent



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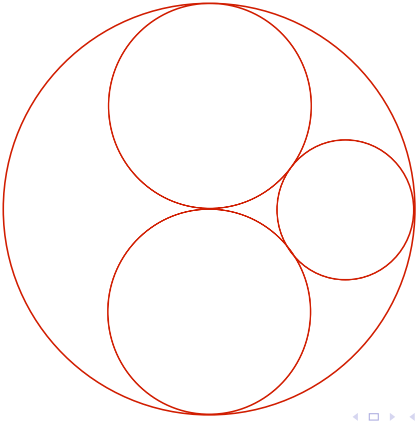

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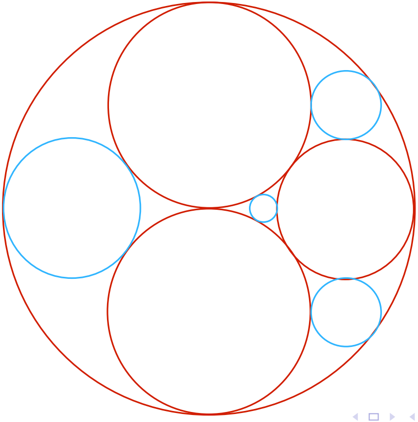

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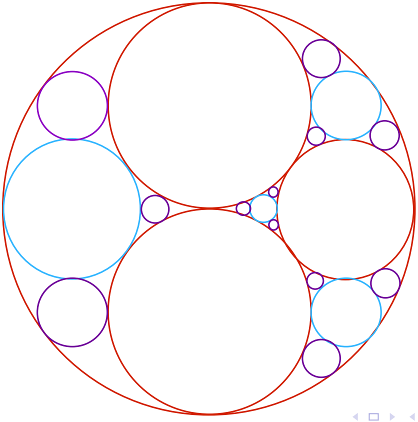
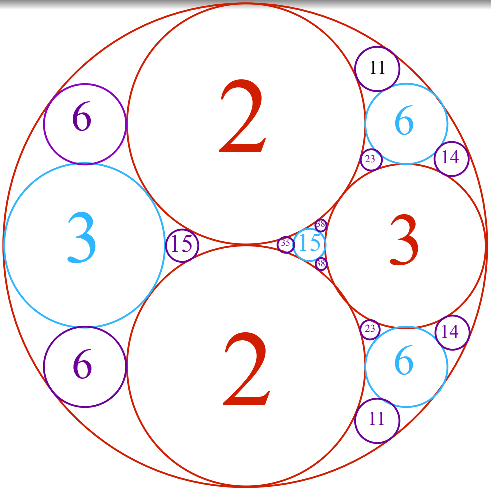


Curvatures



Each circle is labeled with its curvature : $curv(C) = \frac{1}{radius(C)}$
The curvature of the exterior circle is -1 (oriented to the exterior so interiors are dijoints).

Descartes' theorem

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Theorem (Descartes 1643) a, b, c, d are the curvatures of four pairwise tangent circles if and only if they verify the quadratic equation $2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$

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The proof written in a letter to the Princess Elisabeth of Bohemia

Related results

Theorem (Soddy 1936) If the first four circles of an Apollonian packing $\mathcal{P}$ have integer curvatures then $\mathcal{P}$ is integral.

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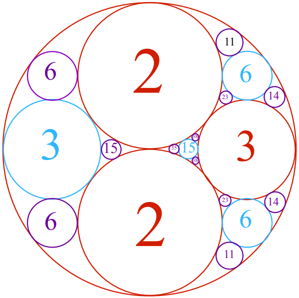
Example

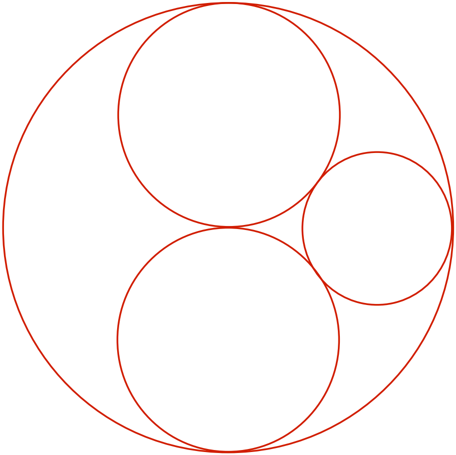
$$2((-1)^2 + 2^2 + 2^2 + 3^2) = 36 = (-1 + 2 + 2 + 3)^2$$

$$2(2^2 + 6^2 + 3^2 + 23^2) = 1156 = (2 + 6 + 3 + 23)^2$$



Packings by using inversions

From a Tetrahedron



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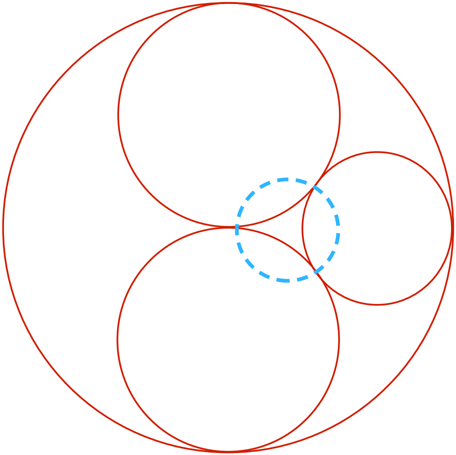

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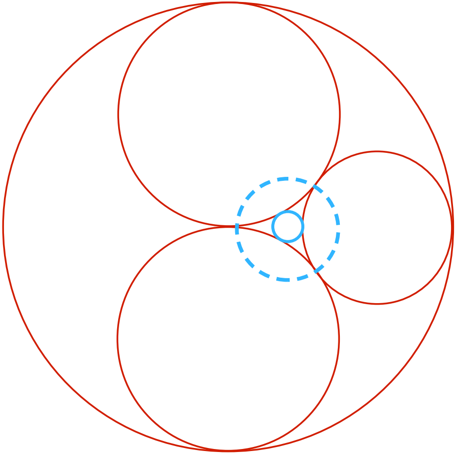

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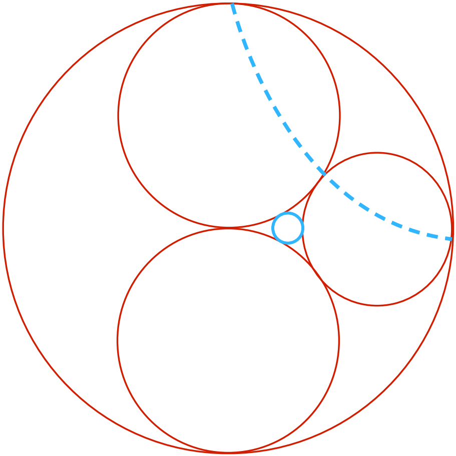

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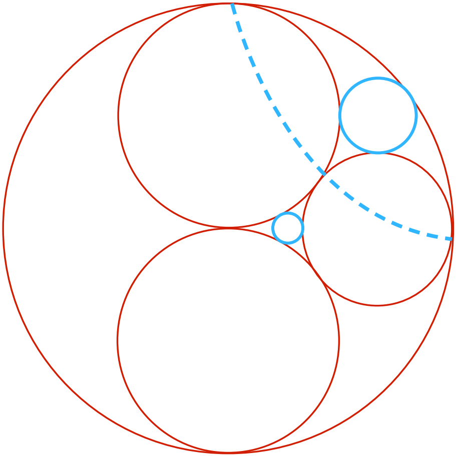
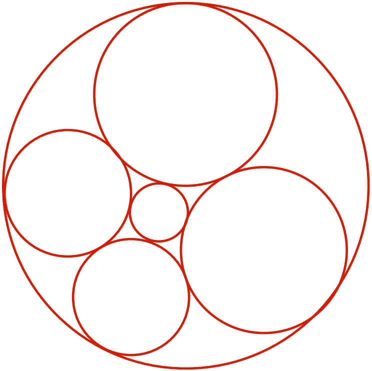


Other packings by using inversions

From an Octahedron



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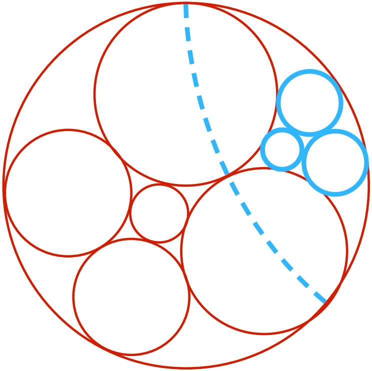

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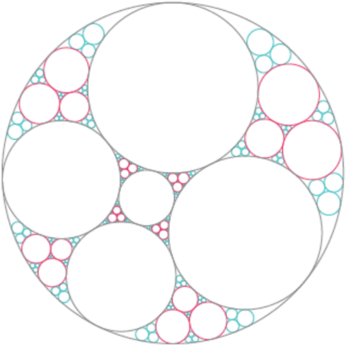
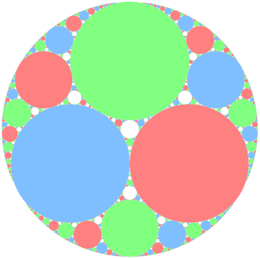
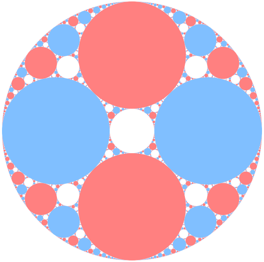
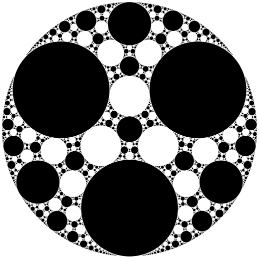


Three packings

From a Tetrahedron an Octahedron and a Cube







Figures done by a software created by I. Rasskin

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Question 3 Are there others Descartes-type identities ?

A bit of Lorenzian's theory

Let $d \geq 1$. The *Lorentzian space* $\mathbb{L}^{d+1,1}$, of dimension $d+2$, is the vector espace of dimension $d+2$ endowed with *Lorentzian product*

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 y_1 + \cdots + x_{d+1} y_{d+1} - x_{d+2} y_{d+2}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{L}^{d+1,1}$$

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The vector $\boldsymbol{v} \in \mathbb{L}^{d+1,1}$ is called *space-like* if $\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0$ and *normalized* if $|\langle \boldsymbol{v}, \boldsymbol{v} \rangle| = 1$.

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• The *Möbius group* $\text{Möb}(\hat{\mathbb{R}}^d)$ is the group generated by the inversions of d-balls.

Arrangement of balls

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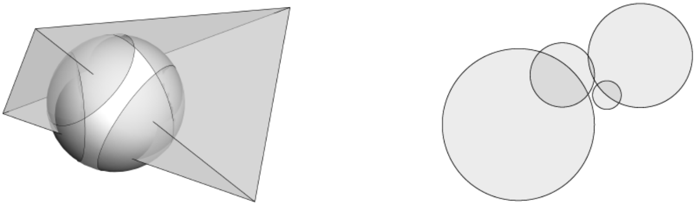
The *arrangement of balls projected* from P, $B(P)$, is the collection of d-balls where the *source light* are the vertices of P.



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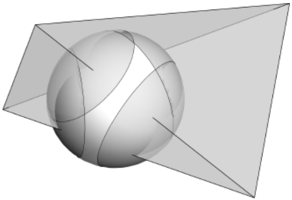

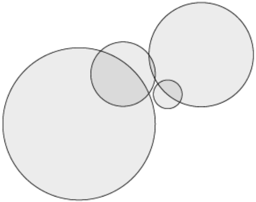

If P is a $(d+1)$-polytope *edge-inscribed* (i.e., all the edges of P are tangent to $\mathbb{S}^d$) then $B(P)$ is a *packing of d-balls B_P*.

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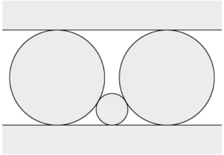

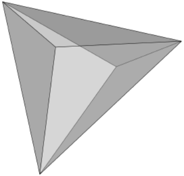

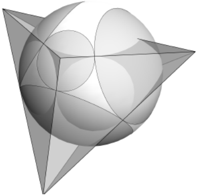

Central projections of packings

Take an edge-inscribed realisation of a regular $(d+1)$-polytope. A *central projection* is the colection of projections of the balls with one k-face which barycentre is in the ray generated from the North pole.

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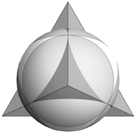
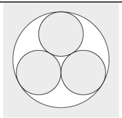
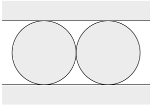
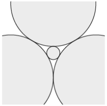
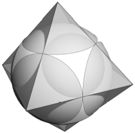
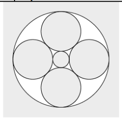
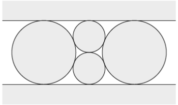
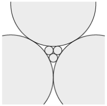
Edge-scribed realization	Vertex centered at ∞	Edge centered at ∞	Face centered at ∞
			
$\ell_{\{3,3\}} = \sqrt{2}$	(table A)	(table B)	(table C)
			
$\ell_{\{3,4\}} = 1$	(table D)	(table E)	(table F)

Table A:

n	κ
1	$\frac{1}{\sqrt{2}}(1-\sqrt{3})$
3	$\frac{1}{\sqrt{2}}(1+\sqrt{1/3})$

Table B:

n	κ
2	0
2	$\sqrt{2}$

Table C:

n	κ
3	$\frac{1}{\sqrt{2}}(1-\sqrt{1/3})$
1	$\frac{1}{\sqrt{2}}(1+\sqrt{3})$

Table D:

n	κ
1	$1-\sqrt{2}$
4	1
1	$1+\sqrt{2}$

Table E:

n	κ
2	0
2	1
2	2

Table F:

n	κ
3	$1-\sqrt{2/3}$
3	$1+\sqrt{2/3}$

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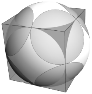
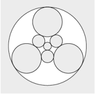
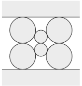


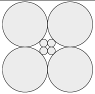
$\ell_{\{4,3\}} = \frac{1}{\sqrt{2}}$



n	κ
1	$\sqrt{2}-\sqrt{3}$
3	$\sqrt{2}-\sqrt{1/3}$
3	$\sqrt{2}+\sqrt{1/3}$
1	$\sqrt{2}+\sqrt{3}$

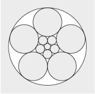


n	κ
2	0
4	$\sqrt{2}$
2	$2\sqrt{2}$

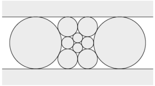


n	κ
4	$\sqrt{2}-1$
4	$\sqrt{2}+1$

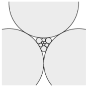
$\ell_{\{3,5\}} = \frac{1}{\varphi}$



n	κ
1	$\varphi-\sqrt{\varphi+2}$
5	$\varphi-\sqrt{\frac{\varphi+2}{5}}$
5	$\varphi+\sqrt{\frac{\varphi+2}{5}}$
1	$\varphi+\sqrt{\varphi+2}$

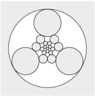


n	κ
2	0
2	$\varphi-1$
4	φ
2	$\varphi+1$
2	2φ

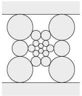


n	κ
3	$\varphi-\varphi^2\sqrt{1/3}$
3	$\varphi-\varphi^{-1}\sqrt{1/3}$
3	$\varphi+\varphi^{-1}\sqrt{1/3}$
3	$\varphi+\varphi^2\sqrt{1/3}$

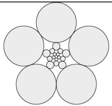
$\ell_{\{5,3\}} = \frac{1}{\varphi^2}$



n	κ
1	$\varphi^2-\varphi\sqrt{3}$
3	$\varphi^2-\varphi\sqrt{5/3}$
6	$\varphi^2-\varphi\sqrt{1/3}$
6	$\varphi^2+\varphi\sqrt{1/3}$
3	$\varphi^2+\varphi\sqrt{5/3}$
1	$\varphi^2+\varphi\sqrt{3}$



n	κ
2	0
4	$\varphi^2-\varphi$
2	φ^2-1
4	φ^2
2	φ^2+1
4	$\varphi^2+\varphi$
2	$2\varphi^2$



n	κ
5	$\varphi^2-\sqrt{\frac{1}{5}(7+11\varphi)}$
5	$\varphi^2-\sqrt{\frac{1}{5}(3-\varphi)}$
5	$\varphi^2+\sqrt{\frac{1}{5}(3-\varphi)}$
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- The union of $B_P \cup B_P^*$ is called *primal-dual representation* of P.

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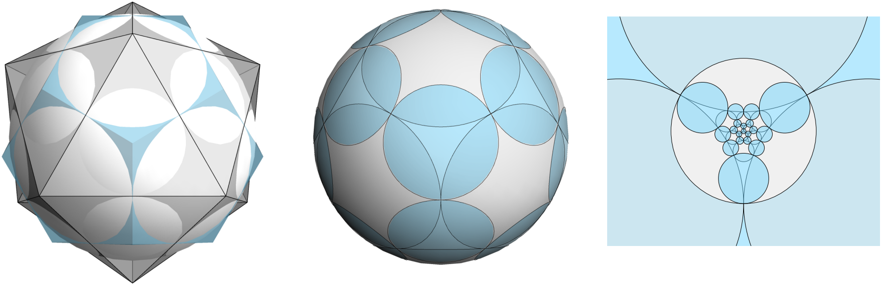
An edge-inscribed icosahedron and its polar (in blue).



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Proof (idea) : B is Möbius equivalent to B' iff $Gram(B) = Gram(B')$ where

$$Gram(B) = \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_n \rangle \\ \vdots & \ddots & \vdots \\ \langle v_n, v_1 \rangle & \cdots & \langle v_1, v_n \rangle \end{pmatrix}$$

$B = \{v_1, \ldots v_n\} \subset \mathbb{L}^{d+1,1}$

Let B_P be a polytopal d-balls packing.
The *Symmetric* group : $Sym(B_P) := \langle \mu \in \text{Möb}(\hat{\mathbb{R}}) | \mu(B_P) = B_P \rangle$

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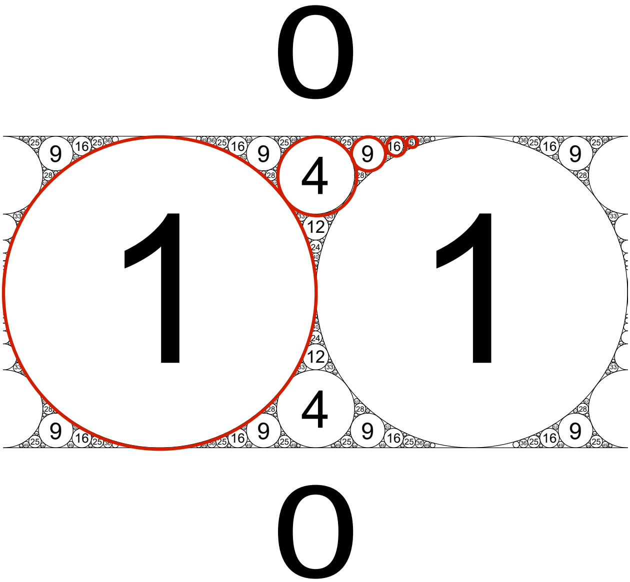
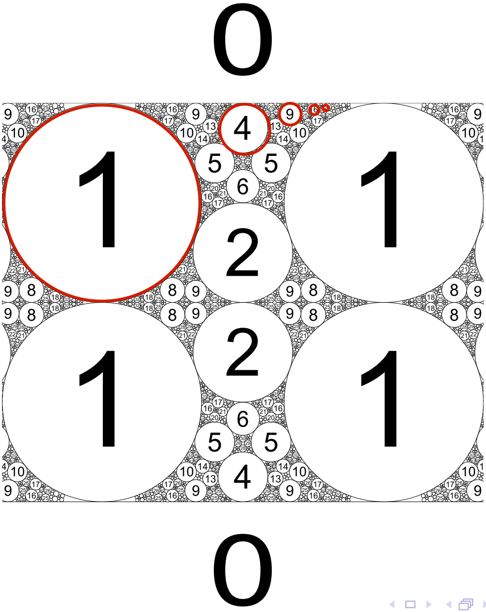
Theorem (Rasskin + R.A. 2021)
There exist tetrahedral, cubic and dodechaedral Apollonian packings with curvatures containing all the perfect squares.

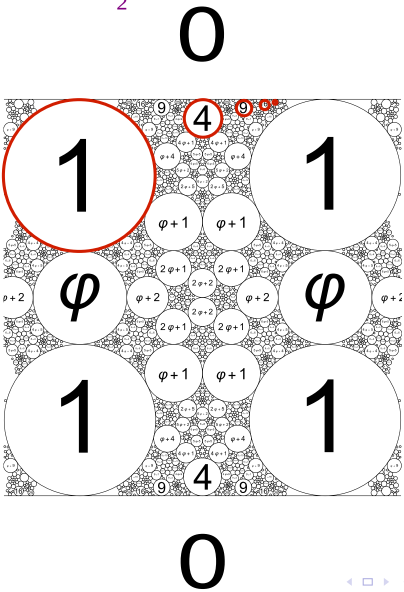
Tetrahedral packing



Cubic packing



φ is the golden number $\frac{1+\sqrt{5}}{2}$



For all $\boldsymbol{x} \in \mathbb{L}^{d+1,1}$, we define

$$\kappa(\boldsymbol{x}) = -\langle \boldsymbol{x}_N, \boldsymbol{x} \rangle$$

where $\boldsymbol{x}_N = (e_{d+1} + e_{d+2})$ with e_i canonical vector of $\mathbb{L}^{d+1,1}$.

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Let $P \subset E^{d+1}$ be sphere-exterior. The *Lorentzian barycenter* of P is

$$\boldsymbol{x}_P := \frac{1}{|\mathcal{F}_0(P)|} \sum_{v \in \mathcal{F}_0(P)} \boldsymbol{x}_{b(v)}$$

where $b(v)$ is the illuminated region from vertex v.

Lorentzian curvature of polytopes

The *Lorentzian curvature* of P is defined as

$$\kappa_P := \kappa(\boldsymbol{x}_P)$$

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By linearity, we have that

$$\kappa_P := \frac{1}{|\mathcal{F}_0(P)|} \sum_{v \in \mathcal{F}_0(P)} \kappa(b(v))$$

Theorem (Rasskin + R.A. 2021) Let $d \geq 1$. We take an edge-scribed realization of a regular $(d+1)$-polytope P with Schläfli's symbols $\{p_1, \ldots, p_d\}$. The Lorentzianas' curvatures of the *flag* $(f_0, f_1, \ldots, f_d, f_{d+1} = P)$ verifies

$$\kappa_P^2 = L_P(d+1) \sum_{i=0}^{d} \frac{(\kappa_{f_i} - \kappa_{f_{i+1}})^2}{L_P(i+1) - L_P(i)}$$

con

$$L_P(i) := \begin{cases} -1 & i = 0 \\ 0 & i = 1 \\ \ell_{\{p_1,\ldots,p_{i-1}\}}^{-2} & \text{if } 2 \leq i \leq d+1 \end{cases}$$

where $\ell_{\{p_1,\ldots,p_{i-1}\}}$ is the half of the edge length of the edge-scribed realization of the regular face f_i.

Integral Octahedral packing

Proposition (Rasskin + R.A. 2021) Let $\kappa_1, \kappa_2, \kappa_3$ be the curvatures of three discs pairwise tangent in a octahedral polytopal packing B_O. If $\kappa_1, \kappa_2, \kappa_3$ and $\sqrt{2(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3)}$ are integers then the Apollonian packing generated from B_O is integral.

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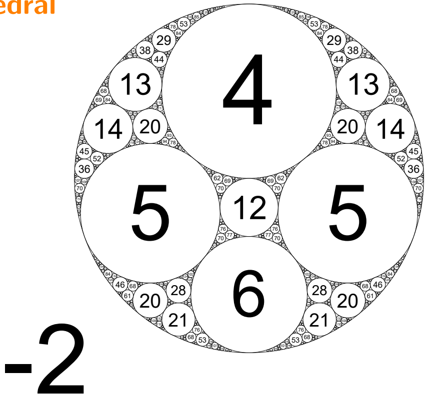
Octahedral



Integral Cubic packing

Proposición (Rasskin + R.A. 2021) Let $\kappa_{i-1}, \kappa_i, \kappa_{i+1}$ be the curvatures of three discs consecutively tangent in a cubic polytopal packing B_C. If $\kappa_{i-1}, \kappa_i, \kappa_{i+1}$ and $\sqrt{-\kappa_i^2 + \kappa_i\kappa_{i+1} + \kappa_i\kappa_{i-1} + \kappa_{i-1}\kappa_{i+1}}$ are integers then the Apollonian packing generated from B_C is integral.

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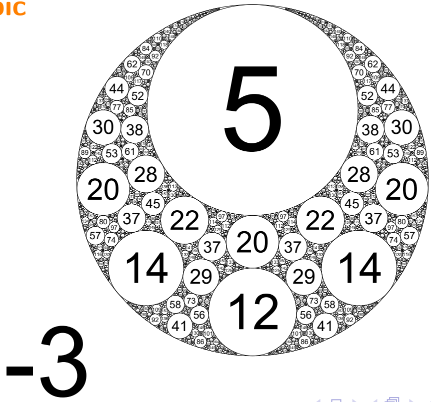

Integral Icosahedral packing

Proposition (Rasskin + R.A. 2021) Let $\kappa_1, \kappa_2, \kappa_3$ be the curvatures of three discs pairwise tangent in a icosahedral polytopal packing B_I. If $\kappa_1, \kappa_2, \kappa_3$ and $\sqrt{\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3}$ are in $\mathbb{Z}[\varphi]$ then the Apollonian packing generated from B_I is φ-integral.

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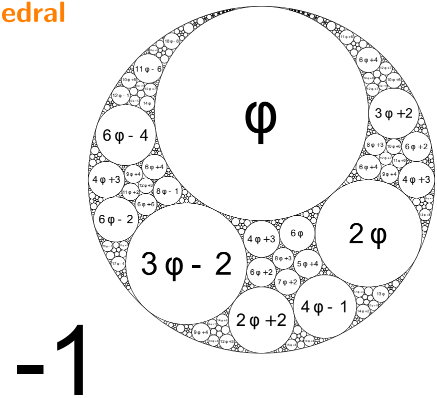
Icosahedral



Integral Dodecahedral packing

Proposition (Rasskin + R.A. 2021) Let $\kappa_{i-1}, \kappa_i, \kappa_{i+1}$ be the curvatures of three discs consecutively tangent in a dodecahedral polytopal packing B_D. If $\kappa_{i-1}, \kappa_i, \kappa_{i+1}$ and $\sqrt{-\varphi^2\kappa_i^2 + \kappa_i\kappa_{i+1} + \kappa_i\kappa_{i-1} + \kappa_{i-1}\kappa_{i+1}}$ are in $\mathbb{Z}[\varphi]$ then the Apollonian packing generated from B_D es φ-integral.

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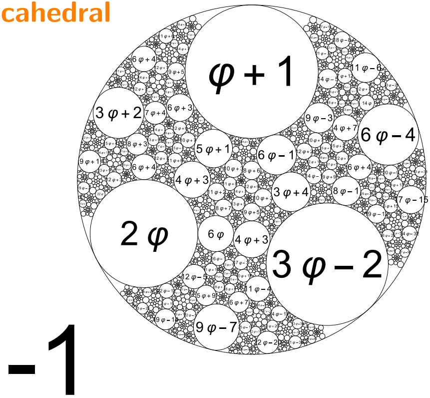
Dodecahedral



Thanks

for your attention

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CUMPLEAÑOS