Cubic graphs, their Ehrhart quasi-polynomials, and a scissors congruence phenomenon*

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Abstract

The scissors congruence conjecture for the unimodular group is an analogue of Hilbert's third problem, for the equidecomposability of polytopes. Liu and Osserman studied the Ehrhart quasi-polynomials of polytopes naturally associated to graphs whose vertices have degree one or three. In this paper, we prove the scissors congruence conjecture, posed by Haase and McAllister, for this class of polytopes. The key ingredient in the proofs is the nearest neighbor interchange (NNI) move on graphs and a naturally arising piecewise unimodular transformation. We provide a generalization of the context in which the NNI moves appear, to connected graphs with the same degree sequence. We also show that, up to a dilation factor of 4, and an integer translation, all of these Liu-Osserman polytopes are reflexive.

1 Introduction

A *cubic graph* is a graph whose vertices have degree three and a $\{1,3\}$-*graph* is a graph whose vertices have degree either one or three. The graphs are allowed to have loops and parallel edges. Motivated by a result of Mochizuki [14], Liu and Osserman [13] associated a polytope $\mathcal{P}_G$ to each $\{1,3\}$-graph G and studied its Ehrhart quasi-polynomial.

For each degree three vertex v of a $\{1,3\}$-graph $G=(V,E)$, let a, b, and c be the three edges incident to v. Denote by $S(v)$ the linear system consisting of a perimeter inequality and three metric inequalities defined on the variables w_a, w_b, and w_c as follows:

$$w_a + w_b + w_c \leq 1$$

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$$\begin{aligned} w_a &\leq w_b + w_c \\ w_b &\leq w_a + w_c \\ w_c &\leq w_a + w_b. \end{aligned}$$

From the three metric inequalities, one can immediately conclude that w_a, w_b, and w_c are nonnegative. Consider the union of all the linear systems $S(v)$, taken over all degree three vertices v of G. The polytope $\mathcal{P}_G$ is defined by the set of all real solutions for this linear system.

Let $E = \{1, \ldots, m\}$ and $w : E \to \mathbb{R}$ be a weight function defined on the edges of G. We use the vector notation $w = (w_1, \ldots, w_m) \in \mathbb{R}^m$. In particular, when w is a solution for the linear system defining $\mathcal{P}_G$, we write $w \in \mathcal{P}_G$.

Given a rational polytope $\mathcal{P}$, Eugène Ehrhart defined the function $L_{\mathcal{P}}(t) := |t\mathcal{P} \cap \mathbb{Z}^m|$, which is the number of lattice points in the closed *dilated polytope* $t\mathcal{P}$, for a nonnegative integer parameter t. Ehrhart showed that this function is a polynomial in t when $\mathcal{P}$ is an integral polytope. More generally, if $\mathcal{P}$ is a rational polytope, the function $L_{\mathcal{P}}(t)$ is a quasi-polynomial whose period is closely related to the denominators appearing in the coordinates of the vertices of $\mathcal{P}$ [6, 2].

Liu and Osserman conjectured ([13, Conj. 4.2]) that polytopes associated to connected $\{1, 3\}$-graphs with the same number of vertices and edges have the *same* Ehrhart quasi-polynomial. They partially proved their conjecture, by showing that these quasi-polynomials coincide for all nonnegative odd values of the dilation parameter t. In 2013, Wakabayashi [18, Thm. A(ii)] proved their conjecture.

An ingredient in Wakabayashi's proof for Liu and Osserman's conjecture is a local transformation performed in $\{1, 3\}$-graphs. This transformation is called an A-move by Wakabayashi and it is also known as a *nearest neighbor interchange* (NNI).

We present the following general result for connected graphs with the same degree sequence, which might be of interest on its own. We refer to a vertex of degree one in a graph simply as a *leaf*. An edge is *external* if it is incident to a leaf, otherwise it is *internal*.

Theorem 1. *Let G and G' be connected graphs with the same degree sequence and the same set of external edges. Then*

(a) G can be transformed into G' through a series of NNI moves.

(b) One can choose a spanning tree in G and a spanning tree in G' and require that all the pivots of the NNI moves are internal edges of both of these spanning trees.

In particular, for the special case of $\{1, 3\}$-graphs, the proof provided for Theorem 1(a) constitutes an alternative graph theoretic proof of a proposition by Wakabayashi [18, Prop. 6.2].

The NNI has been studied mainly for binary trees [4, 15], cubic graphs [17], and $\{1, 3\}$-graphs [18]. Culik and Wood [4, Thm. 2.4] used the shortest length of a series of NNI moves transforming a $\{1,3\}$-tree into another $\{1,3\}$-tree with the same number of leaves as a measure of distance between the two trees. This has relevance for instance in the study of phylogenetic networks [10], where some graphs with circuits are being compared by a similar metric. Theorem 1 suggests a distance based on NNI moves to a more broad class of graphs.

Another possible use for Theorem 1 is of a more structural flavour. It is conceivable that Theorem 1 can be used to prove invariants on graphs with the same degree sequence. For instance, one can use Theorem 1 to prove that all connected graphs with the same degree sequence have the same number of Eulerian subgraphs.

One of the topics in this paper is the scissors congruence conjecture for the unimodular group, which is an analogue of Hilbert's third problem (equidecomposability). Concretely, this was stated as the following question by Haase and McAllister [8]. An integral matrix U is *unimodular* if it has determinant ± 1. An *affine unimodular transformation* is defined by $x \to Ux + b$, where U is a unimodular matrix and b is a real vector.

Question 2. [8, Question 4.1] *Suppose that $\mathcal{P}$ and $\mathcal{P}'$ are polytopes with the same Ehrhart quasi-polynomial. Is it true that there is a decomposition of $\mathcal{P}$ into relatively open simplices $\mathcal{P}^1, \ldots, \mathcal{P}^k$ and affine unimodular transformations $U^1, \ldots, U^k$ such that $\mathcal{P}'$ is the disjoint union of $U^1(\mathcal{P}^1), \ldots, U^r(\mathcal{P}^r)$?*

We show that for polytopes associated to $\{1,3\}$-graphs such a scissors congruence decomposition holds. Namely, we have the following.

Theorem 3. *Let G and G' be two connected $\{1,3\}$-graphs with the same number of vertices and edges. Then there is a dissection of $\mathcal{P}_G$ into smaller polytopes $\mathcal{P}_G^1, \ldots, \mathcal{P}_G^k$ and affine unimodular transformations $U^1, \ldots, U^k$ such that $\mathcal{P}_{G'}$ is the union of $U^1(\mathcal{P}_G^1), \ldots, U^k(\mathcal{P}_G^k)$.*

The proof of Theorem 3 relies on a piecewise unimodular transformation associated to a weighted version of the NNI move.

The rational polytope $\mathcal{P}_G$, associated to a $\{1,3\}$-graph G, enjoys some fascinating symmetry. Linke [12] considered the extension of $L_{\mathcal{P}}(t)$ for all nonnegative real numbers t. Royer [16] defined a polytope to be *semi-reflexive* if $L_{\mathcal{P}}(s) = L_{\mathcal{P}}(\lfloor s \rfloor)$ for every nonnegative real number s. One can verify that $\mathcal{P}_G$ is semi-reflexive. A polytope is *reflexive* if it is integral, the origin is in its interior, and it is semi-reflexive [2]. We prove the following.

Theorem 4. *For each $\{1,3\}$-graph G, the polytope $4\mathcal{P}_G - \mathbb{1}$ is reflexive.*

The paper is organized as follows. Section 2 contains the results involving the NNI move in graphs with the same degree sequence, including Theorem 1, while Section 3 discusses the extension of the NNI move to weighted graphs. Section 4 describes the unimodular decomposition of $\mathcal{P}_G$ and presents the proof of Theorem 3. In Section 5, we prove Theorem 4. Finally, Section 6 contains some concluding remarks.

2 Nearest neighbor interchange

With an eye towards Section 3, where we consider weighted graphs, we think of a graph as defined by Bondy and Murty [3]. A graph G is an ordered pair (V, E) consisting of a set V of vertices and a set E, disjoint from V, of edges, together with an *incidence function* ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. The *degree* of a vertex is the number of edges incident with the vertex, with loops counted twice. The *degree sequence* of G is the monotonic nonincreasing sequence consisting of the degrees of its vertices. A *trail* is a sequence $v_1e_1v_2e_2 \cdots v_ke_kv_{k+1}$, where each v_i is a vertex and each e_i is an edge whose ends are v_i and v_{i+1}, and in which no edge is repeated. However, some vertices of the trail may be repeated.

A *nearest neighbor interchange (NNI)* is a local move performed in G on a trail W of length three. We refer to the central edge of W as the *pivot* of the NNI move. The move interchanges the ends of the two extreme edges of W on the pivot edge. We consider the resulting graph G' as having the same set of vertices and edges of G, and that only the incidence function $\psi_{G'}$ is adjusted accordingly (Figure 1). We think of an NNI move as a function γ_W that associates the graph G to the graph G'. In symbols, $\gamma_W(G) = G'$. Observe that W is also a trail in G' and $\gamma_W(G') = G$.

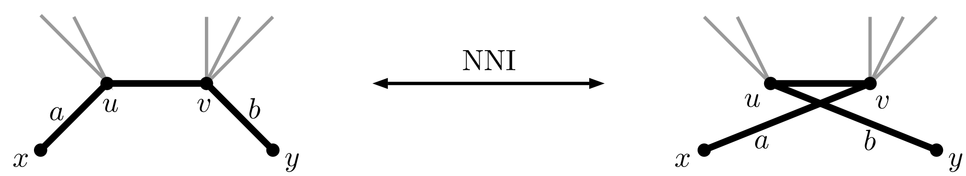


Figure 1: An NNI move on the trail marked in bold. One of the incidences of the edges a and b were interchanged, that is, $\psi_G(a) = \{x, u\}$ and $\psi_{G'}(a) = \{x, v\}$ while $\psi_G(b) = \{v, y\}$ and $\psi_{G'}(b) = \{u, y\}$. For every $f \notin \{a, b\}$, $\psi_{G'}(f) = \psi_G(f)$. Intuitively, one can think of the edges as sticks, so that edges a and b are sliding along the pivot edge c, from G to G'.

We note that an external edge can never be a pivot of an NNI move, and hence any external edge can never become an internal edge under an NNI move, and vice-versa. This simple observation will be used throughout the proofs below.

The well-known rotation, used in data structures to balance binary trees, is a particular NNI move, performed on a $\{1, 3\}$-tree (Figure 2).

Culik and Wood [4, Thm. 2.4] proved that any two $\{1, 3\}$-trees with ℓ (labelled) leaves can be transformed into one another through a finite series of NNI moves. They additionally gave an upper bound of $4\ell - 12 + 4\ell \log_2(\ell/3)$ on the number of NNI moves needed for this transformation. In this section, first we extend Culik and Wood's theorem to trees with the same degree sequence (Lemma 8), which we then use to extend their result further to connected graphs with the same degree sequence (Theorem 1).

A *caterpillar* is a tree for which the removal of all leaves results in a path, called its

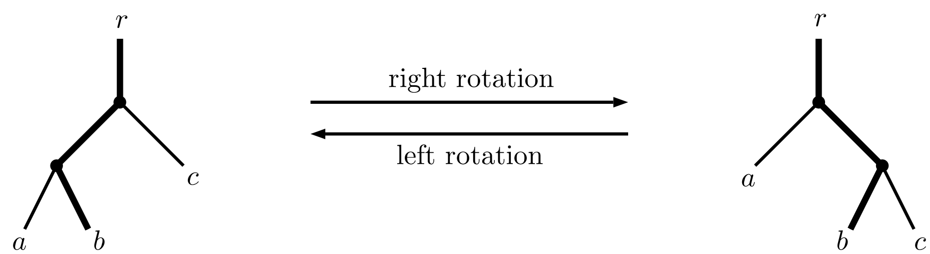


Figure 2: Right and left rotations applied to a binary tree.

central path, or in the empty graph. For the later, we define that the central path is empty.

Lemma 5. *Any tree can be transformed into a caterpillar with the same degree sequence through a series of NNI moves.*

Proof. Let T be a tree. If T is a caterpillar, there is nothing to prove. So, we may assume T is not a caterpillar. Let P be a longest path in T and uv an internal edge of T not in P such that u is a vertex in P. The vertex u has two neighbors in P, otherwise P would not be a longest path, as v is not in P. Let w and z be the two neighbors of u in P. Let W be a trail with edges wu, uv, and vv', where v' is a neighbor of v other than u. Perform an NNI on the trail W as in Figure 3, to insert v in P, obtaining another tree T' with the same degree sequence and a path longer than P. By repeating this process, we obtain a desired caterpillar after a finite number of NNI moves. □

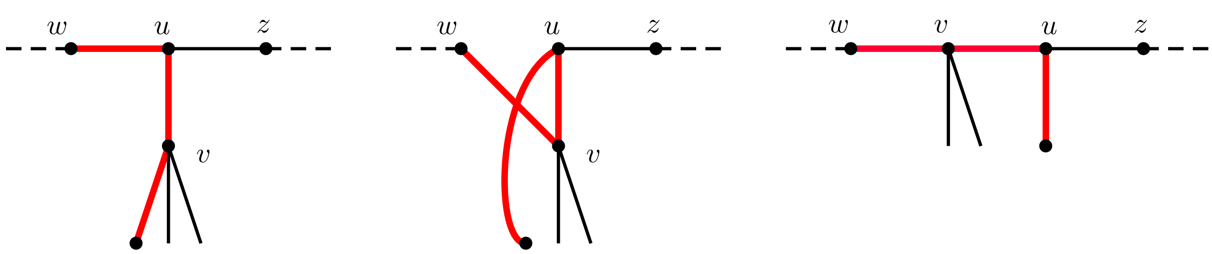


Figure 3: An NNI to insert v in the central path.

The *spine* of a caterpillar is the sequence of the degrees of the vertices in the central path. We say the caterpillar is *ordered* if its spine is a monotonic nonincreasing sequence (Figure 4).

Lemma 6. *Any caterpillar can be transformed into an ordered caterpillar with the same degree sequence through a series of NNI moves.*

Figure 4: (a) A caterpillar with spine $(4, 5, 2, 3)$ and central path highlighted in bold. (b) An ordered caterpillar.

Proof. An NNI can be used to swap any two adjacent vertices in the central path of a caterpillar, as in Figure 5(a). So we use an NNI to decrease, one by one, the number of inversions in the spine of a caterpillar until we obtain an ordered caterpillar. □

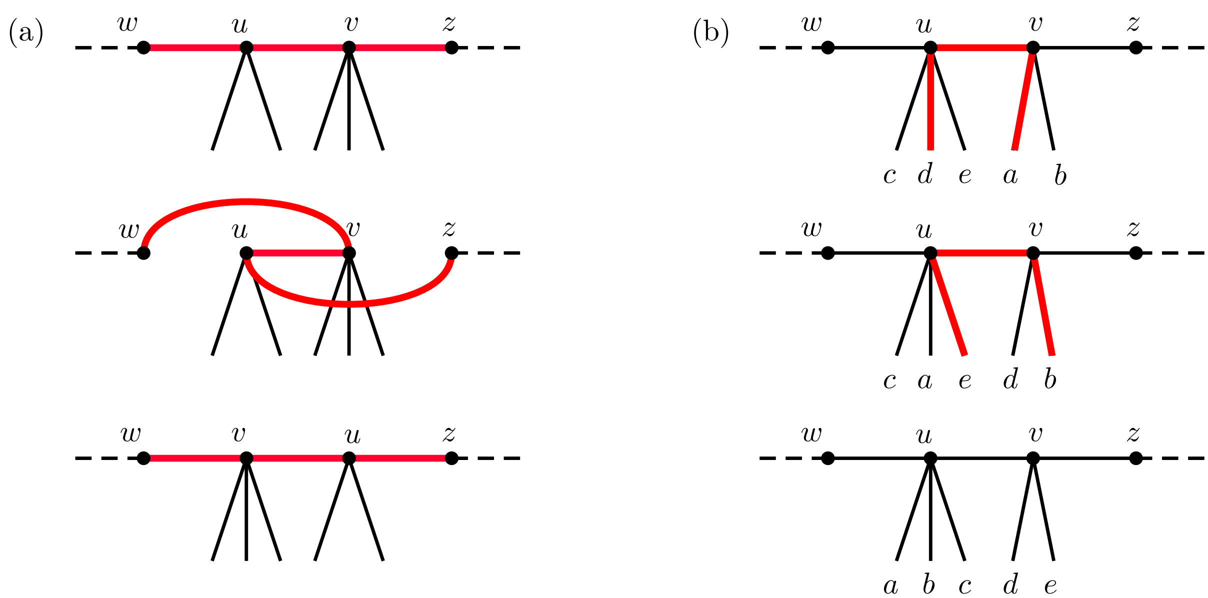


Figure 5: (a) An NNI to swap two adjacent vertices in the central path of a caterpillar. (b) NNIs to swap labels on leaves hanging from adjacent vertices in the central path.

Lemma 7. *The external edges of a caterpillar can be sorted arbitrarily through a series of NNI moves.*

Proof. An NNI can be used to swap any two external edges incident to two adjacent vertices in the central path of a caterpillar, as in Figure 5(b). □

The next lemma is an extension of previous works in the literature [4, 15, 17, 18] that might be of independent interest.

Lemma 8. *Any two trees with the same degree sequence and the same set of external edges can be transformed into one another through a series of NNI moves.*

Proof. Let T and T' be two trees with the same degree sequence and the same set of external edges. Using Lemmas 5 and 6, we obtain a series of NNI moves that transforms T into an ordered caterpillar with the same degree sequence and the same set of external edges of T. Similarly, we obtain another series for T'. Using Lemma 7, we extend the series of NNI moves for T to sort the external edges of the caterpillar coming from T into the order they appear in the caterpillar coming from T'. The composition of these two series of NNI moves, with the series for T' inverted, gives a series of NNI moves that transforms T into T'. □

Let G be a connected graph that is not a tree, and let e be an edge that is in a cycle. The graph obtained from G by *cutting* e is the graph G' resulting from the splitting of e into two edges, each connecting one of the ends of e to one of two new leaves (Figure 6).

Figure 6: Graph G' obtained from G by cutting an edge e.

Proof of Theorem 1. Let n be the number of vertices and m be the number edges in G and G'. The proof of the theorem is by induction on $r(G) := m - n + 1$. If $r(G) = 0$, then G and G' are trees and the theorem follows from Lemma 8. Hence we may assume that $r(G) > 0$.

Choose an edge in a cycle of G and an edge in a cycle of G'. Let us denote these two edges by e. Let H and H' be the graphs obtained from G and G', respectively, by cutting their edge e. The number of vertices in H and H' is $n + 2$ and the number of edges is $m + 1$. Call e' and e'' the two new edges in both H and H'. Since e is in a cycle in G and in a cycle in G', the graphs H and H' are connected and $r(H) = r(H') = (m + 1) - (n + 2) + 1 = r(G) - 1$. Also, H and H' have the same degree sequence and the same set of external edges. By induction, H can be transformed into H' through a series of NNI moves. The same series of NNI moves transforms G into G'. We note that it may occur that the two external edges e' and e'' are involved in the same NNI move, but in this case they will both remain external edges, and this does not affect the process of transforming G into G' using NNI moves. Indeed, the set of external edges in all graphs obtained during the application of this series of NNI moves, transforming H into H', contain e' and e''. Glue e' and e'' into an edge in each of these graphs, obtaining a sequence of connected graphs that is the result of a series of NNI moves starting at G and ending at G'. This ends the proof of (a).

$$
\begin{array}{ccccc}
G & & & & G \\
\Downarrow_{\text{cut}} & & & & \Downarrow_{\text{cut}} \\
H = H_0 & \overset{\text{NNI}}{\longleftrightarrow} & \cdots & \overset{\text{NNI}}{\longleftrightarrow} & H_k = H' \\
\Downarrow_{\text{glue}} & & \Downarrow_{\text{glue}} & & \Downarrow_{\text{glue}} \\
G = G_0 & \overset{\text{NNI}}{\longleftrightarrow} & \cdots & \overset{\text{NNI}}{\longleftrightarrow} & G_k = G'
\end{array}
$$

Similarly, by induction on $r(G)$ (which is equal to $r(G')$), one can prove (b). Indeed, it is sufficient to choose in each step of the induction an edge to be cut which is not in either of the spanning trees. □

For the case of cubic graphs, the proof of Theorem 1 provides an alternative proof for a theorem by Tsukui [17, Thm. II], which refers to NNI moves as $\tilde{S}$-transformations, where the 'S' stands for *slide*. For the slightly more general case of $\{1, 3\}$-graphs, the proof of Theorem 1 provides an alternative proof for a proposition by Wakabayashi [18, Prop. 6.2], which refers to NNI moves as A-moves. Wakabayashi's proof uses a topological pants decomposition for compact, oriented surfaces of finite genus.

Figure 8 shows a series of NNI moves from the complete graph K_4 to a tree with a loop added to each of its leaves.

3 Weighted NNIs for $\{1, 3\}$-graphs

To deal with weights on the edges of a $\{1, 3\}$-graph, we now enhance the NNI move. This was achieved by a bijection defined by Wakabayashi [18, Prop. 6.3].

Let $G = (V, E)$ be a $\{1, 3\}$-graph and w be a weight function defined on the edges of G. A *weighted NNI* is a local move performed in (G, w) on a trail W of length three induced by an NNI move in G on W. The result of the move is the graph G' obtained from G by applying an NNI move on W, and the weight function w' defined on the edges of G' as follows.

Let e be the central edge of W, that is, the pivot of the NNI move. Let a and b be the other edges in W, and c and d be the remaining edges adjacent to e, as depicted in Figure 7. Possibly a, b, c, and d are not pairwise distinct. The weight function w' is such that $w'_f = w_f$ for every $f \neq e$ and

$$w'_e = w_e + \max\{w_a + w_c, w_b + w_d\} - \max\{w_b + w_c, w_a + w_d\}.$$

Note that, if w is integer valued, then so is w'. Moreover, since pivots are always internal edges and an NNI move does not affect the partition of the edges into internal and external, a weighted NNI move may only change the weights of internal edges.

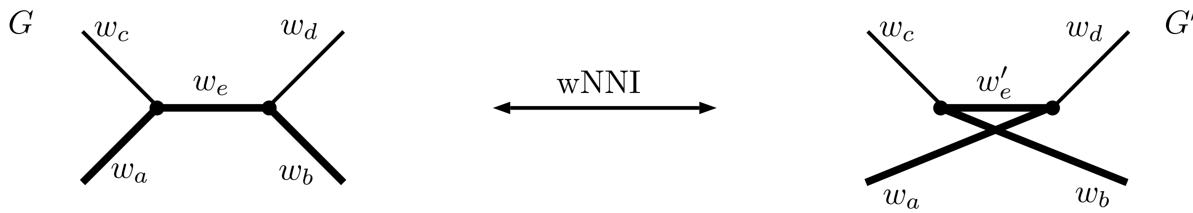


Figure 7: Edges and weights in a weighted nearest neighbor interchange.

We think of a weighted NNI move as a function $\phi_W(G, w) = (G', w')$, which extends the previously defined NNI move, $\gamma_W(G) = G'$. Note that $\phi_W(G', w') = (G, w)$, because $\gamma_W(G') = G$ and

$$\begin{aligned} &w'_e + \max\{w'_b + w'_c, w'_a + w'_d\} - \max\{w'_a + w'_c, w'_b + w'_d\} = \\ &w'_e + \max\{w_b + w_c, w_a + w_d\} - \max\{w_a + w_c, w_b + w_d\} = \\ &w_e + \max\{w_a + w_c, w_b + w_d\} - \max\{w_b + w_c, w_a + w_d\} \\ &\quad + \max\{w_b + w_c, w_a + w_d\} - \max\{w_a + w_c, w_b + w_d\} = w_e. \end{aligned}$$

Wakabayashi [18, Prop. 6.3] proved that $w \in t\mathcal{P}_G$ if and only if $w' \in t\mathcal{P}_{G'}$ for every integer $t \geq 0$, which implies Liu and Osserman's conjecture. We observe that $w \in t\mathcal{P}_G$ if and only if $w' \in t\mathcal{P}_{G'}$ for every real $t \geq 0$. This is stated below as we will use it in the next section. The proof by Wakabayashi [18, Prop. 6.3] can be easily modified to include all real $t \geq 0$.

Lemma 9. *Let G be a $\{1, 3\}$-graph and w be a weight function defined on the edges of G. Let W be a trail in G of length three and suppose that $\phi_W(G, w) = (G', w')$. Then $w \in t\mathcal{P}_G$ if and only if $w' \in t\mathcal{P}_{G'}$ for every real $t \geq 0$.* □

Therefore there are distinct rational polytopes whose Ehrhart quasi-polynomials coincide for all real t.

As a weighted NNI changes only the weight of the pivot, which is always an internal edge, the series of NNI moves from G to G' changes only the weights of internal edges. In fact, by Theorem 1(b), one can choose a spanning tree T in G and a spanning tree T' in G' and require that the series of NNI moves uses as pivots only internal edges of T and T'. As a consequence, the bijection from $\mathcal{P}_G$ to $\mathcal{P}_{G'}$ keeps fixed the majority of the coordinates of the points. Namely, it changes only coordinates that correspond to internal edges of the chosen spanning trees. Formally, the latter discussion provides the following as a corollary of Theorem 1(b) and Lemma 9. Let $E = \{1, \ldots, m\}$ and $w : E \to \mathbb{R}$ be a weight function defined on the edges of G. If X is a subset of E, then the *restriction of w to X* is the function $w|_X : X \to \mathbb{R}$ such that $w|_X(x) = w(x)$ for all $x \in X$.

Corollary 10. *Let G and G' be connected $\{1, 3\}$-graphs with the same number of vertices and on the same set E of edges. Let $X \subseteq E$ be such that $E \setminus X$ is the set of internal*

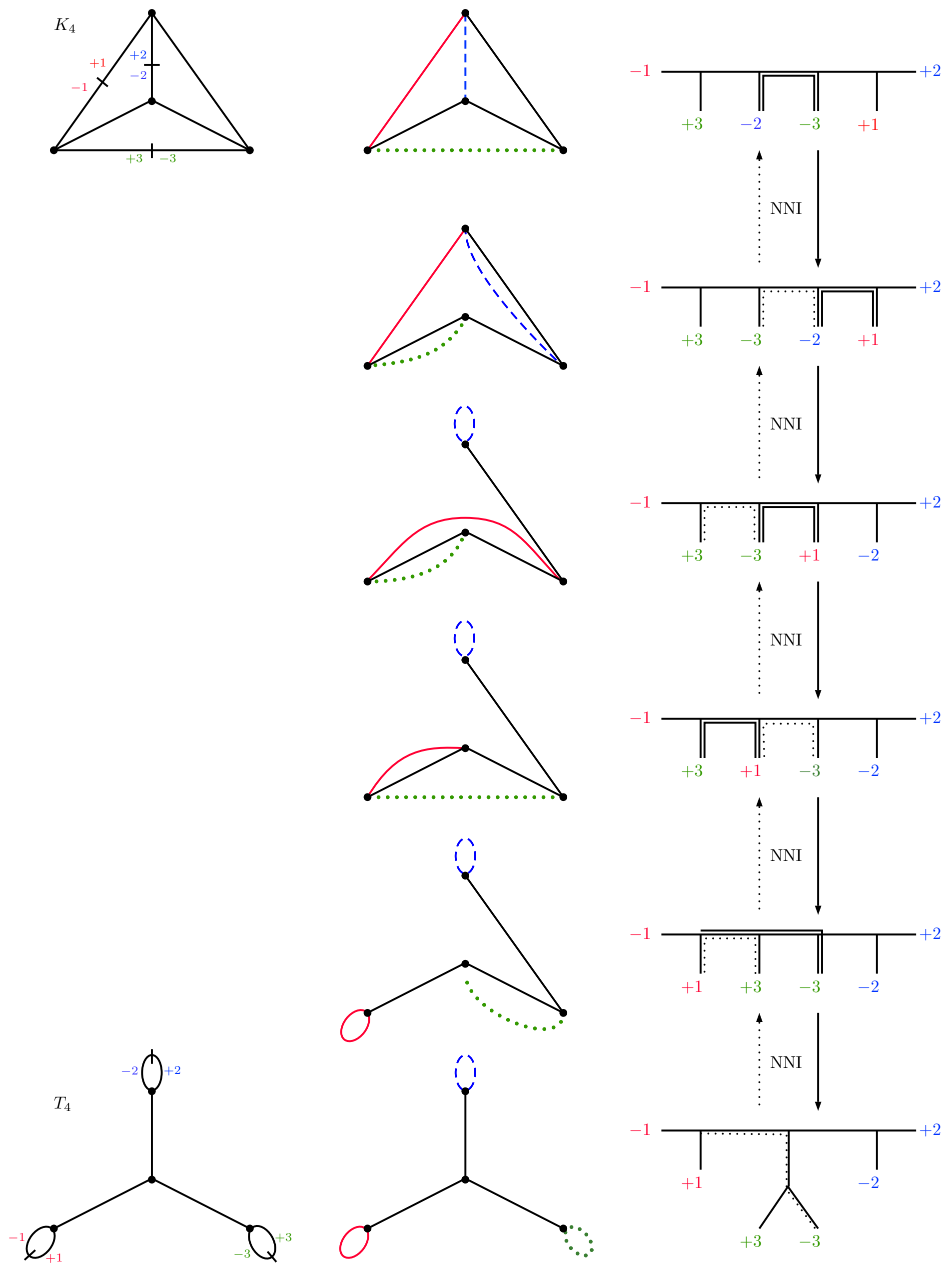


Figure 8: The second column shows a series of NNI moves from K_4 to the graph T_4. The third column keeps track of the corresponding NNI moves on the underlying trees that are obtained by the induction process, after the colored edges of K_4 and T_4 are cut. The labels on the leaves of these trees record the ends of the colored edge that connected the two leafs of the tree in the corresponding graphs.

edges of arbitrary spanning trees in G and G'. Then there exists a bijection ϕ between $\mathcal{P}_G$ and $\mathcal{P}_{G'}$ such that, for $w' = \phi(w)$, we have that

$$w'|_X = w|_X.$$

4 Scissors congruence

Haase and McAllister [8] have raised Question 2 that can be thought of as an analogue of Hilbert's third problem (equidecomposability) for the unimodular group. We show that for polytopes associated to $\{1,3\}$-graphs a statement similar to Question 2 holds.

Let G be a $\{1,3\}$-graph with edge set $E = \{1,\ldots,m\}$. Let W be a trail in G of length three and suppose that $\phi_W(G,w) = (G',w')$. As argued ahead, the function $\phi_W(G,\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is associated to one or two hyperplanes in $\mathbb{R}^m$ and two or four unimodular transformations. For short, let $\phi(w) = \phi_W(G,w)$ for every $w \in \mathbb{R}^m$. Let a, b, c, d, and e be as in Figure 7, with a, e, and b being the edges of W. Clearly ϕ is piecewise linear, namely, for $w \in \mathbb{R}^m$, $w'_f = w_f$ for every $f \neq e$ and

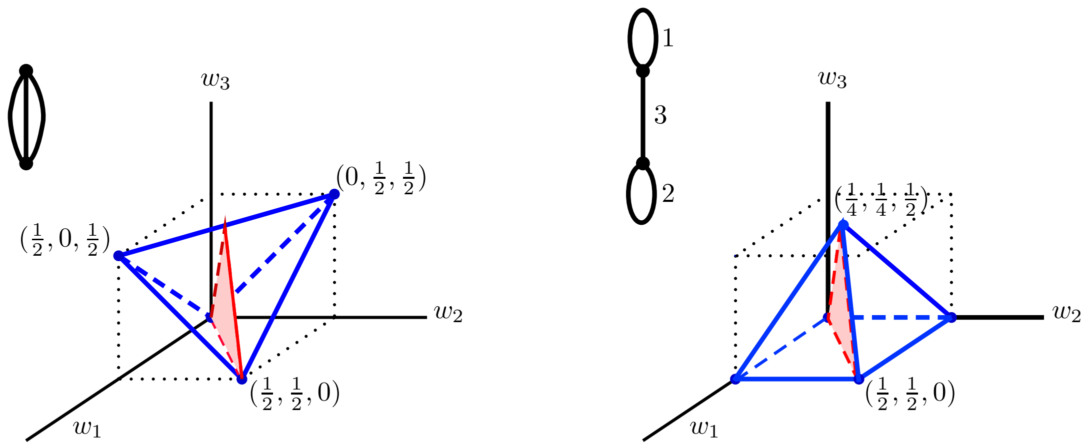
$$w'_e = w_e + w_b - w_d \quad \text{if } w_a + w_b \geq w_c + w_d \text{ and } w_a + w_d \geq w_b + w_c, \tag{1a}$$
$$w'_e = w_e + w_a - w_c \quad \text{if } w_a + w_b \geq w_c + w_d \text{ and } w_a + w_d < w_b + w_c, \tag{1b}$$
$$w'_e = w_e + w_c - w_a \quad \text{if } w_a + w_b < w_c + w_d \text{ and } w_a + w_d \geq w_b + w_c, \tag{1c}$$
$$w'_e = w_e + w_d - w_b \quad \text{if } w_a + w_b < w_c + w_d \text{ and } w_a + w_d < w_b + w_c. \tag{1d}$$

The hyperplanes associated to ϕ are $w_a + w_b - w_c - w_d = 0$ and $w_a - w_b - w_c + w_d = 0$, which are either the same hyperplane (if $a = b$ or $c = d$) or two orthogonal hyperplanes. Moreover, the matrix that gives the linear transformation in each case is unimodular. Indeed, the matrix for case (1a) is obtained from the identity matrix by substituting the row corresponding to the edge e by the row $\chi^e + \chi^b - \chi^d$, the matrix for case (1b) is obtained from the identity matrix by substituting the row corresponding to the edge e by the row $\chi^e + \chi^a - \chi^c$, and similarly for the other two cases. Thus the determinant of each such matrix is always 1. Therefore each of them is unimodular.

Figure 9 shows an example with the two 3-regular graphs on two vertices and their polytopes. The two graphs differ by one NNI move. The function $\phi(w) = w'$ between the two polytopes in Figure 9 is defined by $w'_1 = w_1$, $w'_2 = w_2$, and

$$\begin{aligned} w'_3 &= w_3 + \max\{w_1 + w_2, w_1 + w_2\} - \max\{2w_1, 2w_2\} \\ &= w_3 + w_1 + w_2 - 2\max\{w_1, w_2\} \\ &= \begin{cases} w_3 - w_1 + w_2 & \text{if } w_1 \geq w_2 \\ w_3 + w_1 - w_2 & \text{if } w_1 \leq w_2, \end{cases} \end{aligned}$$

which is piecewise unimodular. In this case, only one hyperplane (the one containing the shaded triangle inside the polytopes, $w_1 = w_2$) splits the polytopes into two, each part be-



$$w_1 + w_2 + w_3 \leq 1 \qquad 2w_1 + w_3 \leq 1$$
$$w_1 \leq w_2 + w_3 \qquad w_3 \leq 2w_1$$
$$w_2 \leq w_1 + w_3 \qquad w_3 \geq 0$$
$$w_3 \leq w_1 + w_2 \qquad 2w_2 + w_3 \leq 1$$
$$w_3 \leq 2w_2$$

Figure 9: The polytopes of the two cubic graphs on two vertices. The shaded triangles are the intersection of the polytopes with the hyperplane $w_1 = w_2$.

ing unimodularly equivalent to one of the parts of the other polytope. The corresponding unimodular transformations are given by

$$U_{w_1 \leq w_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \qquad U_{w_1 \geq w_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}.$$

Note that the polytope on the right side of Figure 9 has one more vertex than the other one, so that the combinatorial types of these piecewise unimodularly equivalent polytopes may differ. Observe how the extra vertex $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ is "formed" when going from the polytope on the left to the one on the right, and how it "disappears" when going in the other direction. Also, the edge between the origin and the vertex $(\frac{1}{2}, \frac{1}{2}, 0)$ in the polytope on the left is not an edge in the polytope on the right.

Theorem 3 extends this for graphs that differ by a series of NNI moves. It relates to Question 2.

Proof of Theorem 3. The proof is constructive. By Theorem 1, there exists a finite sequence of NNI moves that transforms G to G', say $G = G_0 \overset{\text{NNI}}{\longleftrightarrow} \cdots \overset{\text{NNI}}{\longleftrightarrow} G_k = G'$. We shall explain the procedure for the first two NNI moves assuming that both underlying weighted NNI moves are associated to only one hyperplane and consequently two unimodular transformations. The other cases and the rest of the NNI moves follow in an analogous but more cumbersome fashion.

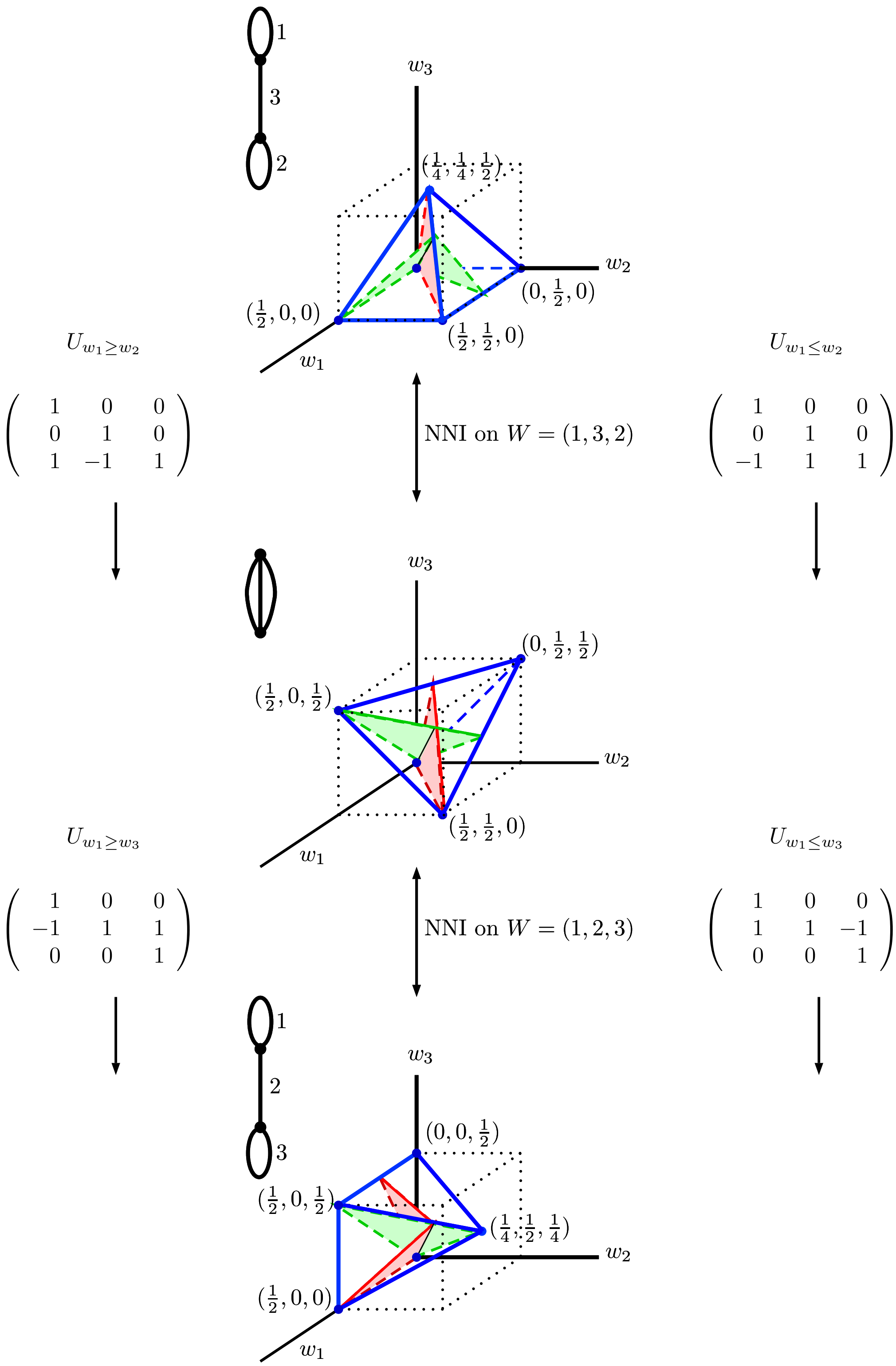


Figure 10: Example for Theorem 3.

Let W_0 be the trail used in the first NNI move, that goes from $G = G_0$ to G_1, and let $\phi_0(w) = \phi_{W_0}(G_0, w)$, for $w \in \mathbb{R}^E$, be the corresponding weighted NNI move. Let H_0 be the hyperplane associated to ϕ_0, and let U_1, U_2 be the two unimodular transformations, one for each side of the hyperplane H_0. The hyperplane H_0 dissects the polytope $\mathcal{P}_G$ into two smaller polytopes, $\mathcal{Q}_G^1$ and $\mathcal{Q}_G^2$, so that $\mathcal{P}_G = \mathcal{Q}_G^1 \cup \mathcal{Q}_G^2$. We now have that $\mathcal{P}_{G_1} = U_1(\mathcal{Q}_G^1) \cup U_2(\mathcal{Q}_G^2)$. In words, we presented a dissection of $\mathcal{P}_G$ into two smaller polytopes and two affine unimodular transformations U_1 and U_2 that, if applied to the two smaller polytopes, result in $\mathcal{P}_{G_1}$. Now we will proceed one more step, and present a dissection of $\mathcal{P}_G$ into four smaller polytopes, and four affine unimodular transformations that, if applied to the four smaller polytopes, will result in $\mathcal{P}_{G_2}$ (Figure 10).

Let W_1 and H_1 be defined similarly for the second NNI move, that goes from G_1 to G_2. Let T_1, T_2 be the two unimodular transformations, one for each side of the hyperplane H_1. The hyperplane H_1 dissects the polytope $\mathcal{P}_{G_1}$ into two smaller polytopes, $\mathcal{Q}_{G_1}^1$ and $\mathcal{Q}_{G_1}^2$, so that $\mathcal{P}_{G_2} = T_1(\mathcal{Q}_{G_1}^1) \cup T_2(\mathcal{Q}_{G_1}^2)$. Note that H_1 dissects $U_1(\mathcal{Q}_G^1)$ and $U_2(\mathcal{Q}_G^2)$ each into two smaller polytopes, obtaining the dissection

$$\mathcal{P}_{G_1} = (\mathcal{Q}_{G_1}^1 \cap U_1(\mathcal{Q}_G^1)) \cup (\mathcal{Q}_{G_1}^1 \cap U_2(\mathcal{Q}_G^2)) \cup (\mathcal{Q}_{G_1}^2 \cap U_1(\mathcal{Q}_G^1)) \cup (\mathcal{Q}_{G_1}^2 \cap U_2(\mathcal{Q}_G^2)).$$

The latter naturally induces the following dissection:

$$\begin{aligned}\mathcal{P}_G &= U_1^{-1}(\mathcal{Q}_{G_1}^1 \cap U_1(\mathcal{Q}_G^1)) \cup U_2^{-1}(\mathcal{Q}_{G_1}^1 \cap U_2(\mathcal{Q}_G^2)) \cup U_1^{-1}(\mathcal{Q}_{G_1}^2 \cap U_1(\mathcal{Q}_G^1)) \cup U_2^{-1}(\mathcal{Q}_{G_1}^2 \cap U_2(\mathcal{Q}_G^2)) \\ &= \left(U_1^{-1}(\mathcal{Q}_{G_1}^1) \cap \mathcal{Q}_G^1\right) \cup \left(U_2^{-1}(\mathcal{Q}_{G_1}^1) \cap \mathcal{Q}_G^2\right) \cup \left(U_1^{-1}(\mathcal{Q}_{G_1}^2) \cap \mathcal{Q}_G^1\right) \cup \left(U_2^{-1}(\mathcal{Q}_{G_1}^2) \cap \mathcal{Q}_G^2\right).\end{aligned}$$

Name the four pieces of this dissection $\mathcal{P}^{11}$, $\mathcal{P}^{12}$, $\mathcal{P}^{21}$, $\mathcal{P}^{22}$, where $\mathcal{P}^{11} = U_1^{-1}(\mathcal{Q}_{G_1}^1) \cap \mathcal{Q}_G^1$ and the other three are defined similarly. Then

$$\begin{aligned}\mathcal{P}_G &= \mathcal{P}^{11} \cup \mathcal{P}^{12} \cup \mathcal{P}^{21} \cup \mathcal{P}^{22}, \text{ and} \\ \mathcal{P}_{G_2} &= T_1(\mathcal{Q}_{G_1}^1) \cup T_2(\mathcal{Q}_{G_1}^2) \\ &= \left(T_1(\mathcal{Q}_{G_1}^1 \cap U_1(\mathcal{Q}_G^1)) \cup T_1(\mathcal{Q}_{G_1}^1 \cap U_2(\mathcal{Q}_G^2))\right) \cup \left(T_2(\mathcal{Q}_{G_1}^2 \cap U_1(\mathcal{Q}_G^1)) \cup T_2(\mathcal{Q}_{G_1}^2 \cap U_2(\mathcal{Q}_G^2))\right) \\ &= \left(T_1U_1(\mathcal{P}^{11}) \cup T_1U_2(\mathcal{P}^{12})\right) \cup \left(T_2U_1(\mathcal{P}^{21}) \cup T_2U_2(\mathcal{P}^{22})\right).\end{aligned}$$

This completes the proof for the two first weighted NNI moves assuming that both are associated to only one hyperplane and consequently two unimodular transformations. For the remaining cases, whenever a weighted NNI move is associated to two hyperplanes (and consequently four unimodular transformations), the polytopes would be dissected into up to four smaller polytopes, but the process would be essentially the same. □

5 Reflexivity

An equivalent way to define a reflexive polytope is to require that it is integral and has the hyperplane description $\{x \in \mathbb{R}^d \mid Ax \leq \mathbb{1}\}$ for some integral matrix A. Another equivalent definition of reflexivity is to say that a polytope $\mathcal{P}$ is reflexive if and only if the origin is in $\mathcal{P}^\circ$ and $(t+1)\mathcal{P}^\circ \cap \mathbb{Z}^d \;=\; t\mathcal{P} \cap \mathbb{Z}^d$ for all $t \in \mathbb{Z}_{\geq 0}$, where $\mathcal{P}^\circ$ is the interior of $\mathcal{P}$.

Proof of Theorem 4. The polytope $4\mathcal{P}_G - \mathbb{1}$ consists of the vectors $w \in \mathbb{R}^m$ satisfying

$$\begin{aligned} w_a + w_b + w_c &\leq 1 \\ w_a - w_b - w_c &\leq 1 \\ -w_a + w_b - w_c &\leq 1 \\ -w_a - w_b + w_c &\leq 1, \end{aligned}$$

for each degree three vertex v and edges a, b, and c incident to v, for a total of $4n_3$ inequalities, where n_3 is the number of degree three vertices in G. From Liu and Osserman [13, Prop. 3.5], the polytope $4\mathcal{P}_G$ is integral, and thus so is $4\mathcal{P}_G - \mathbb{1}$. □

The reflexivity of the integral polytope $4\mathcal{P}_G - \mathbb{1}$ has some interesting consequences, as follows. Let m denote the number of edges of G. Since $4\mathcal{P}_G$ is an m-dimensional integral polytope, we have that for $t \equiv 0 \mod 4$:

$$L_{\mathcal{P}_G}(t) = \binom{t+m}{m} + h_1^* \binom{t+m-1}{m} + \cdots + h_{d-1}^* \binom{t+1}{m} + h_d^* \binom{t}{d},$$

where $h_1^*, \ldots, h_{d-1}^*, h_d^*$ are the coefficients of the numerator of the rational function given by the Ehrhart series of $4\mathcal{P}_G$ [2, Lemma 3.14].

The polytopes $4\mathcal{P}_G$ and $4\mathcal{P}_G - \mathbb{1}$ have the same Ehrhart polynomial. Hence it follows from the reflexivity of $4\mathcal{P}_G - \mathbb{1}$ and from Hibi's palindromic theorem [2, Thm. 4.6] that $h_k^* = h_{m-k}^*$ for all $0 \leq k \leq m/2$. Therefore it follows that to describe $L_{4\mathcal{P}_G}(t)$, we only need to compute half of its coefficients, thus lowering the computational complexity required to compute $L_{4\mathcal{P}_G}(t)$.

6 Concluding remarks and open problems

In the process of proving the main theorem, a great deal of structure of the polytopes $\mathcal{P}_G$, and their corresponding cones has been revealed. The polytopes $\mathcal{P}_G$ and their corresponding cones may be related to the well-known metric polytopes and metric cones [5]. In particular, in the case that G is a planar graph, the polytope $\mathcal{P}_G$ can be thought of as a restricted type of "metric polytope" [11], associated to the dual graph G^*, and it would be very interesting to determine more precisely this relation to metric polytopes.

In the theory of reflexive polytopes, Haase and Melnikov [9] showed that every integer polytope is lattice equivalent to a face of some reflexive polytope. It is natural to wonder if such a universality condition also holds for our reflexive polytopes from Theorem 4, as follows (for details about lattice equivalence, see [9]).

Problem 1. *Is every integer polytope lattice equivalent to a face of one of the reflexive polytopes* $4\mathcal{P}_G - \mathbb{1}$*?*

Problem 2. *For a fixed* n*, what proportion of all reflexive polytopes in* $\mathbb{R}^n$ *is captured by the reflexive polytopes* $4\mathcal{P}_G - \mathbb{1}$*?*

It is not difficult to show that each NNI move preserves the number of Eulerian subgraphs of a graph. Since a Hamiltonian graph is in particular an Eulerian subgraph, it is tempting to look for a new type of NNI move (with some added constraints) that would leave the number of Hamiltonian cycles invariant. If such a restricted NNI move can be found, perhaps the following two problems, related to Theorem 1, may be within reach.

Problem 3. *Suppose* G *and* H *are two graphs with the same degree sequence, and that* G *has a Hamiltonian cycle. Can we use restricted NNI moves to determine whether or not* H *also has a Hamiltonian cycle?*

Problem 4. *Consider the graph* G*, each of whose nodes is a connected graph with a fixed degree sequence. Two vertices of* G *are defined to be adjacent if they differ by an NNI move. Is there a Hamiltonian path in* G*?*

A special case of Problem 4 was proved in [7], where each node of G was a binary tree. As a first step, Theorem 1 proves that the latter graph G is at least connected.

Wakabayashi [18] gave a formula for two of the four constituent polynomials of the Ehrhart quasi-polynomial for $\mathcal{P}_G$ and for the volume of $\mathcal{P}_G$ when G is a connected cubic graph on n vertices. It is rather surprising that the Verlinde formula makes an appearance here, namely, Wakabayashi showed that, for odd t,

$$L_{\mathcal{P}_G}(t) \;=\; \frac{(t+2)^{n/2}}{2^{n+1}} \sum_{j=1}^{t+1} \frac{1}{\sin^n\left(\frac{\pi j}{t+2}\right)} \quad \text{and} \quad \mathrm{vol}(\mathcal{P}_G) \;=\; \frac{|B_n|}{2\, n!},$$

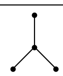
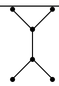
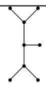
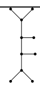
where B_n is the n-th Bernoulli number. Zagier [19] showed that the above Verlinde formula is the polynomial

$$L_{\mathcal{P}_G}(t) \;=\; \frac{(t+2)^{n/2}}{2^{n+1}} \sum_{k=0}^{n/2} \frac{(-1)^{k-1} 2^{2k} B_{2k}}{(2k)!} c_k (t+2)^{2k},$$

where c_k is the coefficient of x^{-2k} in the Laurent expansion of $\sin^{-n} x$ at $x = 0$. It would be interesting to determine if a similar formula holds for all connected $\{1,3\}$-graphs, or even for trees.

Problem 5. *Let* G *be a* $\{1,3\}$*-tree. Determine explicitly the Ehrhart quasi-polynomial of* $\mathcal{P}_G$*.*

We determined the following quasi-polynomials for some small binary trees with the software LattE [1].

G	Ehrhart polynomial $\mathcal{P}_G$
	$\frac{1}{24}t^3 + \frac{1}{4}t^2 + \begin{cases} \frac{5}{6}t + 1, & \text{if } t \text{ is even} \\ \frac{11}{24}t + \frac{1}{4}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{1}{240}t^5 + \frac{1}{24}t^4 + \begin{cases} \frac{5}{24}t^3 + \frac{7}{12}t^2 + \frac{11}{10}t + 1, & \text{if } t \text{ is even} \\ \frac{1}{6}t^3 + \frac{1}{3}t^2 + \frac{79}{240}t + \frac{1}{8}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{17}{40320}t^7 + \frac{17}{2880}t^6 + \begin{cases} \frac{59}{1440}t^5 + \frac{25}{144}t^4 + \frac{179}{360}t^3 + \frac{173}{180}t^2 + \frac{93}{70}t + 1, & \text{if } t \text{ is even} \\ \frac{103}{2880}t^5 + \frac{35}{288}t^4 + \frac{1439}{5760}t^3 + \frac{893}{2880}t^2 + \frac{791}{3360}t + \frac{1}{16}, & \text{if } t \text{ is odd} \end{cases}$
	$\frac{31}{725760}t^9 + \frac{31}{40320}t^8 + \begin{cases} \frac{829}{120960}t^7 + \frac{37}{960}t^6 + \frac{653}{4320}t^5 + \frac{103}{240}t^4 + \frac{20413}{22680}t^3 + \frac{1723}{1260}t^2 + \frac{193}{126}t + 1, & \text{if } t \text{ is even} \\ \frac{43}{6912}t^7 + \frac{19}{640}t^6 + \frac{3181}{34560}t^5 + \frac{123}{640}t^4 + \frac{39205}{145152}t^3 + \frac{9923}{40320}t^2 + \frac{379}{2880}t + \frac{1}{32}, & \text{if } t \text{ is odd} \end{cases}$

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