SELF-DUAL MAPS II: LINKS AND SYMMETRY

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ABSTRACT. In this paper, we investigate representations of links that are either *centrally symmetric* in $\mathbb{R}^3$ or *antipodally symmetric* in $\mathbb{S}^3$. By using the notions of *antipodally self-dual* and *antipodally symmetric* maps, introduced and studied by the authors in [9], we are able to present sufficient combinatorial conditions for a link L to admit such representations. The latter naturally arises sufficient conditions for L to be *amphichiral*.

We also introduce another (closely related) method yielding again to sufficient conditions for L to be amphichiral. We finally prove that a link L, associated to a map G, is amphichiral if the *self-dual pairing* of G is not one of 6 specific ones among the classification of the 24 *self-dual pairing* $Cor(G) \rhd Aut(G)$.

CONTENTS

1. Introduction

Finding symmetrical diagrams for a link L can be a challenging task. In this paper, we are interesting in questions concerning the *symmetry* and the *amphichirality* of L.

We focus our attention to study the existence of both *centrally symmetric* embeddings in $\mathbb{R}^3$ and *antipodally symmetric* embeddings in $\mathbb{S}^3$ of L. It turns out that the notions of *antipodally self-dual map* and *antipodally symmetric map*, introduced and studied by the authors in [9], are helpful combinatorial tools for the above mentioned embeddings. These able us to come up with a number of contributions on the amphichirality of links.

The paper is organized as follows. In the next two sections we give a brief overview of some basic notions and definitions on knots and maps needed for the rest of the paper.

In Section 4, we present combinatorial conditions for L to admit a *centrally symmetric* embedding in $\mathbb{R}^3$ when the *Tait graph*, associated to L, is antipodally self-dual (Theorem 1). As a straightforward consequence, we obtain sufficient combinatorial conditions for a link to be *amphichiral* (Corollary 1). The latter allows us to present infinite families of amphichiral *alternating* links (Corollaries 2, 3, 4, 5, 6 and 7). For instance, we show the amphichirality of the $(3, n)$-*Turk's head* links for any integer $n \geq 1$. We also present conditions on L to admit a *antipodally symmetric* embedding in $\mathbb{S}^3$ when the *Tait graph*, associated to L, is antipodally self-dual (Theorem 2). This allows us to present infinite families of links antipodally symmetric in $\mathbb{S}^3$ (Corollaries 8, 9, 10 and 11). For instance, we show that the torus knot $T(2, n)$ admits such embedding for any $n \geq 2$.

The approach given in Section 4 does not detect amphichirality when the Tait graphs is not antipodally self-dual. For instance, it is not able to determine the amphichirality of the *Figure-eight* knot (whose associated Tait graph is not antipodally self-dual), see Remark 6. In Section 6, we propose a different method, by introduce and investigate the notion of Γ-*curve* in an incidence graph $I(G)$ where G is a Tait's graph associated to a link L. This tool allows us to give sufficient combinatorial conditions on $I(G)$ for L to be amphichiral. The latter is done via sphere isometries (Theorem 3). This approach shows the amphichirality of new families of links (in particular the Figure-eight knot) and lead to an easy way to construct infinite many amphichiral links. We also discuss a closely related method in connection to *rigid* and *invertible* knots. We finally introduce the *amphichiral number* of a link L (the minimal number of crossing switches to become L amphichiral) and present a first rough upper bound for some particular families (Proposition 2).

In Section 7, we focus our attention to self-dual pairings. After recalling basic notions and definitions, we study some properties of *equivalent* maps and we give necessary conditions for the associated link to be amphichiral (Theorem 4). We then consider the groups $Cor(G)$ (generated by the set of all self-dualities of G) and $Aut(G)$ (generated by the set of all automorphisms of G). It is known that any map of G belong to one of the 24 *self-dual pairings* $Cor(G) \rhd Aut(G)$ classification. We show that the link L, associated to a self-dual map G which its self-pairing is other than 6 specific ones, is amphichiral (Theorem 5).

We finally end with some concluding remarks.

2. Knot : preliminaries

We review some knot notions. We refer the reader to [1, 7] for standard background on knot theory.

A *link* with k components consists of k disjoint simple closed curves ($\mathbb{S}^1$) in $\mathbb{R}^3$. A *knot* K is a link with one component. A *link diagram* $D(L)$ of a link L is a regular projection of L into $\mathbb{R}^2$ in such a way that the projection of each component is smooth and at most two curves intersect at any point. At each crossing point of the link diagram the curve which goes over the other is specified, see Figure 1.
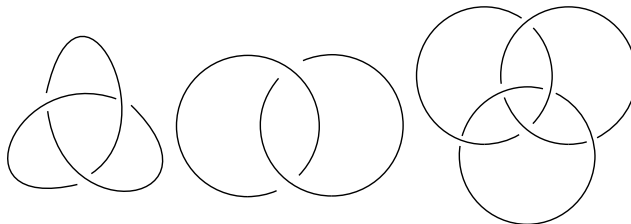

FIGURE 1. (From left to right) Trefoil, Hopf link, denoted by 2_1, and Borromean rings (3 components).

A *shadow* of a link diagram D is a 4-regular graph if the over/under passes of D are ignored. Since the shadow is Eulerian (4-regular) then its faces can be 2-colored, say with colors black and white. We thus have that each vertex is incident to 4 faces alternatively colored around the vertex, see Figure 2.
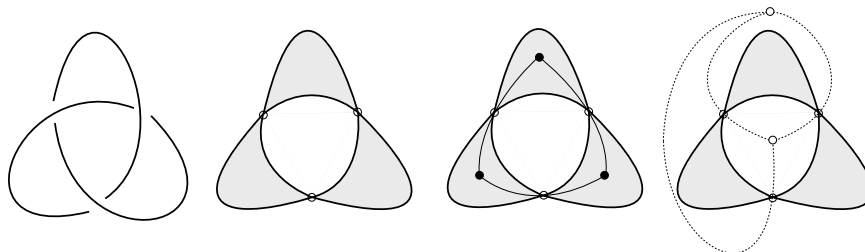

FIGURE 2. (From left to right) A diagram of the Trefoil, its shadow with a 2-colored faces (vertices on white crossed circles), corresponding Black graph (bold edges and black circles) and White graph (dotted edges and white circles).

Given such a coloring, we can define two graphs, one on the faces of each color. Let B_D denote the graph with black faces as its vertices and two vertices are joined if the corresponding faces share a vertex (B_D is called the *checkerboard graph* of D). We define the graph W_D on the white faces of the shadow analogously. The two faces are called *faces graphs* of the shadow. Since the shadow of a knot is connected then B_D and W_D are also connected, and it is not hard to see that $W_D = B_D^*$ and $B_D = W_D^*$, that is, the two faces graphs are duals of each other.

It is not hard to convince oneself that a connected 4-regular planar graph G is determined by either of its face graphs. For, recall that the *medial graph* of H, denoted by $med(H)$ is the graph obtained by placing one vertex on each edge of H and joining two vertices if the corresponding edges are consecutive on a face of H. We notice that $med(H)$ is 4-regular since each edge is shared by exactly two faces. It is thus suffices to notice that $med(B_D)$ and

$med(W_D)$ are the same (since they B_D and W_D are duals) and that $med(W_D)$ is exactly the shadow of D.

If D is a link diagram with more than one component, then the shadow of D may not be connected. In this case, only one of the corresponding faces graphs would be connected and $B_D^* \neq W_D$. We thus would not be able to determine uniquely the shadow of L from the faces graphs. The latter can be overcome by considering each component of the shadow separately, and define the corresponding black face graph to be the union of the black face graphs of its components, and analogously for the white face graph.

An *edge-signed* planar graph, denoted by (G, S_E), is a planar graph G equipped with a signature on its edges $S_E : E \to \{+, -\}$. We will denote by $-S_E$ the signature of G satisfying $-S_E(e) = -(S_E(e))$ for every $e \in E$. We write S_E^+ (resp. S_E^-) when all the signs of S_E are $+$ (resp. $-$).

Given a crossing of the link diagram we sign *positive* or *negative* according to the *left-over-right* and *right-over-left* rules from point of view of black around the crossing, see Figure 3 (a). The latter induce an opposite signing on each crossing by the same rules but now from point of view of white around the crossing, see Figure 3 (b).
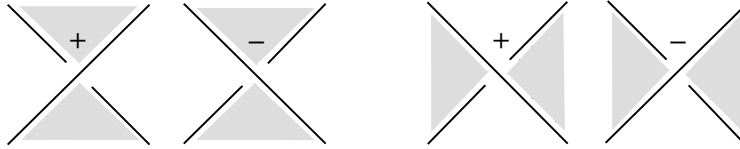

FIGURE 3. (Left) Left-over-right rule from black point of view. (Right) Right-over-left rule from white point of view.

If the crossing is positive, relative to the black faces, then the corresponding edge is declared to be positive in B_D and negative in W_D. Therefore, in this fashion, a link diagram D determine a dual pair of signed planar graphs (B_D, S_E) and $(W_D, -S_E)$ where the signs on edges are swapped on moving to the dual.

Remark 1. *A link diagram can be uniquely recovered from either (B_D, S_E) or $(W_D, -S_E)$.*

We thus have that given an edge-signed planar graph (G, S_E), we can associate to it (in a canonical way) a link diagram $D(L)$ such that (B_D, S_E) (and $(W_D, -S_E)$) gives (G, S_E). The unsigned graph G is called the *Tait graph* of the link L with diagram D. The construction is easy, we just consider the $med(H)$ with signatures on the vertices (induced by the edge-signature S_E of G). The desired diagram, denoted by $D(G, S_E)$, is obtained by determining the under/over pass at each crossing according to Left-over-right (or Right-over-left) rule associated to the sign of the corresponding edge of G, see Figure 4.
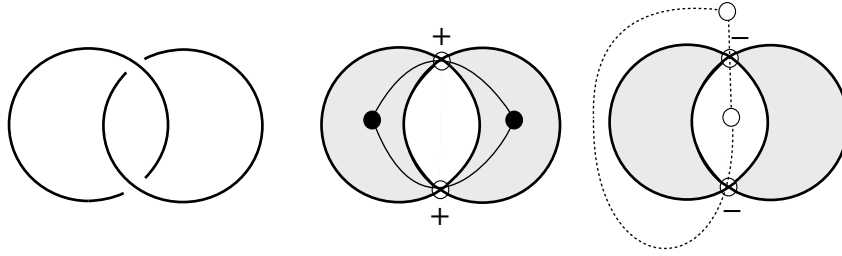
FIGURE 4. (From left to right) diagram D of link 2_1, signed Black graph (B_D, S_E) and signed White graph $(W_D, -S_E)$.

A link L is *alternating* if it admits a diagram D such that the crossings alternate under/over while we go through the link. We notice that a link L is alternating if and only if $L = D(G, S_E^+) = D(G, S_E^-)$ where G is the Tait graph of link L, see Figure 5.
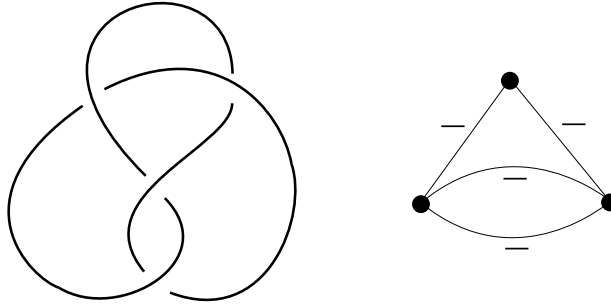

FIGURE 5. A diagram D of the *Figure-eight* knot (alternating knot, denoted by 4_1) and its signed Black graph (B_D, S_E^-).

2.1. **Amphichirality.** Two links L_1 and L_2 are *equivalent* if there is an orientation-preserving homeomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with $\varphi(L_1) = L_2$.

We notice that, in the same way as infinitely many signed plane graphs represent equivalent links for any link L, there are infinitely many link diagrams that can represent equivalent links. We shall denote by $[L]$ (called *link-type*) the class of links equivalent to L.

The mirror-image L^* of a link L is obtained by reflecting it in a plane through the origin in $\mathbb{R}^3$. It may also be defined given a diagram D of L by simply exchanging all the crossing of D (this is clear if one consider reflecting the knot in the plane where the diagram is drawn). Although such a reflection is a bijective map, it is not orientation preserving (its determinant is negative). Therefore, it may happen that L and L^* are not equivalent (that is, $L \notin [L^*]$ and $L^* \notin [L]$).

Link L is called *amphichiral* (also known as *achiral*) if there is $L \in [L^*]$. Equivalently, L is amphichiral if there is an automorphism of $\mathbb{R}^3$ (or $\mathbb{S}^3$) preserving L and reversing the orientation. To see this equivalence, we only need to compose the preserving automorphism with a reflection in a plane to get L^* (such composition reverses twice the orientation).

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3. Maps: background

We review some notions and properties on maps needed for the rest of the paper.

A *map* of $G = (V, E, F)$ is the image of an embedding of G into $\mathbb{S}^2$ where the set of vertices are a collection of distinct points in $\mathbb{S}^2$ and the set of edges are a collection of Jordan curves joining two points in V satisfying that $\alpha \cap \alpha'$ is either empty or a point in the endpoints for any pair of Jordan curves α and α'. Any embedding of the topological realization of G into $\mathbb{S}^2$ partitions the 2-sphere into simply connected regions of $\mathbb{S}^2 \setminus G$ called the *faces F* of the embedding.

Let us define the *antipodal* function as

$$\begin{aligned}
\alpha_n : \quad \mathbb{S}^n &\rightarrow \mathbb{S}^n \\
x &\mapsto -x
\end{aligned}$$

We say that $Y \subseteq \mathbb{S}^n$ is *n-antipodally symmetric* if $\alpha_n(Y) = Y$.

A self-dual map G is called *antipodally self-dual* if the dual map G^* is antipodally embedded in $\mathbb{S}^2$ with respect to G, that is, $\alpha_2(G) = G^*$. We say that G is *2-antipodally symmetric* map if it admits an embedding in $\mathbb{S}^2$ such that $\alpha_2(G) = G$.

Lemma 1. [9, Lemma 1] *If G is an antipodally self-dual map then $med(G)$ is 2-antipodally symmetric.*

Let G be a 2-antipodally symmetric map. If $v \in V(G)$ then its *antipodal* vertex is given by $\alpha_2(v) = -v$. We call them *antipodal pair* of vertices.

Remark 2. *If G is 2-antipodally symmetric map then its number of faces must be even. Moreover, the function α_2 naturally matches the pairs of* antipodal faces*, say f and $\alpha_2(f)$ (we may refer $\alpha_2(f)$ as the f-antipodal face of f). The latter naturally induces a permutation of the faces that turns out to be an automorphism of G^* (that is, $\alpha_2 \in Aut(G)$, and thus G^* is also 2-antipodally symmetric).*

Proposition 1. *Let G be 2-antipodally symmetric map where its faces are 2-colored properly (that is, two faces sharing an edges have different colors). Then, if one pair of antipodal faces have the same (resp. different color) then all pairs of antipodal faces have the same (resp. different color).*

Proof. Suppose that f and f-antipodal are both colored with the same color, say black, then any face f_1 adjacent to f is colored white and thus f_1-antipodal must also be colored white since it is adjacent to f-antipodal, by carry on this argument, we end with f and f-antipodal having the same color for all faces f. Similar argument can be applied when f and f-antipodal have different colors. $\square$

A *bicolored map* is a map $G = (V, E, F)$ together with a coloring $C_X : X \rightarrow \{black, white\}$ where X is either $V(G), E(G)$ or $F(G)$. A *signed map* is map $G = (V, E, F)$ together with a signature $S_Y : Y \rightarrow \{+, -\}$ where Y is either $V(G), E(G)$ or $F(G)$.

Throughout the paper, we will consider bicolored signed maps (G, C_X, S_Y), that is, maps together with both a *vertex-, edge-* or *face-coloring C_X* and a *vertex-, edge-* or *face-signature S_Y*.

Let (G, C_F, S_V) be a colored-face vertex-signed map. We say that an automorphism $\sigma(G) \in Aut(G)$ is *color-preserving* (resp. *color-reversing*) if each pair of faces f and $\sigma(f)$ have the same (resp. different) color. Similarly, σ is said to be *sign-preserving* (resp. *sign-reversing*) if each pair of vertices v and $\sigma(v)$ have the same (resp. different) sign.

Notice that in the case when (G, C_F, S_V) is a 2-antipodally symmetric map, the automorphism α_2 can be either color-preserving or color-reversing, see Figure 6.
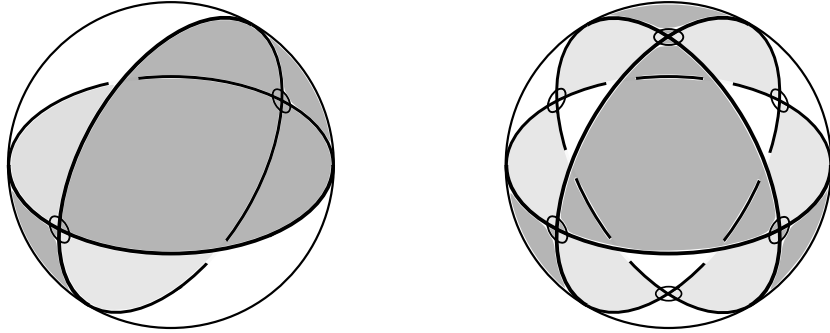

FIGURE 6. (Left) A 2-antipodally symmetric map where the antipodal mapping is color-preserving. (Right) A 2-antipodally symmetric map where the antipodal mapping is color-reversing.

Remark 3. *If G is antipodally self-dual then $med(G)$ admits a 2-antipodally symmetric embedding α_2 in $\mathbb{S}^2$ with α_2 coloring-reversing. Indeed, since G is antipodally self-dual then antipodal faces of $med(G)$ correspond to a pair of antipodal vertices, say $v \in V(G)$ and $\alpha_2(v) \in V(G^*)$. Therefore, if we color all faces corresponding to vertices in G (resp. in G^*) in black (resp. in white) we obtain that α_2 is coloring-reversing.*

3.1. **Incidence graph.** Let G be a plane graph and G^* its geometric dual. We recall that the *(vertex-face) incidence* graph $I(G) = (V^I, E^I)$ has as vertices $V^I = V(G) \cup V(G^*)$ and two vertices $v \in V(G)$ and $v^* \in V(G^*)$ are adjacent if v is a vertex of the face corresponding to v^*, see Figure 7
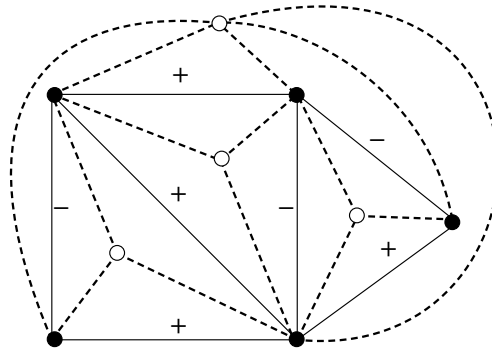

FIGURE 7. Edge-signed graph (G, S_E) (light edges) and its face-signed, vertex-colored incidence graph $(I(G), S_F)$ (dashed edges).

Remark 4. *Let G be a plane graph.*

(a) $I(G) = med(G)^*$.

(b) $I(G)$ is bipartite, the color of a vertex x is black (resp. white) if $x \in V(G)$ (resp. $x \in V(G^*)$).

(c) Each face of $I(G)$ is of length four (each face contains exactly one edge of G as a diagonal).

We have that $(I(G), C_V, S_F)$ also determines (in a canonical way) the link diagram $D(G, S_E)$. The construction is easy, we just determine the under/over pass at each each crossing (in each face) according to either the Left-over right rule or Right-over left rule, see Figure 8.
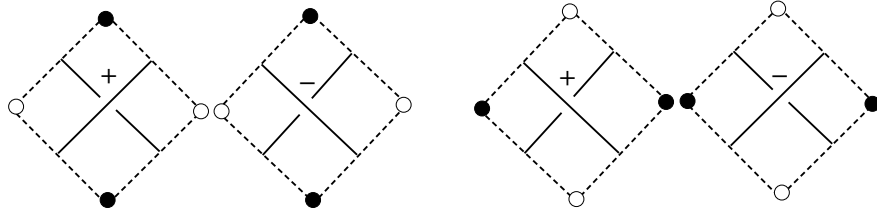

FIGURE 8. (From left to right) Left-over-right rule from black point of view and Right-over-left rule from white point of view.

We denote by C_V^o the *opposite* vertex-coloring of C_V, that is, white vertices become black and black ones become white.

Remark 5. *Let (G, S_E) be an edge-signed map and let $(I(G), C_V, S_F)$ be its vertex-face incidence graph equipped with a face-signature S_F (arising from S_E) and vertex-coloring C_V. We have that if $(I(G), C_V, S_F)$ determines link $D(G, S_E)$ then both $(I(G), C_V^o, S_F)$ and $(I(G), C_V, -S_F)$ determine $D(G, S_E)^*$.*

3.2. **Special embedding of a link.** Let $D(G, S_E)$ be the link diagram in $\mathbb{S}^2$ obtained from a map $(med(G), C_F, S_V)$. We shall construct a specific embedding of $D(G, S_E)$ in $\mathbb{R}^3$ by modifying (locally) the diagram around each crossing. We proceed as follows. Take a small sphere $\mathbb{S}_1$ around each crossing (say, with center the crossing itself). and move (locally) the piece of arc of the diagram passing over (resp. passing under) around $\mathbb{S}_1$ outside (resp. inside) of $\mathbb{S}^2$ according with the *crossing sphere rules*, see Figure 9.
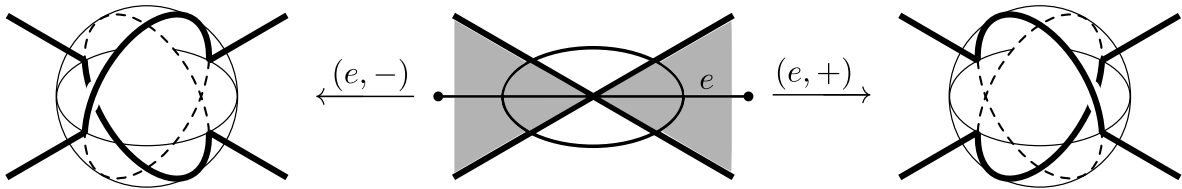

FIGURE 9. The crossing spheres rules.

The rest of the diagram $D(G, S_E)$ remains the same in $\mathbb{S}^2$. We denote by $L(G, S_E)$ such embedding, see Figure 10.
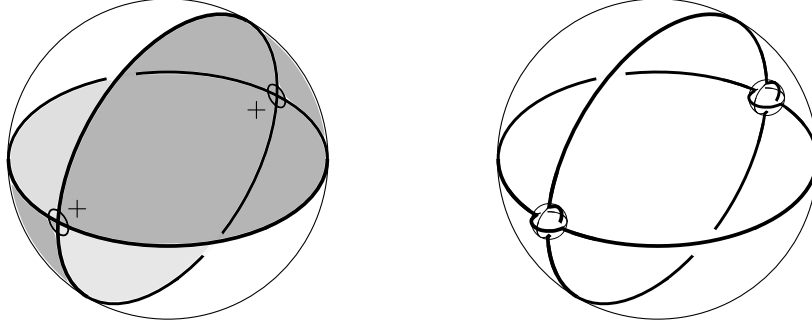
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FIGURE 10. (Left) A 2-antipodally symmetric map of the graph on 2 vertices and four parallel edges. This graph corresponds to a shadow of the Hopf link, see Figure 4 (second row). In this case, α_2 is color-preserving and sign-preserving. (Right) $L(2_1)$.

From the above discussion we have the following diagram.

$$(med(G), C_F, S_V) \quad \longleftrightarrow \quad (I(G), C_V, S_F)$$
$$\searrow \qquad\qquad \nearrow$$
$$(G, S_E)$$
$$\downarrow$$
$$D(G, S_E)$$
$$\updownarrow$$
$$L(G, S_E)$$

where $A \to B$ means B *can be constructed from* A.

4. CENTRAL SYMMETRY IN $\mathbb{R}^3$

Let us define the *centrally symmetric* function as

$$c_n : \quad \mathbb{R}^n \quad \to \quad \mathbb{R}^n$$
$$x \quad \mapsto \quad -x$$

We say that $Y \subseteq \mathbb{R}^n$ is *n-centrally symmetric* if $c_n(Y) = Y$. We say that L is 3-centrally symmetric if there is $\hat{L} \in [L], \hat{L} \in \mathbb{R}^3$ such that $c_3(\hat{L}) = \hat{L}$.

Theorem 1. *Let (G, S_E) be an face-colored edge-signed self-dual map and suppose that $med(G)$ is a 2-antipodally symmetric map (realized by α_2). If either*

a) α_2 is color-preserving and sign-reversing or
b) α_2 is color-reversing and sign-preserving

then $L(G, S_E)$ is 3-centrally symmetric.

Proof. Let $D(G, S_E)$ be the link diagram induced by $(med(G), C_F, S_V)$. Since $med(G)$ is 2-antipodally symmetric then it is also 3-centrally symmetric. Consider the embedding $L(G, S_E)$. It can be checked that if α_2 is either (a) color-preserving and sign-reversing or (b) color-reversing and sign-preserving then the piece of arc of the diagram passing over (resp.

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passing under) in crossing v correspond to the piece of arc of the diagram passing over (resp. passing under) in crossing $\alpha_2(v)$ in $L(G, S_E)$, see Figure 11.
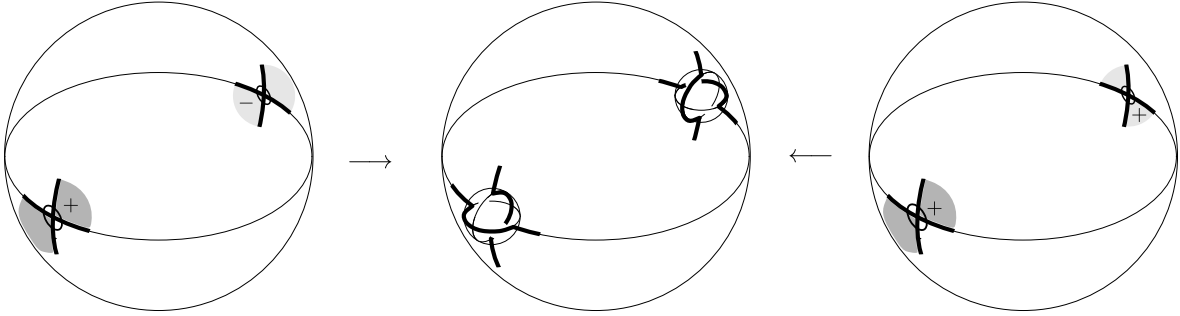

FIGURE 11. (Left) Antipodal pair of vertices of $med(G)$ in case (a) (Right) Antipodal pair of vertices of $med(G)$ in case (b) (Center) The local modifications around an antipodal pair of vertices following the crossing sphere rules.

Therefore, the modifications around v are centrally symmetric with respect to those done around $\alpha_2(v)$. We thus obtain that $L(G, S_E)$ is a 3-centrally symmetric. $\qquad \square$

Figure 10 illustrates an example in which neither condition (a) nor (b) of Theorem 1 are full filled. In this case, $L(2_1)$ does not give a 3-centrally symmetric embedding of 2_1. In fact, it can be showed that the Hopf link does not admit a 3-centrally symmetric embedding.

Corollary 1. *Let (G, S_E) be an edge-signed map and suppose that $med(G)$ is a 2-antipodally symmetric map (realized by α_2). If either*

a) α_2 is color-preserving and sign-reversing or
b) α_2 is color-reversing and sign-preserving

then $L(G, S_E)$ is amphichiral.

Proof. By Theorem 1, $c_3(L(G, S_E)) = L(G, S_E)$. We shall perform 3 reflections to this embedding with respect to the 3 orthogonal planes $x = 0, y = 0$ and $z = 0$. Let $(x, y, z) \in L'$. We have that the first reflection maps (x, y, z) into $(-x, y, z)$ the second maps $(-x, y, z)$ into $(-x, -y, z)$ and the third one maps $(-x, -y, z)$ into $(-x, -y, -z)$. Since $c_3(L(G, S_E)) = L(G, S_E)$ then these three reflections give a nonpreserving orientation homeomorphism mapping $L(G, S_E)$ into itself, implying that $L(G, S_E)$ is amphichiral.

$\qquad \square$

4.1. **Families of amphichiral links.**

Corollary 2. *Let (G, S_E^+) be an edge-signed map with G antipodally self-dual. Then, $L(G, S_E^+)$ is amphichiral.*

Proof. Since G is antipodally self-dual then, by Lemma 1, $med(G)$ is a 2-antipodally symmetric map realized by, say α_2. Since all the signs in S_E are the same then, in particular, each pair of antipodal vertices of $med(G)$ have the same sign, that is, α_2 is sign-preserving. Moreover, by Remark 3, α_2 is color-reversing with respect to $med(G)$. The result follows by Corollary 1 (b). $\qquad \square$

Let $n \geq 1$ be an integer. An *n-wheel*, denoted by W_n, is the graph consisting of an *n*-cycle with a center joined to each vertex of the cycle, see Figure 12
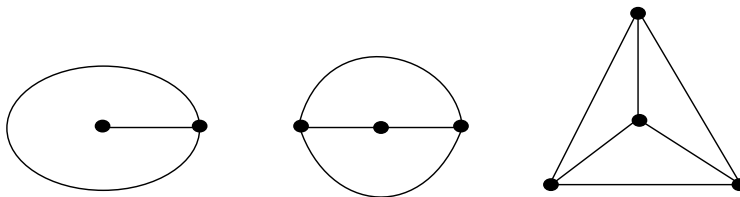

FIGURE 12. W_1, W_2 and W_3.

Corollary 3. *Let $n \geq 1$ be an odd integer and let (W_n, S_E) be a map such that $S_E(v) = S_E(-v)$ for each pair of antipodal vertices of $med(W_n)$. Then, $L(W_n, S_E)$ is amphichiral.*

Proof. By [9, Proposition 1], W_n is an antipodally self-dual map and thus, by Lemma 1, $med(W_n)$ is a 2-antipodally symmetric map realized by, say α_2. Each pair of antipodal vertices of $med(W_n)$ have the same sign, that is, α_2 is sign-preserving. Moreover, by Remark 3, α_2 is color-reversing with respect to $med(W_n)$). The result follows by Corollary 1 (b). $\square$

In the case $n = 3$, $L(W_3, S_E^+)$ is the well-known *Borromean rings*, a 3-centrally symmetric embedding is illustrated in Figure 13.
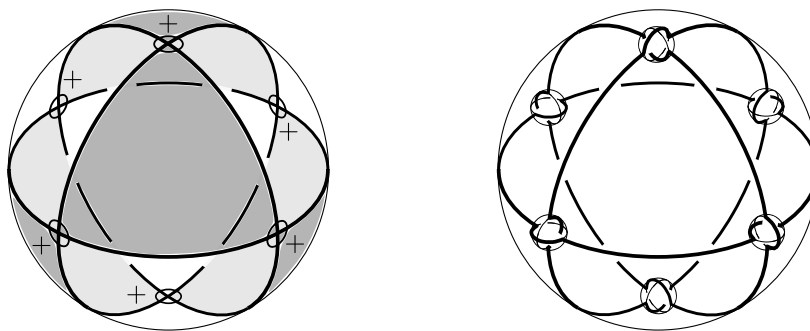

FIGURE 13. A 2-colored map (W_3, S_E^+) and $L(W_3, S_E^+)$.

An appropriate slightly straightening around the crossings arise the IMU's logo[1], see Figure 14 .
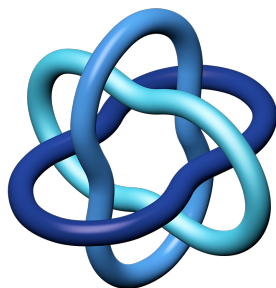
FIGURE 14. International Mathematical Union logo.

Let A_m be the m-strand braid as shown in Figure 15
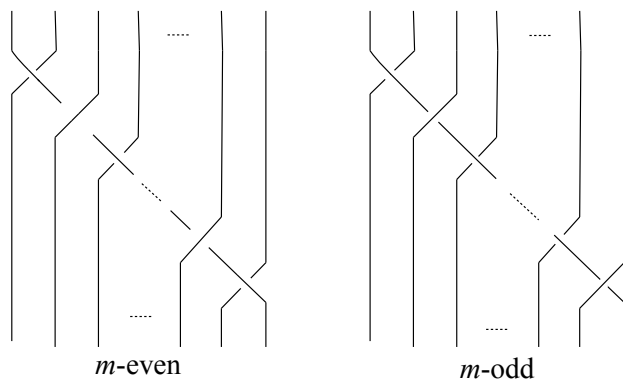

m-even   *m*-odd

FIGURE 15. Braid A_m according with the parity of m.

For $m, n \geq 1$, the (m, n)-*Turk's head* link, denoted by $TH(m, n)$, is the closure of the m-strand braid $(A_m)^n$, see Figure 16.
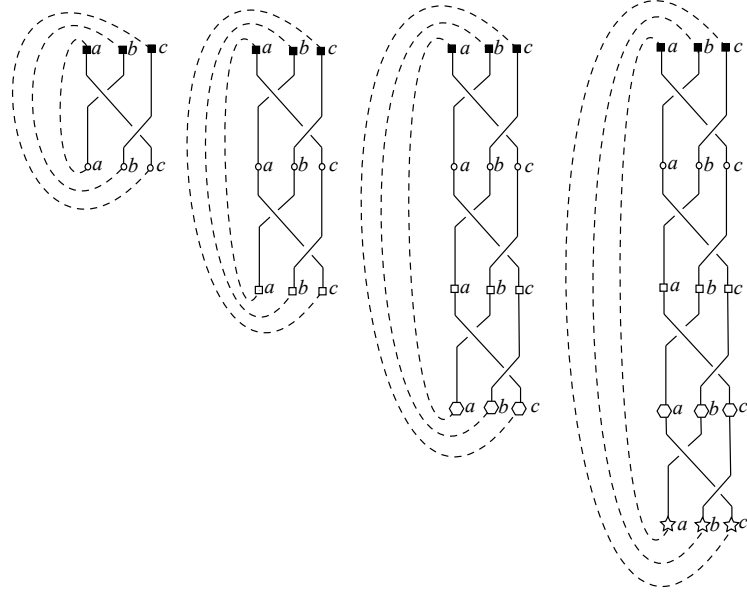
FIGURE 16. $TH(3,1), TH(3,2), TH(3,3)$ and $TH(3,4)$. We label the extremes of each A_3 brick.

Corollary 4. *Let $n \geq 1$ be an integer. Then, $TH(3,n)$ is amphichiral.*

Proof. It can be checked that $D(W_n, S_E^+) = TH(3,n)$, see Figure 17 where the first four cases are illustrated. The result then follows by Corollary 3. $\square$
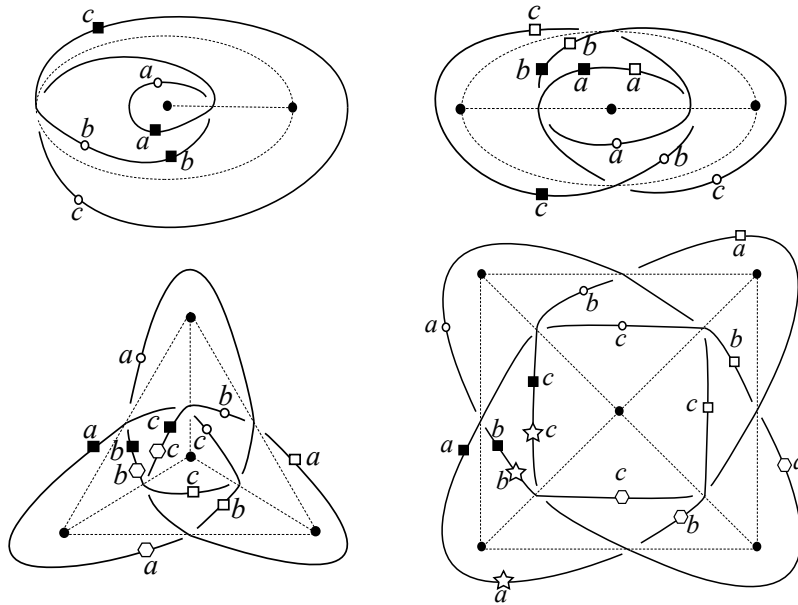

FIGURE 17. Wheel links with labelings showing the equivalence to the corresponding Turk's head links.

Let $n \geq 3$ be an integer. The *n-ear*, denoted by E_n, is the graph consisting of a n-cycle with an ear added on each edge and a center joined to each ear, see Figure 18.
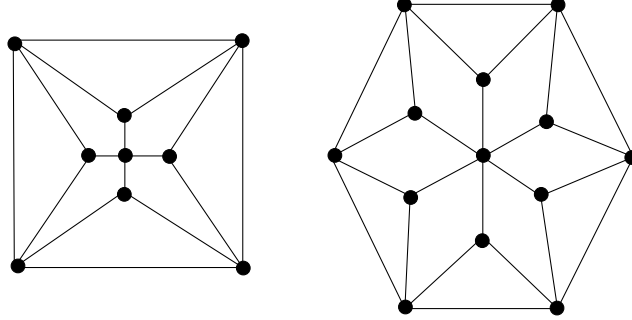

FIGURE 18. 4-ear and 6-ear.

Corollary 5. *Let $n \geq 4$ be an even integer. Let $L(E_n, S_E)$ be a map such that $S_E(v) = S_E(-v)$ for each pair of antipodal vertices of $med(E_n)$. Then, $L(E_n, S_E)$ is amphichiral.*

Proof. By [9, Proposition 2], E_n is an antipodally self-dual map and thus, by Lemma 1, $med(E_n)$ is a 2-antipodally symmetric map realized by, say α_2. Each pair of antipodal vertices of $med(E_n)$ have the same sign and, that is, α_2 is sign-preserving. Moreover, by Remark 3, α_2 is color-reversing with respect to $med(E_n)$. The result follows by Corollary 1 (b). $\square$

Let $n \geq 3$ and $\ell \geq 1$ be integers. The (n, ℓ)-pancake, denoted by P_n^ℓ, is the graph consisting of ℓ cycles $\{v_1^1, \ldots, v_n^1\}, \ldots, \{v_1^\ell, \ldots, v_n^\ell\}$, a vertex v_0 and edges $\{v_i^{j-1}, v_i^j\}$ for each $j = 1, \ldots, n$ and all i, see Figure 19.
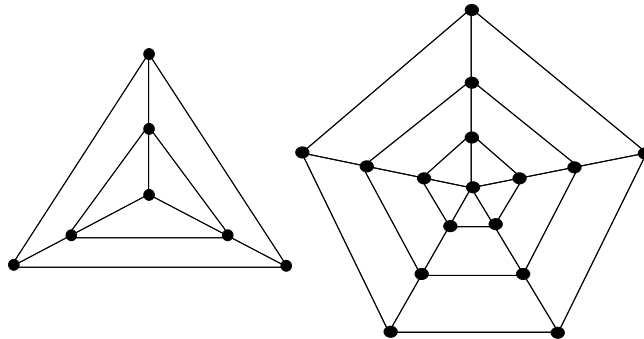

FIGURE 19. P_3^2 and P_5^3.

Corollary 6. *Let $n \geq 3$ and $\ell \geq 1$ be integers with n odd. Let $L(P_n^\ell, S_E)$ be a map such that $S_E(v) = S_E(-v)$ for each pair of antipodal vertices of $med(P_n^\ell)$. Then, $L(P_n^\ell, S_E)$ is amphichiral.*

Proof. By [9, Proposition 3], P_n^ℓ is an antipodally self-dual map and thus, by Lemma 1, $med(P_n^\ell)$ is a 2-antipodally symmetric map realized by, say α_2. Each pair of antipodal vertices of $med(P_n^\ell)$ have the same sign and, that is, α_2 is sign-preserving. Moreover, by Remark 3, α_2 is color-reversing with respect to $med(P_n^\ell)$. The result follows by Corollary 1 (b). □

Let $K_1 \# K_2$ denote the *sum* of knots K_1 and K_2.

Corollary 7. *Let K be a knot. Then, $K \# K^*$ is amphichiral.*

Proof. Let $K = D(G, S_E)$ be the knot diagram obtained from some edge-signed map (G, S_E). We have that $K^* = D(G, -S_E) = D(G^*, S_E)$. We notice that $K \# K^* = D(G \diamond G^*, S_E \cup S_E)$. Now, by [9, Theorem 3], $G \diamond G^*$ is antipodally self-dual thus, by Lemma 1, $med(G \diamond G^*)$ is a 2-antipodally symmetric map realized by, say α_2. Each pair of antipodal vertices of $med(G \diamond G^*)$ have the same sign and, that is, α_2 is sign-preserving. Moreover, by Remark 3, α_2 is color-reversing with respect to $med(G \diamond G^*)$. The result follows by Corollary 1 (b). □

Remark 6. *There are amphichiral knots that are not detected by Theorem 1 (when the Tait graph of the diagram is not antipodally self-dual). For instance, $L(W_2, S_E^+)$ turns out to be the Figure-eight knot which is well-known to be amphichiral however the 2-wheel W_2 is not antipodally self-dual map (and thus Theorem 1 cannot be applied).*

In Section 6, we propose a different method to detect amphichirality. This new approach shows, in particular, the amphichirality of the Figure-eight knot (see first example in Figure 34).

5. Antipodal symmetry in $\mathbb{S}^3$

Let us recall that $\mathbb{S}^3$ can be thought as the *1-point compactification* of $\mathbb{R}^3$, that is, we take $\mathbb{R}^3$ and an additional point denoted by ∞. By using the *stereographic projection*, $\phi_n : \mathbb{S}^n \to \mathbb{R}^n$, it can be showed that $\mathbb{R}^3 \cup \{\infty\}$ is equivalent to $\mathbb{S}^3$. Indeed, the stereographic projection sends the *South pole* $(0,0,0,-1)$ of $\mathbb{S}^3$ to $(0,0,0)$, the *Equator* of $\mathbb{S}^3$ to the unit 2-sphere, the Southern hemisphere to the region inside the unit 2-sphere, and the Northern hemisphere to the region outside of it. The stereographic projection is not defined at the projection point $(0,0,0,1)$ (the *North pole* of $\mathbb{S}^3$) and the closer a point in $\mathbb{S}^3$ is to $(0,0,0,1)$, the more distant its image is from $(0,0,0)$ in the space. For this reason, we may speak of $(0,0,0,1)$ as mapping to *infinity* in the 3-dimensional space, and $\mathbb{S}^3$ as completing $\mathbb{R}^3$ by adding a point at infinity.

We say that L is 3-antipodally symmetric if there is $\hat{L} \in [L]$, $\hat{L} \in \mathbb{S}^3$ such that $\alpha_3(\hat{L}) = \hat{L}$.

Theorem 2. *Let (G, S_E) be an edge-signed self-dual map and suppose that $med(G)$ is a 2-antipodally symmetric map (realized by α_2). If either*

   a) α_2 is color-preserving and sign-preserving or
   b) α_2 is color-reversing and sign-reversing

then $L(G, S_E)$ is 3-antipodally symmetric.

Proof. Since $med(G)$ is 2-antipodally symmetric then it is also 3-centrally symmetric. Consider the embedding $L(G, S_E)$. It can be checked that if α_2 is either (a) color-preserving and sign-preserving or (b) color-reversing and sign-reversing then the piece of arc of the diagram

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passing over (resp. passing under) in crossing v correspond to the piece of arc of the diagram passing under (resp. passing overer) in crossing $\alpha_2(v)$ in $L(G, S_E)$ see Figure 20 (center).
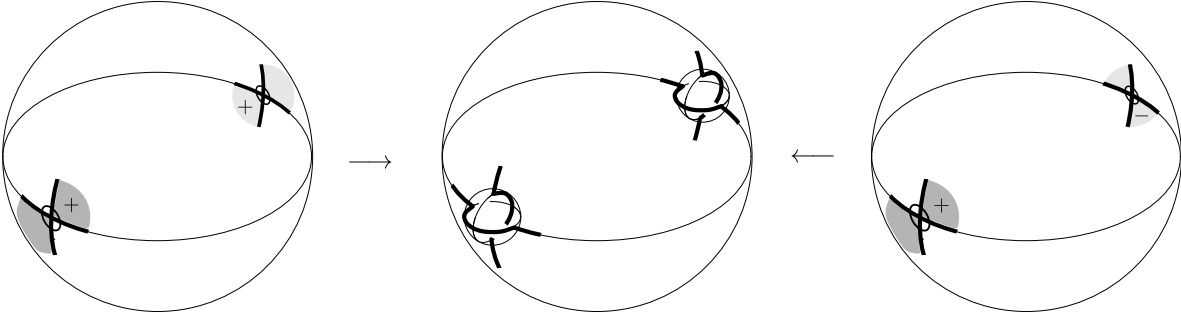

FIGURE 20. (Left) Antipodal pair of vertices of $med(G)$ in case (a). (Right) Antipodal pair of vertices of $med(G)$ in case (b) (Center) The local modifications around the antipodal pair of vertices following the color-sphere rules.

Let H be the hyperplane in $\mathbb{R}^4$ containing the Equator of $\mathbb{S}^3$. Let π be the stereographic projection of $\mathbb{S}^3$ into H. We suppose that the Equator is the unit 2-sphere. Notice that

$$\|\pi(x)\| \begin{cases} > 1 & \text{if } x \text{ is in the Northern hemisphere of } \mathbb{S}^3, \\ = 1 & \text{if } x \text{ is in the Equator of } \mathbb{S}^3, \\ < 1 & \text{if } x \text{ is in the Southern hemisphere of } \mathbb{S}^3. \end{cases}$$

We shall lift $L(G, S_E)$ into $\mathbb{S}^3$ by taking the inverse π^{-1}. We claim that $\pi^{-1}(L(G, S_E)$ gives the desired 3-antipodally symmetric embedding of $L(G, S_E)$. To see this, we define the *inversion* function (with respect to $\mathbb{S}^{n-1}$) as

$$\begin{array}{rccc} i_n : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & x & \mapsto & \frac{x}{\|x\|^2} \end{array}$$

Notice that

$$\|i(x)\| \begin{cases} > 1 & \text{if } \|x\| < 1, \\ = 1 & \text{if } \|x\| = 1, \\ < 1 & \text{if } \|x\| > 1. \end{cases}$$

Let $x \in \mathbb{S}^3$. We have that

$$\pi \circ \alpha_3(x) = \pi(-x) = c_3 \circ i_3 \circ \pi(x).$$

We thus have that the perturbed pieces in each pair of antipodal vertices are mapped into diametrically antipodal pieces in $\mathbb{S}^3$. Indeed, for instance, let us take the piece p, say inside of $\mathbb{S}^2$ around crossing v. We have then that $i_3 \circ c_3(p)$ gives the piece p' outside of $\mathbb{S}^2$ around crossing $\alpha_2(v)$. Therefore, π^{-1} maps p and p' to the Northern and Southern hemisphere of $\mathbb{S}^3$ respectively such that $\pi^{-1}(p) = \alpha_3 \circ \pi^{-1}(p) = \pi^{-1}(p')$.

We thus obtain that $\pi^{-1}(L(G, S_E)$ is an antipodally symmetric embedding in $\mathbb{S}^3$.

$\square$

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The following result is easily derived from Theorem 2 (a).

Corollary 8. *Let (G, S_E^+) be an edge-signed map and suppose that $med(G)$ is a 2-antipodally symmetric map (realized by α_2). If α_2 is color-preserving then $L(G, S_E^+)$ is 3-antipodally symmetric.*

A straightforward consequence of the above corollary, we obtain that the Hopf link is 3-antipodally symmetric, see Figure 10.

Corollary 9. *Let $n \geq 1$ be an odd integer and let $L(W_n, S_E)$ be a map such that $S_E(v) = -S_E(-v)$ for each pair of antipodal vertices of $med(W_n)$. Then, $L(W_n, S_E)$ is 3-antipodally symmetric.*

Proof. By [9, Proposition 1], W_n is an antipodally self-dual map and thus, by Lemma 1, $med(W_n)$ is a 2-antipodally symmetric map, realized by, say α_2. By hypothesis, each pair of antipodal vertices of $med(W_n)$ have opposite signs and thus α_2 is sign-reversing. Moreover, by Remark 3, α_2 is color-reversing with respect to $med(W_n)$. The result follows by Theorem 2 (b). $\square$

Corollary 10. *Let $n \geq 2$ be an odd integer. Then, the* torus knot $T(2, n)$ *is 3-antipodally symmetric.*

Proof. Let S_E' be the edge-signature of W_n where the edges of the exterior cycle have $+$ sign and edges incident with the center have $-$ sign. It can be checked that the diagram $D(W_n, S_E')$ is the same as the one for $T(2, n)$. The result then follows by Corollary 9. $\square$

Corollary 11. *Let K be a knot. Then, $K \# K$ is 3-antipodally symmetric.*

Proof. Let $K = D(G, S_E)$ for some edge-signed map (G, S_E). We have that $D(G, S_E) = D(G^*, -S_E)$. We notice that $K \# K = D(G \diamond G, S_E \cup -S_E)$. Now, by [9, Theorem 3], $G \diamond G^*$ is antipodally self-dual thus, by Lemma 1, $med(G \diamond G^*)$ is a 2-antipodally symmetric map realized by, say α_2. Each pair of antipodal vertices of $med(G \diamond G^*)$ have the opposite signs, that is, α_2 is sign-reverserving. Moreover, by Remark 3, α_2 is color-reversing with respect to $med(G \diamond G^*)$. The result follows by Theorem 2 (b). $\square$

6. Γ-CURVE

Let $G = (V, E, F)$ be a map. Let Γ be a non self-intersecting curve in the plane starting at one vertex in $V(G)$ verifying the following conditions :

a) Γ may passes throughout vertices,
b) Γ may contain edges (if it contains an edge it also contains its extreme vertices). Γ does not intersect edges otherwise.

We have that Γ is composed of paths and intervals of consecutive vertices, called *vertex-interval*. If Γ has not vertex-interval then Γ is just a cycle of G, called Γ-*cycle*. It will be called Γ-*noncycle* otherwise.
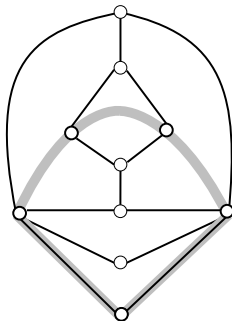
FIGURE 21. A Γ-noncycle (in thick gray) of G consisting of two edges and a vertex-interval formed by four consecutive vertices.

We say that Γ is *even* if it contains an even number of vertices of G. If $V(G)$ is 2-colored, we say that Γ is *adequate* if the vertices, in each interval of consecutive vertices are monochromatic (they have the same color). We notice that the color of the vertices in each path contained in Γ alternates, see Figure 22.
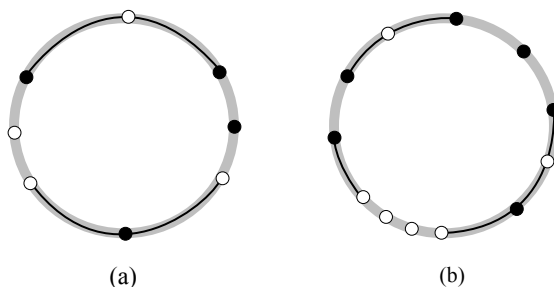

(a)   (b)

FIGURE 22. (a) An even nonadequate Γ-noncycle consisting of two paths (each of length 2) and of two vertex-interval formed by three consecutive vertices (none of them monochromatic) (b) An even adequate Γ-noncycle consisting of three paths (two of length 2 and one of length 1) and three vertex-interval (all monochromatic), formed by two consecutive black vertices, three consecutive black vertices and four consecutive white vertices).

For a curve Γ in G, we denote by $V_{int(\Gamma)}$ (resp. $V_{ext(\Gamma)}$) the set of vertices in the interior (resp. exterior) of Γ. Let $G_{int(\Gamma)} = [V \setminus V_{ext(\Gamma)}]$ (resp. $G_{ext(\Gamma)} = [V \setminus V_{int(\Gamma)}]$), that is, $G_{int(\Gamma)}$ (resp. $G_{ext(\Gamma)}$) is the the graph induced by vertices $V \cup V_{int(\Gamma)}$ (resp. $V \cup V_{ext(\Gamma)}$).

A curve Γ of G is said to be *symmetric* if there is a bijection

$$\sigma : V(G_{int(\Gamma)}) \longrightarrow V(G_{ext(\Gamma)})$$

such that $\{u, v\} \in E(G_{int(\Gamma)})$ if and only if $\{\sigma(u), \sigma(v)\} \in E(G_{ext(\Gamma)})$ with $\sigma(\Gamma) = \Gamma$.

In other words, Γ is symmetric if there is an isomorphism σ between $G_{int(\Gamma)}$ and $G_{ext(\Gamma)}$ where Γ is fixed. We notice that σ can be thought as an automorphism of G with $\sigma|_\Gamma = \Gamma$. In this case, σ is color-preserving (resp. color-reversing) if each pair of vertices $v \in V_{int(\Gamma)}$ and $\sigma(v) \in V_{ext(\Gamma)}$ have the same (resp. opposite) color and it is sign-preserving (resp. sign-reversing) if each pair of faces $f \in F(G_{int(\Gamma)})$ and $\sigma(f) \in F(G_{ext(\Gamma)})$ have the same (resp. different) sign.
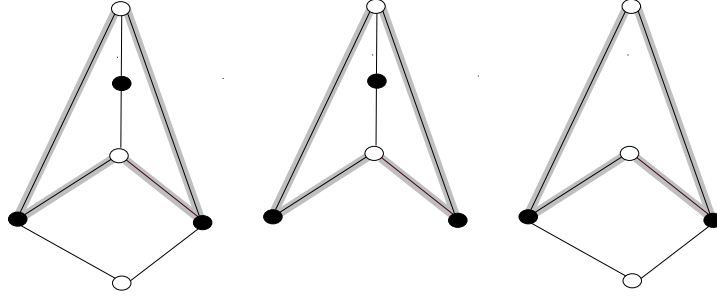
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FIGURE 23. (From left to right) Graph G with a color-reversing symmetric Γ-cycle (in gray), $G_{int(\Gamma)}$ and $G_{ext(\Gamma)}$.

Let us suppose that Γ is an even adequate curve of $(I(G), C_V, S_F)$. Let I_Γ be an interval of vertices in Γ. Since the vertices in I_Γ are monochromatic and the faces of $I(G)$ are squares then two consecutive vertices in I_Γ belong to a face with two edges in $G_{int(\Gamma)}$ and two edges in $G_{ext(\Gamma)}$. We call such face a *shared face* with respect to Γ, see Figure 24.
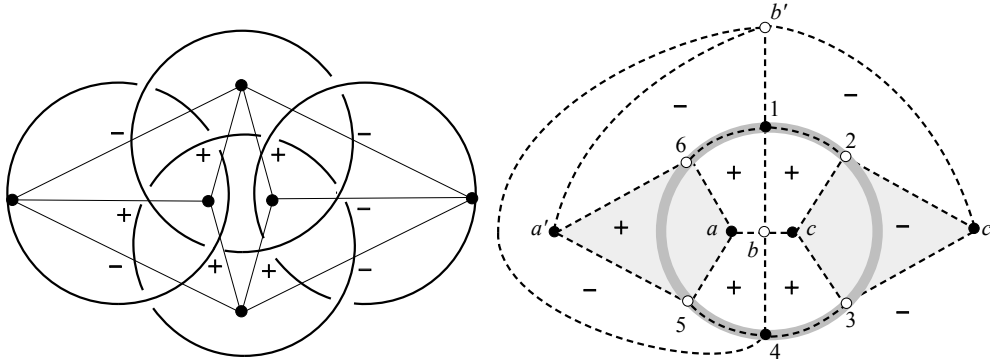

FIGURE 24. (Left) Edge-signed graph map (G, S_E) together with $D(G, S_E)$ (Right) Face-signed incidence graph $(I(G), C_V, S_F)$ with an even adequate sign-reversing curve Γ (gray) with $\sigma(a) = a', \sigma(b) = b', \sigma(c) = c'$ and $\sigma(\Gamma) = \Gamma$ (shared faces are shaded).

Since σ keeps Γ invariant then we should have that $\sigma|_\Gamma$ is either a rotation or a reflection of Γ. Let Γ be symmetric curve of a graph G. Suppose that we draw $G_{int(\Gamma)}$ in the plane with Γ as regular polygon. Let r_Δ be the reflection of $G_{int(\Gamma)}$ with respect to a line Δ. Since $r_\Delta(G_{int(\Gamma)}) = G_{int(\Gamma)}$ and Γ is symmetric then $r_\Delta(G_{int(\Gamma)})$ can be thought as $G_{ext(\Gamma)}$. The latter induce an automorphism $\sigma(G)$ if the graph obtained by identifying the two regular polygones (together with their labels) gives G, see Figure 25.
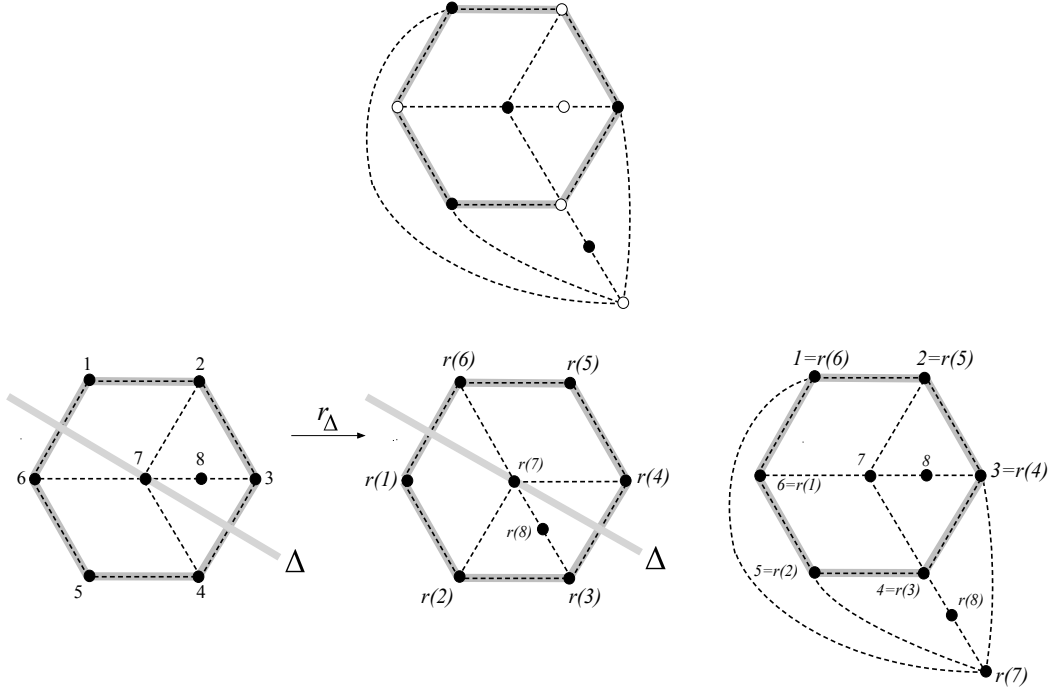
FIGURE 25. (Top) Bicolored incidence graph $I(G)$ (dashed edges) with a symmetric Γ-cycle (in gray). (Bottom) A drawing of $I(G)_{int(\Gamma)}$ with a reflection line Δ, a drawing of $I(G)_{ext(\Gamma)}$ (given by r_Δ) and the graph obtained by identifying Γ (in this case we obtain G and thus r_Δ induces an automorphism $\sigma(I(G))$.

We notice that there might be reflections of a symmetric Γ curve of graph G not inducing an automorphism of G, see Figure 26.
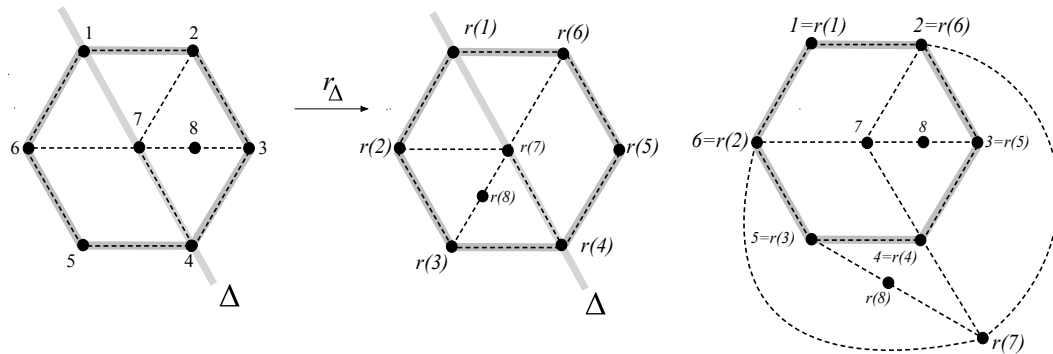
FIGURE 26. A drawing of $I(G)_{int(\Gamma)}$ (of the incident graph of Figure 25) with a reflection line Δ, a drawing of $I(G)_{ext(\Gamma)}$ (given by r_Δ) and the graph obtained by identifying Γ. In this case such a graph is not isomorphic to $I(G)$.

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We say that Γ is *reflexive* if σ arises on this way. In this case, we have that either Δ cuts twice Γ between two vertices (either on an edge or between two vertices in a vertex-interval) or Δ passes either through two vertices or through one vertex and a shared face. We notice that there are graphs admitting a symmetric Γ curve which is not reflexive, see Figure 27.
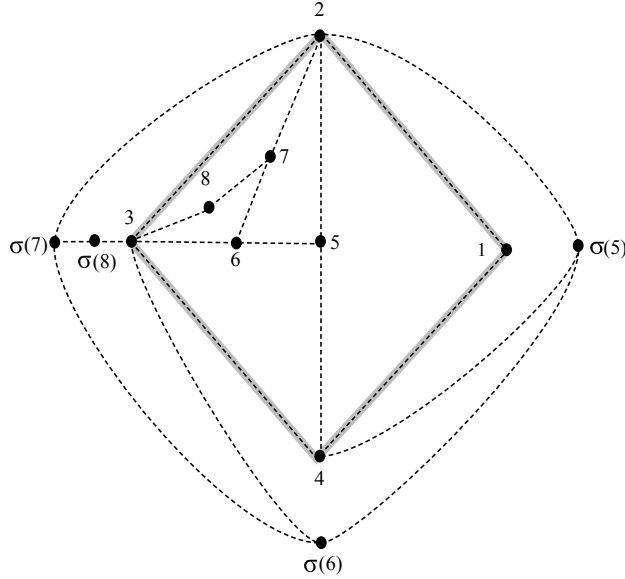

FIGURE 27. A symmetric nonreflexive Γ cycle.

Theorem 3. *Let (G, S_E) be an edge-signed map. If $(I(G), C_V, S_F)$ admits either*
  (a) a sign-preserving color-reversing reflexive curve or
  (b) a sign-reversing color-preserving reflexive curve
  then $L(G, S_E)$ is amphichiral.
  Moreover, $L(G, S_E)$ and its mirror can be embedded such that one can be obtained from the other by performing a rotation π degrees.

Proof. (a) Let Γ be a sign-preserving color-reversing reflexive curve of $(I(G), C_V, S_F)$ realized by σ. We shall construct an embedding of $(I(G), C_V, S_F)$ in $\mathbb{S}^2$ from which we can pass to $(I(G), C_V^o, S_F)$ by performing a rotation of π degrees. Since $(I(G), C_V, S_F)$ determines $L(G, S_E)$ then, by Remark 5, $(I(G), C_V^o, S_F)$ determines $L(G, S_E)^*$. Therefore, this rotation of π degrees would give the desired orientation-preserving homeomorphism from $L(G, S_E)$ to $L(G, S_E)^*$.

The construction of the embedding is in three steps, we will be illustrating these steps by applying them to the map given in Figure 25.

Step 1) We draw $G_{int(\Gamma)}$ in the Northern hemisphere of $\mathbb{S}^2$ by taking the drawing of $G_{int(\Gamma)}$ together with the reflection line in $\mathbb{B}^1$ with Γ forming the vertices of a regular polygon lying in the equator of $\mathbb{S}^2$. We then project this to the Northern hemisphere of $\mathbb{S}^2$, see Figure 28.
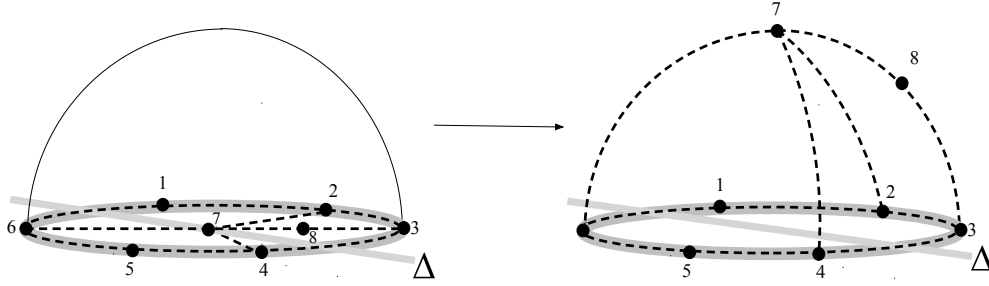
FIGURE 28. Step 1.

Step 2) We draw $G_{ext(\Gamma)}$ in the Southern hemisphere by taking the drawing of $r_\Delta(G_{int(\Gamma)})$ with $r_\Delta(\Gamma)$ forming the vertices of a regular polygon lying in the equator of $\mathbb{S}^2$. We then project this to the Southern hemisphere of $\mathbb{S}^2$. We notice that the same drawing can be obtained by performing a rotation of π degrees around Δ of the drawing of $G_{int(\Gamma)}$ in the Northern hemisphere, see Figure 29.
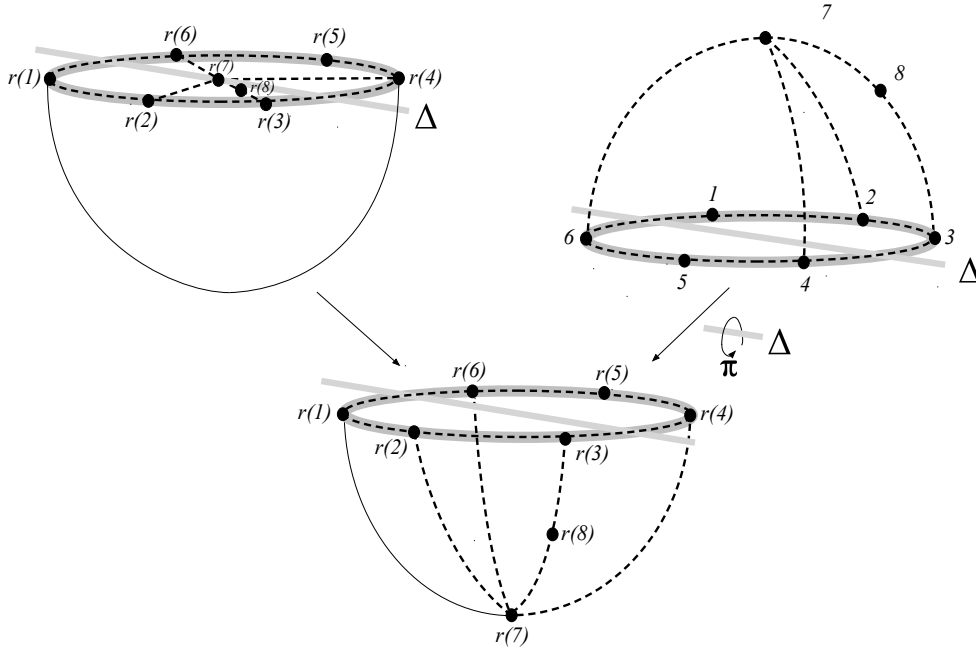

FIGURE 29. Step 2.

Step 3) We color the vertices of both embeddings as given by C_V and update labels according with r_Δ. We finally glue together the two hemispheres. Notice that vertices in Γ with the same labels are aligned one above the other, see Figure 30.
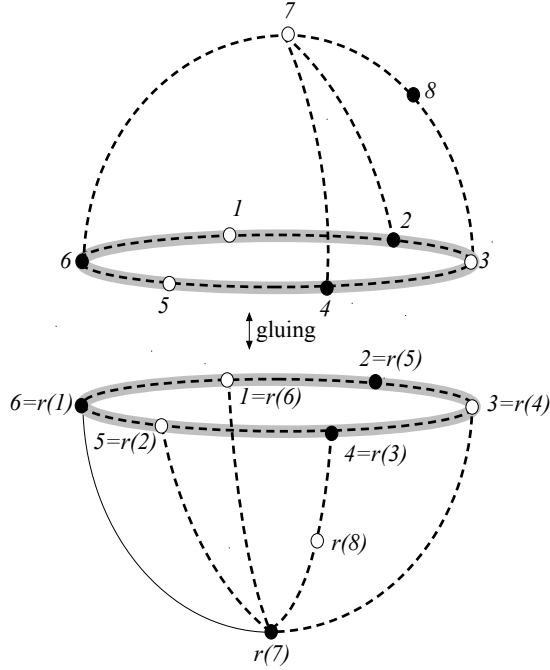
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FIGURE 30. Step 3.

We observe that a rotation of π degrees around Δ of the drawing of $(I(G), C_V, S_F)$ exchanges faces $f \in G_{int(\Gamma)}$ and $\sigma(f) \in G_{ext(\Gamma)}$ both having the same signs (since Γ is sign-preserving) but swapping the coloring (since Γ is color-reversing). We thus have that we can pass from $(I(G), C_V, S_F)$ to $(I(G), C_V^o, S_F)$ by applying a rotation of π degrees, see Figure 31.
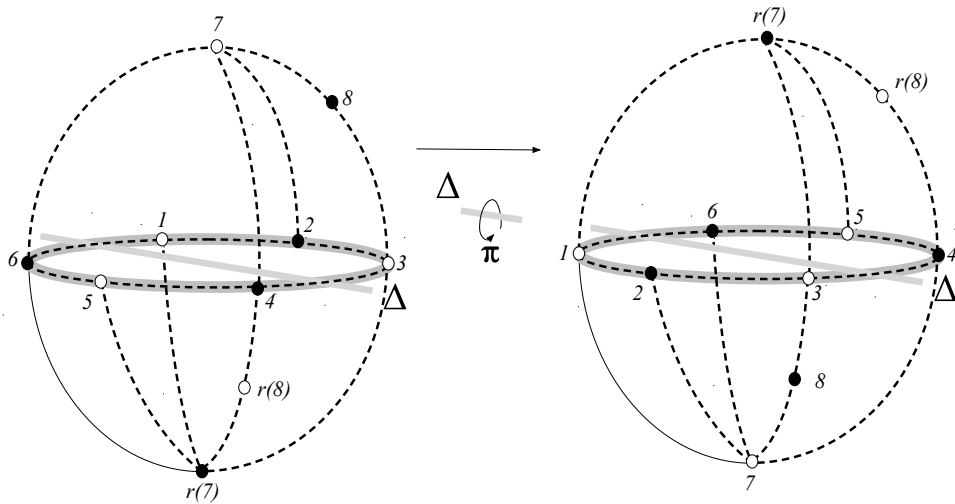

FIGURE 31. Rotation around Δ from $(I(G), C_V, S_F)$ to $(I(G), C_V, S_F)$.

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(b) Let Γ be a sign-reversing color-preserving reflexive curve of $(I(G), C_V, S_F)$. We construct an embedding of $(I(G), C_V, S_F)$ in $\mathbb{S}^2$ from which we can pass to $(I(G), C_V, -S_F)$ by performing a rotation of π degrees. Since $(I(G), C_V, S_F)$ determines $L(G, S_E)$ then, by Remark 5, $(I(G), C_V, -S_F)$ determines $L(G, S_E)^*$. therefore, this rotation of π degrees would give the desired orientation-preserving homeomorphism from $L(G, S_E)$ to $L(G, S_E)^*$. same We construct such a map in the way as in the case (a). This is illustrated in Figure 32 (applied to the incidence graph given in Figure 24).
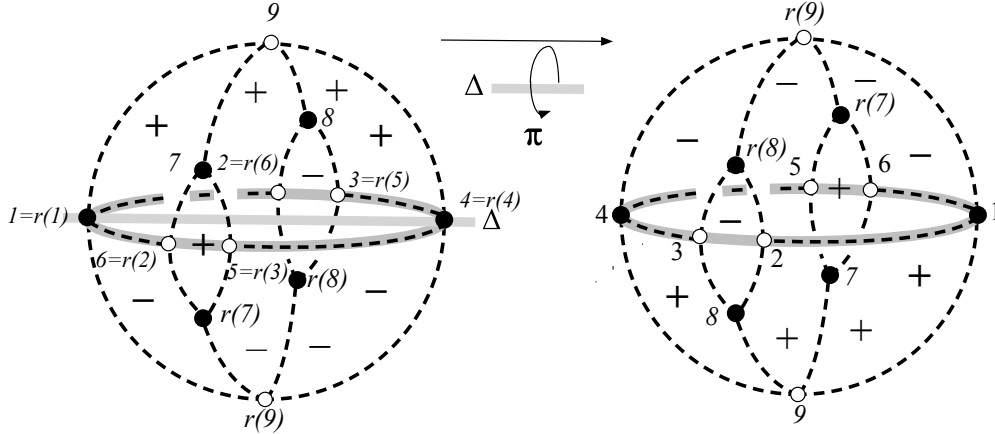

FIGURE 32. Rotation Δ from $(I(G), C_V, S_F)$ to $(I(G), C_V, -S_F)$.

We observe that a rotation of π degrees around Δ of the drawing of $(I(G), C_V, S_F)$ exchanges faces $f \in G_{int(\Gamma)}$ and $\sigma(f) \in G_{ext(\Gamma)}$ having different signs (since Γ is sign-reversing) and keeping the coloring (since Γ is color-preserving). We thus have that we can pass from $(I(G), C_V, S_F)$ to $(I(G), C_V, -S_F)$ by applying the rotation of π degrees. $\square$

Corollary 12. *Let $n \geq 2$ be an even integer and let (W_n, S_E^+) be an edge-signed map. Then, $L(W_n, S_E^+)$ is amphichiral.*

Proof. The result follows by Theorem 3 (a) by noticing that $(I(W_n), S_F^+)$ with $n \geq 2$ even always admits a sign-preserving color-reversing reflexive cycle, see Figure 33.
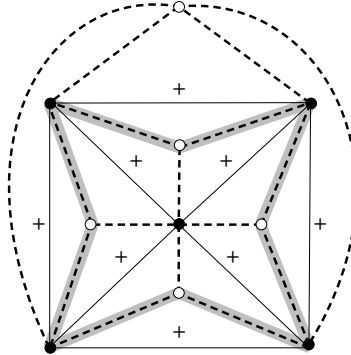

FIGURE 33. (W_4, S_E^+) (light edges) and $(I(W_4), S_F^+)$ (dashed edges) together with a sign-preserving color-reversing reflexive Γ-cycle (gray).
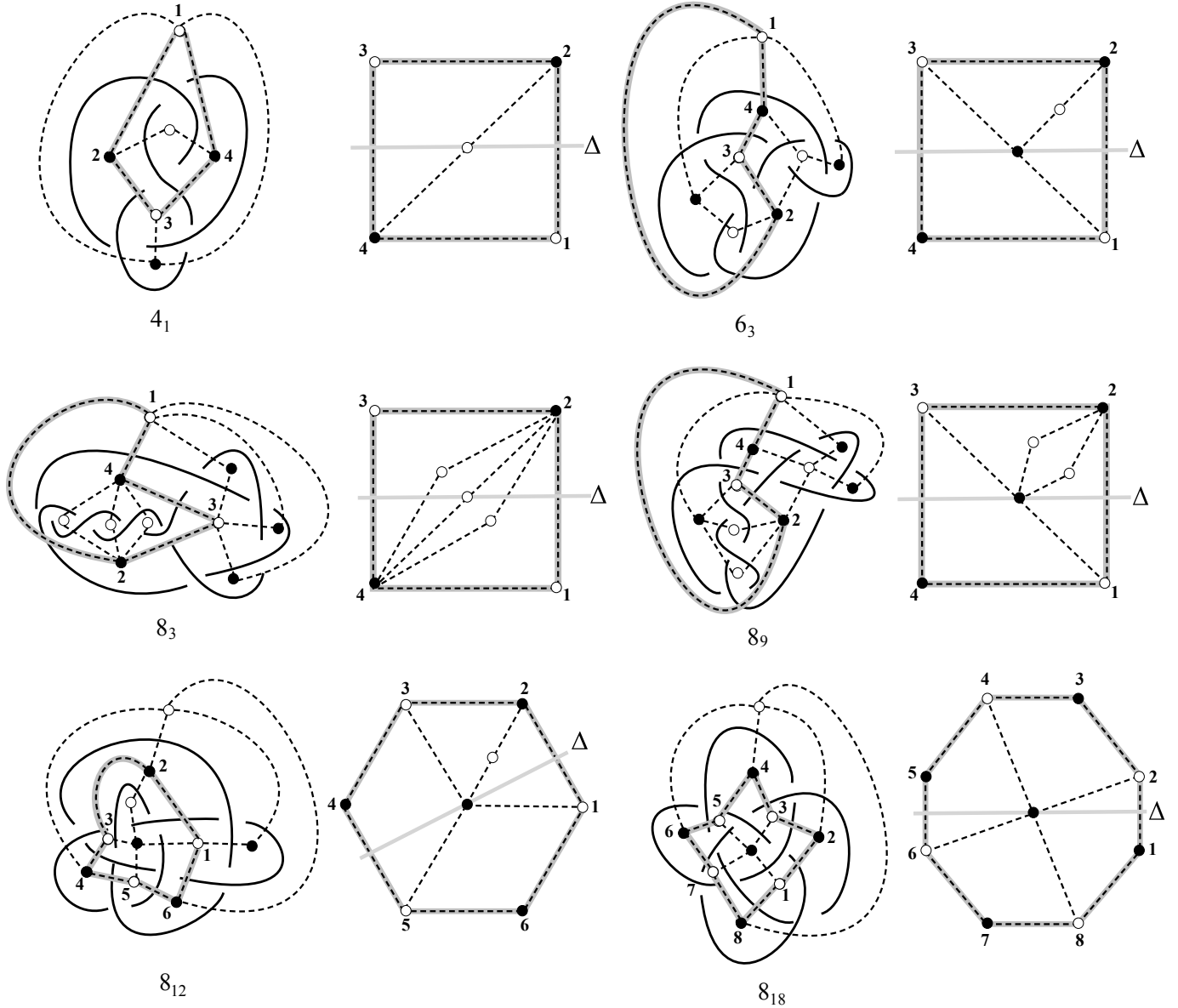
FIGURE 34. Knots (bold) with the incidence graph (dashed edges) admitting a color-preserving reflexive Γ-cycle (gray) together with a drawing of $I(G)_{int(\Gamma)}$ with a reflection line.

$\square$

This corollary implies, in particular, that the Eight-figure knot is amphichiral, see first diagram in Figure 34.

6.1. **Constructing many amphichiral links.** Let $(I(G), C_V, S_F)$ be a bicolored face-signed map where $I(G)$ is the incident graph of G. If $I(G)$ admits a sign-preserving color-reversing reflexive cycle Γ_σ then we are able to construct another bicolored face-signed map $(I'(G), C_V, S'_F)$ also admitting a sign-preserving color-reversing reflexive cycle $\Gamma_{\sigma'}$ where the

isomorphism σ' would be a natural extension of σ. We proceed as follows. Let f be a face in the interior of Γ_σ. We partition f into 5 square faces by inserting a new square in the interior and joining corresponding corners (with the appropriate vertex coloring) and assign a sign to each of the new faces, see Figure 35.
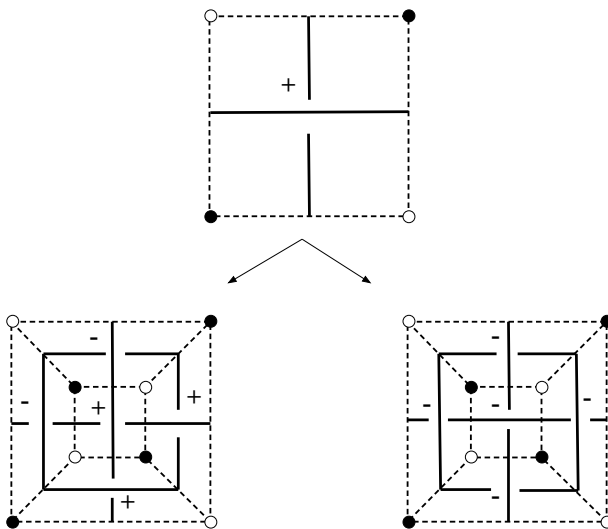

FIGURE 35. Decomposition of a square into 5 squares. Two different signatures are illustrated inducing a different local modification of the link diagram (in bold lines).

Symmetrically, we do the same to decomposition to $\varphi(f)$ where the new squares are properly signed, that is, with the same sign as the corresponding new squares in f. Let $\Gamma_{\varphi'}$ be the same cycle as Γ_σ where $\sigma(v)' = \sigma(v)$ for any $v \in V(I(G))$ and $\sigma'(v) = v'$ for each new vertex $v \in f$ and $v' \in \sigma(f)$. We clearly have that $\Gamma_{\sigma'}$ is a sign-preserving color-reversing reflexive cycle in $(I'(G), C_V, S'_F)$. Therefore, by Theorem 3, $(I'(G), C_V, S'_F)$ determines another amphichiral link $L(I'(G), S'_F)$.

The above construction can be mimic with a different decomposition of a face. For instance, a second decomposition is illustrated in Figure 36.
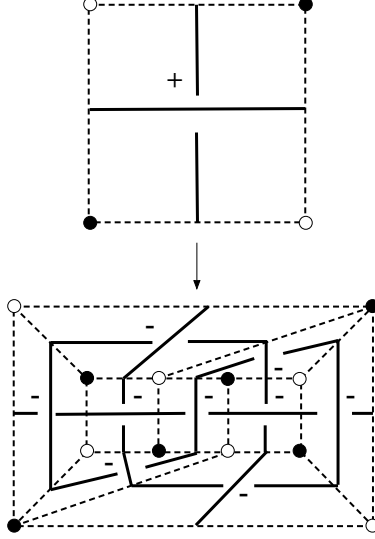
FIGURE 36. A decomposition of a square into 7 squares together with the link diagram changes (bold lines).

6.2. **Amphichiral number.** Let $D(L)$ be a diagram of link L. We define the *amphichiral number* of L, denoted by $\mathrm{Amph}(L)$, as

$\mathrm{Amph}(L) :=$ the minimal number of crossing switches in $D(L)$ to become L amphichiral.

We clearly have that $\mathrm{Amph}(L) = 0$ if L is already amphichiral and $\mathrm{Amph}(K) \leq U(K)$ where $U(K)$ is the *unknotting number* of K, that is, the minimum number of crossing switch to untie K.

Proposition 2. *Let (G, S_E) be an edge-signed map with G antipodally self-dual map. Then,*

$$\mathrm{Amph}(D(G, S_E)) \leq \frac{n}{2}$$

where n is the number of crossings of $D(G, S_E)$.

Proof. In [9, Theorem 1] was proved that if G is an antipodally self-dual map then $I(G)$ admits an isomorphism $\sigma(I(G))$ such that $\sigma(\Gamma) = \Gamma$ and $\sigma(I(G)_{int(\Gamma)}) = I(G)_{ext(\Gamma)}$. Moreover, by Remark 3, Γ_σ is color-reversing.

We may now force Γ_σ to become sign-preversing by switching the signs of the faces in $I(G)_{ext(\Gamma)}$ properly, that is, such that the face $\sigma(f) \in I(G)_{ext(\Gamma)}$ has the same sign as face $f \in I(G)_{int(\Gamma)}$. We might need to make at most $\frac{n}{2}$ such switches in the worse case (notice that n is even since G is antipodally self-dual). Notice that such switchings correspond to crossing switchings of the diagram. The result follows by Theorem 3 (a). $\square$

6.3. **Invertible knots.** It turns out that closely related drawings to those considered above for reflexive curves are useful for some other issues. For instance, they are useful to investigate whether a link is rigidly amphichiral in $\mathbb{S}^3$ but not necessarily rigidly amphichiral in $\mathbb{R}^3$ (the other direction is always true). Indeed, we may construct appropriate drawings to show the latter as follows. Take a drawing of $G_{int(\Gamma)}$ together with the reflection line Δ in $\mathbb{B}^1$ with Γ_σ forming the vertices of a regular polygon lying in the equator of $\mathbb{S}^2$. We then project this to both the Northern and the Southern hemispheres of $\mathbb{S}^2$ and make a rotation of $\mathbb{S}^2$ such that

line Δ going through the North pole. We finally take a stereographic projection in which Γ becomes an infinity line (the ends of the string are closed up at infinity in $\mathbb{S}^3$), see Figure 37.
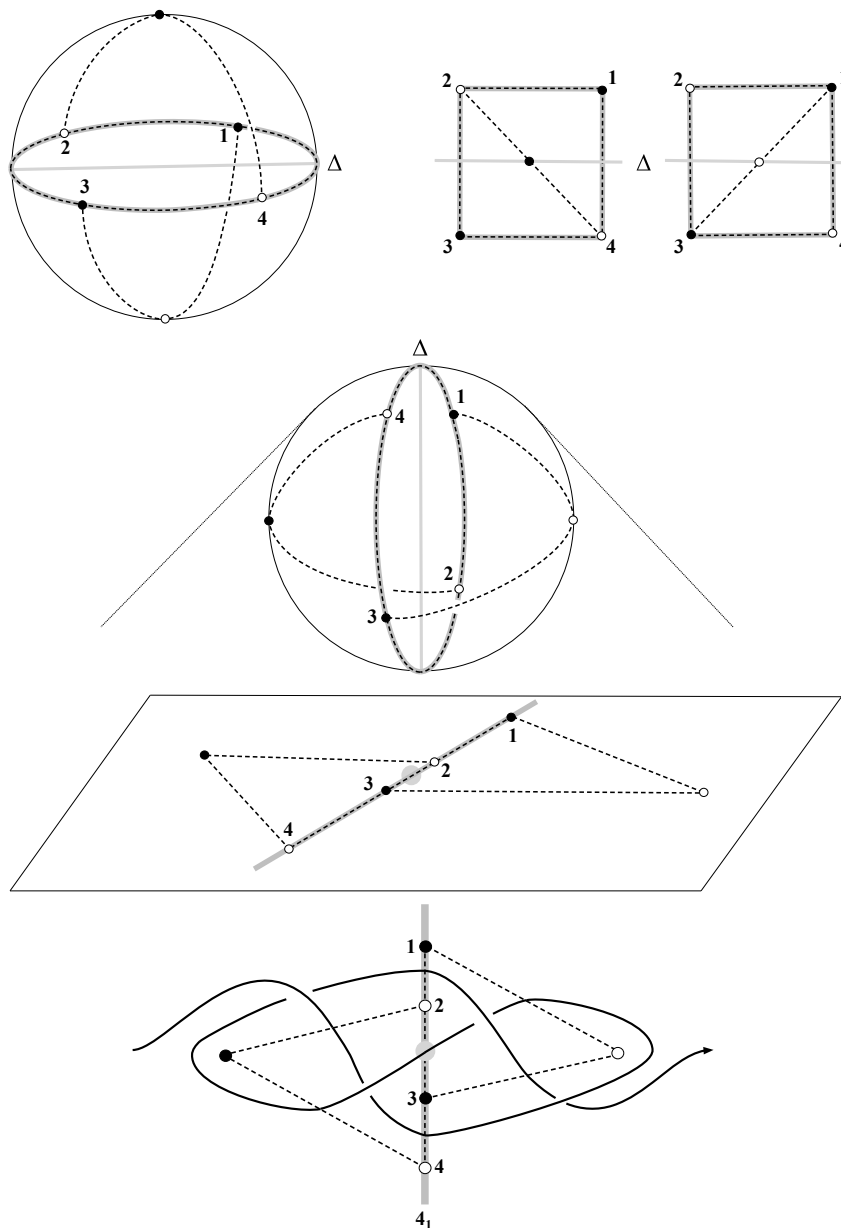
FIGURE 37. (Top) drawing of $I(G)$ together with the projections of both the Northern and the Southern hemispheres. (Middle) Rotation of drawing with the reflection line Δ going through the North pole together with the stereographic projection. (Bottom) Diagram of the induced knot 4_1.

Observe that these embeddings are invariant under the orientation-preserving homeomorphism of $\mathbb{S}^3$ given by a rotation by π degrees about an axis going through the center point of the knot and the point at infinity, composed with a reflection through the plane of the

FIGURE 38. Symmetry representation for knots $6_3, 8_3$ and 8_9 in $\mathbb{S}^3$ that can be rotated by π degrees to obtain their mirror images.
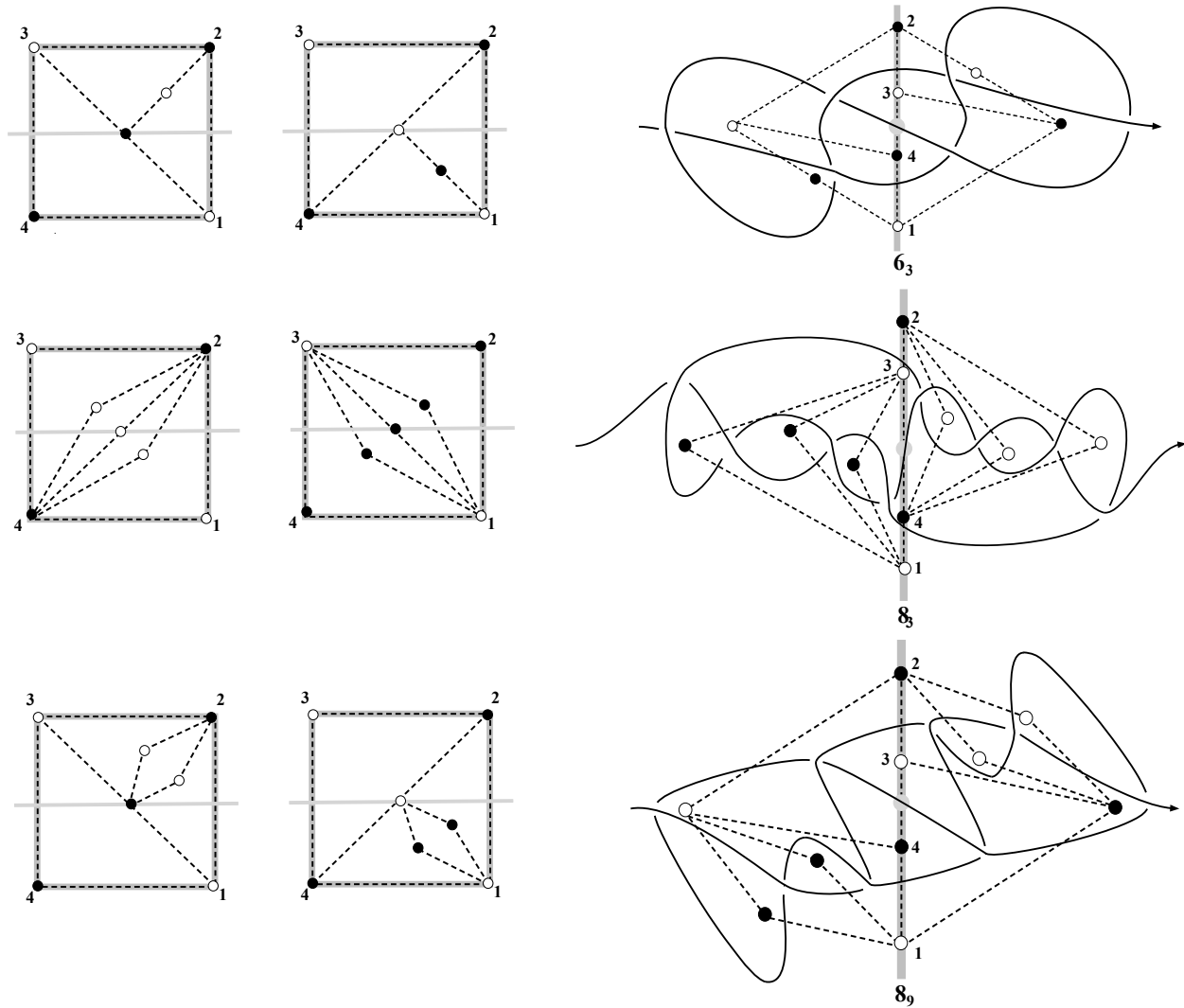
paper. We notice that if we want to restrict a homeomorphism of $\mathbb{S}^3$ to a homeomorphism of $\mathbb{R}^3$, we must select a point that is fixed by the homeomorphism to be our point at infinity in $\mathbb{S}^3$. Both of the fixed points of the homeomorphism in our construction are on the knot, so there is no way to restrict this homeomorphism of $\mathbb{S}^3$ to a homeomorphism of the knot in $\mathbb{R}^3$, see [4, Chapter 4] for a nice discussion on this topic.

We also observe that a rotation of π degree about an axe going through the center point (of the plane) of any of these oriented diagrams give the diagrams of their mirrors with a reversed orientation, that is, the corresponding knots are negative amphichiral.

7. SELF-DUAL PAIRING

Let us quickly recall some notions on the classification of the self-dual maps.
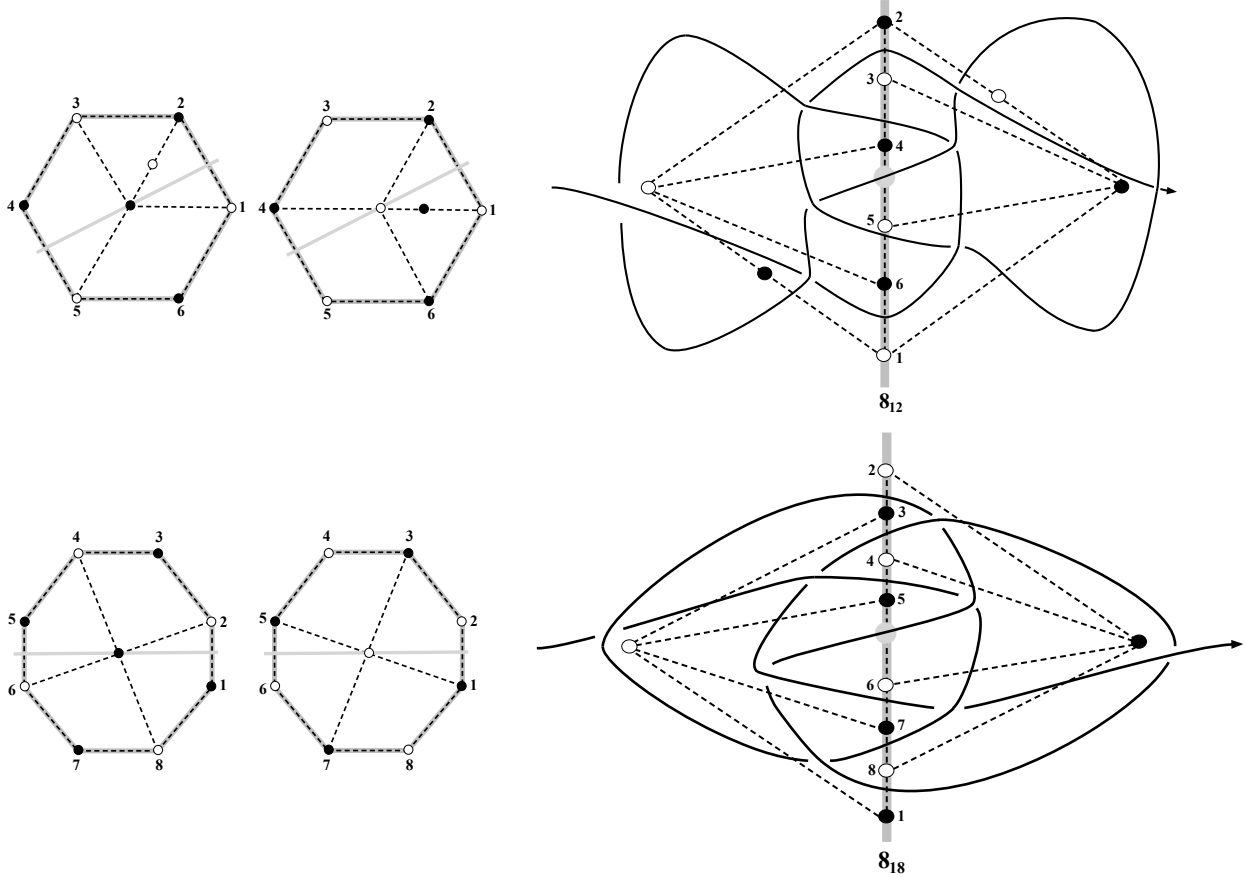
FIGURE 39. Symmetry representation for knots 8_{12} and 8_{18} in $\mathbb{S}^3$ that can be rotated by π degrees to obtain their mirror images.

Let $Aut(G)$ be the group formed by the set of all *automorphism* of G (i.e., the set of isomorphisms of G into itself). Let $Dual(G)$ be the set of all *duality isomorphisms* of G into G^*. We notice that $Dual(G)$ is not a group since the composition of any two of them is an automorphism.

Let us suppose that $G = (V, E, F)$ is a self-dual map so that there is a bijection $\phi :$ $(V, E, F) \rightarrow (F^*, E^*, V^*)$. Following ϕ with the correspondence $*$ gives a permutation on $V \cup E \cup F$ which preserve incidences but reverses dimension of the elements. The collection of all such permutations or *self-dualities* generate a group $Cor(G) = Aut(G) \cup Dual(G)$ in which the automorphisms $Aut(G)$ are contained as a subgroup of index 2.

It is known [11, Lemma 1] that for a given map G there is an homeomorphism ρ of $\mathbb{S}^2$ to itself such that for every $\sigma \in Aut(G)$ we have that $\rho\sigma \in Isom(\mathbb{S}^2)$ where $Isom(\mathbb{S}^2)$ is the group of isometries of the 2-sphere. In other words, any planar graph G can be drawn on the 2-sphere such that any automorphism of G act as an isometry of the sphere. This was extended in [11] by showing that given any self-dual graph G there are maps G and G^* so that $Cor(G)$ is realized as a group of spherical isometries.

From now on, we will denote by $\widehat{G} = \rho(G)$ and $\widehat{\sigma} = \rho\sigma$ for a certain homeomorphism ρ satisfying the above property.

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The couple $Cor(G) \rhd Aut(G)$ is called the *self-dual pairing* of the map G. In [11] were enumerated and classified all self-dual maps. In the notation of [2] the possible 24 pairings are :

- among the infinite classes $[2, q] \rhd [q], [2, q]^+ \rhd [q]^+, [2^+, 2q] \rhd [2q], [2, q^+] \rhd [q]^+$ and $[2^+, 2q^+] \rhd [2q]^+$ or

- among the special pairings $[2] \rhd [1], [2] \rhd [2]^+, [4] \rhd [2], [2]^+ \rhd [1]^+, [4]^+ \rhd [2]^+, [2, 2] \rhd [2, 2,]^+, [2, 4] \rhd [2^+, 4], [2, 2] \rhd [2, 2^+], [2, 4] \rhd [2, 2], [2, 4]^+ \rhd [2, 2]^+, [2^+, 4] \rhd [2, 2]^+, [2^+, 4] \rhd [2^+, 4^+], [2, 4^+] \rhd [2^+, 4^+], [2, 2^+] \rhd [2^+, 2^+], [2, 4^+] \rhd [2, 2^+], [2, 2^+] \rhd [1], [3, 4] \rhd [3, 3], [3, 4]^+ \rhd [3, 3]^+$ and $[3^+, 4] \rhd [3, 3]^+$.

We may translate the notion of symmetric Γ curve, of the precede section, into a dual-pairing terms. To this end, we first notice that $Isom(\mathbb{S}^2)$ consists of three types of elements: (i) rotations, (ii) reflexions and (iii) rotary reflexion (a rotation and a reflexion). We also observe that there is always a plane H stable under these isometries: for a rotation H is the plane perpendicular to an axe, for a reflexion H is either the plane used for the reflexion or any plane perpendicular to it and for a rotary reflexion H is either the plane used for the reflexion or any plane perpendicular to it if $\theta = 0$ where θ is the rotation angle or any plane containing the origin if $\theta = \pi$.

Let G be self-dual map and let $\widetilde{med}(G)$ be a redraw of $med(G)$ such that $Aut(\widetilde{med}(G)) < Isom(\mathbb{S}^2)$. For $\sigma \in Aut(\widetilde{med}(G))$, we define $T_{\sigma,H} = H \cap \mathbb{S}^2$ where H is a stable plane. We have that if $T_{\sigma,H}$ does not go through any vertex of $med(G)$ then $T_{\sigma,H}$ induces a symmetric cycle Γ in $I(G)$ and if $T_{\sigma,H}$ goes through vertices of $med(G)$ then $T_{\sigma,H}$ induce a curve Γ curve with interval vertices of $I(G)$.

7.1. **Equivalent maps.** We say that two maps G_1 and G_2 of a same graph are *equivalent* if there exists an orientation-preserving homeomorphism $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ with $\varphi(G_1) = G_2$. Similarly as done for links, we shall write $[G]$ (called *map-type*) the class of maps equivalent to G.

Remark 7. *We have that $[G]$ is an equivalent relation on the set of all embeddings of G and it depends entirely on the embedding and not on the abstract graph itself.*
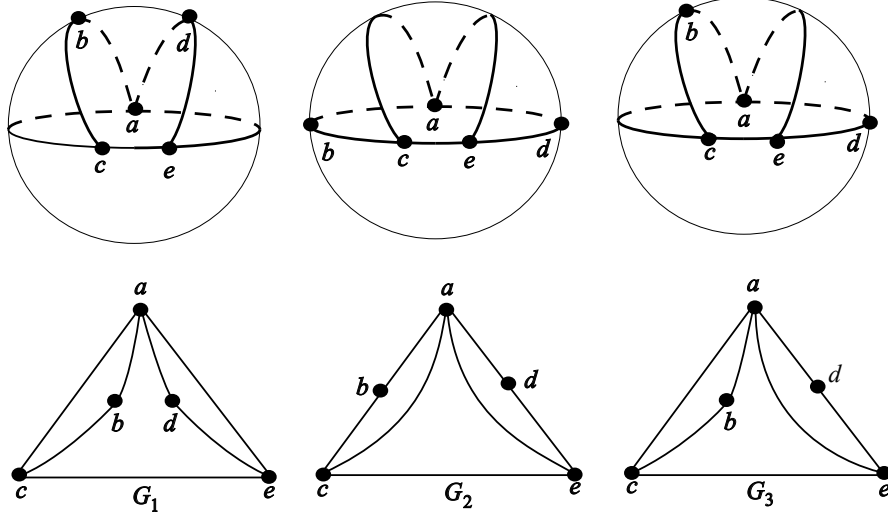
FIGURE 40. Graphs G_1, G_2 and G_3 are isomorphic to each other (bijections are given by the same labeled vertices). However, $G_1 \in [G_2]$ and $G_1 \notin [G_3]$.

Lemma 2. *Let G_1 and G_2 be two maps of a same graph. Then, $G_1 \in [G_2]$ if and only if $G_1^* \in [G_2^*]$.*

Proof. Let F_1 and F_2 be the set of faces of map G_1 and map G_2 respectively. For every homeomorphism h of $\mathbb{S}^2$ to itself sending G_1 to G_2 we have that $h|_f \in F_2$ for every face $f \in F_1$. Furthermore, if two faces f_1 and f_2 of G_1 share an edge e in their boundaries then f_1^* and f_2^* share the edge $h|_e \in E(G_2)$. It follows that the image of every geometric dual of G_1 is also a geometric dual of G_2. $\square$

We have the following easy consequence of the above lemma.

Corollary 13. *Let $G_1 \in [G_2]$. Then, $med(G_1) \in [med(G_2)]$ and $I(G_1) \in [I(G_2)]$.*

Lemma 3. *Let (G_1, S_E) and (G_2, S_E) be two maps such that $[G_1] = [G_2]$. Then, $[L(G_1, S_E)] = [L(G_2, S_E)]$.*

Proof. Let $D(G_1, S_E)$ be the link diagram induced by $(med(G_1), S_V, C_F)$ and let θ be the map from $D(G_1, S_E)$ to $L(G_1, S_E)$. By Corollary 13, there is an orientation-preserving homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ such that $h(med(G_1)) = med(G_2)$. We clearly have that h preserves edge-signs and face-colorings.

The composition $\theta \circ h \circ \theta^{-1}$ induce an homeomorphism from $\mathbb{R}^3$ to itself such that

$$\theta \circ h \circ \theta^{-1}(L(G_1, S_E)) = L(G_2, S_E).$$

$\square$

Theorem 4. *Let (G, S_E) be an edge-signed map such that $G \in [G^*]$. Then, $L(G, S_E)$ is amphichiral for every signature S_E.*

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Proof.

$$
\begin{aligned}
L(G, S_E)^* &\in [L(G, -S_E)] &&\text{(by definition of mirror)} \\
&= [L(G^*, S_E)] &&\text{(by the sign rules)} \\
&= [L(G, S_E)] &&\text{(by Lemma 3 since } G \in [G^*]).
\end{aligned}
$$

$\square$

7.2. Self-dual maps.

We shall give an amphichirality result by using the classification of the self-dual maps.

Lemma 4. *Let G be a self-dual map. If either*
(a) there exists $\tau \in Dual(G)$ such that the isometry $\widehat{\tau}$ is oriented-preserving or

(b) there exists $\sigma \in Aut(G)$ such that the isometry $\widehat{\sigma}$ is not oriented-preserving

then $L(G, S)$ is amphichiral for every signature S.

Proof. (a) The fact that the isometry $\widehat{\tau}$, arising from the duality isomorphism $\tau(G) = G^*$, is oriented-preserving gives an isotopy between $\widehat{G}$ and $\widehat{G^*}$. The result follows by Theorem 4.

(b) Let $\eta \in Dual(G)$ be a duality isomorphism. If the isometry $\widehat{\eta}$ is oriented preserving then the result follows by case a). Otherwise, we consider

$$
\widehat{\sigma} \circ \widehat{\eta}(\widehat{G}) = \widehat{\sigma}(\widehat{G^*}) \overset{(G \text{ self-dual})}{=} \widehat{\sigma}(\widehat{G}) \overset{(\sigma \in Aut(G))}{=} \widehat{G} \overset{(G \text{ self-dual})}{=} \widehat{G^*}
$$

which is clearly oriented-preserving isometry (composition of two non oriented-preserving isometries) implying that $\widehat{G} \in [\widehat{G^*}]$. The result follows by Theorem 4. $\square$

Theorem 5. *Let (G, S_E) be an edge-signed self-dual map. If the self-dual pairing of the map G is other than $[2, q^+] \triangleright [q]^+, [2^+, 2q^+] \triangleright [2q]^+, [2] \triangleright [2]^+, [2, 2] \triangleright [2, 2]^+, [2^+, 4] \triangleright [2, 2]^+, [3^+, 4] \triangleright [3, 3]^+$ then $L(G, S_E)$ is amphichiral for every signature S_E.*

Proof. It can checked that in any self-dual pairing other than the 6 given in the hypothesis we always have that there is either $\rho \in Dual(G)$ with $\widehat{\rho}$ oriented-preserving or $\sigma \in Aut(G)$ with $\widehat{\sigma}$ non oriented-preserving. Table 1 points out the appropriate isometry in each case. The result then follows by Lemma 4.

Self-dual pairing	Type of isometry
$[2,q] \triangleright [q]$	(2)
$[2,q]^+ \triangleright [q]^+$	(1)
$[2^+,2q] \triangleright [2q]$	(2)
$[2] \triangleright [1]$	(1)
$[4] \triangleright [2]$	(1)
$[2]^+ \triangleright [1]^+$	(1)
$[4]^+ \triangleright [2]^+$	(1)
$[2,4] \triangleright [2^+,4]$	(2)
$[2,2] \triangleright [2,2^+]$	(2)
$[2,4] \triangleright [2,2]$	(2)
$[2,4]^+ \triangleright [2,2]^+$	(1)
$[2^+,4] \triangleright [2^+,4^+]$	(2)
$[2,4^+] \triangleright [2^+,4^+]$	(2)
$[2,2^+] \triangleright [2^+,2^+]$	(2)
$[2,4^+] \triangleright [2,2^+]$	(2)
$[2,2^+] \triangleright [1]$	(2)
$[3,4] \triangleright [3,3]$	(2)
$[3,4]^+ \triangleright [3,3]^+$	(1)

TABLE 1. Self-dual pairings having an isometry of type either (1) if there is $\rho \in Dual(G)$ with $\widehat{\rho}$ oriented-preserving or (2) if there is $\sigma \in Aut(G)$ with $\widehat{\sigma}$ not oriented-preserving.

$\square$

We notice that in each of the 6 special self-dual pairings in Theorem 5, we have that both all $\rho \in Dual(G)$ arise non oriented-preserving isometries and all $\sigma \in Aut(G)$ arise oriented-preserving isometries.

8. Concluding Remarks

We say that a link $L(G, S_E)$ is *antipodal* if the map G is antipodally self-dual. In the process of our investigations a great deal of structure of antipodal links has been revealed and some problems and questions arose. We notice that there are antipodal links 3-centrally symmetric with a component not necessarily 2-centrally symmetric to itself, see Figure 41.
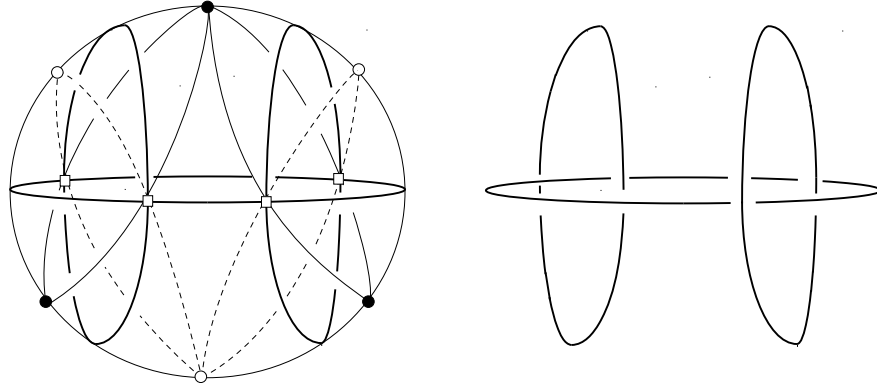
FIGURE 41. (Right) An antipodally self-dual map G (black vertices and straight edges) and G^* (white vertices and dotted edges) and a map of $med(G)$ 3-antipodally symmetric (squared vertices and bold edges). (Left) Induced link 2-antipodally symmetric admitting two noncentrally symmetric components but antipodally symmetric between them.

Question 1. *Is true that conditions (a) and (b) in Theorem 1 are also necessary if a knot is 3-centrally symmetric ?*

Question 2. *Let (G, S_E^+) be an antipodally self-dual edge-signed map. Is it true that alternating link $L(G, S_E^+)$ has always an odd number of components ?*

Problem 1. *Characterize all the antipodal links $L(G, S_E)$, in particular, when G is 1-, 2- or 3-connected.*

In view of our results, the following natural question arise.

Question 3. *Let L be an amphichiral link. Does there exists a diagram D of L such that G_D is self-dual ?*

This question is on the same flavor as a conjecture due to Kauffman [6] (see also [8]) claiming that for any alternating amphichiral knot there is a reduced alternating diagram D of the knot such that G_D is self-dual. Dasbach and Hougardy [3] provided a 14-crossing alternating knot that is a counterexample. Nevertheless, the latter does not seem to be also a counterexample for the above question since it might admits a nonalternating or a no reduced diagram whose Tait's graph is self-dual.

The conditions given in Theorem 3 may lead to an algorithm to detect whether a link is amphichiral. However, its computational complexity depends on how efficiently the desired symmetric cycle is obtained.

Problem 2. *Let G be a bipartite, planar graph having only square faces. Can it be found either a consistent color-reversing symmetric cycle or an opposite color-preserving symmetric cycle in polynomial time ?*

We finally point out that both antipodally self-dual and antipodally symmetric maps play an important rôle in *real projective* links, that is, links embedded in $\mathbb{RP}^3$ (work in progress [10]).

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