SELF-DUAL MAPS I : ANTIPODALITY

LUIS MONTEJANO[1], JORGE L. RAMÍREZ ALFONSÍN[2], AND IVAN RASSKIN

Abstract. A self-dual map G is said to be *antipodally self-dual* if the dual map G^* is antipodal embedded in $\mathbb{S}^2$ with respect to G. In this paper, we investigate necessary and/or sufficient conditions for a map to be antipodally self-dual. In particular, we present a combinatorial characterization for map G to be antipodally self-dual in terms of certain *involutive labelings*. The latter lead us to obtain necessary conditions for a map to be *strongly involutive* (a notion relevant for its connection with convex geometric problems). We also investigate the relation of antipodally self-dual maps and the notion of *antipodally symmetric* maps. It turns out that the latter is a very helpful tool to study questions concerning the *symmetry* as well as the *amphicheirality* of *links*.

1. Introduction

Let G be a *map*, that is, a graph cellularly embedded in the sphere. Then $G = (V, E, F)$ has a natural *geometric dual* $G^* = (V^*, E^*, F^*)$ where each face in F correspond to a vertex in V^* and two vertices in V^* are adjacent if the corresponding faces in G share an edge. A map G is called *self-dual* if there is a bijection from V and F to V^* and F^* which reverses inclusion.

Self-dual maps have been the subject of numerous investigations in different fronts : self-dual polyhedra and ranks [3], isometries in $\mathbb{S}^2$ [8], eigenvalues of h-graphs [11], rigidity [7], tilings [9], etc.

A self-dual map G is said to be *antipodally self-dual* if the dual map G^* is antipodally embedded with respect to G. In other words, the map G is antipodally self-dual if the following holds for any $x \in \mathbb{S}^2$

1) if $x \in V(G)$ then $-x \in V(G^*)$ and

2) if $x \in e \in E(G)$ then $-x \in e^* \in E(G^*)$, that is, e^* is antipodally embedded in $\mathbb{S}^2$ with respect to the embedding of e.

Antipodally self-dual maps are closely related with the notion of *strongly involutive* maps (see beginning of Section 3.2) and thus relevant for their connection with convex geometric problems as the well-known Vázsonyi's problem on *ball polyhedra* (as reported in [2], see also [10]), the chromatic number of *distance graphs* on the sphere [4] and *Reuleaux polyhedra* [5].

The main goal of this paper is to investigate necessary and/or sufficient conditions for a map to be antipodally self-dual.

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The paper is organized as follows. In the next section, we give a brief overview of some notions on self-dual maps needed for the rest of the paper. Given a map G, we recall three special close related maps (*medial graph* $med(G)$, *square graph* $G^\square$ and *vertex-face incidence graph* $I(G)$) that turn out to be very useful for our propose.

In Section 3, we first recall some classical results between isometries in $\mathbb{S}^2$ and maps. We then present a result giving necessary conditions of an antipodally self-dual map G in terms of *symmetric cycles* in $I(G)$ (Theorem 1). In order to prove this, we introduce and study the notion of *antipodally symmetric* maps. Besides many properties, we show that if G is an antipodally self-dual map then both $med(G)$ and $I(G)$ are antipodally symmetric maps (Lemma 1). Afterwards, we discuss the connection between antipodally self-dual maps and *strongly involutive* maps and give a combinatorial characterization for a map G to be antipodally self-dual in terms of certain *involutive labelings* of $I(G)^\square$ (Theorem 2). As a consequence, we obtain necessary conditions for a map G to be strongly involutive in terms of $I(G)^\square$ (Corollary 2).

In Section 4, we characterize three different infinite families of antipodally self-dual maps (Propositions 1, 2 and 3). We also present a more general construction (Theorem 3).

Finally, in Section 5, we briefly discuss some consequences on problems concerning knot theory.

2. Maps preliminaries

Let G be a planar graph. A *map* of $G = (V, E, F)$ is the image of an embedding of G into $\mathbb{S}^2$ where the set of vertices are a collection of distinct points in $\mathbb{S}^2$ and the set of edges are a collection of Jordan curves joining two points in V satisfying that $\alpha \cap \alpha'$ is either empty or a point in the endpoints for any pair of Jordan curves α and α'. Any embedding of the topological realization of G into $\mathbb{S}^2$ partitions the 2-sphere into simply connected regions of $\mathbb{S}^2 \setminus G$ called the *faces* F of the embedding.

Given a map G, we may construct the *dual map* $G^* = (V^*, E^*, F^*)$ by placing a vertex f^* in the interior of each face f of G, and for each edge e of M draw a *dual edge* e^* connecting the vertices f_1^* and f_2^* (corresponding to the two faces f_1 and f_2 sharing edge e) by crossing e transversely.

Two maps $G_1 = (V_1, E_1, F_1)$ and $G_2 = (V_2, E_2, F_2)$ of the same graph are *isomorphic* if there is an isomorphism $\phi : (V_1, E_1, F_1) \to (V_2, E_2, F_2)$ preserving incidences, that is, $\{u, v\} \in E_1$ if and only if $\{\phi(u), \phi(v)\} \in E_2$ (and thus ϕ preserve facial walks). We say that a map $G = (V, E, F)$ is a *self-dual map* if the maps $G = (V, E, F)$ and $G^* = (V^*, E^*, F^*)$ are isomorphic, that is, there is an isomorphism $\phi : (V, E, F) \to (F^*, E^*, V^*)$ preserving incidences.

Given maps $G = (V, E, F)$ and $G^* = (V^*, E^*, F^*)$ we define the following auxiliaries maps.

- The *squares graph of* G is the map $G^\square$ obtained by the simultaneous drawing of $G \cup G^*$ with all the edges split at the intersection points of an edge e with its dual edge e^*. We thus have that every face of $G^\square$ is a square formed by half-edges of G and G^*.

For each square face in $G^\square$, we define two types of diagonals: the *intersecting diagonal* which is the edge joining the intersections points and the *incidence diagonal* which is the edge joining a vertex in $V(G)$ to a vertex in $V(G^*)$.

- The *vertex-face incidence graph* is the map $I(G)$ having as vertices $V \cup V^*$ and as edges are all the incidence diagonals of $G^\square$.

- The *medial of* G is the map $med(G)$ having as vertices the set of intersections points of $E \cap E^*$ and as edges the set of all the intersecting diagonals of $G^\square$.

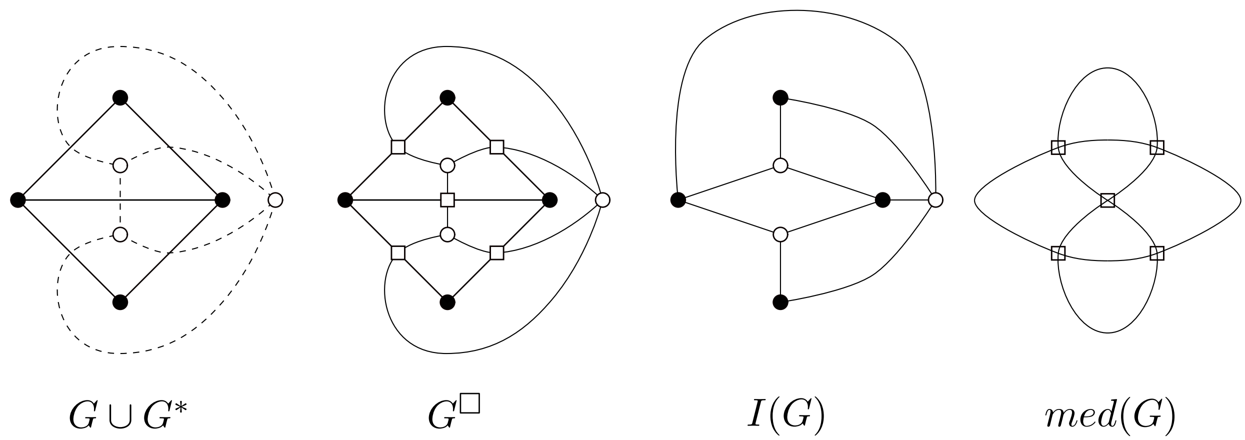

FIGURE 1. A map and its dual, the squares graph, the vertex-face incidence graph and the medial.

Throughout the paper, we will represent the vertices of G with black circles, the vertices of G^* with white circles and the intersection points with white squares and the vertices of the medial with transparent squares.

Notice that $I(G)$ and $med(G)$ are dual from each other for any map G. Hence, we can construct the squares graph of the vertex-face incidence graph which it turns out to be very useful for our propose.

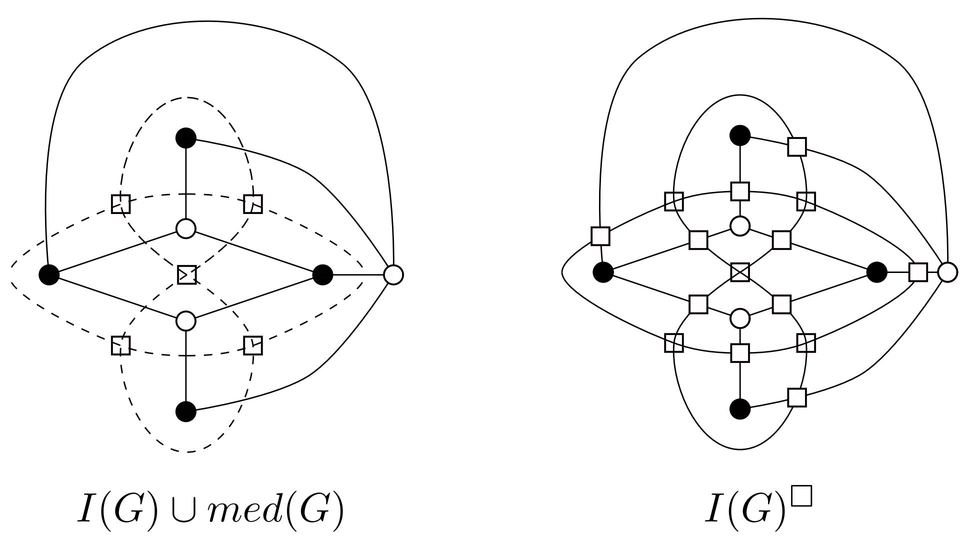

FIGURE 2. (Right) $I(G)$ (straight edges and black and vertices in white circles) and $med(G)$ (dashed edges and vertices in transparent squares) (Left) Graph $I(G)^\square$.

3. ANTIPODALLY SELF-DUAL MAPS

We recall that $Aut(G)$ is the group formed by the set of all *automorphism* of G (i.e., the set of isomorphisms of G into itself). We will denote by $Cor(G)$ the set of all *duality isomorphisms* of G into G^*. We recall that a duality isomorphism is an application ϕ which sends vertices to faces and faces to vertices while preserving incidence, that is, if vertex v is incident with face f then face $\phi(f)$ is incident with vertex $\phi(f)$). Note that ϕ^2 maps vertices to vertices and faces to faces. However, despite the fact that 'the dual of the dual is the original', ϕ^2 need not be the identity. We notice that $Cor(G)$ is not a group since the composition of any two of them is an automorphism.

Let us suppose that $G = (V, E, F)$ is a self-dual map so that there is a bijection $\phi : (V, E, F) \to (F^*, E^*, V^*)$. Following ϕ with the correspondence $*$ gives a permutation on $V \cup E \cup F$ which preserve incidences but reverses dimension of the elements. The collection of all such permutations or *self-dualities* generate a group $Dual(G) = Aut(G) \cup Cor(G)$ in which the automorphisms $Aut(G)$ are contained as a subgroup of index 2.

It is known [8, Lemma 1] that for a given map G there is an homeomorphism ρ of $\mathbb{S}^2$ to itself such that for every $\sigma \in Aut(G)$ we have that $\rho\sigma \in Iso(\mathbb{S}^2)$ where $Iso(\mathbb{S}^2)$ is the group of isometries of the 2-sphere. In other words, any planar graph G can be drawn on the 2-sphere such that any automorphism of G act as an isometry of the sphere. This was extended in [8] by showing that given any self-dual graph G there are maps G and G^* so that $Dual(G)$ is realized as a group of spherical isometries.

From now on, we will denote by $\widehat{G} = \rho(G)$ and $\widehat{\sigma} = \rho\sigma$ for a certain homeomorphism ρ satisfying the above property.

A self-dual map G is *antipodally self-dual* if $-\widehat{G} = \widehat{G}^*$ where $-G$ is the map consisting of the set of points $\{-x \in \mathbb{S}^2 \mid x \in G\}$, that is, for each $x \in \mathbb{S}^2$

- if $x \in V(\widehat{G})$ then $-x \in V(\widehat{G^*})$
- if $x \in E(\widehat{G})$ then $-x \in E(\widehat{G^*})$, i.e., $e^* \in E(\widehat{G^*})$ is antipodally embedded in $\mathbb{S}^2$ with respect to $e \in E(\widehat{G})$
- if $x \in F(\widehat{G})$ then $-x \in F(\widehat{G^*})$, i.e., $f^* \in F(\widehat{G^*})$ is antipodally embedded in $\mathbb{S}^2$ with respect to $f \in F(\widehat{G})$

Let us present a result giving necessary combinatorial conditions for a map to be antipodally self-dual. By a *symmetric cycle* C of a planar graph G we mean that there is an automorphism $\sigma(G)$ such that $\sigma(C) = C$ and $\sigma(int(C)) = ext(C)$, that is, the induced graph in the interior of C is isomorphic to the induced graph in the exterior of C.

Theorem 1. *Let G be antipodally self-dual. Then, $I(G)$ always admits at least one symmetric cycle. Moreover, all symmetric cycles in $I(G)$ are of length $2n$ with $n \geq 1$ odd.*

The proof of Theorem 1 is presented in the next subsection where the notion of *antipodally symmetric* map (needed for the proof) is introduced and discussed.

3.1. **Antipodally symmetric maps.** A map $G = (V, E, F)$ is said to be *antipodally symmetric* if $-\widehat{G} = \widehat{G}$ where $-G$ is the map consisting of points $\{-x \in \mathbb{S}^2 \mid x \in G\}$, that is, for each $x \in \mathbb{S}^2$, if $x \in V(\widehat{G})$ (resp. $x \in E(\widehat{G})$ or $x \in F(\widehat{G})$) then $-x \in V(\widehat{G})$ (resp. $-x \in E(\widehat{G})$ or $-x \in F(\widehat{G})$). We thus have that any vertex, edge or face in an antipodally symmetric $\widehat{G}$ is antipodally embedded in $\mathbb{S}^2$ with respect to another vertex, edge and face of $\widehat{G}$.

Remark 1. *(a)* $med(G) = med(G^*)$.

(b) If G *is self-dual then* $|V(med(G))|$ *is even. Indeed, by Euler's formula we have* $|V(G)| + |F(G)| = 2 + |E(G)|$ *where* $F(G)$ *denote the set of faces of* G*. Since* G *is self-dual then* $|V(G)| = |V(G^*)| = |F(G)|$ *and thus* $2|V(G)| = 2 + |E(G)|$ *implying that* $|E(G)| = |V(med(G))|$ *is even.*

Lemma 1. *Let* G *be an antipodally self-dual map. Then,* $med(G)$ *and* $I(G)$ *are antipodally symmetric.*

Proof. We first show that $med(G)$ is antipodally symmetric. For, let us consider an antipodally self-dual map G, that is, the dual map G^* is antipodally embedded with respect to the map G. The latter induces a map $G^\square$ in which square faces of $G^\square$ are partitioned into pairs that are antipodally embedded in $\mathbb{S}^2$. Indeed, let $F = \{e_1, e_2, e_1^*, e_2^*\}$ be a face of $G^\square$ where e_1, e_2 (resp. e_1^*, e_2^*) are the two half-edge induced by $e \in E(G)$ (resp. induced by $e^* \in E(G^*)$). Since G is antipodally self-dual then there is an edge $f^* \in G^*$ (resp. an edge $f \in G$) which is antipodally embedded to $e \in G$ (resp. to $e^* \in G^*$). We thus have that the corresponding half-edges f_1^*, f_2^* (resp. f_1, f_2) are also antipodally embedded with respect to e_1, e_2 (resp. to e_1^*, e_2^*). Obtaining an other face $F^* = \{f_1, f_2, f_1^*, f_2^*\}$ which is antipodally embedded with respect to F.

We thus have that the intersecting diagonals corresponding to faces F and F^* can also be antipodally embedded. The results follows by recalling that $med(G)$ is given by all the intersecting diagonals of $G^\square$, see Figure 3.

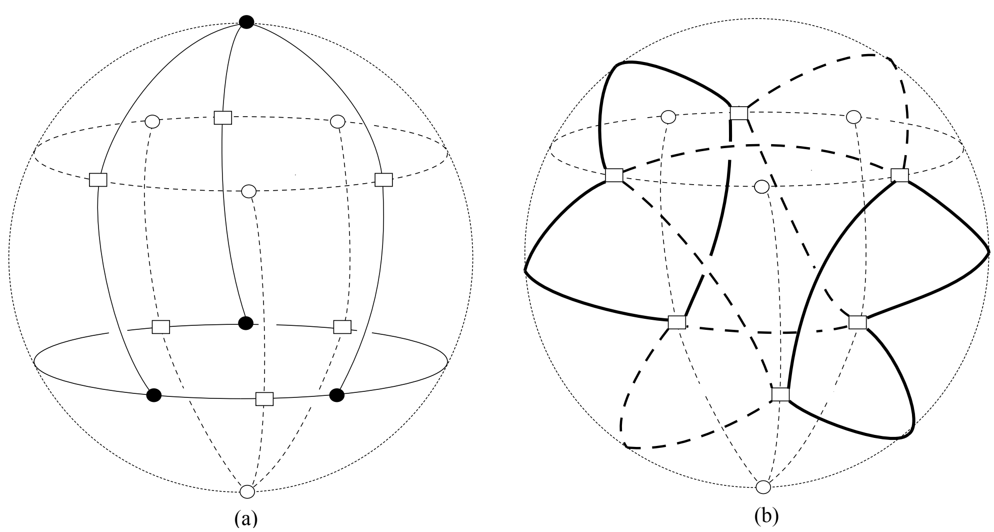

FIGURE 3. (a) Embedding of K_4 (black vertices and straight edges), its dual (white vertices and dashed edges) and the vertices of the medial graph (little squares vertices) (b) K_4 (dashed edges) and its medial graph (in bold dashed edges) with two antipodal faces (in bold).

For $I(G)$, the proof goes in the same way as above but, this time, by considering the incidence diagonals instead of the intersecting diagonals. □

We may now prove Theorem 1.

Proof of Theorem 1. Let G be an antipodally self-dual map. Let $\widehat{med(G)}$ be the drawing of $med(G)$ where all the automorphisms are isometries and let E be the equator of $\mathbb{S}^2$.

Suppose that E does not contain any vertex of $\widehat{med(G)}$; Then, E passes from a face f of $\widehat{med(G)}$ to another face f' that shares and edge with f. Since $med(G)^* = I(G)$ the pair faces $\{f, f'\}$ corresponds to a pair of adjacent vertices $\{v, v'\}$ in $V(I(G))$. Thus, the sequence of faces $(f_1, \ldots, f_n = f_1)$ intersected by E (with the order induced by E) corresponds to a cycle C in $I(G)$. Let $int(C)$ (resp. $ext(G)$) be the subgraph of $I(G)$ corresponding to the faces of $\widehat{med(G)}$ lying on the northern (resp. southern) hemisphere. By Lemma 1, $med(G)$ is antipodally symmetric so the northern faces and the southern faces of $\widehat{med(G)}$ are antipodally drawn. Thus, $int(C)$ is map isomorphic to $ext(C)$ and thus C is a symmetric cycle of $I(G)$.

Now, let us suppose that E passes through a vertex of $med(G)$. Since the set of vertices of $med(G)$ is finite there exists a point $x \in E$ such that x and $-x$ are not vertices of $med(G)$. Let E_α be the the rotation of E of angle α on the line passing through x and $-x$. Let

$$\beta = \min_{\alpha > 0}\{E_\alpha \text{ contains a vertex of } \widehat{med(G)}\}.$$

Then, $E_{\beta/2}$ is a great circle of $\mathbb{S}^2$ which does not contain any vertex of $\widehat{med(G)}$. By taking $E_{\beta/2}$ as equator we can apply the above arguments to show that there exists a symmetric cycle of $I(G)$.

Finally, if C is a symmetric cycle of $I(G)$ then we can draw $\widehat{I(G)}$ with C being the equator of $\mathbb{S}^2$. Since G is antipodally self-dual, a black vertex v of $\widehat{I(G)}$ is antipodal to a white vertex $-v$. Thus, the length of C must be $2n$ with $n \geq 1$ odd. $\square$

Remark 2. *An antipodally self-dual map G induces an involutive self-dual isomorphism $\sigma : V(G) \longrightarrow V^*(G)$. The converse is not necessarily true, there are self-dual graphs not admitting an antipodally self-dual map. For instance, the graph G' illustrated in Figure 4 is self-dual but it is not antipodally self-dual. Indeed, it can be easily checked that $I(G')$ admits a symmetric cycle of length 8 (implying that G' is not antipodally self-dual, by Theorem 1), see Figure 4.*

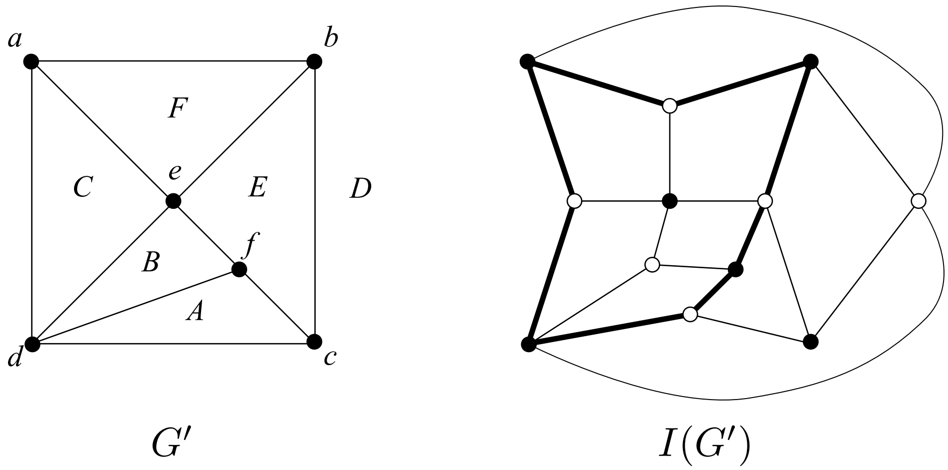

FIGURE 4. (Left) A nonstrongly involutive self-dual map G' (the isomorphism σ is given by $\sigma(a) = A, \sigma(b) = B, \sigma(c) = C$, etc.) (Right) $I(G')$ admitting a symmetric cycle of length 8 (bold edges).

3.2. **Strongly involutive maps and involutive labelings.** Let G be a self-dual graph with duality isomorphism $\sigma : G \longrightarrow G^*$. We say that G is *strongly involutive* if the following conditions are satisfied:

a) for each pair of vertices $u, v \in V(G), u \in \sigma(v)$ if and only if $v \in \sigma(u)$ and

b) for every vertex $v \in V(G)$, we have that $v \notin \sigma(v)$.

We notice that a) is equivalent to say that $\sigma^2 = id$.

The above conditions are the combinatorial counterpart (in the 3-dimensional case) of a more general geometric object called *strong self-dual polytopes*, first introduced by Lovász in [4]. Antipodally self-dual maps are closely related with strongly involutive isomorphism. Indeed, in [1, Theorem 9], it was proven that if G is strongly involutive then G is antipodally self-dual. As we will see below, the latter is a straightforward consequence of Theorem 2 (see Corollary 2).

Let $G = (V, E, F)$ be a map and let $X = \{x_1, \ldots, x_m\}$ and $\bar{X} = \{\overline{x}_1, \ldots, \overline{x}_m\}$ be two sets of labels with $1 \le m \le |V|$ and the property $\overline{\overline{x_i}} = x_i$. Let $\mathcal{P}(X \cup \bar{X})$ be the set of subsets of $X \cup \bar{X}$. An *involutive labeling* of G is a function $\Lambda : V \to \mathcal{P}(X \cup \bar{X})$ satisfying the following properties:

(i) $|\Lambda(v)| \in \{1, 2\}$ for every $v \in V$.
(ii) If $|\Lambda(v)| = 2$ then $\Lambda(v) = \{x_i, \overline{x}_i\}$ for some $1 \le i \le m$. In this case, we say that v is a *fixed vertex* of Λ and we write $x_i = \overline{x}_i$ (instead of $\{x_i, \overline{x}_i\}$).
(iii) $\Lambda(u) \cap \Lambda(v) \neq \emptyset$ if and only if $u = v$.
(iv) $\{\Lambda^{-1}(x_i), \Lambda^{-1}(x_j)\} \in E$ if and only if $\{\Lambda^{-1}(\overline{x}_i), \Lambda^{-1}(\overline{x}_j)\} \in E$ where $\Lambda^{-1}(x_i) := \{v \in V \mid x_i \in \Lambda(v)\}$.

Let $G^\Box = (V^\Box, E^\Box, F^\Box)$ be the square graph associated to a map $G = (V, E, F)$. Recall that $V^\Box = V_V \cup V_E \cup V_F$ where V_V are the vertices of G, V_E are the vertices on the edges of G and V_F are the vertices of G^* (one for each face of G).

Remark 3. *An involutive labeling of $I(G)^{\square}$ naturally induces an automorphism of $I(G)$*

$$\begin{array}{rccc} \sigma_\Lambda: & V\cup V^* & \to & V\cup V^* \\ & v & \mapsto & u \end{array}$$

where $\Lambda(u)=\overline{\Lambda(v)}$ (the adjacency preserving property of σ_Λ is obtained from (ii)).

a) If vertex v was assigned labels k and $\bar{k}$ (and thus $k=\bar{k}$) then it will be a fixed vertex under σ_Λ.

b) $\sigma_\Lambda^2=Id$.

c) σ_Λ corresponds to an involutive duality isomorphism $\sigma: G\to G^*$ if and only if the labels of the black vertices are the opposite to those of the white vertices in $I^{\square}(G)$.

Remark 4. *Let G be a self-dual map. We have that G is strongly involutive if and only if $I(G)$ admits an involutive labeling without edges whose ends are labeled by k and $\bar{k}$.*

3.3. **Characterizing antipodally self-dual maps.** We are interested in giving necessary and sufficient combinatorial conditions for a map to be antipodally self-dual.

Remark 5. *We have that any $\sigma\in Aut(G)$ naturally induces $\sigma^{\square}\in Aut(G^{\square})$ with $\sigma^{\square}$ preserving incidences, that is, if $v_V\in V_V$ is adjacent to $v_E\in V_E$ (resp. $v_E\in V_E$ is adjacent to $v_F\in V_F$) then $\sigma^{\square}(v_V)$ is adjacent to $\sigma^{\square}(v_E)$ (resp. $\sigma^{\square}(v_E)$ is adjacent to $\sigma^{\square}(v_F)$) and where V_V,V_E and V_F are mapped to V_V,V_E and V_F respectively. We finally notice that there might exist $\gamma\in Aut(G^{\square})$ not necessarily arising from an automorphism of G.*

Lemma 2. *Let H be a map and let $\sigma\in Aut(H)$. Then, $\widehat{\sigma}$ has a fixed point in $\mathbb{S}^2$ if and only if $\sigma^{\square}$ has a fixed vertex in $H^{\square}$.*

Proof. Let $x\in\mathbb{S}^2$. A point x corresponds to a vertex on $H^{\square}$, say $x^{\square}$, which lies properly on either V,E or F. If $\widehat{\sigma}(x)=x$ then $\sigma^{\square}(x^{\square})=x^{\square}$.

Conversely, let $v\in V^{\square}=\{V_V\cup V_E\cup V_F\}$ such that $\sigma^{\square}(v)=v$. We have three cases.

Case 1) $v\in V_V$. Then, the point $v\in\mathbb{S}^2$ is such that $\widehat{\sigma}(v)=v$.

Case 2) $v\in V_E$. Suppose v lies properly on an edge e. We know that the isometry $\widehat{\sigma}$ maps e into itself. Since e is topologically equivalent to $\mathbb{B}^1$ then $\widehat{\sigma}$ is a continuous function sending $\mathbb{B}^1$ to itself. Therefore, by the Brouwer fixed-point theorem there is $x\in e$ such that $\widehat{\sigma}(x)=x$.

Case 3) $v\in V_F$. Suppose v lies properly on a face f. We proceed as in the Case 2. The isometry $\widehat{\sigma}$ maps f into itself. Since f is topologically equivalent to $\mathbb{B}^2$ then $\widehat{\sigma}$ is a continuous function sending $\mathbb{B}^2$ to itself. Therefore, by the Brouwer fixed-point theorem there is $x\in f$ such that $\widehat{\sigma}(x)=x$. □

Theorem 2. *Let $G=(V,E,F)$ be a self-dual map. Then, G is antipodally self-dual if and only if $I(G)^{\square}$ admits an involutive labeling without fixed vertices.*

Proof. Suppose that G is antipodally self-dual. Therefore, there is $\widehat{G}$ isomorphic to G such that $-\widehat{G}=\widehat{G}^*$. Let $a:x\mapsto -x$ be the antipodal mapping of $\mathbb{S}^2$. We have that a naturally induces the automorphisms $a_I\in Aut(I(\widehat{G}))$ and $a^{\square}\in Aut(I(\widehat{G})^{\square})$. Furthermore, since a is the antipodal mapping then

- $a_I^2=Id$ (implying that $I(G)^{\square}$ admits an involutive labeling on its vertices) and

- a_I has no fixed points of $\mathbb{S}^2$. Therefore, by Lemma 2, $a^\square$ has no fixed vertices and thus the above involutive labeling of $I(G)^\square$ has no fixed vertices.

We finally notice that an involutive labeling of $I(\widehat{G})^\square$ is also an involutive labeling of $I(G)^\square$.

Conversely, suppose that $I(G)^\square$ admits an involutive labeling without fixed vertices. By Lemma 2, $\widehat{\sigma}(I(G))$ has not a fixed point in $\mathbb{S}^2$. Now, there are three sphere isometries such that $\sigma^2 = Id$: rotation of π degree, reflexion on a hyperplane and the antipodal function. Among them, it is the antipodal function the only without fixed points. Moreover, since $\sigma : G \to G^*$ then σ sends vertices of G to vertices of G^*. Therefore, G is antipodally self-dual. □

For the involutive labelings of squares graphs, we shall use integers (and their opposites) for vertices of type V_V, letters (and their opposites) for vertices of type V_F and greek letters (and their opposites) for vertices of type V_E. On one hand Figure 5 illustrates a self-dual map G and $I(G)^\square$ together with an involutive labeling without fixed vertices. Therefore, as a consequence of Theorem 2, G is antipodally self-dual. On the other hand, Figure 6 illustrates an involutive labeling of the 4-wheel W_4 with $I(W_4)^\square$ admitting two fixed vertices. In fact, it can be checked that any involutive labeling of $I(W_4)^\square$ admits at least one fixed vertex since W_4 is not antipodally self-dual (see Proposition 1).

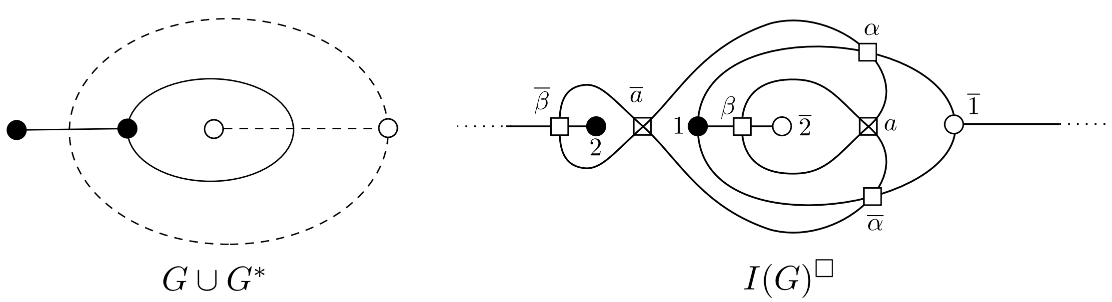

FIGURE 5. (Left) A self-dual map G (straight edges and black vertices) and G^* (dashed edges and white vertices). (Right) An involutive labeling of $I(G)^\square$ without fixed vertices.

It can easily be checked that the graph G (in Figure 5) does not admit a strongly involutive isomorphism.

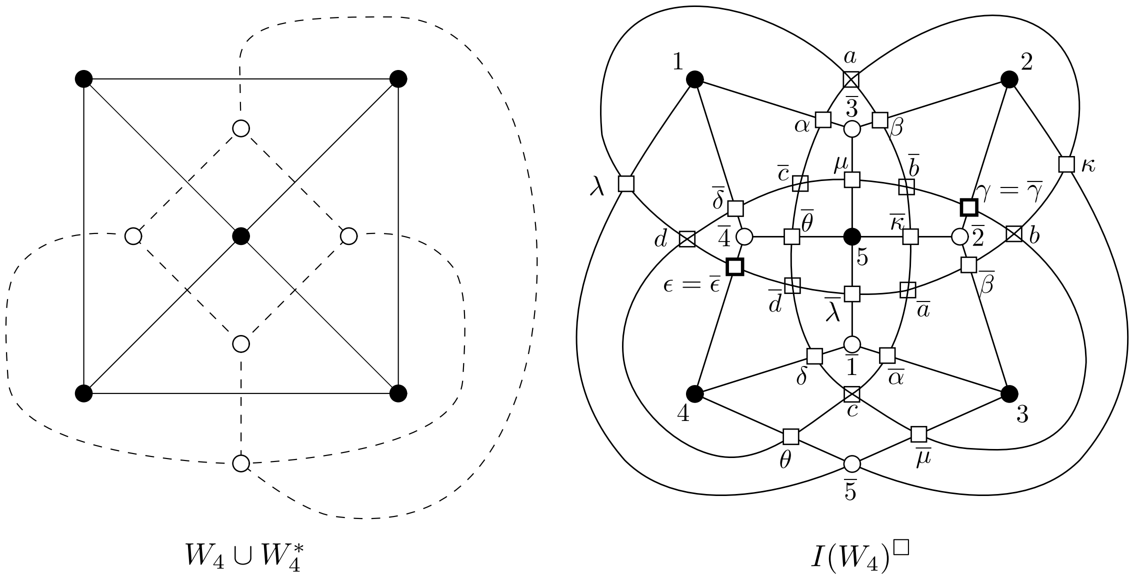

FIGURE 6. (Left) The self-dual map W_4 with its dual. (Right) $I^\square(W_4)$ together with an involutive labelling with two fixed vertices of type V_E: $\gamma = \bar{\gamma}$ and $\epsilon = \bar{\epsilon}$ (bold squares).

Corollary 1. *Let G be a self-dual map. If there is a black vertex of $I(G)$ with odd degree which is connected to all white vertices then G is not antipodally self-dual.*

Proof. Let v be such a black vertex. Since v is connected to all white vertices then for any involutive labeling Λ of $I(G)^\square$ there is an edge in $I(G)$ with ends labeled with k and $\overline{k}$. By Remark 3, the automorphism $\sigma_\Lambda(G)$ maps an edge with ends labeled $\{k, \overline{k}\}$ to an edge with ends labeled $\{k, \overline{k}\}$ (not necessarily the same edge, see Figure 5). Since, by hypothesis, the degree of v is odd then there must be an edge mapped to itself which correspond to a fixed vertex in $V(I(G)^\square)$. Therefore, by Theorem 2, G is not antipodally self-dual. $\square$

Figure 7 illustrates a graph in which $I(G)$ has a vertex in $V(G)$ adjacent to each vertex of G^* by an odd number of edges (and thus, by Corollary 1, G is not antipodally self-dual).

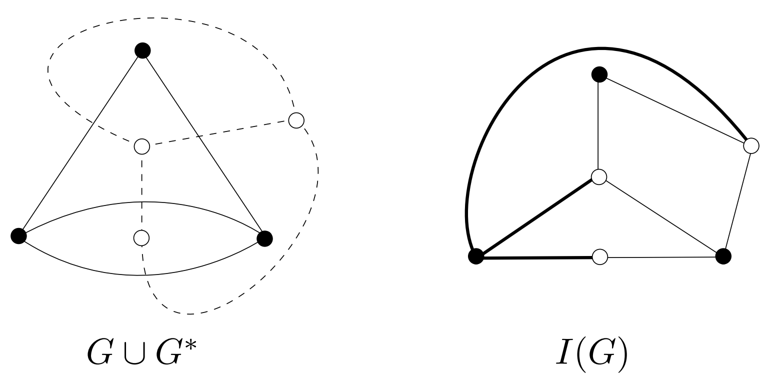

FIGURE 7. (Left) A self-dual map G with its dual. (Right) $I(G)$ with a black vertex joined to each white vertex by an odd number of edges (in bold).

Corollary 2. *Let G be a self-dual map. If G is strongly involutive then G is antipodally self-dual.*

Proof. We shall show that $I(G)^{\Box}$ admits an involutive labeling without fixed vertices. The result then follows by Theorem 2.

Let $\sigma : G \longrightarrow G^*$ be a duality isomorphism. We thus have that σ does not fix vertices. Recall that if G is strongly involutive then σ verifies

a) for each pair of vertices $u, v \in V(G), u \in \sigma(v)$ if and only if $v \in \sigma(u)$ and

b) for every vertex $v \in V(G)$, we have that $v \notin \sigma(v)$.

As remarked above, a) is equivalent to say that $\sigma^2 = id$. We clearly have that σ does not fix vertices since it maps vertices of G to vertices of G^*. The latter implies that σ_I does not fix vertices in $I(G)$ and thus neither $\sigma^{\Box}$ in $I(G)^{\Box}$.

Now, by combining conditions (a) and (b) we obtain that $u \notin \sigma(u)$ for every vertex u in G^*. The latter implies that $I(G)$ does not admit an edge with extremes labeled with k and $\bar{k}$ and so $\sigma^{\Box}$ does not fix vertices of type V_E (i.e., arising from edges of $I(G)$) in $I(G)^{\Box}$.

We finally claim that $\sigma^{\Box}$ does not fix vertices of type V_F (i.e., arising from faces of $I(G)$) in $I(G)^{\Box}$. We proceed by contradiction, suppose that $\sigma\Box$ fixes a vertex u_f arising from a face f of $I(G)$. Let f be the face in $I(G)$ corresponding to $\sigma(u_f)$. Recall that all the faces in $I(G)$ are squares, suppose that $f = \{w, x, y, z\}$ with $w, y \in V(G)$ and $x, z \in V(G^*)$ and $f' = \{w', x', y', z'\}$ with $w', y' \in V(G)$ and $x', z' \in V(G^*)$.

Since $\sigma^{\Box}$ fixes u_f then $\sigma(u_f) = u_f$ but this happen only if $\{\sigma(w), \sigma(y)\} = \{w', y')\}$ and $\{\sigma(x), \sigma(z)\} = \{x', z')\}$. The latter implies the existence of an edge with extremes labeled k and $\bar{k}$, which is not possible. □

We notice that the converse of Corollary 2 is not necessarily true. Indeed, there might be a non strongly involutive map G with $I(G)^{\Box}$ admitting an involutive labeling without fixed vertices (and thus G antipodally self-dual, by Theorem 2), see Figure 5.

4. Infinite families

We give below some infinite families having antipodally self-dual maps. We do so by applying the previous results.

4.1. **The wheel.** Let $n \geq 3$ be an integer. The *n-wheel*, denoted by W_n, is the graph consisting of an n-cycle with a center joined to each vertex of the cycle.

Proposition 1. *The n-wheel is antipodally self-dual if and only if $n \geq 3$ is odd.*

Proof. It can be easily checked that W_n admits a strongly involutive duality-isomorphism for any odd integer $n \geq 3$. To this end, we label the center with n and the vertices around the center with $0, \ldots, n-1$. Let φ be the duality-isomorphism defined as

$$\varphi(i) = \begin{cases} \text{face with vertices } \{n, i + \frac{n-1}{2}, i + \frac{n-1}{2} + 1\} & \text{if } 0 \leq i \leq n-1, \\ \text{exterior face} & \text{if } i = n. \end{cases}$$

where the sum is taken mod n.

It can be easily checked that φ is strongly involutive, see Figure 8.

Moreover, if n is even then $I(W_n)$ admits a symmetric cycle of length $2k$ with k even, see Figure 9. Thus, by Theorem 1, W_n is not antipodally self-dual. □

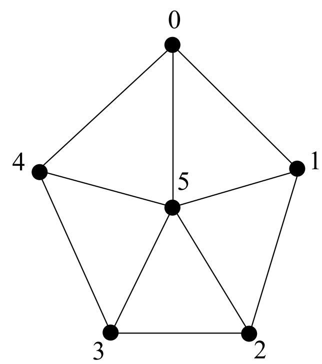

FIGURE 8. 5-wheel together with a labeling used by the strongly involutive duality-isomorphism φ. For instance, $\varphi(3)$ is the face formed by vertices $\{5, 0, 1\}$.

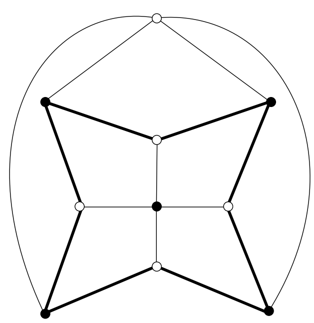

FIGURE 9. $I(W_4)$ admitting a symmetric cycle (bold edges) of length 8.

Figure 3 (a) shows that W_3 admits a antipodally self-dual map. One can easily mimic this embedding for any odd integer $n \geq 3$. Figure 10 illustrates the case $n = 5$.

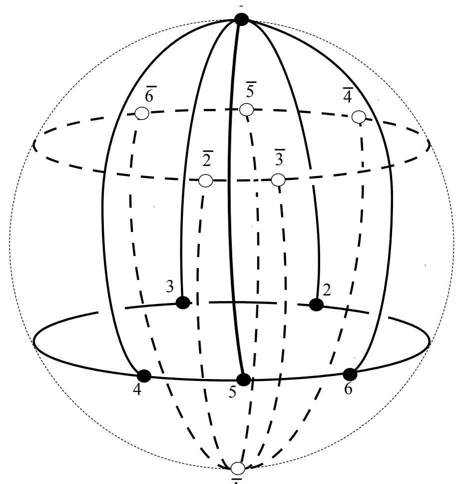

FIGURE 10. A antipodally self-dual map of W_5 (straight edges and black vertices) and its dual (dashed edges and white vertices). Antipodal vertices are given by k and $\bar{k}$

4.2. **The n-ear.** Let $n \geq 3$ be an integer. The *n-ear*, denoted by E_n is the graph consisting of a n-cycle with an *ear* added on each edge and a center is joined to each ear, see Figure 11

Proposition 2. *The n-ear is antipodally self-dual if and only if $n \geq 4$ is even.*

Proof. It can be easily checked that E_n admits a strongly involutive duality-isomorphism for any even integer $n \geq 4$, see Figure 11. Moreover, if n is odd then $I(E_n)$ admits a symmetric cycle of length $2k$ with k even, see Figure 12. Thus, by Theorem 1, W_n is not antipodally self-dual. □

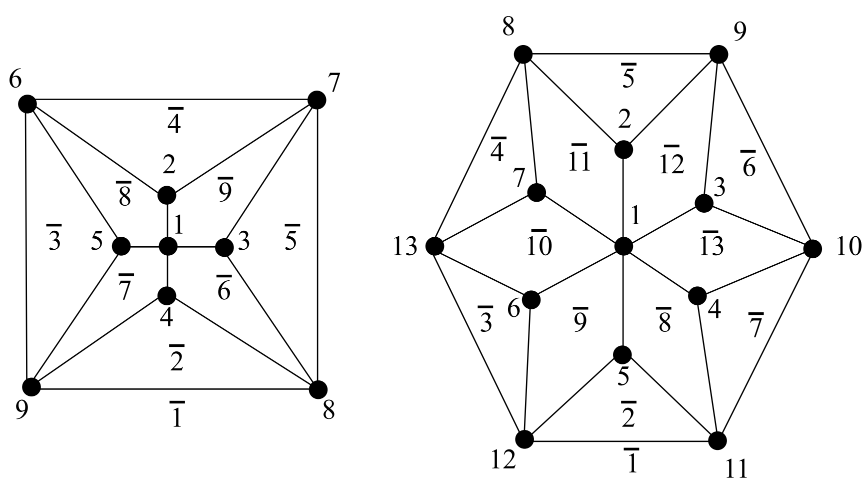

FIGURE 11. The 4-ear and 6-ear graphs together with strongly involutive duality-isomorphisms given by $\sigma(k) = \bar{k}$.

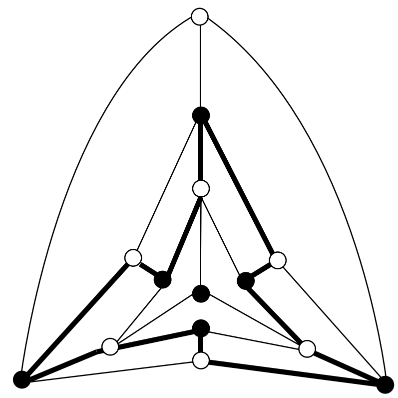

FIGURE 12. I(3-ear) admitting a symmetric cycle (bold edges) of length 12.

The map E_4 given in Figure 13 shows that 4-ear graph is antipodally self-dual. One can easily mimic this embedding for any even integer $n \geq 4$.

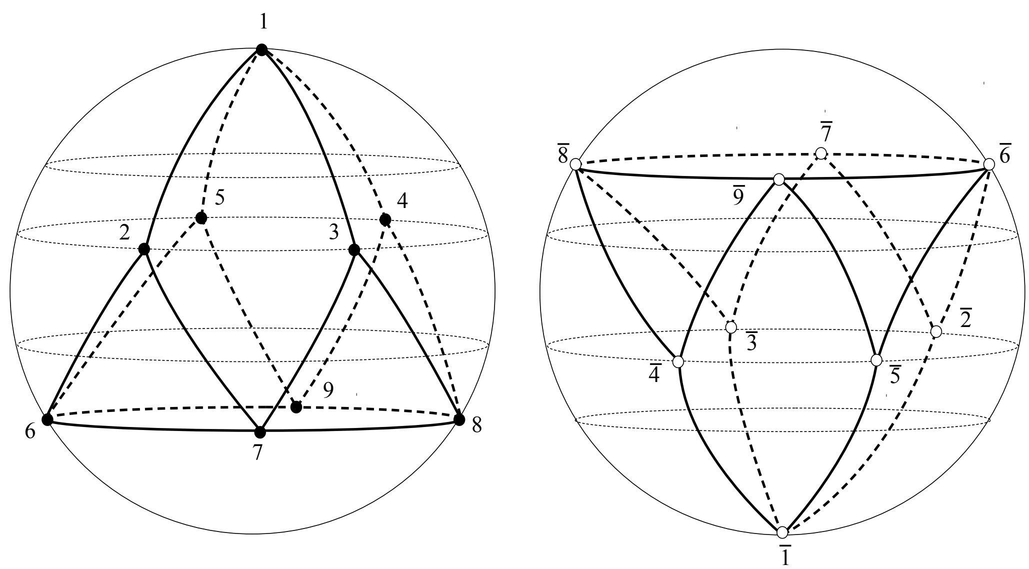

FIGURE 13. A antipodally self-dual map of E_4 (black vertices) and its dual (white vertices). Antipodal vertices are given by k and $\bar{k}$.

4.3. **The** (n, ℓ)**-pancake.** Let $n \geq 3$ and $l \geq 1$ be integers. The (n, ℓ)-*pancake*, denoted by P_n^ℓ, is the graph consisting of ℓ cycles $\{v_1^1, \ldots, v_n^1\}, \ldots, \{v_1^\ell, \ldots, v_n^\ell\}$, a vertex v_i^0 and edges $\{v_i^{j-1}, v_i^j\}$ for each $j = 1, \ldots, n$ and all i, see Figure 14.

Proposition 3. *The* (n, ℓ)*-pancake is antipodally self-dual if and only if* $n \geq 3$ *is odd for all* $\ell \geq 1$.

Proof. It can be easily checked that (n, ℓ)-pancake admits a strongly involutive duality-isomorphism for all integers $n \geq 3, \ell \geq 1$ with n odd, see Figure 14. Moreover, if n is even then $I((n, \ell)$pancake) admits a symmetric cycle of length $2k$ with k even, see Figure 15. Thus, by Theorem 1, (n, ℓ)-pancake is not antipodally self-dual. □

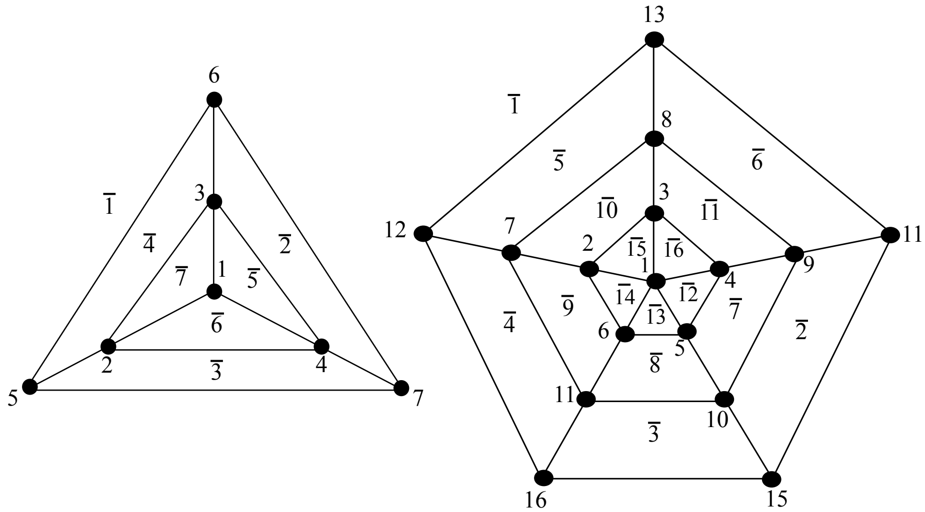

FIGURE 14. P_3^2 and P_5^3 together with a strongly involutive duality-isomorphism given by $\sigma(k) = \bar{k}$.

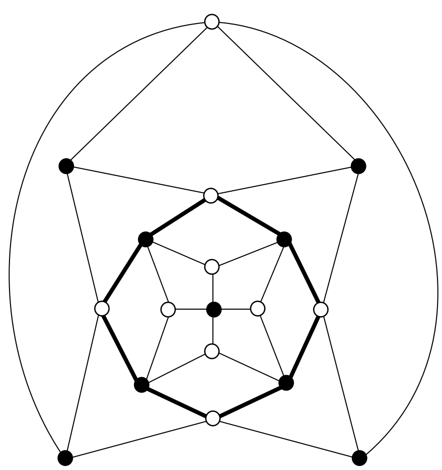

FIGURE 15. $I(P_4^2)$ admitting a symmetric cycle (bold edges) of length 8.

The map of P_5^2 given in Figure 16 shows that $(3,2)$-pancake is self-dual antipodal. One can easily mimic this embedding for any odd integer $n \geq 3$ and any $\ell \geq 1$.

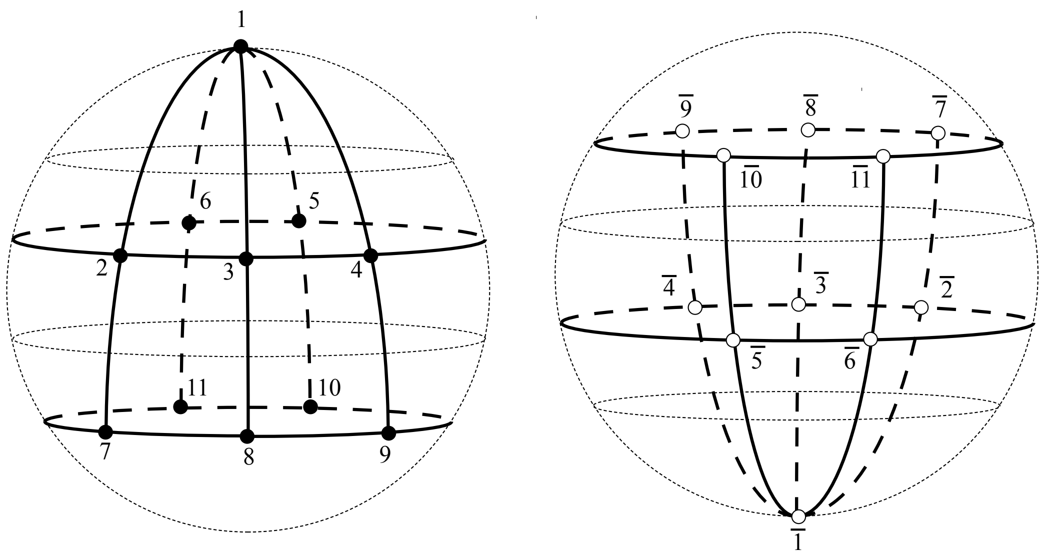

FIGURE 16. A antipodally self-dual map of P_5^2 (black vertices) and its dual (white vertices). Antipodal vertices are given by k and $\bar{k}$.

4.4. **Adhesion construction.** Let us give a way to construct infinite families of antipodally self-dual graphs. The latter is based on a procedure to construct self-dual graphs called the *adhesion*, given in [7]. Let G be a planar connected graph and let G^* be its geometric dual. Let x (resp. x^*) be the vertex corresponding to the exterior face of G^* (resp. exterior face of $G^{**} = G$). We define the graph $G \diamond G^*$ obtained by identifying x and x^*, see Figure 17.

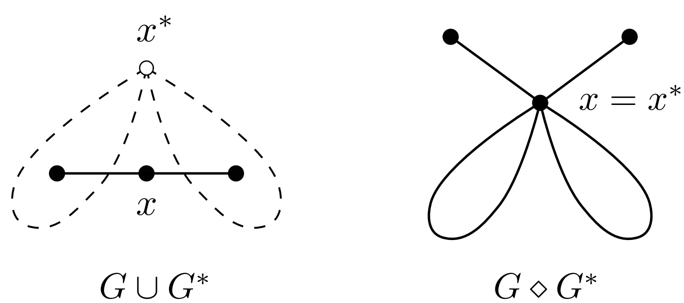

FIGURE 17. (Left) Drawing of G and its dual. (Right) The adhesion of G.

Lemma 3. [7] *Let G be a planar connected graph. Then, the graph $G \diamond G^*$ is self-dual.*

Proof. $H = G \diamond G^*$ is clearly self-dual since $H^* = (G \diamond G^*)^* = G^* \diamond G = G \diamond G^*$. □

Notice that in the construction of $G \diamond G^*$ the couple x and x^* cannot be replaced by <u>any</u> pair of vertices since we may end up with a not self-dual graph, see Figure 18.

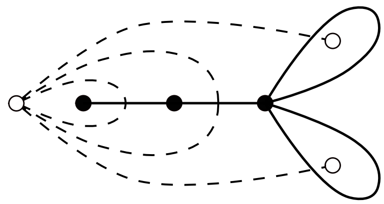

FIGURE 18. A glueing of G and G^* by another pair of vertices which leads to a map which is clearly not self-dual.

Theorem 3. *Let G be a planar connected graph. Then, $G \diamond G^*$ is antipodally self-dual.*

Proof. By Lemma 3 $H = G \diamond G^*$ is self-dual. Let us show that H admits an antipodal map. Let x (resp. x^*) be the vertex corresponding to the exterior face of G^* (resp. exterior face of $G = G^{**}$). We first draw G and its dual within a circle C such that x and x^* are antipodal points on C and no other edge or vertex (of G or G^*) lie on C, see Figure 19.

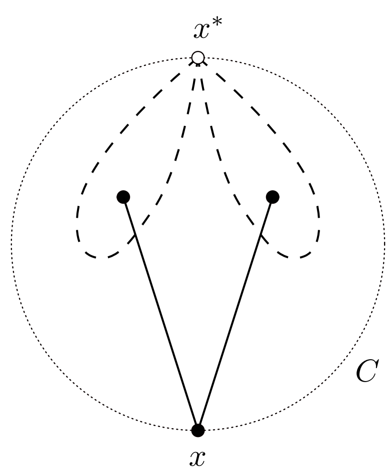

FIGURE 19. Embedding of G and G^* inside circle C with vertex b diametrically opposed to vertex b^*.

We shall construct two embeddings (one in the Northern hemisphere and the other in the Southern one) that will be glued together giving the desired antipodally self-dual embedding of H. For, we consider C as the equator of $\mathbb{S}^2$ and project our drawing perpendicularly to the Northern hemisphere of $\mathbb{S}^2$. We then take the antipodal of the latter embedding, obtaining an embedding in the Southern hemisphere.

We finally glue together both embeddings along the equator (x and x^* are the only vertices that are identified twice on the equator). By construction, this is an antipodal map of H, see Figure 20. $\square$

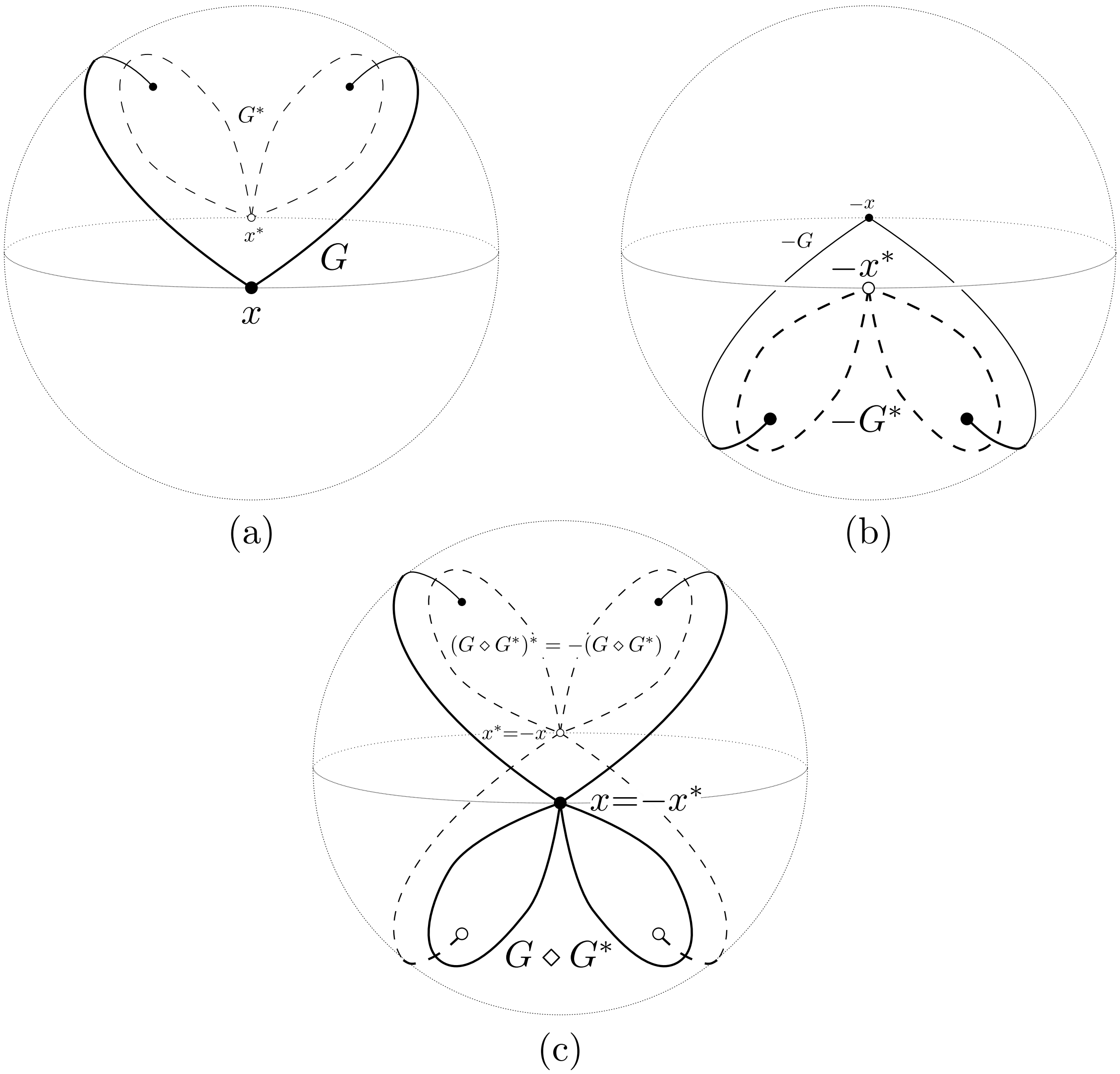

FIGURE 20. (a) Embedding of the drawings of G and G^* in the Northern hemisphere (b) Antipodal embedding of the drawing of the Northern hemisphere (c) Antipodal embedding of $H = G \diamond G^*$ (bold edges) and H^* (dashed edges).

Question 1. *Let H be a antipodally self-dual graph with a cut-vertex. Is it true that $H = G \diamond G^*$ where G is a planar connected graph and G^* its geometric dual ?*

5. Links: quick discussion

The notions of antipodally self-dual map and antipodally symmetric map turns out to be very helpful combinatorial tools to investigate special representations of links. These allow to give sufficient conditions for a link L to admit an embedding *centrally symmetric* in $\mathbb{R}^3$ or *antipodally symmetric* in $\mathbb{S}^3$.

This approach lead to sufficient conditions for L to be *amphichiral*. Moreover, the notion of symmetric cycle is also useful to study the amphichirality of a link from another closely related point of view (also arising other combinatorial sufficient conditions).

All these require further (much technical) extra work which is done in [6].

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Instituto de Matemáticas, Universidad Nacional A. de México at Querétaro Querétaro, México, CP. 07360

Email address: luis@im.unam.mx

UMI2924 - J.-C. Yoccoz, CNRS-IMPA, Brazil and IMAG, Univ. Montpellier, CNRS, Montpellier, France

Email address: jorge.ramirez-alfonsin@umontpellier.fr

IMAG, Univ. Montpellier, CNRS, Montpellier, France

Email address: ivan.rasskin@umontpellier.fr