The Diophantine Frobenius Problem

J.L. Ramírez Alfonsín
Equipe Combinatoire et Optimisation
Université Pierre et Marie Curie, Paris 6
4 Place Jussieu, 75252 Paris Cedex 05

Preface

During the early part of the last century, Ferdinand Georg Frobenius (1849-1917) raised, in his lectures (according to [55]), the following problem (called the *Frobenius Problem* **FP**): given relatively prime positive integers $a_1, \ldots, a_n$, find the largest natural number (called the *Frobenius number* and denoted by $g(a_1, \ldots, a_n)$) that is not representable as a nonnegative integer combination of $a_1, \ldots, a_n$.

At first glance, **FP** may look deceptively specialized. Nevertheless it crops up again and again in the most unexpected places. It turned out that the knowledge of $g(a_1, \ldots, a_n)$ has been extremely useful to investigate many different problems.

A number of methods, from several areas of mathematics, have been used in the hope of finding a formula giving the Frobenius number and algorithms to calculate it. The main intention of this book is to highlight such 'methods, ideas, viewpoints and applications' for as wide an audience as possible. The results on **FP** are quite scattered in the literature and, at present, there is no complete or accessible source summarizing the progress on it. This book aims to provide a comprehensive exposition of what is known today on **FP**.

Chapter 1 is devoted to the computational aspects of the Frobenius number. After discussing a number of methods to solve **FP** when $n = 3$ (some of these procedures make use of diverse concepts, such as the *division remainder, continued fractions* and *maximal lattice free bodies*) we present a variety of algorithms to compute $g(a_1, \ldots, a_n)$ for general n. The main ideas of these algorithms are based on concepts from *graph theory, index of primitivity* of nonnegative matrices (see Appendix 9.6) and *mathematical programming.* While the running times of these algorithms are super-polynomial, there does exists a method, due to R. Kannan, that solves **FP** in polynomial time for any *fixed* n. We describe this method, in which the *covering radius* concept is introduced. We finally prove that **FP** is **NP**-*hard* under Turing reductions.

FP is easy to solve when $n = 2$. Indeed,

$$g(a_1, a_2) = a_1 a_2 - a_1 - a_2. \tag{1}$$

However the computation of a (simple) formula when $n = 3$ is much more difficult and has been the subject of numerous research papers over a long period. F. Curtis has proved that the search for such a formula is, in some sense, doomed to failure since the Frobenius number cannot be given by 'closed' formulas of a certain type. Recently, an explicit formula for computing $g(a_1, a_2, a_3)$ has been found. After presenting four different proofs of equality (1), one of which uses the well-known *Pick's theorem*, Chapter 2 presents the result of Curtis, the general formula (whose algebraic proof is given in Chapter 4) and summarizes the known upper bounds for $g(a_1, a_2, a_3)$, as well as exact formulas for particular triples.

Chapter 3 provides a systematic exposition of the known formulas, including upper and lower bounds for $g(a_1, \ldots, a_n)$ for general n and for special sequences (for instance, when $a_1, \ldots, a_n$ forms an *arithmetic* sequence). Results on the change in value of $g(a_1, \ldots, a_n)$, when an additional element a_{n+1} is inserted, are also given.

In 1857, while investigating the *partition number* function, James Joseph Sylvester (1814-1897) [434] defined the function $d(m; a_1, \ldots, a_n)$, called the *denumerant*, as the number of nonnegative integer representations of m by $a_1, \ldots, a_n$, that is, the number of solutions of the form

$$m = \sum_{i=1}^{n} x_i a_i$$

with integers $x_i \geq 0$. Chapter 4 is devoted to the study of the denumerant and related functions. After discussing briefly some basic properties of the *partition function* and its relation with denumerants, we analyze the general behaviour of $d(m; a_1, \ldots, a_n)$ and its connection to $g(a_1, \ldots, a_n)$. Two interesting methods for computing denumerants, one based on a decomposition of the *rational fraction* into *partial fractions* and a second due to E.T. Bell, are described. We prove an exact value of $d(m; p, q)$, first found by T. Popoviciu in 1953, and summarize the known results when $n = 2$ and $n = 3$. We shall see how to calculate $g(a_1, \ldots, a_n)$ by using *Hilbert series* via *free resolutions* and use this approache to show an explicit formula for $g(a_1, a_2, a_3)$. We discuss the connection among denumerants, **FP** and *Ehrhart polynomial.* Also, two variants of $d(m; a_1, \ldots, a_n)$ are studied. The first is related to counting the number of lattice points lying in certain polytopes while the second restricts the number of repetitions of the a_i's.

Let $N(a_1, \ldots, a_n)$ be the number of integers without nonnegative integer representations by $a_1, \ldots, a_n$. In Chapter 5, a thorough presentation of the function $N(a_1, \ldots, a_n)$ is given. In 1882, Sylvester [435], obtained the exact value when $n = 2$,

$$N(a_1, a_2) = \frac{1}{2}(a_1 - 1)(a_2 - 1). \qquad (2)$$

Later, in 1884, in the *Educational Times* journal, Sylvester [433] posed (as a recreational problem) the question of finding such a formula. An ingenious solution was given by W.J. Curran Sharp. It remains a mystery why the standard reference to this celebrated formula of Sylvester is the solution given by Curran Sharp rather than its original appearance in [435, page 134]. In this chapter, we reproduce the original page of this famous and much cited manuscript. We also give two other proofs of equality (2). We then discuss the work of M. Nijenhuis and H.S. Wilf connecting $N(a_1, \ldots, a_n)$ to **FP** as well as to other concepts (such as the *Gorenstein* condition). We continue by discussing some general bounds on $N(a_1, \ldots, a_n)$ and exact formulas for special sequences, for instance the formula given by E.S. Selmer for *almost arithmetic* sequences. A generalization of Sylvester's formula due to Ø.J. Rødseth, where the so-called *Bernoulli* numbers (see Appendix 9.5) appeared, is treated. The final section of this chapter is devoted to two 'integer representation' games: the well-known *Sylver Coinage*, invented by J.C. Conway and the *jugs problem* which roots can be traced back at least as far as Tartaglia, an Italian mathematician of the sixteenth century.

Let $g(n, t)$ and $h(n, t)$ be the largest and smallest of the Frobenius numbers when $a_1 < \cdots < a_n = t$ and $t = a_1 < \cdots < a_n$, respectively. Chapter 6 reviews the results on these functions. It also examines an algorithm that solves the *modular change* problem, a generalization of **FP**, due to Z. Skupień, discribes the relation between **FP** and $(a_1, \ldots, a_n)$-*trees*, discusses the *postage stamp* problem as well as a multidimensional generalization of **FP**.

Chapter 7 introduces the concept of *numerical semigroups.* We investigate several properties of the *gaps* and *nongaps* of a semigroup (which are closely related to $N(a_1, \ldots, a_n)$) and point out the importance of the role played by the Frobenius number (also known as *conductor*) in the study of *symmetric* and *pseudo-symmetric* semigroups (and their connection to *monomial curves*). We prove a number of results relating **FP** to *telescopic semigroups*, the famous *Apéry Sets* (used by R. Apéry [13] in the study of *algebroid planar branches*), *type sequences* in semigroups, *complete intersection* semigroups, γ-*hyperelliptic* semigroups (motivated by the study of *Weierstrass* semigroups), the *Möbius* function, and other related concepts.

Chapter 8 presents a number of applications of **FP** to a variety of problems. The complexity analysis of the *Shell-sort* method was not well understood until J. Incerpi and R. Sedgewick nicely observed that **FP** can be used to obtain upper bounds for the running time of this fundamental sorting algorithm. Chapter 8 starts by explaining this application. Then, it is explained how **FP** may be applied to analyse *Petri nets* (a net model for discrete event systems), to study *partitions of vector spaces* (which can be considered as a generalization of partitions of abelian groups), to compute *exact resolutions* via Rødseth's method for finding the Frobenius number when $n = 3$, to investigate *Algebraic Geometric* codes via the properties of special semigroups and their corresponding conductors and to study *tiling* problems. Chapter 8 also discusses three applications of the denumerant. One in relation with the calculation of the number of possible placements of n different balls into r distinct cells under certain restrictions, another to investigate the solution of some *conjugate* problems and the last one in relation with *invariant cubature* formulas. We also present an application of the *modular change* problem to study *nonhypohamiltonian* graphs, and of the *vector* generalization to give a new method for generating *random* vectors.

The book concludes with an appendix where some notation, definitions and basic of various topics results are given.

This book attempts to place the reader at the frontier of what is known on **FP**. In the interests of balance, we have chosen not to give a proof of each and every result (particularly of the numerous bounds and formulas stated in Chapters 2 and 3). However, all the main theorems are either proved or treated in some detail. We illustrate with examples most of the methods explained in Chapter 1. We always try to give exact references and appropriate credits for the proofs and results that have been adapted from printed material. References to the literature where the reader may find more complete treatments of the various topics, and some historical comments, are given at the end of each chapter.

Despite many careful readings, errors will unavoidable remain. We plan to mantain an updated list of corrections at the following WEB site pointer

`http://www.ecp6.math.jussieu.fr/pageperso/ramirez/ramirez.html`

The topics in this book are in a state of continual development. We also plan to note new progress on **FP** in the same site.