

# On the Möbius function of semigroup posets

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# Basics on posets

Let  $(\mathcal{P}, \leq)$  be a **locally finite poset**, i.e.,

- the set  $\mathcal{P}$  is partially ordered by  $\leq$ , and
- for every  $a, b \in \mathcal{P}$  the set  $\{c \in \mathcal{P} \mid a \leq c \leq b\}$  is finite.

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A **chain** of length  $l \geq 0$  between  $a, b \in \mathcal{P}$  is

$$\{a = a_0 < a_1 < \cdots < a_l = b\} \subset \mathcal{P}.$$

We denote by  $c_l(a, b)$  the number of chains of length  $l$  between  $a$  and  $b$ .

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The **Möbius function**  $\mu_{\mathcal{P}}$  is the function

$$\mu_{\mathcal{P}} : \mathcal{P} \times \mathcal{P} \longrightarrow \mathbb{Z}$$

$$\mu_{\mathcal{P}}(a, b) = \sum_{l \geq 0} (-1)^l c_l(a, b).$$



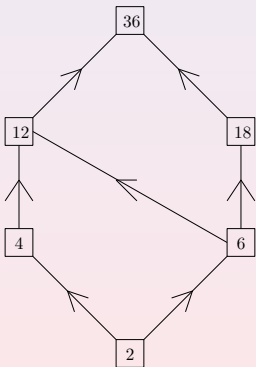
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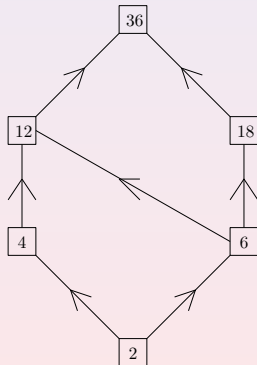
- length 1  $\rightarrow \{2, 36\}$
- length 2  $\left\{ \begin{array}{l} \{2, 4, 36\} \\ \{2, 6, 36\} \\ \{2, 12, 36\} \\ \{2, 18, 36\} \end{array} \right.$
- length 3  $\left\{ \begin{array}{l} \{2, 4, 12, 36\} \\ \{2, 6, 12, 36\} \\ \{2, 6, 18, 36\} \end{array} \right.$





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- length 3  $\left\{ \begin{array}{l} \{2, 4, 12, 36\} \\ \{2, 6, 12, 36\} \\ \{2, 6, 18, 36\} \end{array} \right.$



Thus,

$$\mu_{\mathbb{N}}(2, 36) = -c_1(2, 36) + c_2(2, 36) - c_3(2, 36) = 1 - 4 + 3 = 0.$$

# Möbius classical arithmetic function

Given  $n \in \mathbb{N}$  the *Möbius arithmetic function*  $\mu(n)$  is defined as

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 \cdots p_k \text{ with } p_i \text{ distinct primes,} \\ 0 & \text{otherwise (i.e., } n \text{ admits at least one square} \\ & \text{factor bigger than one).} \end{cases}$$

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The inverse of the  $\zeta$  Riemann function,  $s \in \mathbb{C}$ ,  $\operatorname{Re}(s) > 0$

$$\zeta^{-1}(s) = \left( \sum_{n=1}^{+\infty} \frac{1}{n^s} \right)^{-1} = \prod_{p-\text{primes}} (1 - p^{-s}) = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}$$

There are impressive results using  $\mu$ , for instance, for an integer  $n$

$$\Pr(n \text{ do not contain a square factor}) = \frac{6}{\pi^2}$$

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Example:  $\mu_{\mathbb{N}}(2, 36) = 0$  because  $36/2 = 18 = 2 \cdot 3^2$



# Möbius inversion formula

## Theorem (Rota)

Let  $(\mathcal{P}, \leq)$  be a poset, let  $p$  be an element of  $\mathcal{P}$  and consider  $f : \mathcal{P} \rightarrow \mathbb{R}$  a function such that  $f(x) = 0$  for all  $x \not\leq p$ . Suppose that

$$g(x) = \sum_{y \leq x} f(y) \text{ for all } x \in \mathcal{P}.$$

Then,

$$f(x) = \sum_{y \leq x} g(y) \mu_{\mathcal{P}}(y, x) \text{ for all } x \in \mathcal{P}.$$

Compute the *Euler function*  $\phi(n)$  (the number of integers smaller or equal to  $n$  and coprime with  $n$ )

$$\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}$$

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Let  $D$  be a finite set and consider the **poset of multisets over  $D$  ordered by inclusion**  $\mathcal{P}$ . Then, for all  $A, B$  multisets over  $D$  we have that

$$\mu_{\mathcal{P}}(A, B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subset B \text{ and } B \setminus A \text{ is a set,} \\ 0 & \text{otherwise.} \end{cases}$$

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An immediate consequence is the classical *inclusion-exclusion* counting formula !!



# Semigroup poset

Let  $\mathcal{S} := \langle a_1, \dots, a_n \rangle$  denote the **subsemigroup** of  $\mathbb{Z}^m$  generated by  $a_1, \dots, a_n \in \mathbb{N}^m$ , i.e.,

$$\mathcal{S} := \langle a_1, \dots, a_n \rangle = \{x_1 a_1 + \dots + x_n a_n \mid x_1, \dots, x_n \in \mathbb{N}\}.$$

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It turns out that  $\leq_{\mathcal{S}}$  is an **order** iff  $\mathcal{S}$  is **pointed** (i.e.,  $\mathcal{S} \cap -\mathcal{S} = \{0\}$ ). Moreover, whenever  $\mathcal{S}$  is pointed the poset  $(\mathbb{Z}^m, \leq_{\mathcal{S}})$  is locally finite.



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It is easy to check that  $\mu_S(x, y) = \mu_S(0, y - x)$ , hence we shall only consider the **reduced Möbius function**  $\mu_S : \mathbb{Z}^m \rightarrow \mathbb{Z}$  defined by

$$\mu_S(x) := \mu_S(0, x) \quad \text{for all } x \in \mathbb{Z}^m.$$

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## Proposition (Key)

*If  $S$  is a pointed semigroup,  $x \in \mathbb{Z}^m$ , then*

$$\sum_{b \in S} \mu_S(x - b) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

# Known results about $\mu_S$

① *Deddens* (1979).

For  $S = \langle a, b \rangle \subset \mathbb{N}$  where  $a, b \in \mathbb{Z}^+$  are relatively prime:

$$\mu_S(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \equiv 0 \text{ or } a + b \pmod{ab}, \\ -1 & \text{if } x \geq 0 \text{ and } x \equiv a \text{ or } b \pmod{ab}, \\ 0 & \text{otherwise.} \end{cases}$$

② *Chappelon and R.A.* (2013).

- A **recursive formula** for  $\mu_S$  when  $S = \langle a, a + d, \dots, a + kd \rangle \subset \mathbb{N}$  for some  $a, k, d \in \mathbb{Z}^+$ , and
- a **semi-explicit formula** for  $S = \langle 2q, 2q + d, 2q + 2d \rangle \subset \mathbb{N}$  where  $q, d \in \mathbb{Z}^+$  and  $\gcd\{2q, 2q + d, 2q + 2d\} = 1$ .

# Goals

- 1 Provide **general tools** to study  $\mu_S$  for every semigroup  $S \subset \mathbb{Z}^m$ .

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- 2 Provide **explicit formulas** for certain families of semigroups  $S \subset \mathbb{Z}^m$ .

# Multigraded Hilbert series

- Let  $k$  be a field. A semigroup  $\mathcal{S} = \langle \mathbf{a}_1, \dots, \mathbf{a}_n \rangle \subset \mathbb{N}^m$  induces a **grading** in the **ring of polynomials**  $k[x_1, \dots, x_n]$  by assigning  $\deg_{\mathcal{S}}(x_i) := \mathbf{a}_i$  for all  $i \in \{1, \dots, n\}$ .

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- The  **$\mathcal{S}$ -degree** of the monomial  $m := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$  is  $\deg_{\mathcal{S}}(m) = \sum \alpha_i \mathbf{a}_i$ .



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- For all  $b \in \mathbb{N}^m$ , we denote by  $k[x_1, \dots, x_n]_b$  the  $k$ -vector space formed by all **polynomials  $\mathcal{S}$ -homogeneous of  $\mathcal{S}$ -degree  $b$** .

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- Consider  $I \subset k[\mathbf{x}]$  an ideal generated by  **$\mathcal{S}$ -homogeneous polynomials**. For all  $b \in \mathbb{N}^m$  we denote by  $I_b$  the  $k$ -vector space formed by the  **$\mathcal{S}$ -homogeneous polynomials of  $I$  of  $\mathcal{S}$ -degree  $b$** .

The **(multigraded) Hilbert function** of  $M := k[x_1, \dots, x_n]/I$  is

$$HF_M : \mathbb{N}^m \longrightarrow \mathbb{N},$$

where  $HF_M(b) := \dim_k(k[x_1, \dots, x_n]_b) - \dim_k(I_b)$  for all  $b \in \mathbb{N}^m$ .

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We define the **(multivariate) Hilbert series** of  $M$  as the formal power series in  $\mathbb{Z}[[t_1, \dots, t_m]]$ :

$$\mathcal{H}_M(\mathbf{t}) := \sum_{b \in \mathbb{N}^m} HF_M(b) \mathbf{t}^b$$

where  $\mathbf{t}^b$  denote the monomial  $t_1^{b_1} \cdots t_m^{b_m} \in \mathbb{Z}[t_1, \dots, t_m]$ .

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## Theorem

$$\mathcal{H}_M(\mathbf{t}) = \frac{\mathbf{t}^\alpha h(\mathbf{t})}{(1 - \mathbf{t}^{a_1}) \cdots (1 - \mathbf{t}^{a_n})},$$

where  $\alpha \in \mathbb{Z}^m$  and  $h(\mathbf{t}) \in \mathbb{Z}[t_1, \dots, t_m]$ .



We denote by  $I_{\mathcal{S}}$  the **toric ideal** of  $\mathcal{S}$ , i.e., the kernel of the homomorphism of  $k$ -algebras

$$\varphi : k[x_1, \dots, x_n] \longrightarrow k[t_1, \dots, t_m]$$

induced by  $\varphi(x_i) = \mathbf{t}^{a_i}$  for all  $i \in \{1, \dots, n\}$ .

It is well known that  $I_{\mathcal{S}}$  is generated by  $\mathcal{S}$ -homogeneous polynomials.

We denote by  $I_S$  the **toric ideal** of  $S$ , i.e., the kernel of the homomorphism of  $k$ -algebras

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## Proposition

$$\mathcal{H}_{k[x_1, \dots, x_n]/I_S}(\mathbf{t}) = \sum_{b \in S} \mathbf{t}^b.$$



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### Proposition

$$\mathcal{H}_{k[x_1, \dots, x_n]/I_{\mathcal{S}}}(\mathbf{t}) = \sum_{b \in \mathcal{S}} \mathbf{t}^b.$$

From now on, we denote  $\mathcal{H}_{\mathcal{S}}(\mathbf{t}) := \mathcal{H}_{k[x_1, \dots, x_n]/I_{\mathcal{S}}}(\mathbf{t})$  and we call it the **Hilbert series of  $\mathcal{S}$** .



# Möbius function via Hilbert series

**Example:** For  $\mathcal{S} = \langle 2, 3 \rangle \subset \mathbb{N}$ , we have that  $\mathcal{S} = \{0, 2, 3, 4, 5, \dots\}$

$$\mathcal{H}_{\mathcal{S}}(t) = 1 + t^2 + t^3 + t^4 + t^5 + \dots$$

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$$\text{Then, } (1 - t^2) \mathcal{H}_{\mathcal{S}}(t) = 1 + t^3 \text{ and } \mathcal{H}_{\mathcal{S}}(t) = \frac{1 + t^3}{1 - t^2}.$$

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Theorem (1 (Chappelon, García-Marco, Montejano, R.A. 2014) )

Let  $c_1, \dots, c_k$  nonzero vectors of  $\mathbb{Z}$  and denote

$$(1 - \mathbf{t}^{a_1}) \cdots (1 - \mathbf{t}^{a_n}) \mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \sum_{b \in \mathbb{Z}^m} f_b \mathbf{t}^b.$$

Then,

$$\sum_{b \in \mathbb{Z}^m} f_b \mu(x - b) = 0 \text{ for all } x \notin \left\{ \sum_{i \in A} a_i \mid A \subset \{1, \dots, n\} \right\}.$$



## Example: $\mathcal{S} = \langle 2, 3 \rangle$

We know that,

$$\mathcal{H}_{\mathcal{S}}(t) = \frac{1 + t^3}{1 - t^2}.$$

By **Theorem (1)** we have that

$$\mu_{\mathcal{S}}(x) + \mu_{\mathcal{S}}(x - 3) = 0$$

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It is evident that  $\mu_{\mathcal{S}}(0) = 1$ . A direct computation yields  $\mu_{\mathcal{S}}(2) = -1$ .

Hence

$$\mu_{\mathcal{S}}(x) = \begin{cases} 1 & \text{if } x \equiv 0 \text{ or } 5 \pmod{6}, \\ -1 & \text{if } x \equiv 2 \text{ or } 3 \pmod{6}, \\ 0 & \text{otherwise.} \end{cases}$$



Example:  $\mathcal{S} = \mathbb{N}^m$

$$\mu_{\mathbb{N}^m}(x) = \begin{cases} (-1)^{|A|} & \text{if } x = \sum_{i \in A} e_i \text{ for some } A \subset \{1, \dots, m\} \\ 0 & \text{otherwise} \end{cases}$$



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Proof #1.

We observe that  $\mathbb{N}^m = \langle e_1, \dots, e_m \rangle$  and that

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \sum_{(b_1, \dots, b_m) \in \mathbb{N}^m} t_1^{b_1} \cdots t_m^{b_m} = \frac{1}{(1-t_1) \cdots (1-t_m)}.$$

By Theorem (1) we have that  $\mu_{\mathbb{N}^m}(x) = 0$  for all  $x \notin \{\sum_{i \in A} e_i \mid A \subset \{1, \dots, m\}\}$ .

A direct computation yields  $\mu_{\mathbb{N}^m}(\sum_{i \in A} e_i) = (-1)^{|A|}$  for every  $A \subset \{1, \dots, m\}$ . □



# Möbius function via Hilbert series

We consider  $\mathcal{G}_S$  the **generating function of the Möbius function**, which is

$$\mathcal{G}_S(\mathbf{t}) := \sum_{b \in \mathbb{N}^m} \mu_S(b) \mathbf{t}^b.$$

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Theorem (2 (Chappelon, García-Marco, Montejano, R.A. 2014))

$$\mathcal{H}_S(\mathbf{t}) \mathcal{G}_S(\mathbf{t}) = 1.$$

Example:  $\mathcal{S} = \mathbb{N}^m$

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Proof #2.

$$\mathcal{H}_{\mathcal{S}}(\mathbf{t}) = \sum_{(b_1, \dots, b_m) \in \mathbb{N}^m} t_1^{b_1} \cdots t_m^{b_m} = \frac{1}{(1-t_1) \cdots (1-t_m)}.$$

By Theorem (2) we have that

$$\mathcal{G}_{\mathcal{S}}(\mathbf{t}) = (1-t_1) \cdots (1-t_m) = \sum_{A \subset \{1, \dots, m\}} (-1)^{|A|} \mathbf{t}^{\sum_{i \in A} e_i}.$$



## Explicit formulas for $\mu_S$ .

A semigroup  $S \subset \mathbb{N}^m$  is said to be a **semigroup with a unique Betti element**  $b \in \mathbb{N}^m$  if  $I_S$  is **generated by polynomials of  $S$ -degree  $b$** .

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We denote  $d := \dim(\mathbb{Q}\{a_1, \dots, a_n\})$ . In this setting we have the following result.

Theorem (Chappelon, García-Marco, Montejano, R.A. 2014)

$$\mu_S(x) = \sum_{j=1}^t (-1)^{|A_j|} \binom{k_j + n - d - 1}{k_j},$$

if  $x = \sum_{i \in A_1} a_i + k_1 b = \dots = \sum_{i \in A_t} a_i + k_t b$ .

When  $\mathcal{S} = \langle a_1, \dots, a_n \rangle \subset \mathbb{N}$  is a **semigroup with a unique Betti element** and  $\gcd\{a_1, \dots, a_n\} = 1$ , García-Sánchez, Ojeda and Rosales (2013) proved that there exist **pairwise relatively prime different integers**  $b_1, \dots, b_n \geq 2$  such that  $a_i := \prod_{j \neq i} b_j$  for all  $i \in \{1, \dots, n\}$ .

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In this setting we can refine the previous Theorem to obtain the following result.



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In this setting we can refine the previous Theorem to obtain the following result.

**Corollary (Chappelon, García-Marco, Montejano, R.A. 2014)**

Set  $b := \prod_{i=1}^n b_i$ , then

$$\mu_{\mathcal{S}}(x) = \begin{cases} (-1)^{|A|} \binom{k+n-2}{k} & \text{if } x = \sum_{i \in A} a_i + k b \\ & \text{for some } A \subset \{1, \dots, n\} \\ 0 & \text{otherwise} \end{cases}$$



Whenever  $\mathcal{S} := \langle a_1, a_2, a_3 \rangle \subset \mathbb{N}$  with  $\gcd\{a_1, a_2, a_3\} = 1$ , we say that  $\mathcal{S}$  is a **complete intersection** if there exists **two**  $\mathcal{S}$ -homogeneous polynomials  $f_1, f_2$  such that  $I_{\mathcal{S}} = (f_1, f_2)$ .

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### Theorem (Herzog (1970))

$\mathcal{S}$  is a complete intersection  $\iff \gcd\{a_i, a_j\} a_k \in \langle a_i, a_j \rangle$ , where  $\{i, j, k\} = \{1, 2, 3\}$ .

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We aim at presenting a formula for  $\mathcal{S} = \langle a_1, a_2, a_3 \rangle \subset \mathbb{N}$  complete intersection and  $\gcd\{a_1, a_2, a_3\} = 1$ , so we assume that  $da_1 \in \langle a_2, a_3 \rangle$ , where  $d := \gcd\{a_2, a_3\}$ .

For every  $x \in \mathbb{Z}$  and every  $B = (b_1, \dots, b_k) \subset (\mathbb{Z}^+)^k$ , the **Sylvester denumerant**  $d_B(x)$  is the number of non-negative integer solutions  $(x_1, \dots, x_k) \in \mathbb{N}^k$  to the equation  $x = \sum_{i=1}^k x_i b_i$ .

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For every  $x \in \mathbb{Z}$  we denote by  $\alpha(x)$  the only integer such that  $0 \leq \alpha(x) \leq d - 1$  and  $\alpha(x) a_1 \equiv x \pmod{d}$ .

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For  $\mathcal{S} = \langle a_1, a_2, a_3 \rangle$  complete intersection and  $\gcd\{a_1, a_2, a_3\} = 1$ , we have the following result.

Theorem (Chappelton, García-Marco, Montejano, R.A. 2014)

$\mu_{\mathcal{S}}(x) = 0$  if  $\alpha(x) \geq 2$ , or

$$\mu_{\mathcal{S}}(x) = (-1)^\alpha (d_B(x') - d_B(x' - a_2) - d_B(x' - a_3) + d_B(x' - a_2 - a_3))$$

otherwise, where  $x' := x - \alpha(x) a_1$  and  $B := (da_1, a_2, a_3/d)$ .

## How general are semigroup posets?

Let  $D = \{d_1, \dots, d_m\}$  be a finite set and let us consider  $(\mathcal{P}, \subset)$ , the **poset of all multisets of  $D$  ordered by inclusion**.



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For the semigroup  $\mathcal{S} := \mathbb{N}^m$ , we consider the map

$$\begin{aligned} \psi : (\mathcal{P}, \subset) &\rightarrow (\mathbb{N}^m, \leq_{\mathbb{N}^m}) \\ A &\mapsto (m_A(d_1), \dots, m_A(d_m)), \end{aligned}$$

where  $m_A(d_i)$  denotes the number of times that  $d_i$  belongs to  $A$ .

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It is easy to check that  $\psi$  is an **order isomorphism** (an order preserving and order reflecting bijection).

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Hence,

$$\mu_{\mathcal{P}}(A, B) = \mu_{\mathbb{N}^m}(\psi(A), \psi(B)),$$

and we can recover the formula for  $\mu_{\mathcal{P}}$  by means of  $\mu_{\mathbb{N}^m}$ .

$$\mu_{\mathcal{P}}(A, B) = \begin{cases} (-1)^{|B \setminus A|} & \text{if } A \subset B \text{ and } B \setminus A \text{ is a set} \\ 0 & \text{otherwise} \end{cases}$$

Take  $p_1, \dots, p_m$  the  $m$  first prime numbers, and consider

$$\mathbb{N}_m := \{p_1^{\alpha_1} \cdots p_m^{\alpha_m} \mid \alpha_1, \dots, \alpha_m \in \mathbb{N}\} \subset \mathbb{N}.$$

Let us consider the **poset**  $(\mathbb{N}_m, |)$ , i.e.,  $\mathbb{N}_m$  **partially ordered by divisibility**.

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Hence,

$$\mu_{\mathbb{N}_m}(a, b) = \mu_{\mathbb{N}^m}(\psi(a), \psi(b)),$$

and we can recover the formula for  $\mu_{\mathbb{N}_m}$  by means of  $\mu_{\mathbb{N}^m}$ .

$$\mu_{\mathbb{N}_m}(a, b) = \begin{cases} (-1)^r & \text{if } b/a \text{ is a product of } r \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$



**Roli doing elegant mathematics**



**Roli doing elegant mathematics - Kaleidoscope version**





**Roli supporting 'Pumas'**





**Roli supporting 'Pumas' - Kaleidoscope version**





**Luis & Roli supporting 'Pumas'**





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