Theory of matroids I: basic notions and Tutte polynomial

J.L. Ramírez Alfonsín

I3M, Université Montpellier 2

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Independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

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On one hand, $dim(W) \geq |I_2|$, on the other hand W is contained in the space generated by I_1.

$$|I_2| \leq dim(W) \leq |I_1| < |I_2| \quad !!!$$

Representable Matroids

Let A be the following matrix with coefficients in $\mathbb{R}$.

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$\{\emptyset, \{1\}, \{2\}, \{4\}, \{4\}, \{5\}, \{1,2\}, \{1,5\}, \{2,4\}, \{2,5\}, \{4,5\}\} \subseteq \mathcal{I}(M)$

A matroid obtained form a matrix A with coefficients in $\mathbb{F}$ is denoted by $M(A)$ and is called representable over $\mathbb{F}$ or $\mathbb{F}$-representable.

Circuits

A subset $X \subseteq E$ is said to be minimal dependent if any proper subset of X is independent. A minimal dependent set of matroid M is called circuit of M.

We denote by $\mathcal{C}$ the set of circuits of a matroid.

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We denote by $\mathcal{C}$ the set of circuits of a matroid.

$\mathcal{C}$ is the set of circuits of a matrid on E if and only if $\mathcal{C}$ verifies the following properties :

$(C1)$ $\emptyset \notin \mathcal{C}$,

$(C2)$ $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$,

$(C3)$ *(elimination property)* If $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq \{C_1 \cup C_2\} \backslash \{e\}$.

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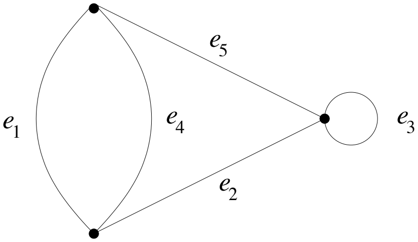
Proof : Verify $(C1), (C2)$ and $(C3)$.

A subset of edges $I \subset \{e_1, \ldots, e_n\}$ of G is independent if the graph induced by I does not contain a cycle.

Graphic Matroid



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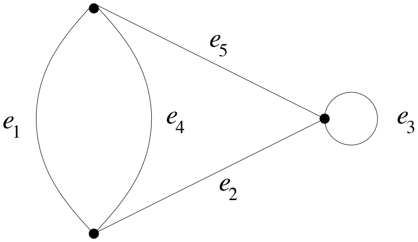

It can be checked that $M(G)$ is isomorphic to $M(A)$ (under the bijection $e_i \to i$).

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Proof (idea) Let $G = (V, E)$ be an oriented graph and let $\{x_i,\ i \in V\}$ be the canonical base of $\mathbb{R}^{|V|}$.

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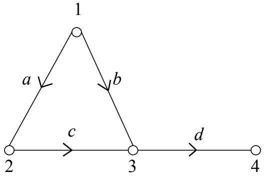
Theorem A graphic matroid is always representable over $\mathbb{R}$.

Proof (idea) Let $G = (V, E)$ be an oriented graph and let $\{x_i,\ i \in V\}$ be the canonical base of $\mathbb{R}^{|V|}$.

Exercice : Verify that the graph $G = (V, E)$ gives the same matroid that the one given by the set of vectors $y_e = x_i - x_j$ where $e = (i, j) \in E$.

Graphic Matroid

G



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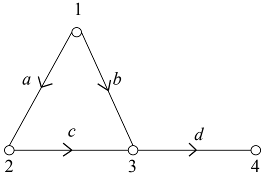

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$M(G)$ is isomorphic to $M(A)$ $(a \to y_a, b \to y_b, c \to y_c, d \to y_d)$.

Graphic Matroid

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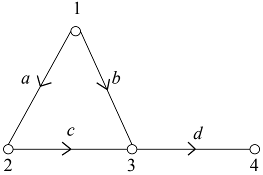

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The cycle formed by the edges $a = \{1,2\}, b = \{1,3\}$ et $c = \{2,3\}$ in the graph correspond to the linear dependency $y_b - y_a = y_c$.

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A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid.

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The family $\mathcal{B}$ verifies the following conditions :

($B1$) $\mathcal{B} \neq \emptyset$,

($B2$) *(exchange propety)* $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

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If $\mathcal{I}$ is the family of subsets contained in a set of $\mathcal{B}$ then $(E, \mathcal{I})$ is a matroid.

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Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

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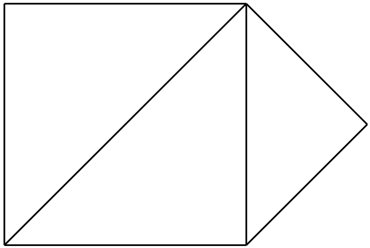

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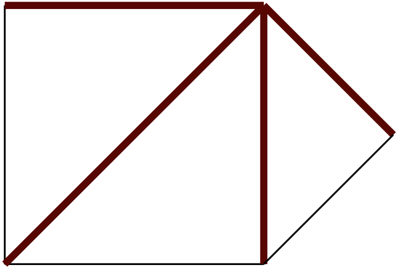

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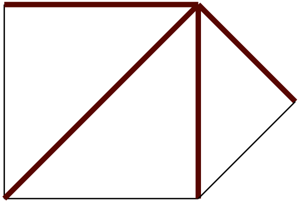

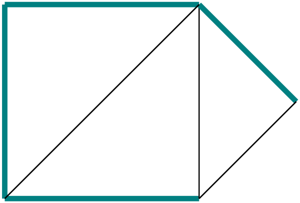

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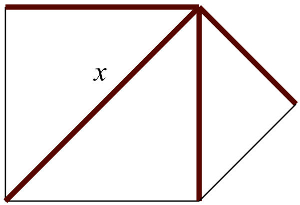

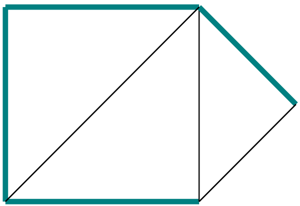

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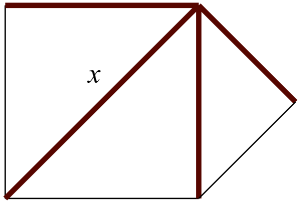

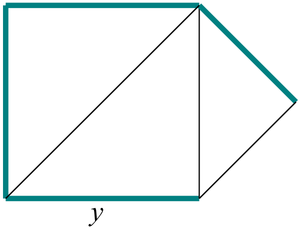

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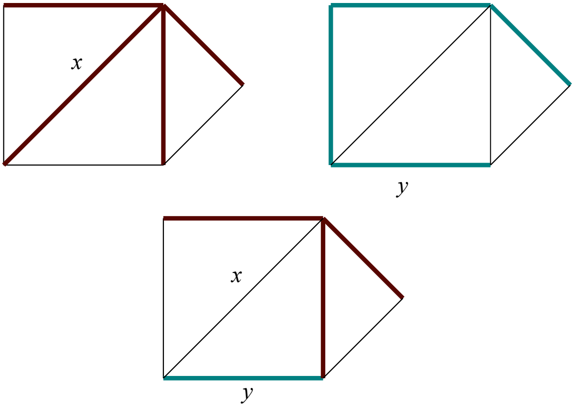

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$r = r_M$ is the rank function of a matroid $(E, \mathcal{I})$ (where $\mathcal{I} = \{I \subseteq E : r(I) = |I|\}$) if and only if r verifies the following conditions :

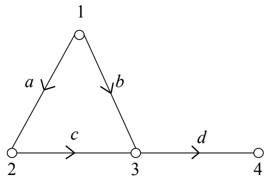
$(R1)$ $0 \leq r(X) \leq |X|$, for all $X \subseteq E$,

$(R2)$ $r(X) \leq r(Y)$, for all $X \subseteq Y$,

$(R3)$ *(sub-modulairity)* $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ for all $X, Y \subset E$.

Rank

Let M be a graphic matroid obtained from G



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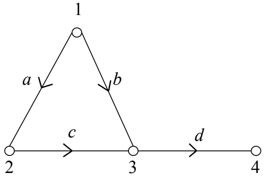

It can be verified that :
$r_M(\{a,b,c\}) = r_M(\{c,d\}) = r_M(\{a,d\}) = 2$ et
$r(M(G)) = r_M(\{a,b,c,d\}) = 3$.

Greedy Algorithm

Let $\mathcal{I}$ be a set of subsets of E verifying $(I1)$ and $(I2)$. Let $w : E \to \mathbb{R}$, and let $w(X) = \sum_{x \in X} w(x), X \subseteq E, w(\emptyset) = 0$.

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An optimization problem consist of finding a maximal set B of $\mathcal{I}$ with maximal weight (or minimal).

Greedy algorithm for $(\mathcal{I}, w)$
$X_0 = \emptyset$
$j = 0$
While $e \in E \backslash X_j \ : \ X_j \cup \{e\} \in \mathcal{I}$ **do**
Choose an element e_{j+1} of maximal weight
$X_{j+1} \leftarrow X_j \cup \{e_{j+1}\}$
$j \leftarrow j + 1$
$B_G \leftarrow X_j$
Return B_G

Greedy Algorithm

Theorem $(\mathcal{I}, E)$ is a matroid if and only if the following conditions are verified :

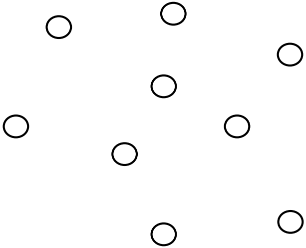
($I1$) $\emptyset \in \mathcal{I}$,

($I2$) $I \in \mathcal{I}, I' \subseteq I \Rightarrow I' \in \mathcal{I}$,

(G) For any function $w : E \to \mathbb{R}$, the greedy algorithm gives a maximal set of $\mathcal{I}$ of maximal weight.

Application : Spanning tree of minimal weight

We want to construct a network (of minimal cost) connecting the 9 cities.



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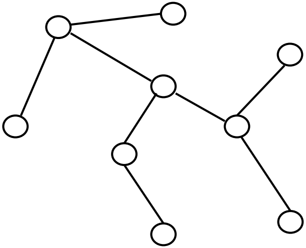

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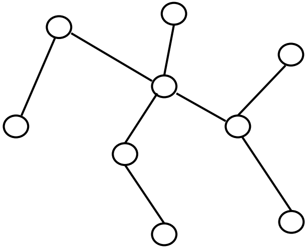

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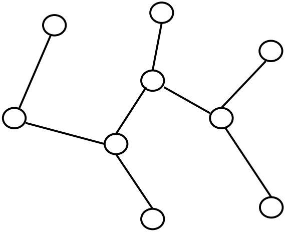

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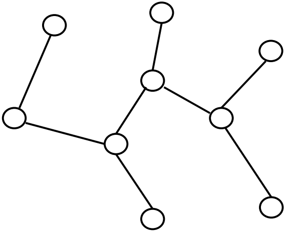

Theorem (Caley) There exist n^{n-2} labeled trees on n vertices.

Application : Spanning tree of minimal weight

Theorem (Kruskal) Given a complete graph with weights on the edges there exist a polynomial time algorithm that finds a spanning tree of minimal weight.

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Theorem (Kruskal) Given a complete graph with weights on the edges there exist a polynomial time algorithm that finds a spanning tree of minimal weight.

Indeed, the greedy algorithm returns a base (maximal independent) of minimal weight by considering the graphic matroid associated to a complete graph and $w(e)$, $e \in E(G)$ is the the weight of each edge.

Duality

Let M be a matroid on the ground set E and let $\mathcal{B}$ the set of bases of M. Then,

$$\mathcal{B}^* = \{E \backslash B \mid B \in \mathcal{B}\}$$

is the set of bases of a matroid on E.

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The matroid on E having $\mathcal{B}^*$ as set of bases, denoted by M^*, is called the dual of M.

A base of M^* is also called cobase of M.

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$\mathcal{I}^* = \{X \mid X \subset E$ such that there exists $B \in \mathcal{B}(M)$ with $X \cap B = \emptyset\}$.

- The rank function of M^* is given by

$$r_{M^*}(X) = |X| + r_M(E \backslash X) - r_M,$$

for $X \subset E$.

Cocycle Matroid

Let $G = (V, E)$ be a graph. A cocycle (or cut) of G is the set of edges joining the two parts of a partition of the set of vertices of the graph.

Cocycle Matroid

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The matroid obtained on this way is called the matroid of cocycle of G or bond matroid, denoted by $B(G)$.

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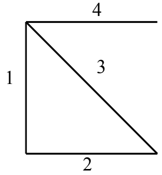

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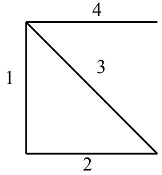

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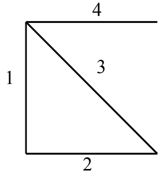

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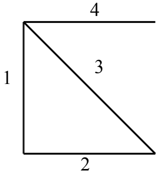

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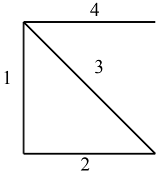

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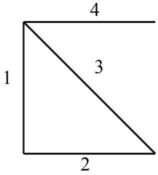

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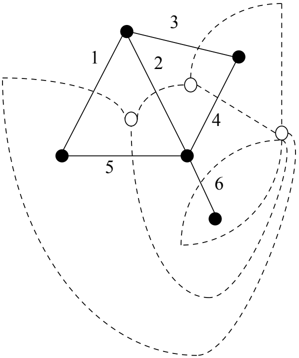

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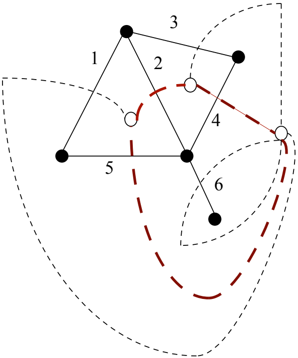

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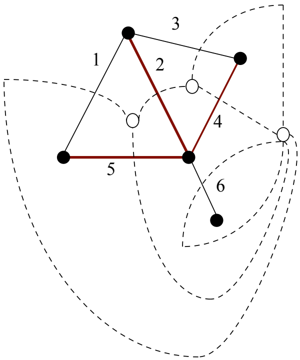

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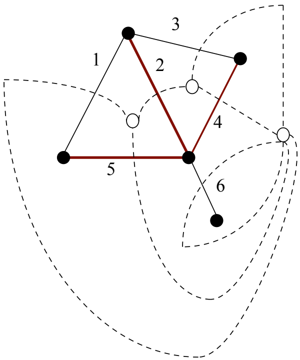

Remark The dual of a graphic matroid is not necessarely graphic.

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(Exercise) M^* can be obtained from the set of columns of the matrix

$$(-{}^tA \mid I_{n-r})$$

where I_{n-r} is the identity $(n-r) \times (n-r)$ and ${}^tA$ is the transpose of A.

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The orthogonal space of the space generated by the columns of $(I \mid A)$ is given by the space generated by the columns of $(-{}^tA \mid I_{n-r})$.

Operation : deletion

Let M be a matroid on the set E and let $A \subset E$. Then,

$$\{X \subset E \backslash A \mid X \text{ is independent in } M\}$$

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Operations : deletion and contraction

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(ii) $(M/A)/A' = M/(A \cup A')$
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Question : Is it true that any family of matroids is closed under deletions/contractions operations ?

Minors - uniform matroids

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Contraction : it follows by using duality.

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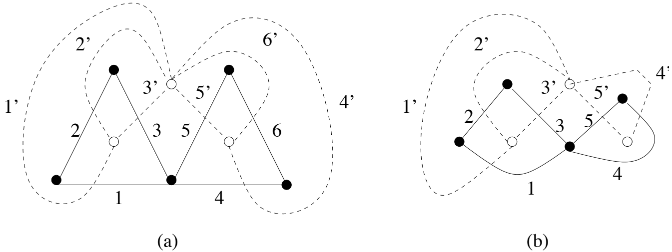

Contracting element 6

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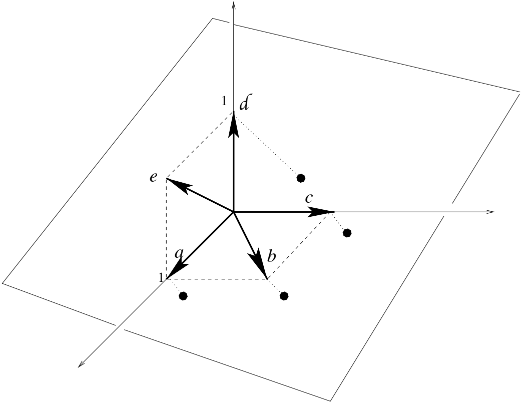
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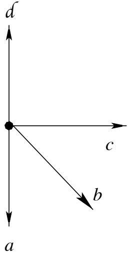
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For $\mathbb{F} = GF(2) = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (binary matroids) : the list has only one matroid $U_{2,4}$ (3 pages proof)

$$\mathcal{B}(U_{2,4}) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

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Theorem A matroid is regular if and only if has neither $U_{2,4}$, F_7 nor F_7^* as minors.

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Example : Graphic matroids are regulars.

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• R_{10} is the matroid of the linear dependencies over $\mathbb{Z}_2$ of the 10 vectors of $\mathbb{Z}_2^5$ having 3 components equal to one and 2 equal to zero.

• M is built with bricks (graphic, cographic and R_{10}) via 3 operations :

1-sum : direct sum of two matroids

2-sum : patching two matroids on one common element

3-sum : patching two binary matroids on 3 common elements forming a 3-circuit in each matroid.

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Remark Most of the combinatorial optimization problems can be realized as a unimodular linear programming.

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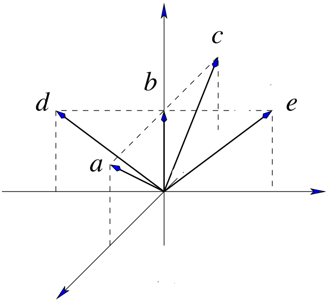
The Minkowski's sum of two sets A and B of $\mathbb{R}^d$ is $A + B = \{a + b \mid a \in A, b \in B\}$.

Let $A = \{v_1, \ldots, v_k\}$ be a finite set of elements of $\mathbb{R}^d$.

A zonotope, generated by A and denoted by $Z(A)$, is a polytope formed by the Minkowski's sum of line segments

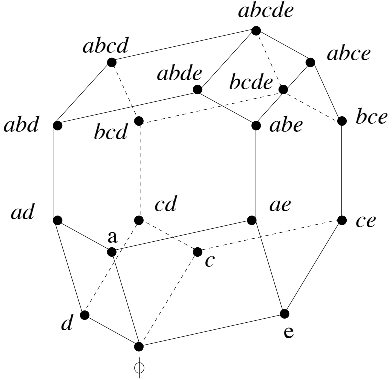
$$Z(A) = \{\alpha_1 + \cdots + \alpha_k | \alpha_i \in [-v_i, v_i]\}.$$

Regular Matroids - Applications



Regular Matroids - Applications

Permutahedron



Regular Matroids - Applications

Theorem (Voronoï - end of the 19th century) There exist exactly 5 types of zonotopes of $\mathbb{R}^3$ tiling the space by translations.

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Voronoï's result : there exist exactly 5 regular matroids of rank 3.

Non Representable Matroids

There exists matroids that are not representable in ANY field.

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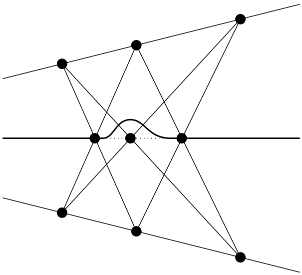
Example (classic) : the rank 3 matroid on 9 elements obtained from the Non-Pappus configuration



Tutte Polynomial

The Tutte polynomial of a matroid M is the generating function defined as follows

$$t(M; x, y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)} (y-1)^{|X|-r(X)}.$$

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$$\begin{aligned} t(U_{2,3};x,y) &= \sum_{X\subseteq E,\ |X|=0} (x-1)^{2-0}(y-1)^{0-0} + \sum_{X\subseteq E,\ |X|=1} (x-1)^{2-1}(y-1)^{1-1} \\ &+ \sum_{X\subseteq E,\ |X|=2} (x-1)^{2-2}(y-1)^{2-2} + \sum_{X\subseteq E,\ |X|=3} (x-1)^{2-2}(y-1)^{3-2} \\ &= (x-1)^2 + 3(x-1) + 3(1) + y - 1 \\ &= x^2 - 2x + 1 + 3x - 3 + 3 + y - 1 = x^2 + x + y. \end{aligned}$$

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The Tutte polynomial can be expressed recursively as follows

$$t(M;x,y) = \begin{cases} t(M \setminus e;x,y) + t(M/e;x,y) & \text{if } e \neq \text{isthmus, loop,} \\ x \cdot t(M \setminus e;x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot t(M/e;x,y) & \text{if } e \text{ is a loop.} \end{cases}$$

Acyclic Orientations

Let $G = (V, E)$ be a connected graph. An orientation of G is an orientation of the edges of G.

We say that the orientation is acyclic if the oriented graph do not contain an oriented cycle (i.e., a cycle where all its edges are oriented clockwise or anti-clockwise).

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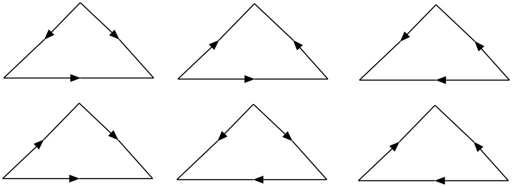
Theorem The number of acyclic orientations of G is equals to

$$t(M(G); 2, 0).$$

Acyclic Orientations

Example : There are 6 acyclic orientations of C_3



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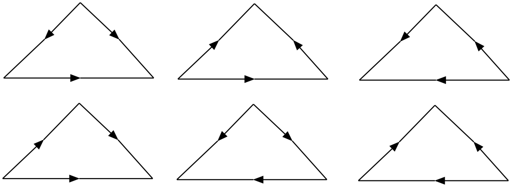

Notice that $M(C_3)$ is isomorphic to $U_{2,3}$.

Since $t(U_{2,3}; x, y) = x^2 + x + y$ then the number of acyclic orientations of C_3 is $t(U_{2,3}; 2, 0) = 2^2 + 2 + 0 = 6$.

Chromatic Polynomial

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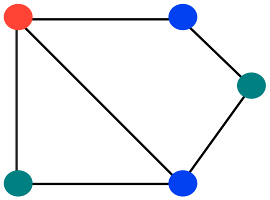

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Let $\chi(G,\lambda)$ be the number of good λ-colorings of G.

Theorem $\chi(G,\lambda)$ is a polynomial on λ. Moreover

$$\chi(G,\lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{\omega(G[X])}$$

where $\omega(G[X])$ denote the number of connected components of the subgraph generated by X.

Proof (idea) By using the inclusion-exclusion formula.

Chromatic Polynomial

The chromatic polynomial has been introduced by Birkhoff as a tool to attack the 4-color problem.

Indeed, if for a planar graph G we have $\chi(G, 4) > 0$ then G admits a good 4-coloring.

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Exemple : $\chi(K_3,3) = 3^1(-1)^{3-1}t(K_3;1-3,0)$
$= 3\cdot 1\cdot t(U_{2,3};-2,0) = 3((-2)^2-2+0) = 6.$

Ehrhart Polynomial

The theory of Ehrhart focuses in counting the number of points with integer coordinates lying in a polytope.

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A polytope is called integer if all its vertices have integer coordinates.
Ehrhart studied the function i_P that counts the number of integer points in the polytope P *dilated* by a factor of t

$$\begin{aligned} i_P: \quad & \mathbb{N} \longrightarrow \mathbb{N}^* \\ & t \mapsto \left|tP \cap \mathbb{Z}^d\right| \end{aligned}$$

Ehrhart Polynomial

Theorem (Ehrhart) i_P is a polynomial on t of degree d,

$$i_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

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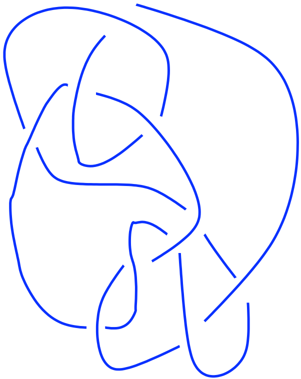
All others coefficients remain a mystery ! !

Ehrhart Polynomial

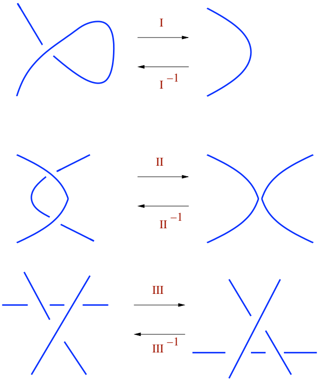
Theorem Let M be a regular matroid and let A be one of its representation matrix. Then, the Ehrhart polynomial associated to the zonotope $Z(A)$ is given by

$$i_{Z(A)}(q) = q^{r(M)} t\left(M; 1 + \frac{1}{q}, 1\right).$$

Knots



Reidemeister moves



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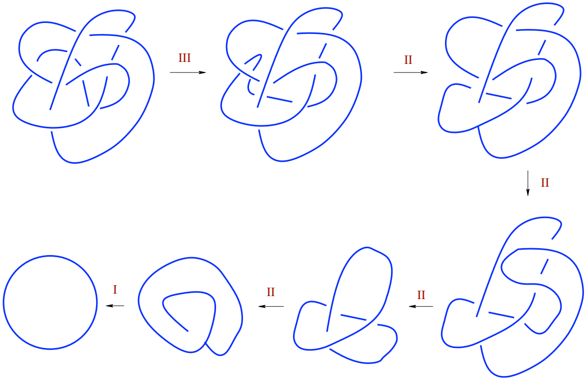

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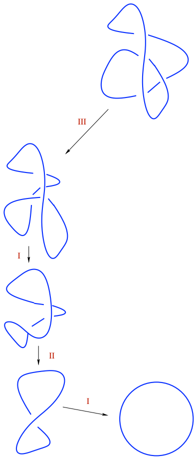

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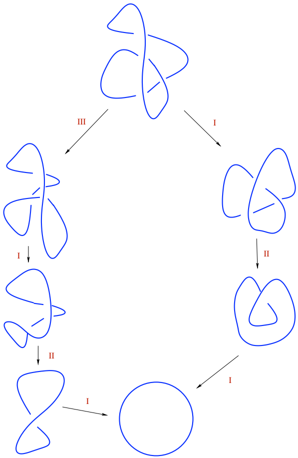

Bracket polynomial

For any link diagram D define a Laurent polynomial $< D >$ in one variable A which obeys the following three rules where U denotes the unknot :

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i) $\langle U \rangle = 1$

ii) $\langle U + D \rangle = -(A^2 + A^{-2}) \langle D \rangle$

iii)

Theorem For any link L the bracket polynomial is independent of the order in which rules $(i)-(iii)$ are applied to the crossings. Further, it is invariant under the Reidemeister moves II and III

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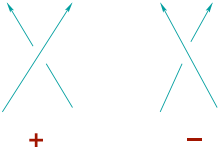
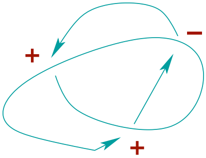
The writhe of an oriented link diagram D is the sum of the signs at the crossings of D (denoted by $\omega(D)$).

Knots





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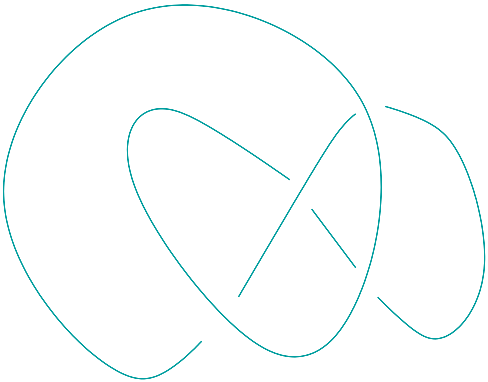
Now, define for any link L

$$V_L(z) = f_D(z^{-1/4})$$

where D is any diagram representing L. Then $V_L(z)$ is the Jones polynomial of the oriented link L.

Knots



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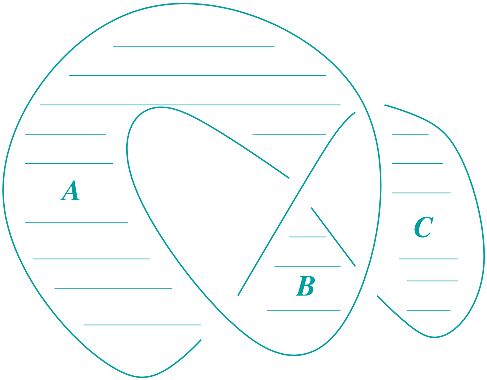

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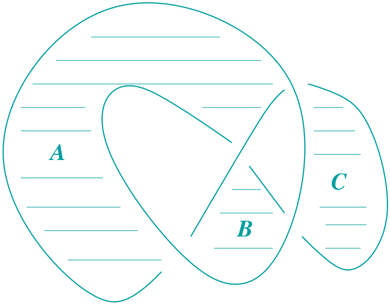

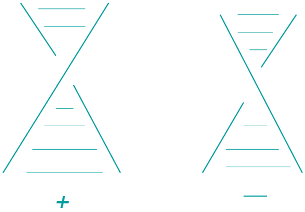

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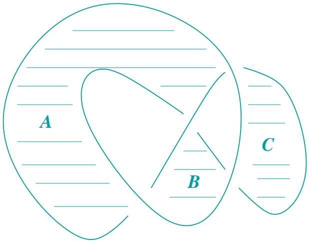

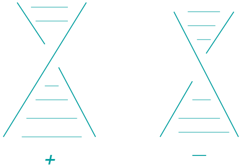

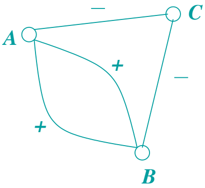

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A link is alternating if there is an alternating link diagram representing L.

Theorem (Thistlethwaite 1987) If D is an oriented alternating link diagram then

$$V_L(z) = (z^{-1/4})^{3\omega(D)-2} t(M(G); -z, -z^{-1})$$

where G is the graph associated to the knot diagram.

More applications

- Code theory
- Flow polynomial
- Bicycle space of a graph
- Statistical mechanics
- Arrangements of hyperplanes

$\vdots$