Theory of matroids II: toric ideals and simplicial complexes

J.L. Ramírez Alfonsín

I3M, Université Montpellier 2

XX Coloquio Latinoamericano de Álgebra "Ingeniero Orlando Eugenio Villamayor"

Lima, Peru, December 12, 2014

Quick recall

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.

Quick recall

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid.

Quick recall

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid. The family $\mathcal{B}$ verifies the following conditions :

(B1) $\mathcal{B} \neq \emptyset$,

(B2) *(exchange propety)* $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

Quick recall

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid. The family $\mathcal{B}$ verifies the following conditions :

$(B1)$ $\mathcal{B} \neq \emptyset$,

$(B2)$ *(exchange propety)* $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

If $\mathcal{I}$ is the family of subsets contained in a set of $\mathcal{B}$ then $(E, \mathcal{I})$ is a matroid.

Toric ideal associated to a matroid

Let M be a matroid on a finite ground set $E = \{1, \ldots, n\}$. We denote by $\mathcal{B}$ the set of bases of M.

Toric ideal associated to a matroid

Let M be a matroid on a finite ground set $E = \{1, \ldots, n\}$. We denote by $\mathcal{B}$ the set of bases of M.

Let k be an arbitrary field and consider $k[x_1, \ldots, x_n]$ a polynomial ring over k.

Toric ideal associated to a matroid

Let M be a matroid on a finite ground set $E = \{1, \dots, n\}$. We denote by $\mathcal{B}$ the set of bases of M.

Let k be an arbitrary field and consider $k[x_1, \dots, x_n]$ a polynomial ring over k.

For each base $B \in \mathcal{B}$, we introduce a variable y_B and we denote by R the polynomial ring in the variables y_B, i.e., $R := k[y_B \mid B \in \mathcal{B}]$.

Toric ideal associated to a matroid

Let M be a matroid on a finite ground set $E = \{1, \dots, n\}$. We denote by $\mathcal{B}$ the set of bases of M.

Let k be an arbitrary field and consider $k[x_1, \dots, x_n]$ a polynomial ring over k.

For each base $B \in \mathcal{B}$, we introduce a variable y_B and we denote by R the polynomial ring in the variables y_B, i.e., $R := k[y_B \mid B \in \mathcal{B}]$.

A binomial in R is a difference of two monomials, an ideal generated by binomials is called a *binomial ideal*.

Toric ideal associated to a matroid

We consider the homomorphism of k-algebras
$\varphi : R \longrightarrow k[x_1, \ldots, x_n]$ induced by

$$y_B \mapsto \prod_{i \in B} x_i.$$

The image of φ is a standard graded k-algebra, which is called the bases monomial ring of the matroid M and it is denoted by S_M.

Toric ideal associated to a matroid

We consider the homomorphism of k-algebras $\varphi : R \longrightarrow k[x_1, \ldots, x_n]$ induced by

$$y_B \mapsto \prod_{i \in B} x_i.$$

The image of φ is a standard graded k-algebra, which is called the bases monomial ring of the matroid M and it is denoted by S_M.

The kernel of φ, which is the presentation ideal of S_M, is called the toric ideal of M and is denoted by I_M.

Toric ideal associated to a matroid

We consider the homomorphism of k-algebras $\varphi : R \longrightarrow k[x_1, \ldots, x_n]$ induced by

$$y_B \mapsto \prod_{i \in B} x_i.$$

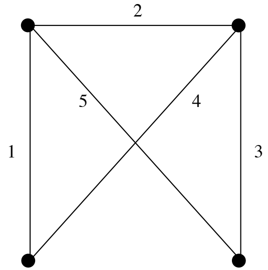
The image of φ is a standard graded k-algebra, which is called the bases monomial ring of the matroid M and it is denoted by S_M.

The kernel of φ, which is the presentation ideal of S_M, is called the toric ideal of M and is denoted by I_M.

Observation Let b be the number of bases of a matroid M on n elements. Then, I_M is generated by the kernel of the integer $n \times b$ matrix whose columns are the zero-one incidence vectors of the bases of M.

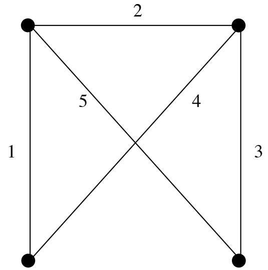
Example

Matroid $M(G)$ associated to graph G. We have $r(M(G)) = 3$.



Example

Matroid $M(G)$ associated to graph G. We have $r(M(G)) = 3$.



$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$

Example

$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$

$$\begin{array}{cccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Example

$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$

$$\begin{array}{cccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

By considering $\varphi : k[y_{B_1}, \ldots, y_{B_8}] \longrightarrow k[x_1, \ldots, x_5]$ we have that

$y_{B_1} \mapsto x_1x_2x_3, \quad y_{B_2} \mapsto x_1x_2x_5, \quad y_{B_3} \mapsto x_1x_3x_4, \quad \ldots$

Example

$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$

$$\begin{array}{cccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 & B_8 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

By considering $\varphi : k[y_{B_1}, \ldots, y_{B_8}] \longrightarrow k[x_1, \ldots, x_5]$ we have that

$y_{B_1} \mapsto x_1x_2x_3, \quad y_{B_2} \mapsto x_1x_2x_5, \quad y_{B_3} \mapsto x_1x_3x_4, \quad \ldots$

An element of the kernel of φ (*i.e.*, $I_{M(G)}$) is : $y_{B_7}y_{B_4} - y_{B_2}y_{B_8}$.

Toric ideal associated to a matroid

- It is well known that I_M is a prime, binomial and homogeneous ideal.

Toric ideal associated to a matroid

• It is well known that I_M is a prime, binomial and homogeneous ideal.

Observation Since $R/I_M \simeq S_M$, it follows that the height of I_M is $\mathrm{ht}(I_M) = |\mathcal{B}| - \dim(S_M) = |\mathcal{B}| - (n - c + 1)$, where c is the number of connected components of M.

White's conjecture

Let $\mathcal{B}$ denote the set of bases of M. By definition $\mathcal{B}$ is not empty and satisfies the following exchange axiom :

For every $B_1, B_2 \in \mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$.

White's conjecture

Let $\mathcal{B}$ denote the set of bases of M. By definition $\mathcal{B}$ is not empty and satisfies the following exchange axiom :

For every $B_1, B_2 \in \mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$.

Brualdi proved that the exchange axiom is equivalent to the symmetric exchange axiom :

For every B_1, B_2 in $\mathcal{B}$ and for every $e \in B_1 \setminus B_2$, there exists $f \in B_2 \setminus B_1$ such that both $(B_1 \cup \{f\}) \setminus \{e\} \in \mathcal{B}$ and $(B_2 \cup \{e\}) \setminus \{f\} \in \mathcal{B}$.

White's conjecture

Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$.

White's conjecture

Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$.

We say that the quadratic binomial $y_{B_1} y_{B_2} - y_{D_1} y_{D_2}$ correspond to a symmetric exchange.

White's conjecture

Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$.

We say that the quadratic binomial $y_{B_1} y_{B_2} - y_{D_1} y_{D_2}$ correspond to a symmetric exchange.

It is clear that such binomial belong to the ideal I_M.

White's conjecture

Suppose that a pair of bases D_1, D_2 is obtained from a pair of bases B_1, B_2 by a symmetric exchange. That is $D_1 = (B_1 \setminus e) \cup f$ and $D_2 = (B_2 \setminus f) \cup e$ for some $e \in B_1$ and $f \in B_2$.

We say that the quadratic binomial $y_{B_1} y_{B_2} - y_{D_1} y_{D_2}$ correspond to a symmetric exchange.

It is clear that such binomial belong to the ideal I_M.

Conjecture (White 1980) For every matroid M its toric ideal I_M is generated by quadratic binomials corresponding to symmetric exchanges.

White's conjecture

Observation for $B_1, \dots, B_s, D_1, \dots, D_s \in \mathcal{B}$, the homogeneous binomial $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$ belongs to I_M if and only if $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$ as multisets.

White's conjecture

Observation for $B_1, \ldots, B_s, D_1, \ldots, D_s \in \mathcal{B}$, the homogeneous binomial $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$ belongs to I_M if and only if $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$ as multisets.

Since I_M is a homogeneous binomial ideal, it follows that

$$I_M = \left(\{y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s} \mid B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s \text{ as multisets}\}\right)$$

White's conjecture

Observation for $B_1, \dots, B_s, D_1, \dots, D_s \in \mathcal{B}$, the homogeneous binomial $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$ belongs to I_M if and only if $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$ as multisets.

Since I_M is a homogeneous binomial ideal, it follows that

$$I_M = \left(\{y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s} \mid B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s \textit{ as multisets}\}\right)$$

White's original formulation Two sets of bases of a matroid have equal union (as multiset), then one can pass between them by a sequence of symmetric exchanges.

White's conjecture

Observation for $B_1, \ldots, B_s, D_1, \ldots, D_s \in \mathcal{B}$, the homogeneous binomial $y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s}$ belongs to I_M if and only if $B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s$ as multisets.

Since I_M is a homogeneous binomial ideal, it follows that

$$I_M = \left(\{y_{B_1} \cdots y_{B_s} - y_{D_1} \cdots y_{D_s} \mid B_1 \cup \cdots \cup B_s = D_1 \cup \cdots \cup D_s \textit{ as multisets}\}\right)$$

White's original formulation Two sets of bases of a matroid have equal union (as multiset), then one can pass between them by a sequence of symmetric exchanges.

Observation White's conjecture does not depend on the field k.

Example continued

We had $\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{245\}, B_8 = \{345\}\}$.

We also had that $y_{B_7}y_{B_4} - y_{B_2}y_{B_8} \in I_{M(G)}$.

We can check that $B_7 \cup B_4 = \{2,4,5,1,3,5\} = B_2 \cup B_8$.

Results of White's conjecture

- Blasiak (2008) has confirmed the conjecture for graphical matroids.

Results of White's conjecture

- Blasiak (2008) has confirmed the conjecture for graphical matroids.
- Kashiwaba (2010) checked the case of matroids of rank ≤ 3.

Results of White's conjecture

- Blasiak (2008) has confirmed the conjecture for graphical matroids.
- Kashiwaba (2010) checked the case of matroids of rank ≤ 3.
- Schweig (2011) proved the case of lattice path matroids which are a subclass of transversal matroids.

Results of White's conjecture

- Blasiak (2008) has confirmed the conjecture for graphical matroids.
- Kashiwaba (2010) checked the case of matroids of rank ≤ 3.
- Schweig (2011) proved the case of lattice path matroids which are a subclass of transversal matroids.
- Bonin (2013) confirmed the conjecture for sparse paving matroids

Results of White's conjecture

- Blasiak (2008) has confirmed the conjecture for graphical matroids.
- Kashiwaba (2010) checked the case of matroids of rank ≤ 3.
- Schweig (2011) proved the case of lattice path matroids which are a subclass of transversal matroids.
- Bonin (2013) confirmed the conjecture for sparse paving matroids
- Lasoń, Michałek (2014) proved for strongly base orderables matroids.

Blasiak's reduction

Let M be a matroid on a ground set E with $|E| = nr(M)$ where $r(M)$ is the rank of M.

Blasiak's reduction

Let M be a matroid on a ground set E with $|E| = nr(M)$ where $r(M)$ is the rank of M.

The n-base graph of M, which is denoted by $G_n(M)$, has as its vertex set the set of all sets of n disjoint bases (a set of n bases $\{B_1, \ldots, B_n\}$ of M is disjoint if and only if

$$|E| = \bigcup_{i=1}^{n} B_i.$$

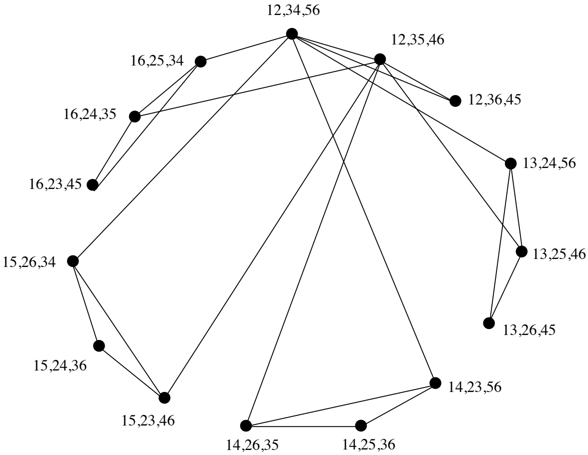
There is an edge between $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ if and only if $B_i = D_j$ for some i, j.

$G_2(U_{2,6})$

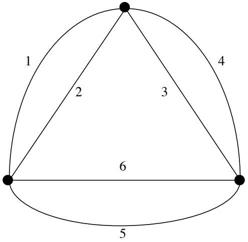
We have that $r(U_{2,6}) = 2$, and let us take $n = 3$.

$G_2(U_{2,6})$

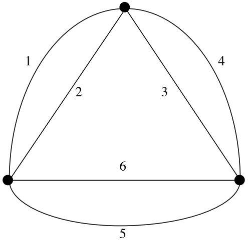
We have that $r(U_{2,6}) = 2$, and let us take $n = 3$. So



$M(G)$

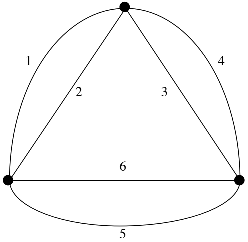


$M(G)$



We have that $r(M(G)) = 2$ and we set $n = 3$.

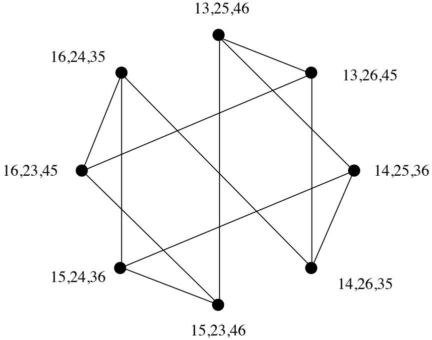
$M(G)$



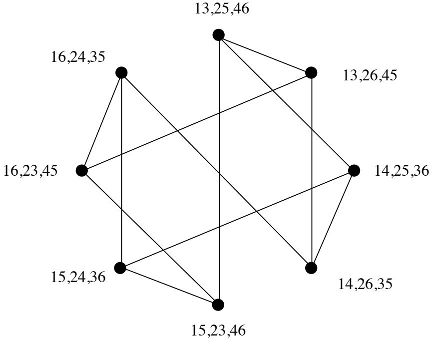
We have that $r(M(G)) = 2$ and we set $n = 3$.

$\mathcal{B}(M(G)) = \{B_1 = \{1,3\}, B_2 = \{1,4\}, B_3 = \{1,5\}, B_4 = \{1,6\}, B_5 = \{2,3\}, B_6 = \{2,4\}, B_7 = \{2,5\}, B_8 = \{2,6\}, B_9 = \{3,5\}, B_{10} = \{3,6\}, B_{11} = \{4,5\}, B_{12} = \{4,6\}\}$.

$G_3(M(G))$

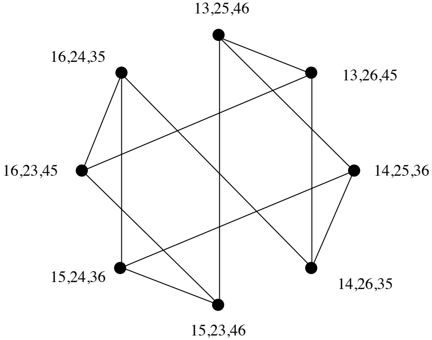


$G_3(M(G))$



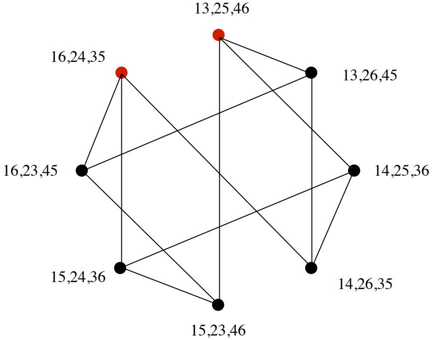
We notice that $y_{B_4}y_{B_6}y_{B_9} - y_{B_1}y_{B_7}y_{B_{12}} \in I_{M(G)}$

$G_3(M(G))$



We notice that $y_{B_4}y_{B_6}y_{B_9} - y_{B_1}y_{B_7}y_{B_{12}} \in I_{M(G)}$
since $B_4 \cup B_6 \cup B_9 = \{1,2,3,4,5,6\} = B_1 \cup B_7 \cup B_{12}$.

$G_3(M(G))$



We notice that $y_{B_4}y_{B_6}y_{B_9} - y_{B_1}y_{B_7}y_{B_{12}} \in I_{M(G)}$
since $B_4 \cup B_6 \cup B_9 = \{1,2,3,4,5,6\} = B_1 \cup B_7 \cup B_{12}$.

Blasiak's reduction

Lemma (Blasiak) Let $\mathfrak{C}$ be a collection of matroids that is closed under deletions and adding parallel elements. Suppose that for each $n \geq 3$ and for every matroid M in $\mathfrak{C}$ on a ground set of size $nr(M)$ the n-base graph of M is connected. Then, for every matroid M in $\mathfrak{C}$, I_M is generated by quadratics polynomials.

Blasiak's reduction

Lemma (Blasiak) Let $\mathfrak{C}$ be a collection of matroids that is closed under deletions and adding parallel elements. Suppose that for each $n \geq 3$ and for every matroid M in $\mathfrak{C}$ on a ground set of size $nr(M)$ the n-base graph of M is connected. Then, for every matroid M in $\mathfrak{C}$, I_M is generated by quadratics polynomials.

Proof (idea) The following statement is proved by induction on n :

Blasiak's reduction

Lemma (Blasiak) Let $\mathfrak{C}$ be a collection of matroids that is closed under deletions and adding parallel elements. Suppose that for each $n \geq 3$ and for every matroid M in $\mathfrak{C}$ on a ground set of size $nr(M)$ the n-base graph of M is connected. Then, for every matroid M in $\mathfrak{C}$, I_M is generated by quadratics polynomials.

Proof (idea) The following statement is proved by induction on n :

for every $M \in \mathfrak{C}$ and every binomial $b \in I_M$ of degree n, b is in the ideal generated by the quadratics of I_M.

Blasiak's reduction

Lemma (Blasiak) Let $\mathfrak{C}$ be a collection of matroids that is closed under deletions and adding parallel elements. Suppose that for each $n \geq 3$ and for every matroid M in $\mathfrak{C}$ on a ground set of size $nr(M)$ the n-base graph of M is connected. Then, for every matroid M in $\mathfrak{C}$, I_M is generated by quadratics polynomials.

Proof (idea) The following statement is proved by induction on n :

for every $M \in \mathfrak{C}$ and every binomial $b \in I_M$ of degree n, b is in the ideal generated by the quadratics of I_M.

This will prove the result because I_M, as a toric ideal, is generated by binomials.

Blasiak's reduction

Proof (continuation ...) $M \in \mathfrak{C}$ and b is binomial of degree n in I_M.

Blasiak's reduction

Proof (continuation ...) $M \in \mathfrak{C}$ and b is binomial of degree n in I_M.

The binomial b is necessarily of the form $b = \prod_{i=1}^{n} y_{B_i} - \prod_{i=1}^{n} y_{D_i}$ for some bases $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ of M such that the B_i and D_i have the same multiset union.

Blasiak's reduction

Proof (continuation ...) $M \in \mathfrak{C}$ and b is binomial of degree n in I_M.

The binomial b is necessarily of the form $b = \prod_{i=1}^{n} y_{B_i} - \prod_{i=1}^{n} y_{D_i}$ for some bases $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ of M such that the B_i and D_i have the same multiset union.

It is proved that b is in the ideal generated by the degree $n-1$ binomials of I_M (this is done by constructing a new matroid M' that depends on the binomial b).

Blasiak's reduction

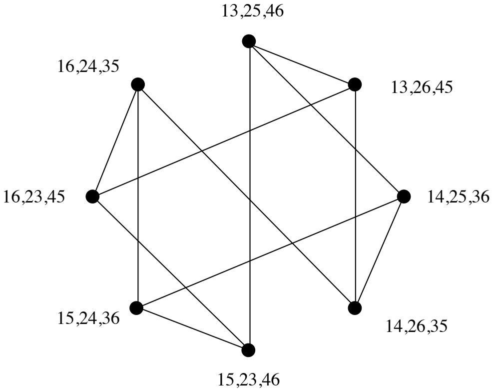
Proof (continuation ...) $M \in \mathfrak{C}$ and b is binomial of degree n in I_M.

The binomial b is necessarily of the form $b = \prod_{i=1}^{n} y_{B_i} - \prod_{i=1}^{n} y_{D_i}$ for some bases $\{B_1, \ldots, B_n\}$ and $\{D_1, \ldots, D_n\}$ of M such that the B_i and D_i have the same multiset union.

It is proved that b is in the ideal generated by the degree $n-1$ binomials of I_M (this is done by constructing a new matroid M' that depends on the binomial b).

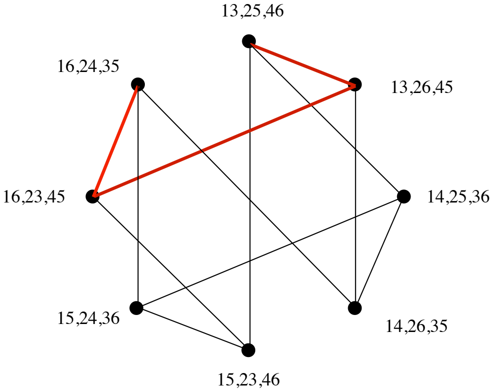
By induction the degree $n-1$ binomials are in the ideal generated by the quadratics of I_M which is properly used the proof de result for n.

Blasiak's reduction



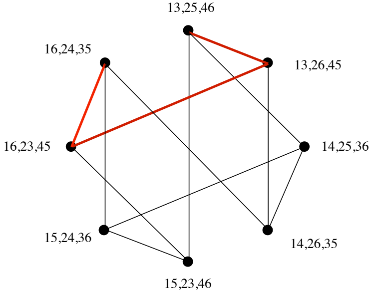
$y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

Blasiak's reduction



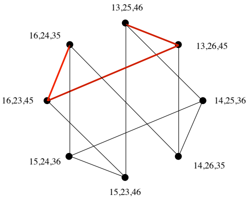
$y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

Blasiak's reduction



By following the path we construct
$y_{16}y_{24}y_{35} - y_{16}y_{23}y_{45} + y_{16}y_{23}y_{45} - y_{13}y_{26}y_{45} + y_{13}y_{26}y_{55} - y_{13}y_{25}y_{46} = y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

Blasiak's reduction



By following the path we construct

$y_{16}y_{24}y_{35} - y_{16}y_{23}y_{45} + y_{16}y_{23}y_{45} - y_{13}y_{26}y_{45} + y_{13}y_{26}y_{55} - y_{13}y_{25}y_{46} = y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

Or equivalently

$y_{16}(y_{24}y_{35} - y_{23}y_{45}) + y_{45}(y_{16}y_{23} - y_{13}y_{26}) + y_{13}(y_{26}y_{55} - y_{25}y_{46}) = y_{16}y_{24}y_{35} - y_{13}y_{25}y_{46} \in I_{M(G)}$.

Strongly base orderable matroid

A matroid is strongly base order able if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \longrightarrow B_2$ satisfying the multiple symmetric exchange property, that is : $(B_1 \setminus A) \cup \pi(A)$ is a basis for every $A \subset B_1$.

Strongly base orderable matroid

A matroid is strongly base order able if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \longrightarrow B_2$ satisfying the multiple symmetric exchange property, that is : $(B_1 \setminus A) \cup \pi(A)$ is a basis for every $A \subset B_1$.

- π restricted to the intersection $B_1 \cap B_2$ is the identity.

Strongly base orderable matroid

A matroid is strongly base order able if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \longrightarrow B_2$ satisfying the multiple symmetric exchange property, that is : $(B_1 \setminus A) \cup \pi(A)$ is a basis for every $A \subset B_1$.

• π restricted to the intersection $B_1 \cap B_2$ is the identity.

• $(B_2 \setminus \pi(A)) \cup A$ is a basis for every $A \subset B_1$ (by the multiple symmetric exchange property for $B_1 \setminus A$).

Strongly base orderable matroid

A matroid is strongly base order able if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \longrightarrow B_2$ satisfying the multiple symmetric exchange property, that is : $(B_1 \setminus A) \cup \pi(A)$ is a basis for every $A \subset B_1$.

• π restricted to the intersection $B_1 \cap B_2$ is the identity.

• $(B_2 \setminus \pi(A)) \cup A$ is a basis for every $A \subset B_1$ (by the multiple symmetric exchange property for $B_1 \setminus A$).

• The class of strongly base orderable matroids is closed under taking minors.

Strongly base orderable matroid

Theorem (Lasoń, M. Michałek) If M is a strong order able base matroid, then the toric ideal I_M is generated by quadratics binomials corresponding to symmetric exchanges.

Strongly base orderable matroid

Theorem (Lasoń, M. Michałek) If M is a strong order able base matroid, then the toric ideal I_M is generated by quadratics binomials corresponding to symmetric exchanges.

Proof (idea) Since I_M, as a toric ideal, is generated by binomials then it is enough to prove that all binomials of I_M belong to the ideal J_M generated by quadratics binomials corresponding to symmetric exchanges.

Strongly base orderable matroid

Theorem (Lasoń, M. Michałek) If M is a strong order able base matroid, then the toric ideal I_M is generated by quadratics binomials corresponding to symmetric exchanges.

Proof (idea) Since I_M, as a toric ideal, is generated by binomials then it is enough to prove that all binomials of I_M belong to the ideal J_M generated by quadratics binomials corresponding to symmetric exchanges.

Fix $n \geq 2$. We shall prove by decreasing induction on the overlap function

$$d(y_{B_1} \cdots y_{B_n}, y_{D_1} \cdots y_{D_n}) := \max_{\pi \in S_n} \sum_{i=1}^{n} |B_i \cap D_{\pi(i)}|$$

that a binomial $y_{B_1} \cdots y_{B_n} - y_{D_1} \cdots y_{D_n} \in I_M$ belongs to J_M.

Variants of White's conjecture

Conjecture 1 For any matroid M, the toric ideal I_M has a Gröbner basis consisting of quadratics binomials.

Variants of White's conjecture

Conjecture 1 For any matroid M, the toric ideal I_M has a Gröbner basis consisting of quadratics binomials.

Sturmfels (1996) proved that Conjecture 1 holds for uniform matroids.

Variants of White's conjecture

Conjecture 1 For any matroid M, the toric ideal I_M has a Gröbner basis consisting of quadratics binomials.

Sturmfels (1996) proved that Conjecture 1 holds for uniform matroids.

Conjecture 2 For any matroid M, the toric ideal I_M is generated by quadratics binomials.

Variants of White's conjecture

Conjecture 1 For any matroid M, the toric ideal I_M has a Gröbner basis consisting of quadratics binomials.

Sturmfels (1996) proved that Conjecture 1 holds for uniform matroids.

Conjecture 2 For any matroid M, the toric ideal I_M is generated by quadratics binomials.

Conjecture 3 For any matroid M, the quadratic binomials of I_M are in the ideal generated by the binomials $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ such that the pair of bases D_1, D_2 can be obtained from the pair B_1, B_2 by a symmetric exchange.

Variants of White's conjecture

Conjecture 1 For any matroid M, the toric ideal I_M has a Gröbner basis consisting of quadratics binomials.

Sturmfels (1996) proved that Conjecture 1 holds for uniform matroids.

Conjecture 2 For any matroid M, the toric ideal I_M is generated by quadratics binomials.

Conjecture 3 For any matroid M, the quadratic binomials of I_M are in the ideal generated by the binomials $y_{B_1}y_{B_2} - y_{D_1}y_{D_2}$ such that the pair of bases D_1, D_2 can be obtained from the pair B_1, B_2 by a symmetric exchange.

Remark : Conjectures 2 and 3 together imply White's conjecture.

Complete Intersection

The toric ideal I_M is a complete intersection if and only if there exists a set of homogeneous binomials $g_1, \ldots, g_s \in R$ such that $s = \mathrm{ht}(I_M)$ and $I_M = (g_1, \ldots, g_s)$.

Complete Intersection

The toric ideal I_M is a complete intersection if and only if there exists a set of homogeneous binomials $g_1, \ldots, g_s \in R$ such that $s = \mathrm{ht}(I_M)$ and $I_M = (g_1, \ldots, g_s)$.

Equivalently, I_M is a complete intersection if

$$\mu(I_M) = \mathrm{ht}(I_M) = |\mathcal{B}| - (n - c + 1)$$

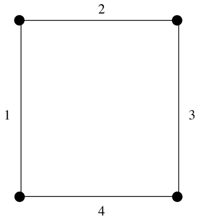
where $\mu(I_M)$ denotes the minimal number of generators of I_M and c the number of connected components of M.

Complete Intersection

The number of connected components of a matroid M is given by the number of equivalent classes induced by the relation $\mathcal{R}$ defined as follows : $a\mathcal{R}b$ if and only if there exist a circuit of M containing both $a, b \in M$.

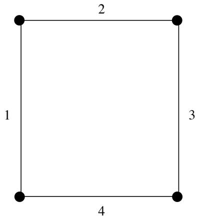
Complete Intersection

The number of connected components of a matroid M is given by the number of equivalent classes induced by the relation $\mathcal{R}$ defined as follows : $a\mathcal{R}b$ if and only if there exist a circuit of M containing both $a, b \in M$.



Complete Intersection

The number of connected components of a matroid M is given by the number of equivalent classes induced by the relation $\mathcal{R}$ defined as follows : $a\mathcal{R}b$ if and only if there exist a circuit of M containing both $a, b \in M$.



We have $\mathcal{B}(M(G)) = \{123, 124, 134, 234\}$. There is one equivalent classe, and thus $\mathrm{ht}(I_M) = 4 - (4 - 1 + 1) = 0$.

Complete Intersection

Recall that

$$I_M = \left(\{y_{B_1}\cdots y_{B_s} - y_{D_1}\cdots y_{D_s} \mid B_1\cup\cdots\cup B_s = D_1\cup\cdots\cup D_s\}\right) \quad (1)$$

Complete Intersection

Recall that

$$I_M = \left(\{y_{B_1}\cdots y_{B_s} - y_{D_1}\cdots y_{D_s} \mid B_1\cup\cdots\cup B_s = D_1\cup\cdots\cup D_s\}\right) \quad (1)$$

• If $r = n$ then $\mathrm{ht}(I_M) = 1-(n-n+1) = 0$, and clearly by (1), we have $I_M = (0)$. So, in this case I_M is complete intersection.

Complete Intersection

Recall that

$$I_M = \left(\{y_{B_1}\cdots y_{B_s} - y_{D_1}\cdots y_{D_s} \mid B_1\cup\cdots\cup B_s = D_1\cup\cdots\cup D_s\}\right) \quad (1)$$

- If $r = n$ then $\mathrm{ht}(I_M) = 1 - (n - n + 1) = 0$, and clearly by (1), we have $I_M = (0)$. So, in this case I_M is complete intersection.
- If $r = n-1$ then $\mathrm{ht}(I_M) = n - (n - 1 + 1) = 0$, and clearly by (1), we have $I_M = (0)$. So, in this case I_M is also complete intersection.

Complete Intersection

Recall that

$$I_M = \left(\{y_{B_1}\cdots y_{B_s} - y_{D_1}\cdots y_{D_s} \mid B_1\cup\cdots\cup B_s = D_1\cup\cdots\cup D_s\}\right) \quad (1)$$

• If $r = n$ then $\mathrm{ht}(I_M) = 1 - (n - n + 1) = 0$, and clearly by (1), we have $I_M = (0)$. So, in this case I_M is complete intersection.

• If $r = n-1$ then $\mathrm{ht}(I_M) = n-(n-1+1) = 0$, and clearly by (1), we have $I_M = (0)$. So, in this case I_M is also complete intersection.

Thus, we only consider the case $r \leq n-2$.

Complete Intersection : duality and minors

We denote by M^* the dual matroid of M.

Complete Intersection : duality and minors

We denote by M^* the dual matroid of M.

σ is the isomorphism of k-algebras $\sigma : R \longrightarrow k[y_{E\setminus B} \mid B \in \mathcal{B}]$ induced by $y_B \mapsto y_{E\setminus B}$.

Complete Intersection : duality and minors

We denote by M^* the dual matroid of M.

σ is the isomorphism of k-algebras $\sigma : R \longrightarrow k[y_{E\setminus B} \mid B \in \mathcal{B}]$ induced by $y_B \mapsto y_{E\setminus B}$.

It is straightforward to check that $\sigma(I_M) = I_{M^*}$

Complete Intersection : duality and minors

We denote by M^* the dual matroid of M.

σ is the isomorphism of k-algebras $\sigma : R \longrightarrow k[y_{E\setminus B} \mid B \in \mathcal{B}]$ induced by $y_B \mapsto y_{E\setminus B}$.

It is straightforward to check that $\sigma(I_M) = I_{M^*}$

Thus, I_M is a complete intersection if and only if I_{M^*} also is.

Complete Intersection : duality and minors

We denote by M^* the dual matroid of M.

σ is the isomorphism of k-algebras $\sigma : R \longrightarrow k[y_{E\setminus B} \mid B \in \mathcal{B}]$ induced by $y_B \mapsto y_{E\setminus B}$.

It is straightforward to check that $\sigma(I_M) = I_{M^*}$

Thus, I_M is a complete intersection if and only if I_{M^*} also is.

Proposition Let M' be a minor of M. If I_M is a complete intersection, then $I_{M'}$ also is.

Complete Intersection : rank 2 case

If M has rank 2 then we associate to M the graph H_M with vertex set E and edge set $\mathcal{B}$.

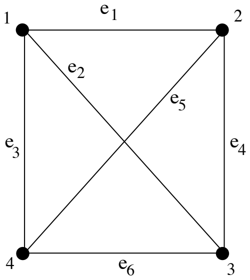
Complete Intersection : rank 2 case

If M has rank 2 then we associate to M the graph H_M with vertex set E and edge set $\mathcal{B}$.

Example :
$\mathcal{B}(U_{2,4}) = \{B_1 = \{1,2\}, B_2 = \{1,3\}, B_3 = \{1,4\}, B_4 = \{2,3\}, B_5 = \{2,4\}, B_6 = \{3,4\}\}$

$$\begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Complete Intersection : rank 2 case



$H_{U_{2,4}}$

$$\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Complete Intersection : rank 2 case

If M has rank 2 then we associate to M the graph H_M with vertex set E and edge set $\mathcal{B}$.

• It turns out that I_M coincides with the toric ideal of the graph H_M.

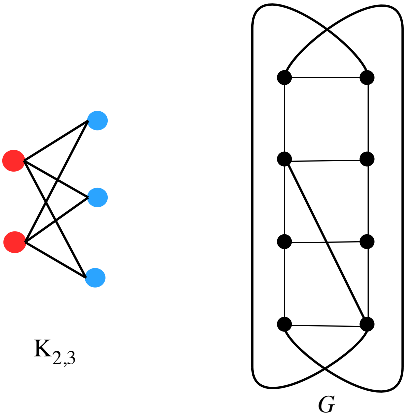
Complete Intersection : rank 2 case

If M has rank 2 then we associate to M the graph H_M with vertex set E and edge set $\mathcal{B}$.

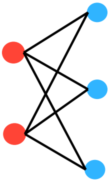
• It turns out that I_M coincides with the toric ideal of the graph H_M.

Theorem (I. Bermejo, I. Garcia-Marco, E. Reyes) Whenever $I_{H(M)}$ is a complete intersection, then H_M does not contain $K_{2,3}$ as subgraph.

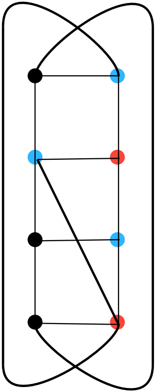
Complete Intersection : rank 2 case



Complete Intersection : rank 2 case

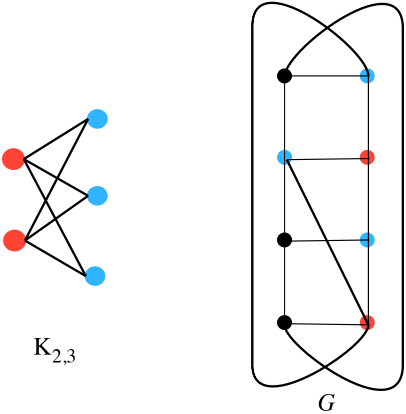


$K_{2,3}$



G

Complete Intersection : rank 2 case



Therefore I_G is not complete intersection.

Complete Intersection : rank 2 case

Proposition Let M be a rank 2 matroid on a ground set of $n \geq 4$ elements without loops or coloops. Then, I_M is a complete intersection if and only if $n = 4$.

Complete Intersection : rank 2 case

Proposition Let M be a rank 2 matroid on a ground set of $n \geq 4$ elements without loops or coloops. Then, I_M is a complete intersection if and only if $n = 4$.

Proof (idea) ($\Rightarrow$) Assume that $n \geq 5$ and let us prove that I_M is not a complete intersection.

Complete Intersection : rank 2 case

Proposition Let M be a rank 2 matroid on a ground set of $n \geq 4$ elements without loops or coloops. Then, I_M is a complete intersection if and only if $n = 4$.

Proof (idea) ($\Rightarrow$) Assume that $n \geq 5$ and let us prove that I_M is not a complete intersection.

Since M has no loops or coloops, we may assume that $B_1 = \{1,2\}, B_2 = \{3,4\}, B_3 = \{1,5\} \in \mathcal{B}$.

Complete Intersection : rank 2 case

Proposition Let M be a rank 2 matroid on a ground set of $n \geq 4$ elements without loops or coloops. Then, I_M is a complete intersection if and only if $n = 4$.

Proof (idea) $(\Rightarrow)$ Assume that $n \geq 5$ and let us prove that I_M is not a complete intersection.

Since M has no loops or coloops, we may assume that $B_1 = \{1, 2\}, B_2 = \{3, 4\}, B_3 = \{1, 5\} \in \mathcal{B}$.

Since $B_1, B_2 \in \mathcal{B}$, by the symmetric exchange axiom, we can also assume that $B_4 = \{1, 3\}, B_5 = \{2, 4\} \in \mathcal{B}$.

Complete Intersection : rank 2 case

Proposition Let M be a rank 2 matroid on a ground set of $n \geq 4$ elements without loops or coloops. Then, I_M is a complete intersection if and only if $n = 4$.

Proof (idea) ($\Rightarrow$) Assume that $n \geq 5$ and let us prove that I_M is not a complete intersection.

Since M has no loops or coloops, we may assume that $B_1 = \{1, 2\}, B_2 = \{3, 4\}, B_3 = \{1, 5\} \in \mathcal{B}$.

Since $B_1, B_2 \in \mathcal{B}$, by the symmetric exchange axiom, we can also assume that $B_4 = \{1, 3\}, B_5 = \{2, 4\} \in \mathcal{B}$.

If $\{4, 5\} \in \mathcal{B}$, then H_M has a subgraph $K_{2,3}$ and I_M is not a complete intersection.

Complete Intersection : rank 2 case

Proposition Let M be a rank 2 matroid on a ground set of $n \geq 4$ elements without loops or coloops. Then, I_M is a complete intersection if and only if $n = 4$.

Proof (idea) ($\Rightarrow$) Assume that $n \geq 5$ and let us prove that I_M is not a complete intersection.

Since M has no loops or coloops, we may assume that $B_1 = \{1, 2\}, B_2 = \{3, 4\}, B_3 = \{1, 5\} \in \mathcal{B}$.

Since $B_1, B_2 \in \mathcal{B}$, by the symmetric exchange axiom, we can also assume that $B_4 = \{1, 3\}, B_5 = \{2, 4\} \in \mathcal{B}$.

If $\{4, 5\} \in \mathcal{B}$, then H_M has a subgraph $K_{2,3}$ and I_M is not a complete intersection.

If $\{4, 5\} \notin \mathcal{B}$ also implies that H_M has a subgraph $K_{2,3}$.

Complete Intersection : rank 2 case

Proposition Let M be a rank 2 matroid on a ground set of $n \geq 4$ elements without loops or coloops. Then, I_M is a complete intersection if and only if $n = 4$.

Proof (idea) $(\Rightarrow)$ Assume that $n \geq 5$ and let us prove that I_M is not a complete intersection.

Since M has no loops or coloops, we may assume that $B_1 = \{1, 2\}, B_2 = \{3, 4\}, B_3 = \{1, 5\} \in \mathcal{B}$.

Since $B_1, B_2 \in \mathcal{B}$, by the symmetric exchange axiom, we can also assume that $B_4 = \{1, 3\}, B_5 = \{2, 4\} \in \mathcal{B}$.

If $\{4, 5\} \in \mathcal{B}$, then H_M has a subgraph $K_{2,3}$ and I_M is not a complete intersection.

If $\{4, 5\} \notin \mathcal{B}$ also implies that H_M has a subgraph $K_{2,3}$.

$(\Leftarrow)$ By computer.

Complete Intersection : general case

Theorem (Garcia-Marco, R.A, 2014) Let M be a matroid without loops or coloops and with $n > r + 1$. Then, I_M is a complete intersection if and only if $n = 4$ and M is the matroid whose set of bases is :

1. $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$,
2. $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\}\}$, or
3. $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\},\{2,3\}\}$, i.e., $M = U_{2,4}$.

Detecting minors

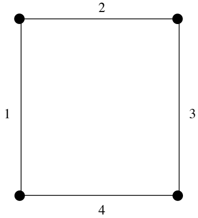
We consider the following binary equivalence relation $\sim$ on the set of pairs of bases :

$$\{B_1, B_2\} \sim \{B_3, B_4\} \iff B_1 \cup B_2 = B_3 \cup B_4 \text{ as multisets,}$$

and we denote by $\Delta_{\{B_1,B_2\}}$ the cardinality of the equivalence class of $\{B_1, B_2\}$.

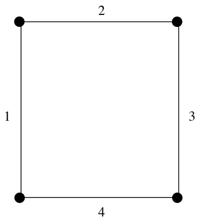
Detecting minors

We consider the graph



Detecting minors

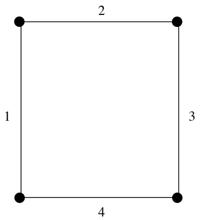
We consider the graph



Therefore,
$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{124\}, B_3 = \{134\}, B_4 = \{234\}\}.$

Detecting minors

We consider the graph



Therefore,
$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{124\}, B_3 = \{134\}, B_4 = \{234\}\}$.
It can be checked that the equivalent class of $\{B_i, B_j\}$ is $\{B_i, B_j\}$, that is, $\Delta_{\{B_i,B_j\}} = 1$ for any pair $1 \leq i \neq j \leq 4$.

Detecting minors

Lemma For every $B_1, B_2 \in \mathcal{B}$, then $2^{d-1} \leq \Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$, where $d := |B_1 \setminus B_2|$.

Detecting minors

Lemma For every $B_1, B_2 \in \mathcal{B}$, then $2^{d-1} \leq \Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$, where $d := |B_1 \setminus B_2|$.

Proof Take $e \in B_1 \setminus B_2$. By the multiple symmetric exchange property, for every A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, there exists $A_2 \subset B_2$ such that both $B_1' := (B_1 \cup A_2) \setminus A_1$ and $B_2' := (B_2 \cup A_1) \setminus A_2$ are bases.

Detecting minors

Lemma For every $B_1, B_2 \in \mathcal{B}$, then $2^{d-1} \leq \Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$, where $d := |B_1 \setminus B_2|$.

Proof Take $e \in B_1 \setminus B_2$. By the multiple symmetric exchange property, for every A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, there exists $A_2 \subset B_2$ such that both $B_1' := (B_1 \cup A_2) \setminus A_1$ and $B_2' := (B_2 \cup A_1) \setminus A_2$ are bases.

Since $B_1 \cup B_2 = B_1' \cup B_2'$ as multisets, we derive that $\Delta_{\{B_1,B_2\}}$ is greater or equal to the number of sets A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, which is exactly 2^{d-1}.

Detecting minors

Lemma For every $B_1, B_2 \in \mathcal{B}$, then $2^{d-1} \leq \Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$, where $d := |B_1 \setminus B_2|$.

Proof Take $e \in B_1 \setminus B_2$. By the multiple symmetric exchange property, for every A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, there exists $A_2 \subset B_2$ such that both $B_1' := (B_1 \cup A_2) \setminus A_1$ and $B_2' := (B_2 \cup A_1) \setminus A_2$ are bases.

Since $B_1 \cup B_2 = B_1' \cup B_2'$ as multisets, we derive that $\Delta_{\{B_1,B_2\}}$ is greater or equal to the number of sets A_1 such that $e \in A_1 \subset (B_1 \setminus B_2)$, which is exactly 2^{d-1}.

$\Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$ easy.

Detecting minors

Lemma Let $B_1, B_2 \in \mathcal{B}$ of a matroid M and consider the matroid $M' := (M/(B_1 \cap B_2))|_{(B_1 \triangle B_2)}$ on the ground set $B_1 \triangle B_2$. Then, the number of bases-cobases of M' is equal to $2\Delta_{\{B_1, B_2\}}$.

Detecting minors

Lemma Let $B_1, B_2 \in \mathcal{B}$ of a matroid M and consider the matroid $M' := (M/(B_1 \cap B_2))|_{(B_1 \triangle B_2)}$ on the ground set $B_1 \triangle B_2$. Then, the number of bases-cobases of M' is equal to $2\Delta_{\{B_1,B_2\}}$.

Theorem (Garcia-Marco, R.A, 2014) If M has a minor $M' \simeq U_{d,2d}$ for some $d \geq 2$, then there exist $B_1, B_2 \in \mathcal{B}$ such that $\Delta_{\{B_1,B_2\}} = \binom{2d-1}{d}$.

Detecting minors

Lemma Let $B_1, B_2 \in \mathcal{B}$ of a matroid M and consider the matroid $M' := (M/(B_1 \cap B_2))|_{(B_1 \triangle B_2)}$ on the ground set $B_1 \triangle B_2$. Then, the number of bases-cobases of M' is equal to $2\Delta_{\{B_1,B_2\}}$.

Theorem (Garcia-Marco, R.A, 2014) If M has a minor $M' \simeq U_{d,2d}$ for some $d \geq 2$, then there exist $B_1, B_2 \in \mathcal{B}$ such that $\Delta_{\{B_1,B_2\}} = \binom{2d-1}{d}$.

Theorem (Garcia-Marco, R.A, 2014) M is binary if and only if $\Delta_{\{B_1,B_2\}} \neq 3$ for every $B_1, B_2 \in \mathcal{B}$.

Detecting minors

Lemma Let $B_1, B_2 \in \mathcal{B}$ of a matroid M and consider the matroid $M' := (M/(B_1 \cap B_2))|_{(B_1 \triangle B_2)}$ on the ground set $B_1 \triangle B_2$. Then, the number of bases-cobases of M' is equal to $2\Delta_{\{B_1,B_2\}}$.

Theorem (Garcia-Marco, R.A, 2014) If M has a minor $M' \simeq U_{d,2d}$ for some $d \geq 2$, then there exist $B_1, B_2 \in \mathcal{B}$ such that $\Delta_{\{B_1,B_2\}} = \binom{2d-1}{d}$.

Theorem (Garcia-Marco, R.A, 2014) M is binary if and only if $\Delta_{\{B_1,B_2\}} \neq 3$ for every $B_1, B_2 \in \mathcal{B}$.

Theorem (Garcia-Marco, R.A, 2014) M has a minor $M' \simeq U_{3,6}$ if and only if $\Delta_{\{B_1,B_2\}} = 10$ for some $B_1, B_2 \in \mathcal{B}$.

System of generators

$\nu(I_M)$ = the number of minimal sets of binomial generators of I_M, where the sign of a binomial does not count

$\mu(I_M)$ = the minimal number of generators of I_M.

System of generators

$\nu(I_M)$ = the number of minimal sets of binomial generators of I_M, where the sign of a binomial does not count

$\mu(I_M)$ = the minimal number of generators of I_M.

Theorem (Garcia-Marco, R.A, 2014) Let $R = \{\{B_1, B_2\}, \ldots, \{B_{2s-1}, B_{2s}\}\}$ be a set of representatives of $\sim$ and set $r_i := \Delta_{\{B_{2i-1}, B_{2i}\}}$ for all $i \in \{1, \ldots, s\}$. Then,

1. $\mu(I_M) \geq (b^2 - b - 2s)/2$, where $b := |\mathcal{B}|$, and
2. $\nu(I_M) \geq \prod_{i=1}^{s} r_i^{r_i - 2}$.

Moreover, in both cases equality holds whenever I_M is generated by quadratics.

System of generators

$\nu(I_M) =$ the number of minimal sets of binomial generators of I_M, where the sign of a binomial does not count

$\mu(I_M) =$ the minimal number of generators of I_M.

Theorem (Garcia-Marco, R.A, 2014) Let $R = \{\{B_1, B_2\}, \ldots, \{B_{2s-1}, B_{2s}\}\}$ be a set of representatives of $\sim$ and set $r_i := \Delta_{\{B_{2i-1}, B_{2i}\}}$ for all $i \in \{1, \ldots, s\}$. Then,

1. $\mu(I_M) \geq (b^2 - b - 2s)/2$, where $b := |\mathcal{B}|$, and
2. $\nu(I_M) \geq \prod_{i=1}^{s} r_i^{r_i - 2}$.

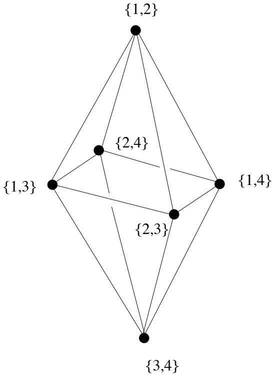
Moreover, in both cases equality holds whenever I_M is generated by quadratics.

Question Can we characterize those matroids M with $\nu(I_M) = 1$?

The basis graph of a matroid M is the undirected graph G_M with vertex set $\mathcal{B}$ and edges $\{B, B'\}$ such that $|B \setminus B'| = 1$. The diameter of a graph is the maximum distance between two vertices of the graph.

The basis graph of a matroid M is the undirected graph G_M with vertex set $\mathcal{B}$ and edges $\{B, B'\}$ such that $|B \setminus B'| = 1$. The diameter of a graph is the maximum distance between two vertices of the graph.

Basis graph $G_{U_{2,4}}$



System of generators

Theorem (Garcia-Marco, R.A, 2014) Let M be a rank $r \geq 2$ matroid. Then, $\nu(I_M) = 1$ if and only if M is binary and the diameter of G_M is at most 2.

System of generators

Theorem (Garcia-Marco, R.A, 2014) Let M be a rank $r \geq 2$ matroid. Then, $\nu(I_M) = 1$ if and only if M is binary and the diameter of G_M is at most 2.

Proof (idea) ($\Rightarrow$) By the previous theorem,we have that $\Delta_{\{B_1,B_2\}} = 1$ or 2 for all $B_1, B_2 \in \mathcal{B}$.

System of generators

Theorem (Garcia-Marco, R.A, 2014) Let M be a rank $r \geq 2$ matroid. Then, $\nu(I_M) = 1$ if and only if M is binary and the diameter of G_M is at most 2.

Proof (idea) ($\Rightarrow$) By the previous theorem,we have that $\Delta_{\{B_1,B_2\}} = 1$ or 2 for all $B_1, B_2 \in \mathcal{B}$.
By above Lemmas and Theorem binary, this is equivalent to M is binary and $|B_1 \setminus B_2| \in \{1, 2\}$ for all $B_1, B_2 \in \mathcal{B}$. Clearly this implies that the diameter of G_M is less or equal to 2.

System of generators

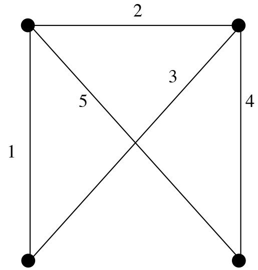
Theorem (Garcia-Marco, R.A, 2014) Let M be a rank $r \geq 2$ matroid. Then, $\nu(I_M) = 1$ if and only if M is binary and the diameter of G_M is at most 2.

Proof (idea) ($\Rightarrow$) By the previous theorem,we have that $\Delta_{\{B_1,B_2\}} = 1$ or 2 for all $B_1, B_2 \in \mathcal{B}$.
By above Lemmas and Theorem binary, this is equivalent to M is binary and $|B_1 \setminus B_2| \in \{1, 2\}$ for all $B_1, B_2 \in \mathcal{B}$. Clearly this implies that the diameter of G_M is less or equal to 2.

($\Leftarrow$) More complicated.

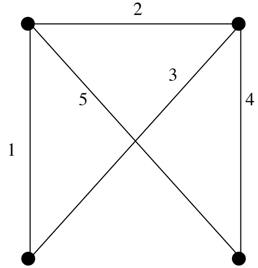
Example

Matroid $M(G)$ associated to graph G.



Example

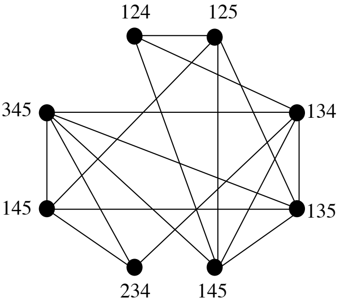
Matroid $M(G)$ associated to graph G.



$\mathcal{B}(M(G)) = \{B_1 = \{124\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{235\}, B_8 = \{345\}\}$

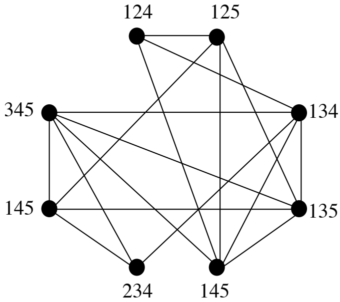
Example

The base graph $G_{M(G)}$



Example

The base graph $G_{M(G)}$



Since diameter of $G_{M(G)}$ is at most two, and $M(G)$ is binary then $\nu(I_M) = 1$.

Definitions

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a simplicial complex if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

Definitions

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a simplicial complex if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

Elements of the simplicial complex Δ are called faces of Δ.

Definitions

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a simplicial complex if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

Elements of the simplicial complex Δ are called faces of Δ.

Maximal faces (under inclusion) are called facets.

Definitions

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a simplicial complex if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

Elements of the simplicial complex Δ are called faces of Δ.

Maximal faces (under inclusion) are called facets.

If $F \in \Delta$ then the dimension of F is $\dim F = |F| - 1$.

Definitions

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a simplicial complex if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

Elements of the simplicial complex Δ are called faces of Δ.

Maximal faces (under inclusion) are called facets.

If $F \in \Delta$ then the dimension of F is $\dim F = |F| - 1$.

The dimension of Δ is defined to be $\dim \Delta = \max\{\dim F | F \in \Delta\}$.

Definitions

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a simplicial complex if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

Elements of the simplicial complex Δ are called faces of Δ.

Maximal faces (under inclusion) are called facets.

If $F \in \Delta$ then the dimension of F is $\dim F = |F| - 1$.

The dimension of Δ is defined to be $\dim \Delta = \max\{\dim F | F \in \Delta\}$.

The complex Δ is said to be pure if all its facets have the same dimension.

Definitions

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a simplicial complex if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

Elements of the simplicial complex Δ are called faces of Δ.

Maximal faces (under inclusion) are called facets.

If $F \in \Delta$ then the dimension of F is dim $F = |F| - 1$.

The dimension of Δ is defined to be dim $\Delta = \max\{\dim F | F \in \Delta\}$.

The complex Δ is said to be pure if all its facets have the same dimension.

If $\{v\} \in \Delta$ then we call v a vertex of Δ.

Definitions

Let $d-1 = \dim\,\Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta|\ \dim\,F = i\}|$ is the number of i-dimensional faces in Δ.

Definitions

Let $d-1 = \dim \Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta|\ \dim F = i\}|$ is the number of i-dimensional faces in Δ.

Let Δ be a simplicial complex with vertex set V.

Definitions

Let $d-1 = \dim \Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta| \dim F = i\}|$ is the number of i-dimensional faces in Δ.

Let Δ be a simplicial complex with vertex set V.

- The k-skeleton of Δ is $[\Delta_k] = \{F \in \Delta | \dim F \leq k\}$.

Definitions

Let $d-1 = \dim \Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta | \dim F = i\}|$ is the number of i-dimensional faces in Δ.

Let Δ be a simplicial complex with vertex set V.

- The k-skeleton of Δ is $[\Delta_k] = \{F \in \Delta | \dim F \leq k\}$.
- If $W \subseteq V$ then the restriction of Δ to W is $\Delta|_W = \{F \in \Delta | F \subseteq W\}$. If $W = V - \{v\}$ then we will write $\Delta_{-v} = \Delta|_W$ and call Δ_{-v} the deletion of Δ with respect to v or the deletion of v from Δ.

Definitions

Let $d-1 = \dim \Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta| \dim F = i\}|$ is the number of i-dimensional faces in Δ.

Let Δ be a simplicial complex with vertex set V.

- The k-skeleton of Δ is $[\Delta_k] = \{F \in \Delta | \dim F \leq k\}$.
- If $W \subseteq V$ then the restriction of Δ to W is $\Delta|_W = \{F \in \Delta | F \subseteq W\}$. If $W = V - \{v\}$ then we will write $\Delta_{-v} = \Delta|_W$ and call Δ_{-v} the deletion of Δ with respect to v or the deletion of v from Δ.
- If $W \subseteq V$ then $link_\Delta(W) = \{F \in \Delta | W \cap F = \emptyset, W \cup F \in \Delta\}$. We call this the link of Δ with respect to W.

Definitions

Let $d-1 = \dim\,\Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta|\ \dim\,F = i\}|$ is the number of i-dimensional faces in Δ.

Let Δ be a simplicial complex with vertex set V.

- The k-skeleton of Δ is $[\Delta_k] = \{F \in \Delta | \dim\,F \leq k\}$.
- If $W \subseteq V$ then the restriction of Δ to W is $\Delta|_W = \{F \in \Delta | F \subseteq W\}$. If $W = V - \{v\}$ then we will write $\Delta_{-v} = \Delta|_W$ and call Δ_{-v} the deletion of Δ with respect to v or the deletion of v from Δ.
- If $W \subseteq V$ then $link_\Delta(W) = \{F \in \Delta | W \cap F = \emptyset, W \cup F \in \Delta\}$. We call this the link of Δ with respect to W.
- If $v \notin V$ then the cone over Δ is $C\Delta = \Delta \cup \{F \cup \{v\} | F \in \Delta\}$

Definitions

Let $d-1 = \dim \Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta | \dim F = i\}|$ is the number of i-dimensional faces in Δ.

Let Δ be a simplicial complex with vertex set V.

- The k-skeleton of Δ is $[\Delta_k] = \{F \in \Delta | \dim F \leq k\}$.
- If $W \subseteq V$ then the restriction of Δ to W is $\Delta|_W = \{F \in \Delta | F \subseteq W\}$. If $W = V - \{v\}$ then we will write $\Delta_{-v} = \Delta|_W$ and call Δ_{-v} the deletion of Δ with respect to v or the deletion of v from Δ.
- If $W \subseteq V$ then $link_\Delta(W) = \{F \in \Delta | W \cap F = \emptyset, W \cup F \in \Delta\}$. We call this the link of Δ with respect to W.
- If $v \notin V$ then the cone over Δ is $C\Delta = \Delta \cup \{F \cup \{v\} | F \in \Delta\}$

v is called the apex of $C\Delta$.

Definitions

Observation Since if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$, the complex Δ is determined completely by those faces that are not contained in any other face, that is the facets of Δ.

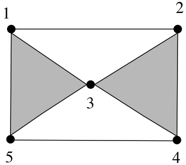
Definitions

Observation Since if $F \in \Delta$ and $G \subseteq F$ then $G \in \Delta$, the complex Δ is determined completely by those faces that are not contained in any other face, that is the facets of Δ.

• Typically, we will describe a simplicial complex by listing its facets.

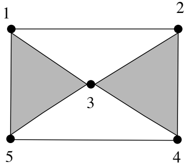
Example

Simplicial complexe Δ of dimension 2



Example

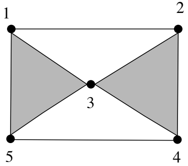
Simplicial complexe Δ of dimension 2



• Δ is not pure as it has facets of dimension 1 (12 and 45) and of dimension 2 (234 and 135).

Example

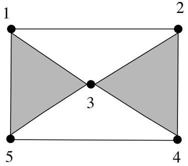
Simplicial complexe Δ of dimension 2



- Δ is not pure as it has facets of dimension 1 (12 and 45) and of dimension 2 (234 and 135).
- $f(\Delta) = (1, 5, 8, 2)$.

Example

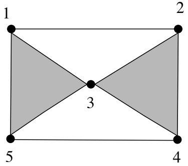
Simplicial complexe Δ of dimension 2



- Δ is not pure as it has facets of dimension 1 (12 and 45) and of dimension 2 (234 and 135).
- $f(\Delta) = (1, 5, 8, 2)$.
- The $link_\Delta(3)$ is the complex with facets 15 and 24, while the $link_\Delta(5)$ has facets 13 and 4.

Example

Simplicial complexe Δ of dimension 2



- Δ is not pure as it has facets of dimension 1 (12 and 45) and of dimension 2 (234 and 135).
- $f(\Delta) = (1, 5, 8, 2)$.
- The $link_\Delta(3)$ is the complex with facets 15 and 24, while the $link_\Delta(5)$ has facets 13 and 4.
- The deletion of 3 has facets 12, 24, 45 and 15. The deletion of 5 has facets 234, 13 and 12.

Matroid complex

Recall that axioms $(I1), (I2)$ for the independent set $\mathcal{I}(M)$ of a matroid M on a set V are equivalent to $\mathcal{I}$ being an abstract simplicial complex on V.

Matroid complex

Recall that axioms $(I1), (I2)$ for the independent set $\mathcal{I}(M)$ of a matroid M on a set V are equivalent to $\mathcal{I}$ being an abstract simplicial complex on V.

The independent sets of M form a simplicial complex, called the independence complex of M.

Matroid complex

Recall that axioms $(I1), (I2)$ for the independent set $\mathcal{I}(M)$ of a matroid M on a set V are equivalent to $\mathcal{I}$ being an abstract simplicial complex on V.

The independent sets of M form a simplicial complex, called the independence complex of M.

Axiom $(I3)$ can be replaced by the following one
$(I3)'$ for every $A \subset E$ the restriction

$$\mathcal{I}|_A = \{I \in \mathcal{I} : I \subset A\}$$

is a *pure* simplicial complex.

Matroid complex

Recall that axioms $(I1), (I2)$ for the independent set $\mathcal{I}(M)$ of a matroid M on a set V are equivalent to $\mathcal{I}$ being an abstract simplicial complex on V.

The independent sets of M form a simplicial complex, called the independence complex of M.

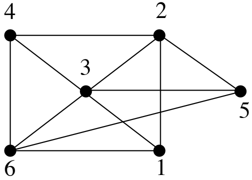
Axiom $(I3)$ can be replaced by the following one
$(I3)'$ for every $A \subset E$ the restriction

$$\mathcal{I}|_A = \{I \in \mathcal{I} : I \subset A\}$$

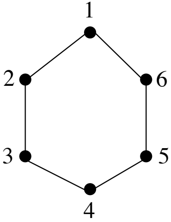
is a *pure* simplicial complex. A simplicial complex Δ over the vertices V is called matroid complex if axiom $(I3)'$ is verified.

Examples

Two 1-dimensional simplicial complexes.



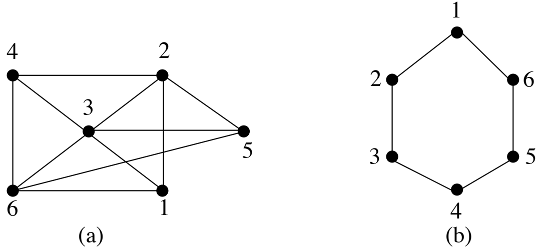
(a)



(b)

Examples

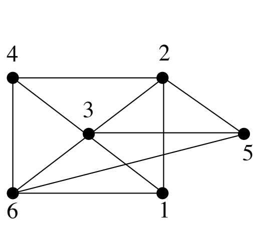
Two 1-dimensional simplicial complexes.



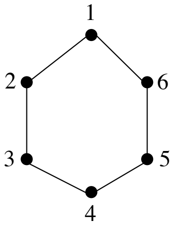
(a) Matroid complex (this can be checked by verifying that every $A \subseteq \{1, \ldots, 6\}$, Δ_A is pure).

Examples

Two 1-dimensional simplicial complexes.



(a)



(b)

(a) Matroid complex (this can be checked by verifying that every $A \subseteq \{1,\ldots,6\}$, Δ_A is pure).

(b) is not a matroid complex since it admits a restriction that is not pure, for instance, the facets of $\Delta_{1,3,4}$ are $\{1\}$ and $\{3,4\}$ as facets so the restriction is not pure.

Standard constructions

Let Δ be a matroid complex with vertex set V. Then, the following complexes are also matroid complexes

Standard constructions

Let Δ be a matroid complex with vertex set V. Then, the following complexes are also matroid complexes

- $\Delta|_W$ for every $W \subseteq V$.

Standard constructions

Let Δ be a matroid complex with vertex set V. Then, the following complexes are also matroid complexes

- $\Delta|_W$ for every $W \subseteq V$.
- $C\Delta$, the cone over Δ.

Standard constructions

Let Δ be a matroid complex with vertex set V. Then, the following complexes are also matroid complexes

- $\Delta|_W$ for every $W \subseteq V$.
- $C\Delta$, the cone over Δ.
- $[\Delta]_k$, the k-skeleton of Δ.

Standard constructions

Let Δ be a matroid complex with vertex set V. Then, the following complexes are also matroid complexes

- $\Delta|_W$ for every $W \subseteq V$.
- $C\Delta$, the cone over Δ.
- $[\Delta]_k$, the k-skeleton of Δ.
- $link_\Delta(F)$ for every $F \in \Delta$.

Standard constructions

Let Δ be a matroid complex with vertex set V. Then, the following complexes are also matroid complexes

- $\Delta|_W$ for every $W \subseteq V$.
- $C\Delta$, the cone over Δ.
- $[\Delta]_k$, the k-skeleton of Δ.
- $link_{\Delta}(F)$ for every $F \in \Delta$.

Remarks : Link and restriction are identical to the contraction and deletion constructions from matroids.

Standard constructions

Let Δ be a matroid complex with vertex set V. Then, the following complexes are also matroid complexes

- $\Delta|_W$ for every $W \subseteq V$.
- $C\Delta$, the cone over Δ.
- $[\Delta]_k$, the k-skeleton of Δ.
- $link_\Delta(F)$ for every $F \in \Delta$.

Remarks : Link and restriction are identical to the contraction and deletion constructions from matroids.

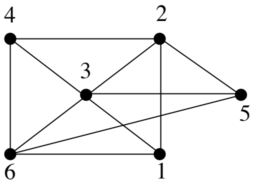
A matroid complex Δ_M is a cone if and only if M has a coloop (or isthme), which corresponds to the apex defined above.

Standard constructions

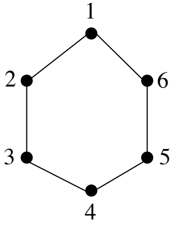
Lemma Let Δ be a 1-dimensional simplicial complex. Then, Δ is matroid if and only if for every vertex v and every edge E, $link_{\Delta}(v) \cap E \neq \emptyset$.

Standard constructions

Lemma Let Δ be a 1-dimensional simplicial complex. Then, Δ is matroid if and only if for every vertex v and every edge E, $link_{\Delta}(v) \cap E \neq \emptyset$.



(a)



(b)

Stanley-Reisner ideal

Let k be a field. We can associate to a simplicial complex Δ, a square free monomial ideal in $S = k[x_1, \ldots, x_n]$,

$$I_\Delta = \left(x_F = \prod_{i \in F} x_i | F \notin \Delta \right) \subseteq S.$$

Stanley-Reisner ideal

Let k be a field. We can associate to a simplicial complex Δ, a square free monomial ideal in $S = k[x_1, \ldots, x_n]$,

$$I_\Delta = \left(x_F = \prod_{i \in F} x_i | F \notin \Delta \right) \subseteq S.$$

The ideal I_Δ is called the Stanley-Reisner ideal of Δ and S/I_Δ the Stanley-Reisner ring of Δ.

Stanley-Reisner ideal

Facts

Stanley-Reisner ideal

Facts

- Hilbert function

$$h_{S/I_\Delta}(h) = dim_k[S/I_\Delta]_h$$

where $[S/I_\Delta]$ is the vector space of degree h homogeneous polynomial outside of I_Δ.

Stanley-Reisner ideal

Facts

- Hilbert function

$$h_{S/I_\Delta}(h) = dim_k[S/I_\Delta]_h$$

where $[S/I_\Delta]$ is the vector space of degree h homogeneous polynomial outside of I_Δ.

- Hilbert series

$$H_{S/I_\Delta}(t) = \sum_{i=1}^{\infty} h_{S/I_\Delta}(i)t^i = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}$$

where $d =$dim I_Δ.

Stanley-Reisner ideal

Facts

- Hilbert function

$$h_{S/I_\Delta}(h) = dim_k[S/I_\Delta]_h$$

where $[S/I_\Delta]$ is the vector space of degree h homogeneous polynomial outside of I_Δ.

- Hilbert series

$$H_{S/I_\Delta}(t) = \sum_{i=1}^{\infty} h_{S/I_\Delta}(i)t^i = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}$$

where d =dim I_Δ.

$h(\Delta) = (h_0, \ldots, h_d)$ is known as the h-vector of Δ.

h-vector of simplicial complexes

Assume that $\dim \Delta = d - 1$.

h-vector of simplicial complexes

Assume that $\dim \Delta = d - 1$.

We may study the h-vector of a simplicial complex of Δ $h(\Delta) = (h_0, \ldots, h_d)$ from its f-vector via the relation

$$\sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} = \sum_{i=0}^{d} h_i t^i$$

h-vector of simplicial complexes

Assume that dim $\Delta = d - 1$.

We may study the h-vector of a simplicial complex of Δ $h(\Delta) = (h_0, \dots, h_d)$ from its f-vector via the relation

$$\sum_{i=0}^{d} f_{i-1} t^i (1-t)^{d-i} = \sum_{i=0}^{d} h_i t^i$$

In particular, for any $j = 0, \dots, d$, we have

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-1} h_i$$

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-1} f_{i-1}.$$

h-vector of simplicial complexes

The h-number of a matroid M may be interpreted combinatorially in terms of certain invariants of M.

h-vector of simplicial complexes

The h-number of a matroid M may be interpreted combinatorially in terms of certain invariants of M.

Fix a total ordering $\{v_1, < v_2 < \cdots < v_n\}$ on $E(M)$.

h-vector of simplicial complexes

The h-number of a matroid M may be interpreted combinatorially in terms of certain invariants of M.

Fix a total ordering $\{v_1, < v_2 < \cdots < v_n\}$ on $E(M)$.

Given a bases B, an element $v_j \in B$ is internally passive in B if there is some $v_i \in E \setminus B$ such that $v_i < v_j$ and $(B \setminus v_j) \cup v_i$ is a bases of M.

h-vector of simplicial complexes

The h-number of a matroid M may be interpreted combinatorially in terms of certain invariants of M.

Fix a total ordering $\{v_1, < v_2 < \cdots < v_n\}$ on $E(M)$.

Given a bases B, an element $v_j \in B$ is internally passive in B if there is some $v_i \in E \setminus B$ such that $v_i < v_j$ and $(B \setminus v_j) \cup v_i$ is a bases of M.

Dually, $v_j \in E \setminus B$ is externally passive in B if there is some $v_i \in B$ such that $v_i < v_j$ and $(B \setminus v_i) \cup v_j$ is a bases of M.

h-vector of simplicial complexes

The h-number of a matroid M may be interpreted combinatorially in terms of certain invariants of M.

Fix a total ordering $\{v_1, < v_2 < \cdots < v_n\}$ on $E(M)$.

Given a bases B, an element $v_j \in B$ is internally passive in B if there is some $v_i \in E \setminus B$ such that $v_i < v_j$ and $(B \setminus v_j) \cup v_i$ is a bases of M.

Dually, $v_j \in E \setminus B$ is externally passive in B if there is some $v_i \in B$ such that $v_i < v_j$ and $(B \setminus v_i) \cup v_j$ is a bases of M.

Remark v_j is externally passive in B if it is internally passive in $E \setminus B$ in M^*.

h-vector of simplicial complexes

Bjorner proved that

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M)} t^{ip(B)}$$

where $ip(B)$ counts the number of internally passive elements in B.

h-vector of simplicial complexes

Bjorner proved that

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M)} t^{ip(B)}$$

where $ip(B)$ counts the number of internally passive elements in B.
Remark This proves that the h-numbers of a matroid complex are nonnegative.

h-vector of simplicial complexes

Bjorner proved that

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M)} t^{ip(B)}$$

where $ip(B)$ counts the number of internally passive elements in B.
Remark This proves that the h-numbers of a matroid complex are nonnegative.

Alternatively,

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M^*)} t^{ep(B)}$$

where $ep(B)$ counts the number of externally passive elements in B.

h-vector of simplicial complexes

Remarks

• Since the f-numbers (and hence the h-numbers) of a matroid depend only on its independent sets, then above equations hold for any ordering of the ground set of M.

h-vector of simplicial complexes

Remarks

• Since the f-numbers (and hence the h-numbers) of a matroid depend only on its independent sets, then above equations hold for any ordering of the ground set of M.

• h-vector of a matroid complex Δ_M is actually a specialization of the Tutte polynomial of the corresponding matroid ; precisely we have $T(M; x, 1) = h_0 x^d + h_1 x^{d_1} + \cdots + h_d$

Example

We consider the matroid complex $\Delta(U_{2,3})$

Example

We consider the matroid complex $\Delta(U_{2,3})$

We have that $\dim\Delta = 1$ and $f_{-1} = 1, f_0 = 3$ and $f_1 = 3$.

Example

We consider the matroid complex $\Delta(U_{2,3})$

We have that $\dim\Delta = 1$ and $f_{-1} = 1, f_0 = 3$ and $f_1 = 3$.

Therefore

$$\begin{aligned}\sum_{i=0}^{2} f_{i-1}t^i(1-t)^{2-i} &= f_{-1}t^0(1-t)^2 + f_0t(1-t) + f_1t^2(1-t)^0 \\ &= (1-t)^2 + 3t(1-t) + 3t^2 \\ &= 1 - 2t + t^2 + 3t - 3t - 3t^2 + 3t^2 \\ &= t^2 + t + 1 = \sum_{i=0}^{2} h_i t^i.\end{aligned}$$

Example

We consider the matroid complex $\Delta(U_{2,3})$

We have that dim$\Delta = 1$ and $f_{-1} = 1, f_0 = 3$ and $f_1 = 3$.

Therefore

$$\begin{aligned}\sum_{i=0}^{2} f_{i-1}t^i(1-t)^{2-i} &= f_{-1}t^0(1-t)^2 + f_0 t(1-t) + f_1 t^2(1-t)^0 \\ &= (1-t)^2 + 3t(1-t) + 3t^2 \\ &= 1 - 2t + t^2 + 3t - 3t - 3t^2 + 3t^2 \\ &= t^2 + t + 1 = \sum_{i=0}^{2} h_i t^i.\end{aligned}$$

Obtaining that $h(\Delta) = (1, 1, 1)$.

Example continuation

Let $\mathcal{B}(U_{2,3}) = \{B_1 = \{1,2\}, B_2 = \{1,3\}, B_3 = \{2,3\}\}$.

Example continuation

Let $\mathcal{B}(U_{2,3}) = \{B_1 = \{1,2\}, B_2 = \{1,3\}, B_3 = \{2,3\}\}$.

We can check that

- there is not internally passive element in B_1

- 3 is internally passive element of B_2

- 2 and 3 are internally passive elements of B_3

Example continuation

Let $\mathcal{B}(U_{2,3}) = \{B_1 = \{1,2\}, B_2 = \{1,3\}, B_3 = \{2,3\}\}$.

We can check that

- there is not internally passive element in B_1

- 3 is internally passive element of B_2

- 2 and 3 are internally passive elements of B_3

Thus

$$\sum_{i=0}^{2} h_i t^i = \sum_{B \in \mathcal{B}(U_{2,3})} t^{ip(B)} = 1 + t + t^2.$$

Example continuation

Let $\mathcal{B}(U^*_{2,3} = U_{1,3}) = \{B_1 = \{1\}, B_2 = \{2\}, B_3 = \{3\}\}$.

Example continuation

Let $\mathcal{B}(U^*_{2,3} = U_{1,3}) = \{B_1 = \{1\}, B_2 = \{2\}, B_3 = \{3\}\}$.

We can check that

- 2 and 3 are externally passive elements of B_1
- 3 is externally passive element of B_2
- there is not externally passive element in B_3

Example continuation

Let $\mathcal{B}(U^*_{2,3} = U_{1,3}) = \{B_1 = \{1\}, B_2 = \{2\}, B_3 = \{3\}\}$.

We can check that

- 2 and 3 are externally passive elements of B_1
- 3 is externally passive element of B_2
- there is not externally passive element in B_3

Thus

$$\sum_{i=0}^{2} h_i t^i = \sum_{B \in \mathcal{B}(U_{1,3})} t^{ep(B)} = t^2 + t + 1.$$

Example continuation

We have that

$$T(U_{3,2}; x, y) = x^2 + x + y,$$

Example continuation

We have that

$$T(U_{3,2}; x, y) = x^2 + x + y,$$

and thus

$$T(U_{3,2}; t, 1) = t^2 + t + 1 = \sum_{i=0}^{2} h_i t^i.$$

Order ideal

An order ideal $\mathcal{O}$ is a family of monomials (say of degree at most r) with the property that if $\mu \in \mathcal{O}$ and $\nu|\mu$ then $\nu \in \mathcal{O}$.

Order ideal

An order ideal $\mathcal{O}$ is a family of monomials (say of degree at most r) with the property that if $\mu \in \mathcal{O}$ and $\nu | \mu$ then $\nu \in \mathcal{O}$.

Let $\mathcal{O}_i$ denote the collection of monomials in $\mathcal{O}$ of degree i. Let $F_i(\mathcal{O}) := |\mathcal{O}_i|$ and $F(\mathcal{O}) = (F_0(\mathcal{O}), F_1(\mathcal{O}), \ldots, F_r(\mathcal{O}))$.

Order ideal

An order ideal $\mathcal{O}$ is a family of monomials (say of degree at most r) with the property that if $\mu \in \mathcal{O}$ and $\nu | \mu$ then $\nu \in \mathcal{O}$.

Let $\mathcal{O}_i$ denote the collection of monomials in $\mathcal{O}$ of degree i. Let $F_i(\mathcal{O}) := |\mathcal{O}_i|$ and $F(\mathcal{O}) = (F_0(\mathcal{O}), F_1(\mathcal{O}), \ldots, F_r(\mathcal{O}))$.

We say that $\mathcal{O}$ is pure if all its maximal monomials (under divisibility) have the same degree.

Order ideal

An order ideal $\mathcal{O}$ is a family of monomials (say of degree at most r) with the property that if $\mu \in \mathcal{O}$ and $\nu | \mu$ then $\nu \in \mathcal{O}$.

Let $\mathcal{O}_i$ denote the collection of monomials in $\mathcal{O}$ of degree i. Let $F_i(\mathcal{O}) := |\mathcal{O}_i|$ and $F(\mathcal{O}) = (F_0(\mathcal{O}), F_1(\mathcal{O}), \ldots, F_r(\mathcal{O}))$.

We say that $\mathcal{O}$ is pure if all its maximal monomials (under divisibility) have the same degree.

A vector $\mathbf{h} = (h_0, \ldots, h_d)$ is a pure O-sequence if there is a pure ideal $\mathcal{O}$ such that $\mathbf{h} = F(\mathcal{O})$.

Example

The pure monomial order ideal (inside $k[x,y,z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3};$$

Example

The pure monomial order ideal (inside $k[x, y, z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3}; y^3z, xy^2z, xy^3, xz^3, x^2z^2, y^2z,$$

Example

The pure monomial order ideal (inside $k[x, y, z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3}; y^3z, xy^2z, xy^3, xz^3, x^2z^2, y^2z, y^3, xyz, xy^2, xz^2, z^3, x^2z,$$

Example

The pure monomial order ideal (inside $k[x,y,z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3}; y^3z, xy^2z, xy^3, xz^3, x^2z^2, y^2z, y^3, xyz, xy^2, xz^2, z^3, x^2z, yz, y^2, xz, xy, z^2, x^2,$$

Example

The pure monomial order ideal (inside $k[x, y, z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3}; y^3z, xy^2z, xy^3, xz^3, x^2z^2, y^2z, y^3, xyz, xy^2, xz^2, z^3, x^2z, yz, y^2, xz, xy, z^2, x^2, z, y, x,$$

Example

The pure monomial order ideal (inside $k[x,y,z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3}; y^3z, xy^2z, xy^3, xz^3, x^2z^2, y^2z, y^3, xyz, xy^2, xz^2, z^3, x^2z, yz, y^2, xz, xy, z^2, x^2, z, y, x, 1\}.$$

Example

The pure monomial order ideal (inside $k[x, y, z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

$$X = \{\mathbf{xy^3z}, \mathbf{x^2z^3}; y^3z, xy^2z, xy^3, xz^3, x^2z^2, y^2z, y^3, xyz, xy^2, xz^2, z^3, x^2z, yz, y^2, xz, xy, z^2, x^2, z, y, x, 1\}.$$

Hence the h-vector of X is the pure O-sequence
$h = (1, 3, 6, 7, 5, 2)$.

Stanley's conjecture

A longstanding conjecture of Stanley suggest that matroid h-vectors are highly structured

Stanley's conjecture

A longstanding conjecture of Stanley suggest that matroid h-vectors are highly structured

Conjecture (Stanley, 1976) For any matroid M, $h(M)$ is a pure O-sequence.

Stanley's conjecture

A longstanding conjecture of Stanley suggest that matroid h-vectors are highly structured

Conjecture (Stanley, 1976) For any matroid M, $h(M)$ is a pure O-sequence.

Conjecture hold for several families of matroid complexes :

Stanley's conjecture

A longstanding conjecture of Stanley suggest that matroid h-vectors are highly structured

Conjecture (Stanley, 1976) For any matroid M, $h(M)$ is a pure O-sequence.

Conjecture hold for several families of matroid complexes :

(Merino, Noble, Ramirez-Ibañez, Villarroel, 2010) Paving matroids

(Merino, 2001) Cographic matroids

(Oh, 2010) Cotranversal matroids

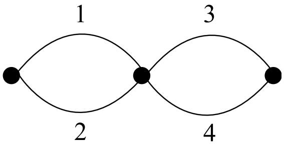
(Schweig, 2010) Lattice path matroids

(Stokes, 2009) Matroids of rank at most three

(De Loera, Kemper, Klee, 2012) for all matroids on at most nine elements all matroids of corank two.

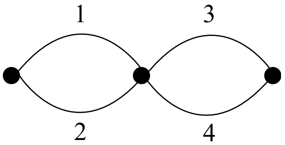
Example

We consider the matroid complexe Δ associated to the rank 2 matroid induced by the graph G



Example

We consider the matroid complexe Δ associated to the rank 2 matroid induced by the graph G



We have that $\dim\Delta = 1$ and $f_{-1} = 1, f_0 = 4$ and $f_1 = 4$.

Example

$$\mathcal{B}(M(G)) = \{B_1 = \{1,3\}, B_2 = \{1,4\}, B_3 = \{2,3\}, B_4 = \{2,4\}\}.$$

Example

$\mathcal{B}(M(G)) = \{B_1 = \{1,3\}, B_2 = \{1,4\}, B_3 = \{2,3\}, B_4 = \{2,4\}\}$.

- there is not internally passive element in B_1

- 4 is internally passive element of B_2

- 2 is internally passive element of B_3

- 2 and 4 are internally passive elements of B_4

Example

$\mathcal{B}(M(G)) = \{B_1 = \{1,3\}, B_2 = \{1,4\}, B_3 = \{2,3\}, B_4 = \{2,4\}\}.$

- there is not internally passive element in B_1
- 4 is internally passive element of B_2
- 2 is internally passive element of B_3
- 2 and 4 are internally passive elements of B_4

Thus,

$$\sum_{i=0}^{2} h_i t^i = \sum_{B \in \mathcal{B}(M(G))} t^{ip(B)} = 1 + t + t + t^2 = 1 + 2t + t^2.$$

Example

$\mathcal{B}(M(G)) = \{B_1 = \{1,3\}, B_2 = \{1,4\}, B_3 = \{2,3\}, B_4 = \{2,4\}\}$.

- there is not internally passive element in B_1
- 4 is internally passive element of B_2
- 2 is internally passive element of B_3
- 2 and 4 are internally passive elements of B_4

Thus,

$$\sum_{i=0}^{2} h_i t^i = \sum_{B \in \mathcal{B}(M(G))} t^{ip(B)} = 1 + t + t + t^2 = 1 + 2t + t^2.$$

Obtaining the h-vector $h(1,2,1)$.

Example

$\mathcal{B}(M(G)) = \{B_1 = \{1,3\}, B_2 = \{1,4\}, B_3 = \{2,3\}, B_4 = \{2,4\}\}$.

- there is not internally passive element in B_1
- 4 is internally passive element of B_2
- 2 is internally passive element of B_3
- 2 and 4 are internally passive elements of B_4

Thus,

$$\sum_{i=0}^{2} h_i t^i = \sum_{B\in\mathcal{B}(M(G))} t^{ip(B)} = 1 + t + t + t^2 = 1 + 2t + t^2.$$

Obtaining the h-vector $h(1,2,1)$. Since $\mathcal{O} = (1, x_1, x_2, x_1x_2)$ is an order ideal then $h(1,2,1)$ is pure O-sequence.