O-sequences and h-vectors of matroid simplicial complexes

J.L. Ramírez Alfonsín

I3M, Université Montpellier 2

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If $\{v\} \in \Delta$ then we call v a vertex of Δ.

Definitions

Let $d-1 = \dim \Delta$. The f-vector of Δ is the vector $f(\Delta) := (f_{-1}, f_0, \ldots, f_{d-1})$, where $f_i = |\{F \in \Delta| \dim F = i\}|$ is the number of i-dimensional faces in Δ.

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v is called the apex of $C\Delta$.

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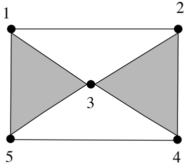
- Typically, we will describe a simplicial complex by listing its facets.

Example

Simplicial complexe Δ of dimension 2



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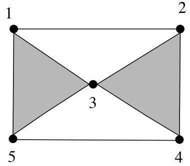

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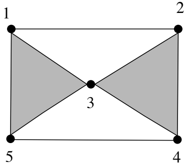

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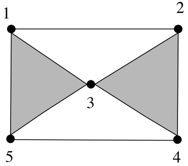

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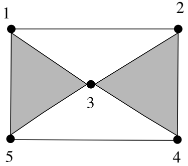

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• The deletion of 3 has facets 12, 24, 45 and 15. The deletion of 5 has facets 234, 13 and 12.

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Axiom $(I3)$ can be replaced by the following one
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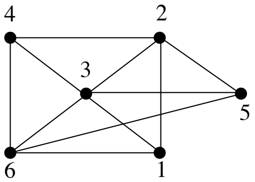
Examples

Two 1-dimensional simplicial complexes.



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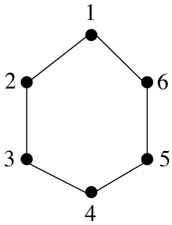

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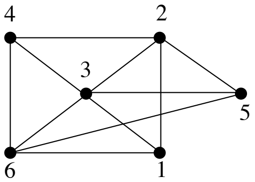

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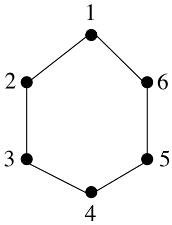

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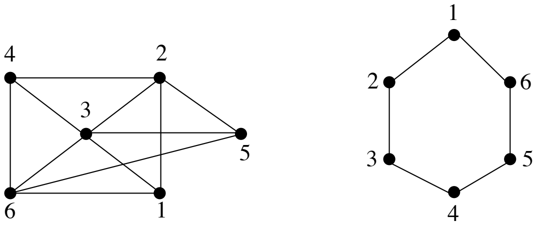

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(b) is not a matroid complex since it admits a restriction that is not pure, for instance, the facets of $\Delta_{1,3,4}$ are $\{1\}$ and $\{3, 4\}$ as facets so the restriction is not pure.

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A matroid complex Δ_M is a cone if and only if M has a coloop (or isthme), which corresponds to the apex defined above.

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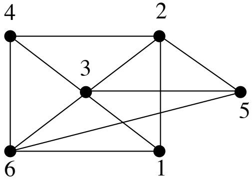

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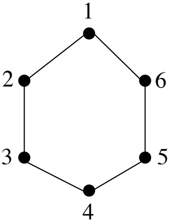

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Stanley-Reisner ideal

Let k be a field. We can associate to a simplicial complex Δ, a square free monomial ideal in $S = k[x_1, \ldots, x_n]$,

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The ideal I_Δ is called the Stanley-Reisner ideal of Δ and S/I_Δ the Stanley-Reisner ring of Δ.

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$h(\Delta) = (h_0, \ldots, h_d)$ is known as the *h*-vector of Δ.

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In particular, for any $j = 0, \ldots, d$, we have

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-1} h_i$$

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-1} f_{i-1}.$$

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Remark v_j is externally passive in B if it is internally passive in $E \setminus B$ in M^*.

h-vector of simplicial complexes

Bjorner proved that

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M)} t^{ip(B)}$$

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Alternatively,

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• h-vector of a matroid complex Δ_M is actually a specialization of the Tutte polynomial of the corresponding matroid ; precisely we have $T(M; x, 1) = h_0x^d + h_1x^{d_1} + \cdots + h_d$

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Obtaining that $h(\Delta) = (1, 1, 1)$.

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A vector $\mathbf{h} = (h_0, \ldots, h_d)$ is a pure O-sequence if there is a pure ideal $\mathcal{O}$ such that $\mathbf{h} = F(\mathcal{O})$.

Example

The pure monomial order ideal (inside $k[x, y, z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

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Hence the *h*-vector of X is the pure O-sequence $h = (1,3,6,7,5,2)$.

Stanley's conjecture

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Conjecture hold for several families of matroid complexes :

(Merino, Noble, Ramirez-Ibañez, Villarroel, 2010) Paving matroids

(Merino, 2001) Cographic matroids

(Oh, 2010) Cotranversal matroids

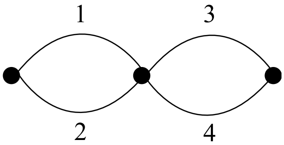
(Schweig, 2010) Lattice path matroids

(Stokes, 2009) Matroids of rank at most three

(De Loera, Kemper, Klee, 2012) for all matroids on at most nine elements all matroids of corank two.

Example

We consider the matroid complexe Δ associated to the rank 2 matroid induced by the graph G



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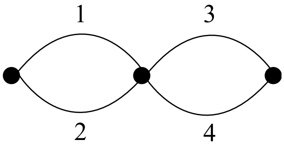

We have that $\dim \Delta = 1$ and $f_{-1} = 1, f_0 = 4$ and $f_1 = 4$.

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Obtaining the h-vector $h(1,2,1)$. Since $\mathcal{O} = (1, x_1, x_2, x_1 x_2)$ is an order ideal then $h(1,2,1)$ is pure O-sequence.