Theory of matroids and Tutte polynomial

J.L. Ramírez Alfonsín

IMAG, Université de Montpellier

San Luis, Argentina
February 2016

Independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

Proof : $(I1)$ et $(I2)$ are trivial.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

Proof : $(I1)$ et $(I2)$ are trivial.
$(I3)$] Let $I_1', I_2' \in \mathcal{I}$ such that the corresponding columns, say I_1 et I_2, are linearly independent with $|I_1| < |I_2|$.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

Proof : $(I1)$ et $(I2)$ are trivial.
$(I3)$] Let $I'_1, I'_2 \in \mathcal{I}$ such that the corresponding columns, say I_1 et I_2, are linearly independent with $|I_1| < |I_2|$.
By contradiction, suppose that $I_1 \cup e$ is linearly <u>dependent</u> for any $e \in I_2 \backslash I_1$.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

Proof : $(I1)$ et $(I2)$ are trivial.
$(I3)$] Let $I_1', I_2' \in \mathcal{I}$ such that the corresponding columns, say I_1 et I_2, are linearly independent with $|I_1| < |I_2|$.
By contradiction, suppose that $I_1 \cup e$ is linearly dependent for any $e \in I_2 \backslash I_1$. Let W the space generated by I_1 and I_2.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

Proof : $(I1)$ et $(I2)$ are trivial.
$(I3)$] Let $I'_1, I'_2 \in \mathcal{I}$ such that the corresponding columns, say I_1 et I_2, are linearly independent with $|I_1| < |I_2|$.
By contradiction, suppose that $I_1 \cup e$ is linearly dependent for any $e \in I_2 \backslash I_1$. Let W the space generated by I_1 and I_2.
On one hand, $dim(W) \geq |I_2|$,

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

Proof : $(I1)$ et $(I2)$ are trivial.
$(I3)$] Let $I_1', I_2' \in \mathcal{I}$ such that the corresponding columns, say I_1 et I_2, are linearly independent with $|I_1| < |I_2|$.
By contradiction, suppose that $I_1 \cup e$ is linearly <u>dependent</u> for any $e \in I_2 \backslash I_1$. Let W the space generated by I_1 and I_2.
On one hand, $dim(W) \geq |I_2|$, on the other hand W is contained in the space generated by I_1.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

Proof : $(I1)$ et $(I2)$ are trivial.
$(I3)$] Let $I'_1, I'_2 \in \mathcal{I}$ such that the corresponding columns, say I_1 et I_2, are linearly independent with $|I_1| < |I_2|$.
By contradiction, suppose that $I_1 \cup e$ is linearly dependent for any $e \in I_2 \backslash I_1$. Let W the space generated by I_1 and I_2.
On one hand, $dim(W) \geq |I_2|$, on the other hand W is contained in the space generated by I_1.

$$|I_2| \leq dim(W) \leq |I_1| < |I_2| \quad !!!$$

Representable Matroids

Let A be the following matrix with coefficients in $\mathbb{R}$.

$$\begin{array}{cc} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ A = & \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$\{\emptyset, \{1\}, \{2\}, \{4\}, \{4\}, \{5\}, \{1,2\}, \{1,5\}, \{2,4\}, \{2,5\}, \{4,5\}\} \subseteq \mathcal{I}(M)$

A matroid obtained form a matrix A with coefficients in $\mathbb{F}$ is denoted by $M(A)$ and is called representable over $\mathbb{F}$ or $\mathbb{F}$-representable.

Circuits

A subset $X \subseteq E$ is said to be minimal dependent if any proper subset of X is independent. A minimal dependent set of matroid M is called circuit of M.
We denote by $\mathcal{C}$ the set of circuits of a matroid.

Circuits

A subset $X \subseteq E$ is said to be minimal dependent if any proper subset of X is independent. A minimal dependent set of matroid M is called circuit of M.
We denote by $\mathcal{C}$ the set of circuits of a matroid.

$\mathcal{C}$ is the set of circuits of a matrid on E if and only if $\mathcal{C}$ verifies the following properties :

$(C1)$ $\emptyset \notin \mathcal{C}$,

$(C2)$ $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$,

$(C3)$ *(elimination property)* If $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq \{C_1 \cup C_2\} \backslash \{e\}$.

Graphic Matroid

Let $G = (V, E)$ be a graph. A cycle in G is a closed walk without repeated vertices.

Graphic Matroid

Let $G = (V, E)$ be a graph. A cycle in G is a closed walk without repeated vertices.

Theorem The set of cycles in a graph $G = (V, E)$ is the set of circuits of a matroid on E.

Graphic Matroid

Let $G = (V, E)$ be a graph. A cycle in G is a closed walk without repeated vertices.

Theorem The set of cycles in a graph $G = (V, E)$ is the set of circuits of a matroid on E.

This matroid is denoted by $M(G)$ and called graphic.

Graphic Matroid

Let $G = (V, E)$ be a graph. A cycle in G is a closed walk without repeated vertices.

Theorem The set of cycles in a graph $G = (V, E)$ is the set of circuits of a matroid on E.

This matroid is denoted by $M(G)$ and called graphic.

Proof : Verify $(C1), (C2)$ and $(C3)$.

Graphic Matroid

Let $G = (V, E)$ be a graph. A cycle in G is a closed walk without repeated vertices.

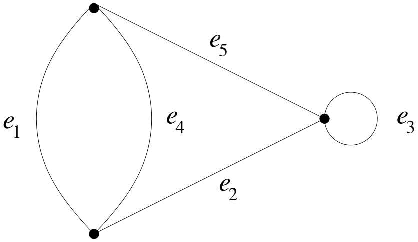
Theorem The set of cycles in a graph $G = (V, E)$ is the set of circuits of a matroid on E.

This matroid is denoted by $M(G)$ and called graphic.

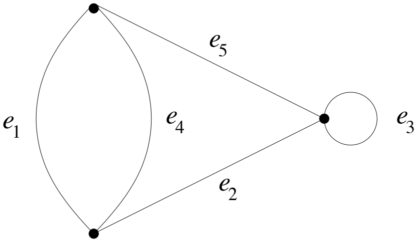
Proof : Verify $(C1), (C2)$ and $(C3)$.

A subset of edges $I \subset \{e_1, \ldots, e_n\}$ of G is independent if the graph induced by I does not contain a cycle.

Graphic Matroid



Graphic Matroid



It can be checked that $M(G)$ is isomorphic to $M(A)$ (under the bijection $e_i \to i$).

$$A = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Graphic Matroid

Theorem A graphic matroid is always representable over $\mathbb{R}$.

Graphic Matroid

Theorem A graphic matroid is always representable over $\mathbb{R}$.

Proof (idea) Let $G = (V, E)$ be an oriented graph and let $\{x_i,\ i \in V\}$ be the canonical base of $\mathbb{R}$.

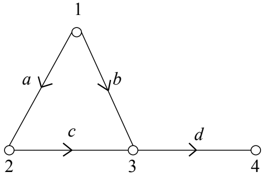
Graphic Matroid

Theorem A graphic matroid is always representable over $\mathbb{R}$.

Proof (idea) Let $G = (V, E)$ be an oriented graph and let $\{x_i,\ i \in V\}$ be the canonical base of $\mathbb{R}$.

Exercice : Verify that the graph $G = (V, E)$ gives the same matroid that the one given by the set of vectors $y_e = x_i - x_j$ where $e = (i, j) \in E$.

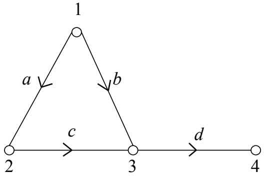
G



$$A = \begin{array}{c} \begin{array}{cccc} y_a & y_b & y_c & y_d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

Graphic Matroid

G

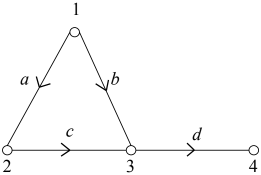


$$A = \begin{array}{c} \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} \;\;y_a & \;\;y_b & \;\;y_c & \;\;y_d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

$M(G)$ is isomorphic to $M(A)$ $(a \to y_a, b \to y_b, c \to y_c, d \to y_d)$.

Graphic Matroid

G



$$A = \begin{array}{c} \begin{array}{cccc} y_a & y_b & y_c & y_d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

$M(G)$ is isomorphic to $M(A)$ $(a \to y_a, b \to y_b, c \to y_c, d \to y_d)$.

The cycle formed by the edges $a = \{1,2\}, b = \{1,3\}$ et $c = \{2,3\}$ in the graph correspond to the linear dependency $y_b - y_a = y_c$.

Bases

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid.

Bases

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid.

Lemma The bases of a matroid have the same cardinality.

Bases

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid.

Lemma The bases of a matroid have the same cardinality.

Proof : exercices.

Bases

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid.

Lemma The bases of a matroid have the same cardinality.

Proof : exercices.

The family $\mathcal{B}$ verifies the following conditions :

$(B1)$ $\mathcal{B} \neq \emptyset$,

$(B2)$ *(exchange propety)* $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

Bases

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid.

Lemma The bases of a matroid have the same cardinality.

Proof : exercices.

The family $\mathcal{B}$ verifies the following conditions :

($B1$) $\mathcal{B} \neq \emptyset$,

($B2$) *(exchange propety)* $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

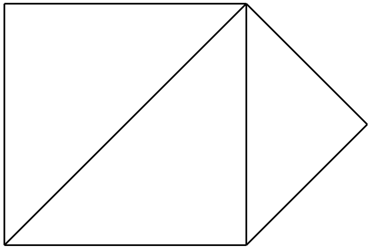
If $\mathcal{I}$ is the family of subsets contained in a set of $\mathcal{B}$ then $(E, \mathcal{I})$ is a matroid.

Bases

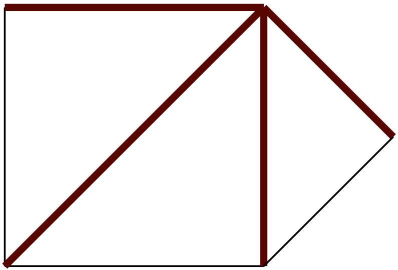
Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

Bases

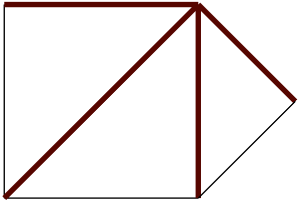
Theorem $\mathcal{B}$ is the set of basis of a matroid if and only if it verifies $(B1)$ and $(B2)$.

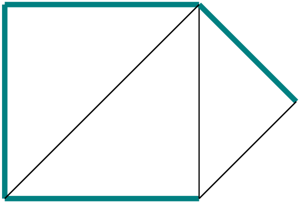


Bases

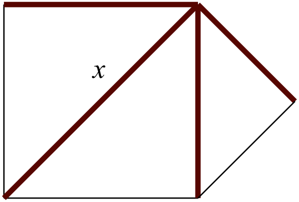


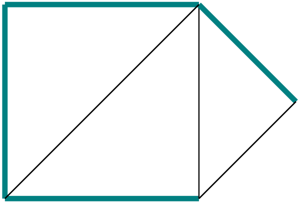
Bases



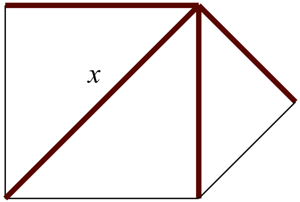


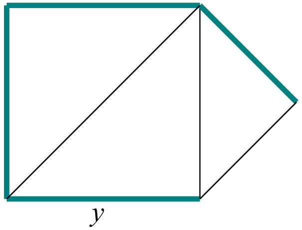
Bases



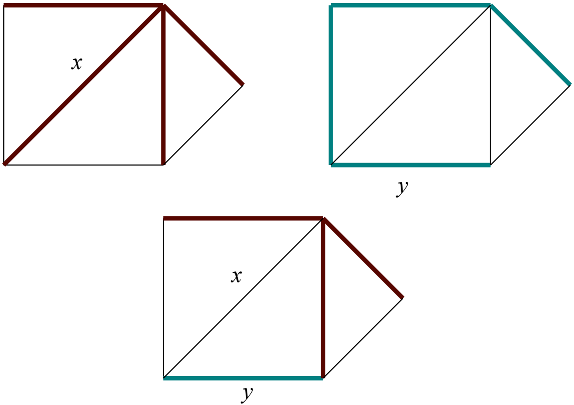


Bases





Bases



Rank

The rank of a set $X \subseteq E$ is defined by

$$r_M(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

Rank

The rank of a set $X \subseteq E$ is defined by

$$r_M(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$

$r = r_M$ is the rank function of a matroid $(E, \mathcal{I})$ (where $\mathcal{I} = \{I \subseteq E : r(I) = |I|\}$) if and only if r verifies the following conditions :

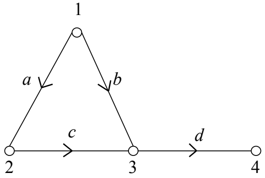
$(R1)$ $0 \leq r(X) \leq |X|$, for all $X \subseteq E$,

$(R2)$ $r(X) \leq r(Y)$, for all $X \subseteq Y$,

$(R3)$ *(sub-modulairity)* $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ for all $X, Y \subset E$.

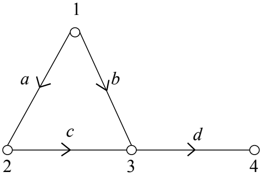
Rank

Let M be a graphic matroid obtained from G



Rank

Let M be a graphic matroid obtained from G



It can be verified that :
$r_M(\{a,b,c\}) = r_M(\{c,d\}) = r_M(\{a,d\}) = 2$ et
$r(M(G)) = r_M(\{a,b,c,d\}) = 3$.

Duality

Let M be a matroid on the ground set E and let $\mathcal{B}$ the set of bases of M. Then,

$$\mathcal{B}^* = \{E \backslash B \mid B \in \mathcal{B}\}$$

is the set of bases of a matroid on E.

Duality

Let M be a matroid on the ground set E and let $\mathcal{B}$ the set of bases of M. Then,

$$\mathcal{B}^* = \{E \backslash B \mid B \in \mathcal{B}\}$$

is the set of bases of a matroid on E.

The matroid on E having $\mathcal{B}^*$ as set of bases, denoted by M^*, is called the dual of M.

A base of M^* is also called cobase of M.

Duality

We have that

- $r(M^*) = |E| - r_M$ and $M^{**} = M$.

Duality

We have that

- $r(M^*) = |E| - r_M$ and $M^{**} = M$.
- The set $\mathcal{I}^*$ of independents of M^* is given by

$$\mathcal{I}^* = \{X \mid X \subset E \text{ such that there exists } B \in \mathcal{B}(M) \text{ with } X \cap B = \emptyset\}.$$

Duality

We have that

- $r(M^*) = |E| - r_M$ and $M^{**} = M$.
- The set $\mathcal{I}^*$ of independents of M^* is given by

$\mathcal{I}^* = \{X \mid X \subset E$ such that there exists $B \in \mathcal{B}(M)$ with $X \cap B = \emptyset\}$.

- The rank function of M^* is given by

$$r_{M^*}(X) = |X| + r_M(E \backslash X) - r_M,$$

for $X \subset E$.

Cocycle Matroid

Let $G = (V, E)$ be a graph. A cocycle (or cut) of G is the set of edges joining the two parts of a partition of the set of vertices of the graph.

Cocycle Matroid

Let $G = (V, E)$ be a graph. A cocycle (or cut) of G is the set of edges joining the two parts of a partition of the set of vertices of the graph.

Theorem Let $\mathcal{C}(G)^*$ be the set of minimal (by inclusion) cocycles of a graph G. Then, $\mathcal{C}(G)^*$ is the set of circuits of a matroid on E.

Cocycle Matroid

Let $G = (V, E)$ be a graph. A cocycle (or cut) of G is the set of edges joining the two parts of a partition of the set of vertices of the graph.

Theorem Let $\mathcal{C}(G)^*$ be the set of minimal (by inclusion) cocycles of a graph G. Then, $\mathcal{C}(G)^*$ is the set of circuits of a matroid on E.

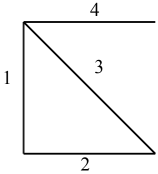
The matroid obtained on this way is called the matroid of cocycle of G or bond matroid, denoted by $B(G)$.

Bond Matroid

Theorem $M^*(G) = B(G)$ and $M(G) = B^*(G)$.

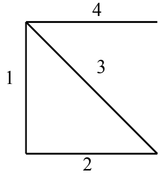
Bond Matroid

Theorem $M^*(G) = B(G)$ and $M(G) = B^*(G)$.



Bond Matroid

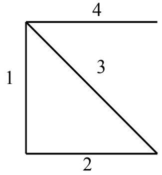
Theorem $M^*(G) = B(G)$ and $M(G) = B^*(G)$.



$\mathcal{B}(M(G)) = \{\{4,1,3\},\{4,1,2\},\{4,2,3\}\}$

Bond Matroid

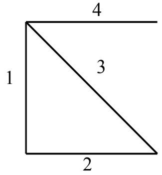
Theorem $M^*(G) = B(G)$ and $M(G) = B^*(G)$.



$\mathcal{B}(M(G)) = \{\{4,1,3\},\{4,1,2\},\{4,2,3\}\}$
$\mathcal{B}(M^*(G)) = \{\{2\},\{3\},\{1\}\}$

Bond Matroid

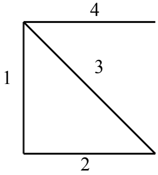
Theorem $M^*(G) = B(G)$ and $M(G) = B^*(G)$.



$\mathcal{B}(M(G)) = \{\{4, 1, 3\}, \{4, 1, 2\}, \{4, 2, 3\}\}$
$\mathcal{B}(M^*(G)) = \{\{2\}, \{3\}, \{1\}\}$
$\mathcal{I}(M^*(G)) = \{\emptyset, \{1\}, \{2\}, \{3\}\}$

Bond Matroid

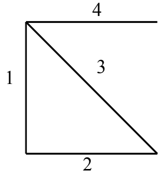
Theorem $M^*(G) = B(G)$ and $M(G) = B^*(G)$.



$\mathcal{B}(M(G)) = \{\{4,1,3\},\{4,1,2\},\{4,2,3\}\}$
$\mathcal{B}(M^*(G)) = \{\{2\},\{3\},\{1\}\}$
$\mathcal{I}(M^*(G)) = \{\emptyset,\{1\},\{2\},\{3\}\}$
The dependents of $M^*(G)$ are $\mathcal{P}(\{1,2,3,4\}) \setminus \{\emptyset,\{1\},\{2\},\{3\}\}$

Bond Matroid

Theorem $M^*(G) = B(G)$ and $M(G) = B^*(G)$.



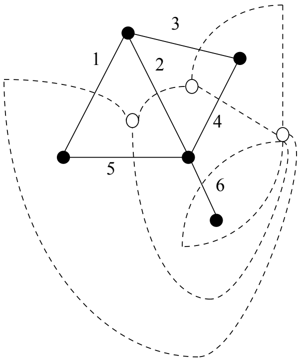
$\mathcal{B}(M(G)) = \{\{4,1,3\},\{4,1,2\},\{4,2,3\}\}$
$\mathcal{B}(M^*(G)) = \{\{2\},\{3\},\{1\}\}$
$\mathcal{I}(M^*(G)) = \{\emptyset,\{1\},\{2\},\{3\}\}$
The dependents of $M^*(G)$ are $\mathcal{P}(\{1,2,3,4\}) \setminus \{\emptyset,\{1\},\{2\},\{3\}\}$
$\mathcal{C}(M^*(G)) = \{\{4\},\{1,2\},\{1,3\},\{2,3\}\}$ that are precisely the cocycles of G.

Planarity

Theorem If G is planar then $M^*(G) = M(G^*)$.

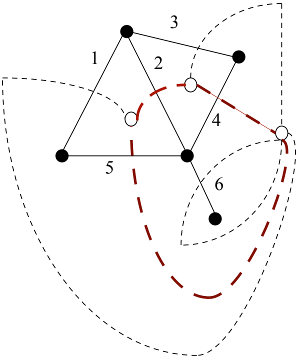
Planarity

Theorem If G is planar then $M^*(G) = M(G^*)$.



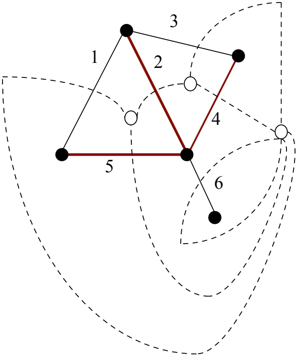
Planarity

Theorem If G is planar then $M^*(G) = M(G^*)$.



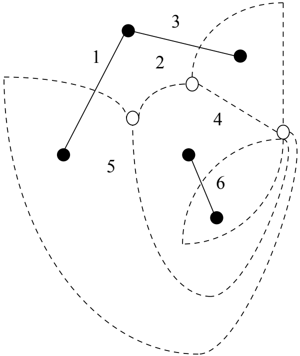
Planarity

Theorem If G is planar then $M^*(G) = M(G^*)$.



Planarity

Theorem If G is planar then $M^*(G) = M(G^*)$.



Remark The dual of a graphic matroid is not necessarly graphic.

Duality - representable matroid

Theorem The dual of a $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable.

Duality - representable matroid

Theorem The dual of a $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable.

Proof. The matrix representing M can always be written as

$$(I_r \mid A)$$

where I_r is the identity $r \times r$ and A is a matrix of size $r \times (n - r)$.

Duality - representable matroid

Theorem The dual of a $\mathbb{F}$-representable matroid is $\mathbb{F}$-representable.

Proof. The matrix representing M can always be written as

$$(I_r \mid A)$$

where I_r is the identity $r \times r$ and A is a matrix of size $r \times (n-r)$.

(Exercise) M^* can be obtained from the set of columns of the matrix

$$(-{}^tA \mid I_{n-r})$$

where I_{n-r} is the identity $(n-r) \times (n-r)$ and ${}^tA$ is the transpose of A.

Duality - representable matroid

The matroid M^* is also called the orthogonal matroid of M since the duality for representable matroids is a generalization of the notion of orthogonality in vector spaces.

Duality - representable matroid

The matroid M^* is also called the orthogonal matroid of M since the duality for representable matroids is a generalization of the notion of orthogonality in vector spaces.

Let V be a subspace of $\mathbb{F}^n$ where $n = |E|$. We recall that the orthogonal space $V^\perp$ is defined from the canonical scalar product $\langle u, v\rangle = \sum_{e\in E} u(e)v(e)$ by

$$V^\perp = \{v \in \mathbb{F}^n \mid \langle u, v\rangle = 0 \text{ for any } u \in V\}.$$

Duality - representable matroid

The matroid M^* is also called the orthogonal matroid of M since the duality for representable matroids is a generalization of the notion of orthogonality in vector spaces.

Let V be a subspace of $\mathbb{F}^n$ where $n = |E|$. We recall that the orthogonal space $V^\perp$ is defined from the canonical scalar product $\langle u, v \rangle = \sum_{e \in E} u(e)v(e)$ by

$$V^\perp = \{v \in \mathbb{F}^n \mid \langle u, v \rangle = 0 \text{ for any } u \in V\}.$$

The orthogonal space of the space generated by the columns of $(I \mid A)$ is given by the space generated by the columns of $(-{}^tA \mid I_{n-r})$.

Operation : deletion

Let M be a matroid on the set E and let $A \subset E$. Then,

$$\{X \subset E \setminus A \mid X \text{ is independent in } M\}$$

is a set of independent of a matroid on $E \setminus A$.

Operation : deletion

Let M be a matroid on the set E and let $A \subset E$. Then,

$$\{X \subset E \backslash A \mid X \text{ is independent in } M\}$$

is a set of independent of a matroid on $E \backslash A$.

This matroid is obtained from M by deleting the elements of A and it is denoted by $M \backslash A$.

Operation : contraction

Let M be a matroid on the set E and let $A \subset E$.
Let $M|_A = \{X \subseteq A | X \in \mathcal{I}(M)\}$ and $X \subseteq E \setminus A$. Then,

$\{X \subseteq E \setminus A|$ there exists a base B of $M|_A$ such that $X \cup B \in \mathcal{I}(M)\}$

is the set of independents of a matroid in $E \setminus A$.

Operation : contraction

Let M be a matroid on the set E and let $A \subset E$.
Let $M|_A = \{X \subseteq A | X \in \mathcal{I}(M)\}$ and $X \subseteq E \setminus A$. Then,

$$\{X \subseteq E \backslash A | \text{ there exists a base } B \text{ of } M|_A \text{ such that } X \cup B \in \mathcal{I}(M)\}$$

is the set of independents of a matroid in $E \setminus A$.

This matroid is obtained from M by contracting the elements of A and it is denoted by M/A.

Operations : deletion and contraction

Properties

(i) $(M\backslash A)\backslash A' = M\backslash(A \cup A')$
(ii) $(M/A)/A' = M/(A \cup A')$
(iii) $(M\backslash A)/A' = (M/A')\backslash A$

Operations : deletion and contraction

Properties
(i) $(M\backslash A)\backslash A' = M\backslash(A \cup A')$
(ii) $(M/A)/A' = M/(A \cup A')$
(iii) $(M\backslash A)/A' = (M/A')\backslash A$

The operations deletion and contraction are duals, that is,

$$(M\backslash A)^* = (M^*)/A \text{ and } (M/A)^* = (M^*)\backslash A$$

Operations : deletion and contraction

Properties

(i) $(M\backslash A)\backslash A' = M\backslash(A \cup A')$

(ii) $(M/A)/A' = M/(A \cup A')$

(iii) $(M\backslash A)/A' = (M/A')\backslash A$

The operations deletion and contraction are duals, that is,

$$(M\backslash A)^* = (M^*)/A \text{ and } (M/A)^* = (M^*)\backslash A$$

and thus $M/A = (M^* \setminus A)^*$

Operations : deletion and contraction

Properties
(i) $(M\backslash A)\backslash A' = M\backslash(A \cup A')$
(ii) $(M/A)/A' = M/(A \cup A')$
(iii) $(M\backslash A)/A' = (M/A')\backslash A$

The operations deletion and contraction are duals, that is,

$$(M\backslash A)^* = (M^*)/A \text{ and } (M/A)^* = (M^*)\backslash A$$

and thus $M/A = (M^* \setminus A)^*$
A minor of a matroid of M is any matroid obtained by a sequence of deletions and contractions.

Operations : deletion and contraction

Properties

(i) $(M\backslash A)\backslash A' = M\backslash(A \cup A')$

(ii) $(M/A)/A' = M/(A \cup A')$

(iii) $(M\backslash A)/A' = (M/A')\backslash A$

The operations deletion and contraction are duals, that is,

$$(M\backslash A)^* = (M^*)/A \text{ and } (M/A)^* = (M^*)\backslash A$$

and thus $M/A = (M^* \setminus A)^*$

A minor of a matroid of M is any matroid obtained by a sequence of deletions and contractions.

Question : Is it true that any family of matroids is closed under deletions/contractions operations ?

Minors - uniform matroids

The uniform matroid (denoted by $U_{n,r}$) is the matroid on E with $|E| = n$ elements where

$$\mathcal{B}(U_{n,r}) = \{X \subset E : |X| = r\}$$

Minors - uniform matroids

The uniform matroid (denoted by $U_{n,r}$) is the matroid on E with $|E| = n$ elements where

$$\mathcal{B}(U_{n,r}) = \{X \subset E : |X| = r\}$$

Proposition Any minor of a uniform matroid is uniform.

Minors - uniform matroids

The uniform matroid (denoted by $U_{n,r}$) is the matroid on E with $|E| = n$ elements where

$$\mathcal{B}(U_{n,r}) = \{X \subset E : |X| = r\}$$

Proposition Any minor of a uniform matroid is uniform.

Proof <u>Deletion</u> : let $T \subseteq E$ with $|T| = t$. Then,

$$U_{n,r} \backslash T = \begin{cases} U_{n-t,n-t} & \text{if } \ n \geq t \geq n-r \\ U_{n-t,r} & \text{if } \ t < n-r. \end{cases}$$

Minors - uniform matroids

The uniform matroid (denoted by $U_{n,r}$) is the matroid on E with $|E| = n$ elements where

$$\mathcal{B}(U_{n,r}) = \{X \subset E : |X| = r\}$$

Proposition Any minor of a uniform matroid is uniform.

Proof Deletion : let $T \subseteq E$ with $|T| = t$. Then,

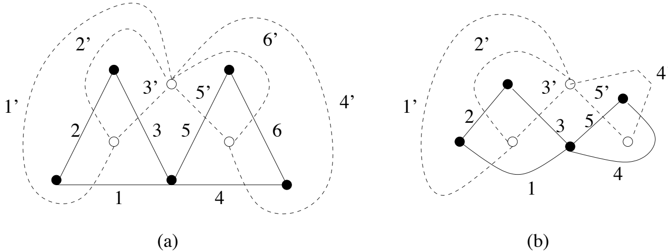
$$U_{n,r} \backslash T = \begin{cases} U_{n-t,n-t} & \text{if } \; n \geq t \geq n-r \\ U_{n-t,r} & \text{if } \; t < n-r. \end{cases}$$

Contraction : it follows by using duality.

Minors - graphic matroids

Proposition The class of graphic matroids is closed under deletions and contractions.

Proposition The class of graphic matroids is closed under deletions and contractions.



Contracting element 6

Minors - representable matroids

Proposition The class of representable matroids over a field $\mathbb{F}$ is closed under deletions and contractions.

Minors - representable matroids

Proposition The class of representable matroids over a field $\mathbb{F}$ is closed under deletions and contractions.

Let M be a matroid obtained from the vectors $(v_e)_{e\in E}$ of $\mathbb{F}^d$.

Deleting : $M \setminus a$ is the matroid obtained from the vectors $(v_e)_{e\in E\setminus a}$

Minors - representable matroids

Proposition The class of representable matroids over a field $\mathbb{F}$ is closed under deletions and contractions.

Let M be a matroid obtained from the vectors $(v_e)_{e\in E}$ of $\mathbb{F}^d$.

Deleting : $M \setminus a$ is the matroid obtained from the vectors $(v_e)_{e\in E\setminus a}$

Remark : Lines sums and scalar multiplications do not change the associated matroid. So, if $v_a \neq \overline{0}$ then we suppose that v_a is the unit vector.

Minors - representable matroids

Proposition The class of representable matroids over a field $\mathbb{F}$ is closed under deletions and contractions.

Let M be a matroid obtained from the vectors $(v_e)_{e\in E}$ of $\mathbb{F}^d$.

Deleting : $M \setminus a$ is the matroid obtained from the vectors $(v_e)_{e\in E\setminus a}$

Remark : Lines sums and scalar multiplications do not change the associated matroid. So, if $v_a \neq \overline{0}$ then we suppose that v_a is the unit vector.

Contracting : M/a is the matroid obtained from the vectors $(v'_e)_{e\in E\setminus a}$ where v'_e is the vector obtained from v_e by deleting the non zero entry of v_a.

Minors - representable matroids

Proposition The class of representable matroids over a field $\mathbb{F}$ is closed under deletions and contractions.

Let M be a matroid obtained from the vectors $(v_e)_{e\in E}$ of $\mathbb{F}^d$.

Deleting : $M \setminus a$ is the matroid obtained from the vectors $(v_e)_{e\in E\setminus a}$

Remark : Lines sums and scalar multiplications do not change the associated matroid. So, if $v_a \neq \overline{0}$ then we suppose that v_a is the unit vector.

Contracting : M/a is the matroid obtained from the vectors $(v'_e)_{e\in E\setminus a}$ where v'_e is the vector obtained from v_e by deleting the non zero entry of v_a.

- If we change the nonzero component we obtain another representation of M/a.

Minors - representable matroids

Proposition The class of representable matroids over a field $\mathbb{F}$ is closed under deletions and contractions.

Let M be a matroid obtained from the vectors $(v_e)_{e\in E}$ of $\mathbb{F}^d$.

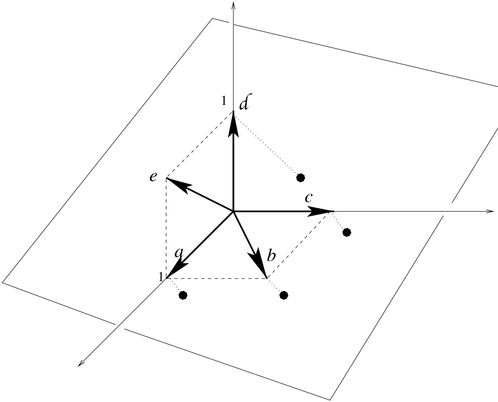
Deleting : $M\setminus a$ is the matroid obtained from the vectors $(v_e)_{e\in E\setminus a}$

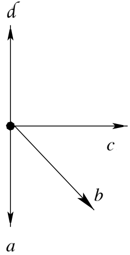
Remark : Lines sums and scalar multiplications do not change the associated matroid. So, if $v_a\neq\overline{0}$ then we suppose that v_a is the unit vector.

Contracting : M/a is the matroid obtained from the vectors $(v'_e)_{e\in E\setminus a}$ where v'_e is the vector obtained from v_e by deleting the non zero entry of v_a.

- If we change the nonzero component we obtain another representation of M/a.
- If $v_a=\overline{0}$ then a is a loop of M and thus $M/a=M\setminus a$.

Minors - representable matroids





Excluded Minors

For any field $\mathbb{F}$, there exists a list of excluded minors, that is, nonrepresentable matroids over $\mathbb{F}$ but any of its proper minors is representable over $\mathbb{F}$.

Excluded Minors

For any field $\mathbb{F}$, there exists a list of excluded minors, that is, nonrepresentable matroids over $\mathbb{F}$ but any of its proper minors is representable over $\mathbb{F}$.

Determining the list of excluded minors over $\mathbb{F}$ gives a characterization of the matroids representables over $\mathbb{F}$.

Excluded Minors

For any field $\mathbb{F}$, there exists a list of excluded minors, that is, nonrepresentable matroids over $\mathbb{F}$ but any of its proper minors is representable over $\mathbb{F}$.

Determining the list of excluded minors over $\mathbb{F}$ gives a characterization of the matroids representables over $\mathbb{F}$.

For $\mathbb{F} = GF(2) = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (binary matroids) : the list has only one matroid $U_{2,4}$ (3 pages proof)

$$\mathcal{B}(U_{2,4}) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

Excluded Minors

For $\mathbb{F} = GF(3) = \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ (ternary matroids) : the list has 4 matroids F_7 F_7^*, $U_{2,5}$ $U_{3,5}$ (10 pages proof)

Excluded Minors

For $\mathbb{F} = GF(3) = \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ (ternary matroids) : the list has 4 matroids F_7 F_7^*, $U_{2,5}$ $U_{3,5}$ (10 pages proof)

For $\mathbb{F} = GF(4)$: the list has 8 matroids explicitly given (50 pages proof)

Excluded Minors

For $\mathbb{F} = GF(3) = \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ (ternary matroids) : the list has 4 matroids F_7 F_7^*, $U_{2,5}$ $U_{3,5}$ (10 pages proof)

For $\mathbb{F} = GF(4)$: the list has 8 matroids explicitly given (50 pages proof)

Theorem A matroid is graphic if and only if has neither $U_{2,4}, F_7, F_7^*, M^*(K_5) = B(K_5)$ nor $M^*(K_{3,3}) = B(K_{3,3})$ as minors.

Excluded Minors

For $\mathbb{F} = GF(3) = \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ (ternary matroids) : the list has 4 matroids F_7 F_7^*, $U_{2,5}$ $U_{3,5}$ (10 pages proof)

For $\mathbb{F} = GF(4)$: the list has 8 matroids explicitly given (50 pages proof)

Theorem A matroid is graphic if and only if has neither $U_{2,4}, F_7, F_7^*, M^*(K_5) = B(K_5)$ nor $M^*(K_{3,3}) = B(K_{3,3})$ as minors.

Theorem A matroid is cographic if and only if has neither $U_{2,4}, F_7, F_7^*, M(K_5)$ nor $M(K_{3,3})$ as minors.

Excluded Minors

For $\mathbb{F} = GF(3) = \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ (ternary matroids) : the list has 4 matroids F_7 F_7^*, $U_{2,5}$ $U_{3,5}$ (10 pages proof)

For $\mathbb{F} = GF(4)$: the list has 8 matroids explicitly given (50 pages proof)

Theorem A matroid is graphic if and only if has neither $U_{2,4}, F_7, F_7^*, M^*(K_5) = B(K_5)$ nor $M^*(K_{3,3}) = B(K_{3,3})$ as minors.

Theorem A matroid is cographic if and only if has neither $U_{2,4}, F_7, F_7^*, M(K_5)$ nor $M(K_{3,3})$ as minors.

Theorem A matroid is regular if and only if has neither $U_{2,4}$, F_7 nor F_7^* as minors.

Tutte Polynomial

The Tutte polynomial of a matroid M is the generating function defined as follows

$$t(M;x,y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)} (y-1)^{|X|-r(X)}.$$

Tutte Polynomial

The Tutte polynomial of a matroid M is the generating function defined as follows

$$t(M;x,y) = \sum_{X\subseteq E}(x-1)^{r(E)-r(X)}(y-1)^{|X|-r(X)}.$$

Let $U_{2,3}$ be the matroid of rank 2 on 3 elements with $\mathcal{B}(U_{2,3}) = \{\{1,2\},\{1,3\},\{2,3\}\}$

Tutte Polynomial

The Tutte polynomial of a matroid M is the generating function defined as follows

$$t(M;x,y)=\sum_{X\subseteq E}(x-1)^{r(E)-r(X)}(y-1)^{|X|-r(X)}.$$

Let $U_{2,3}$ be the matroid of rank 2 on 3 elements with $\mathcal{B}(U_{2,3})=\{\{1,2\},\{1,3\},\{2,3\}\}$

$$\begin{aligned}
t(U_{2,3};x,y) &= \sum_{X\subseteq E,\ |X|=0}(x-1)^{2-0}(y-1)^{0-0}+\sum_{X\subseteq E,\ |X|=1}(x-1)^{2-1}(y-1)^{1-1}\\
&+\sum_{X\subseteq E,\ |X|=2}(x-1)^{2-2}(y-1)^{2-2}+\sum_{X\subseteq E,\ |X|=3}(x-1)^{2-2}(y-1)^{3-2}\\
&=(x-1)^2+3(x-1)+3(1)+y-1\\
&=x^2-2x+1+3x-3+3+y-1=x^2+x+y.
\end{aligned}$$

Tutte Polynomial

A loop of a matroid M is a circuit of cardinality one.
An isthmus of M is an element that is contained in all the bases.

Tutte Polynomial

A loop of a matroid M is a circuit of cardinality one.
An isthmus of M is an element that is contained in all the bases.

The Tutte polynomial can be expressed recursively as follows

$$t(M;x,y) = \begin{cases} t(M \setminus e;x,y) + t(M/e;x,y) & \text{if } e \neq \text{isthmus, loop,} \\ x \cdot t(M \setminus e;x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot t(M/e;x,y) & \text{if } e \text{ is a loop.} \end{cases}$$

Acyclic Orientations

Let $G = (V, E)$ be a connected graph. An orientation of G is an orientation of the edges of G.

We say that the orientation is acyclic if the oriented graph do not contain an oriented cycle (i.e., a cycle where all its edges are oriented clockwise or anti-clockwise).

Acyclic Orientations

Let $G = (V, E)$ be a connected graph. An orientation of G is an orientation of the edges of G.

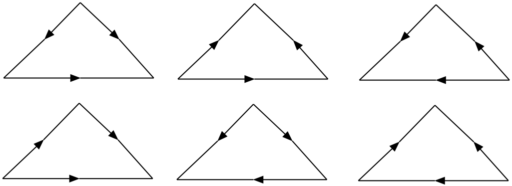
We say that the orientation is acyclic if the oriented graph do not contain an oriented cycle (i.e., a cycle where all its edges are oriented clockwise or anti-clockwise).

Theorem The number of acyclic orientations of G is equals to

$$t(M(G); 2, 0).$$

Acyclic Orientations

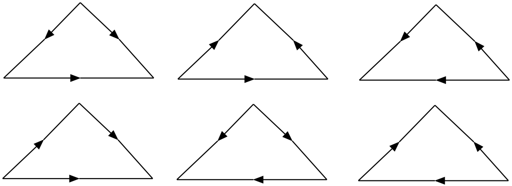
Example : There are 6 acyclic orientations of C_3



Notice that $M(C_3)$ is isomorphic to $U_{2,3}$.

Acyclic Orientations

Example : There are 6 acyclic orientations of C_3



Notice that $M(C_3)$ is isomorphic to $U_{2,3}$.

Since $t(U_{2,3}; x, y) = x^2 + x + y$ then the number of acyclic orientations of C_3 is $t(U_{2,3}; 2, 0) = 2^2 + 2 + 0 = 6$.

Chromatic Polynomial

Let $G = (V, E)$ be a graph and let λ be a positive integer.

Chromatic Polynomial

Let $G = (V, E)$ be a graph and let λ be a positive integer.
A λ-coloring of G is a map $\phi : V \longrightarrow \{1, \ldots, \lambda\}$.

Chromatic Polynomial

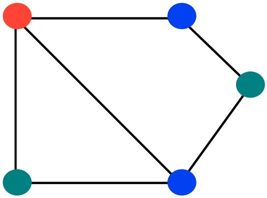
Let $G = (V, E)$ be a graph and let λ be a positive integer. A λ-coloring of G is a map $\phi : V \longrightarrow \{1, \ldots, \lambda\}$.

The coloring is called good if for any edge $\{u, v\} \in E(G)$, $\phi(u) \neq \phi(v)$.

Chromatic Polynomial

Let $G = (V, E)$ be a graph and let λ be a positive integer.
A λ-coloring of G is a map $\phi : V \longrightarrow \{1, \ldots, \lambda\}$.

The coloring is called good if for any edge $\{u, v\} \in E(G)$, $\phi(u) \neq \phi(v)$.



Chromatic Polynomial

Let $\chi(G, \lambda)$ be the number of good λ-colorings of G.

Chromatic Polynomial

Let $\chi(G,\lambda)$ be the number of good λ-colorings of G.

Theorem $\chi(G,\lambda)$ is a polynomial on λ. Moreover

$$\chi(G,\lambda)=\sum_{X\subseteq E}(-1)^{|X|}\lambda^{\omega(G[X])}$$

where $\omega(G[X])$ denote the number of connected components of the subgraph generated by X.

Proof (idea) By using the inclusion-exclusion formula.

Chromatic Polynomial

The chromatic polynomial has been introduced by Birkhoff as a tool to attack the 4-color problem.

Indeed, if for a planar graph G we have $\chi(G,4) > 0$ then G admits a good 4-coloring.

Chromatic Polynomial

The chromatic polynomial has been introduced by Birkhoff as a tool to attack the 4-color problem.

Indeed, if for a planar graph G we have $\chi(G,4) > 0$ then G admits a good 4-coloring.

Theorem If G is a graph with $\omega(G)$ connected components. Then,

$$\chi(G,\lambda) = \lambda^{\omega(G)}(-1)^{|V(G)|-\omega(G)}t(M(G);1-\lambda,0).$$

Chromatic Polynomial

The chromatic polynomial has been introduced by Birkhoff as a tool to attack the 4-color problem.

Indeed, if for a planar graph G we have $\chi(G,4) > 0$ then G admits a good 4-coloring.

Theorem If G is a graph with $\omega(G)$ connected components. Then,

$$\chi(G,\lambda) = \lambda^{\omega(G)}(-1)^{|V(G)|-\omega(G)}t(M(G);1-\lambda,0).$$

Exemple : $\chi(K_3,3) = 3^1(-1)^{3-1}t(K_3;1-3,0)$
$= 3\cdot 1\cdot t(U_{2,3};-2,0) = 3((-2)^2-2+0) = 6.$

Ehrhart Polynomial

The theory of Ehrhart focuses in counting the number of points with integer coordinates lying in a polytope.

Ehrhart Polynomial

The theory of Ehrhart focuses in counting the number of points with integer coordinates lying in a polytope.
A polytope is called integer if all its vertices have integer coordinates.
Ehrhart studied the function i_P that counts the number of integer points in the polytope P *dilated* by a factor of t

$$\begin{aligned} i_P : \quad & \mathbb{N} \longrightarrow \mathbb{N}^* \\ & t \mapsto \left| tP \cap \mathbb{Z}^d \right| \end{aligned}$$

Ehrhart Polynomial

Theorem (Ehrhart) i_P is a polynomial on t of degree d,

$$i_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

Ehrhart Polynomial

Theorem (Ehrhart) i_P is a polynomial on t of degree d,

$$i_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

- c_d is equals to $Vol(P)$ (the volume of P),

Ehrhart Polynomial

Theorem (Ehrhart) i_P is a polynomial on t of degree d,

$$i_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

- c_d is equals to $Vol(P)$ (the volume of P),
- c_{d-1} is equals to $Vol(\partial(P)/2)$ where $\partial(P)$ is the surface of P,

Ehrhart Polynomial

Theorem (Ehrhart) i_P is a polynomial on t of degree d,

$$i_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

- c_d is equals to $Vol(P)$ (the volume of P),
- c_{d-1} is equals to $Vol(\partial(P)/2)$ where $\partial(P)$ is the surface of P,
- $c_0 = 1$ is the Euler's characteristic of P.

Ehrhart Polynomial

Theorem (Ehrhart) i_P is a polynomial on t of degree d,

$$i_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

- c_d is equals to $Vol(P)$ (the volume of P),
- c_{d-1} is equals to $Vol(\partial(P)/2)$ where $\partial(P)$ is the surface of P,
- $c_0 = 1$ is the Euler's characteristic of P.

All others coefficients remain a mystery ! !

Ehrhart Polynomial

The Minkowski's sum of two sets A and B of $\mathbb{R}^d$ is

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Ehrhart Polynomial

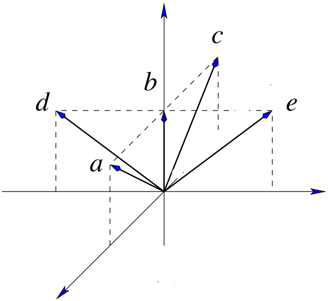
The Minkowski's sum of two sets A and B of $\mathbb{R}^d$ is

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Let $A = \{v_1, \ldots, v_k\}$ be a finite set of elements of $\mathbb{R}^d$.
A zonotope generated by A, denoted by $Z(A)$, is a polytope formed by the Minkowski's sum of line segments

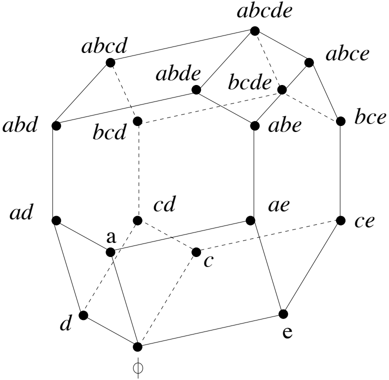
$$Z(A) = \{\alpha_1 + \cdots + \alpha_k | \alpha_i \in [-v_i, v_i]\}.$$

Ehrhart Polynomial



Ehrhart Polynomial

Permutahedron



Ehrhart Polynomial

A matroid is regular if it is representable over any field.

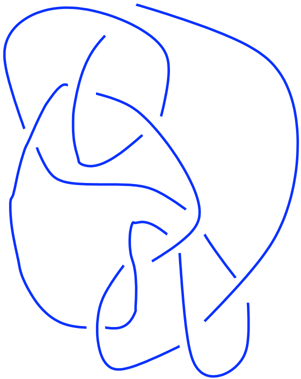
Ehrhart Polynomial

A matroid is regular if it is representable over any field.

Theorem Let M be a regular matroid and let A be one of its representation matrix. Then, the Ehrhart polynomial associated to the zonotope $Z(A)$ is given by

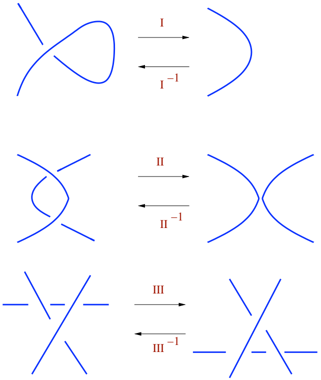
$$i_{Z(A)}(q) = q^{r(M)} t\left(M; 1+\frac{1}{q}, 1\right).$$

Knots

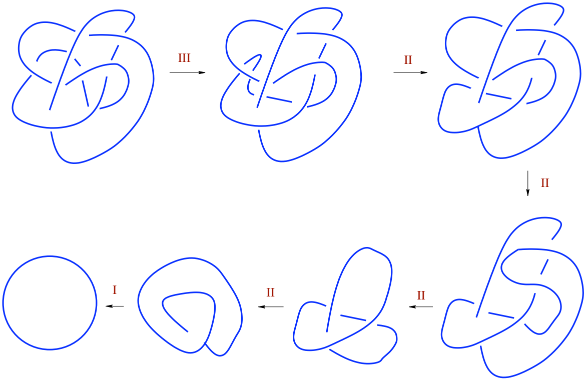


Knots

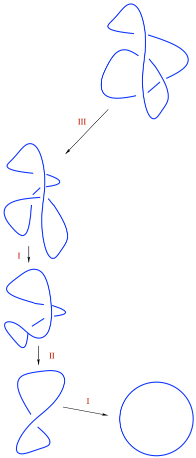
Reidemeister moves



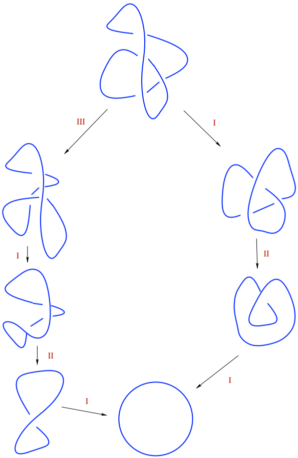
Knots



Knots



Knots



Bracket polynomial

For any link diagram D define a Laurent polynomial $< D >$ in one variable A which obeys the following three rules where U denotes the unknot :

Bracket polynomial

For any link diagram D define a Laurent polynomial $< D >$ in one variable A which obeys the following three rules where U denotes the unknot :

i) $\langle U \rangle = 1$

ii) $\langle U + D \rangle = -(A^2 + A^{-2}) \langle D \rangle$

iii) 

Knots

Theorem For any link L the bracket polynomial is independent of the order in which rules $(i) - (iii)$ are applied to the crossings. Further, it is invariant under the Reidemeister moves **II** and **III**

but it is not invariant under Reidemeister move **I**!!

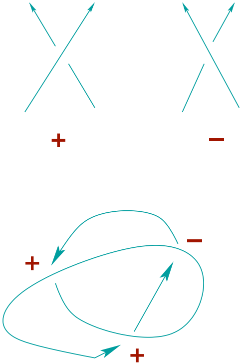
Knots

Theorem For any link L the bracket polynomial is independent of the order in which rules $(i)-(iii)$ are applied to the crossings. Further, it is invariant under the Reidemeister moves II and III

but it is not invariant under Reidemeister move I!!

The writhe of an oriented link diagram D is the sum of the signs at the crossings of D (denoted by $\omega(D)$).

Knots



$\omega(D)=1$

Knots

Theorem For any link L define the Laurent polynomial

$$f_D(A) = (-A^3)^{\omega(D)} < L >$$

Then, $f_D(A)$ is an invariant of ambient isotopy.

Knots

Theorem For any link L define the Laurent polynomial

$$f_D(A) = (-A^3)^{\omega(D)} < L >$$

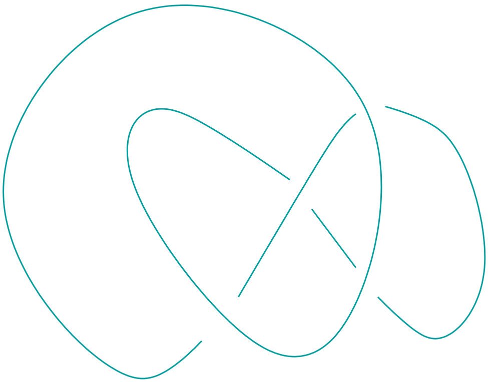
Then, $f_D(A)$ is an invariant of ambient isotopy.

Now, define for any link L

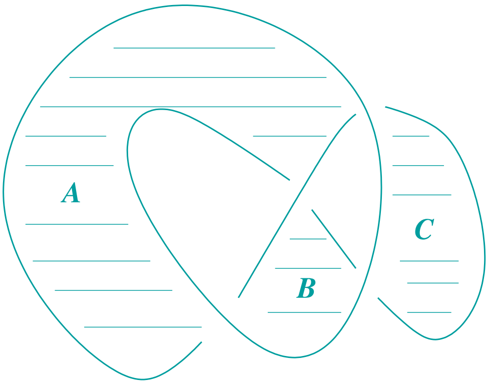
$$V_L(z) = f_D(z^{-1/4})$$

where D is any diagram representing L. Then $V_L(z)$ is the Jones polynomial of the oriented link L.

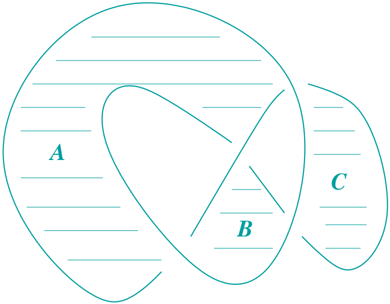
Knots

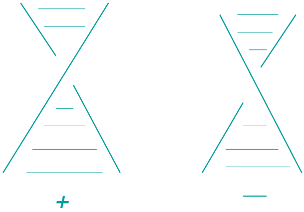


Knots

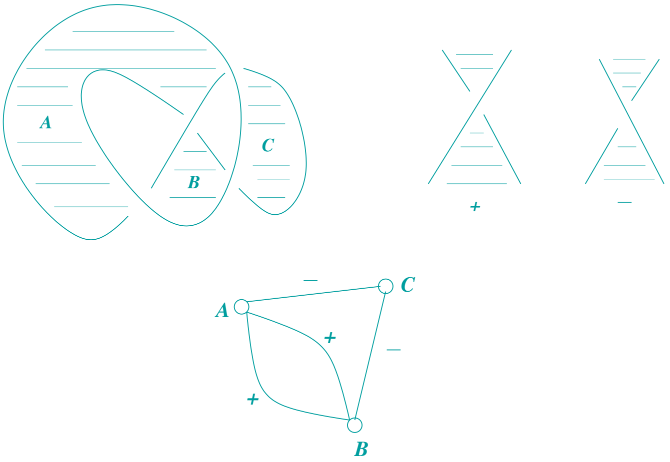


Knots





Knots



Knots

A link diagram is alternating if the crossings alternate under-over-under-over ... as the link is traversed.

Knots

A link diagram is alternating if the crossings alternate under-over-under-over ... as the link is traversed.

A link is alternating if there is an alternating link diagram representing L.

Knots

A link diagram is alternating if the crossings alternate under-over-under-over ... as the link is traversed.

A link is alternating if there is an alternating link diagram representing L.

Theorem (Thistlethwaite 1987) If D is an oriented alternating link diagram then

$$V_L(z) = (z^{-1/4})^{3\omega(D)-2} t(M(G); -z, -z^{-1})$$

where G is the graph associated to the knot diagram.

More applications

- Code theory
- Flow polynomial
- Bicycle space of a graph
- Statistical mechanics
- Arrangements of hyperplanes

$\vdots$