Toric ideals and matroids II

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Complete Intersection

The toric ideal I_M is a complete intersection if and only if there exists a set of homogeneous binomials $g_1, \dots, g_s \in R$ such that $s = \mathrm{ht}(I_M)$ and $I_M = (g_1, \dots, g_s)$.

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Equivalently, I_M is a complete intersection if

$$\mu(I_M) = \mathrm{ht}(I_M) = |\mathcal{B}| - (n - c + 1)$$

where $\mu(I_M)$ denotes the minimal number of generators of I_M and c the number of connected components of M.

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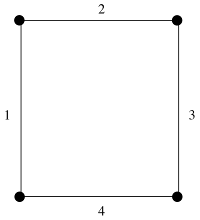

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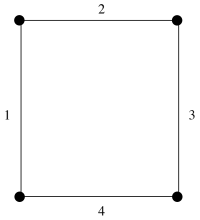

We have $\mathcal{B}(M(G)) = \{123, 124, 134, 234\}$. There is one equivalent classe, and thus $\mathrm{ht}(I_M) = 4 - (4 - 1 + 1) = 0$.

Complete Intersection

Recall that

$$I_M = \left(\{y_{B_1}\cdots y_{B_s} - y_{D_1}\cdots y_{D_s} \mid B_1\cup\cdots\cup B_s = D_1\cup\cdots\cup D_s\}\right) \quad (1)$$

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Thus, we only consider the case $r \leq n-2$.

Complete Intersection : duality and minors

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Proposition Let M' be a minor of M. If I_M is a complete intersection, then $I_{M'}$ also is.

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Example :
$\mathcal{B}(U_{2,4}) = \{B_1 = \{1,2\}, B_2 = \{1,3\}, B_3 = \{1,4\}, B_4 = \{2,3\}, B_5 = \{2,4\}, B_6 = \{3,4\}\}$

$$\begin{array}{cccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 \end{array}$$
$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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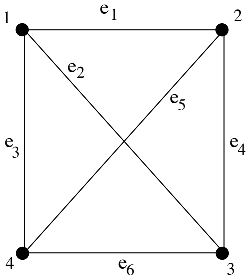

$H_{U_{2,4}}$

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Theorem (I. Bermejo, I. Garcia-Marco, E. Reyes) Whenever $I_{H(M)}$ is a complete intersection, then H_M does not contain $K_{2,3}$ as subgraph.

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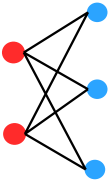

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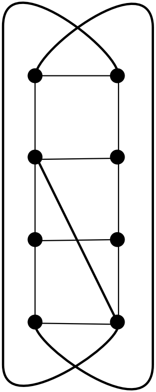

G

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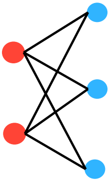

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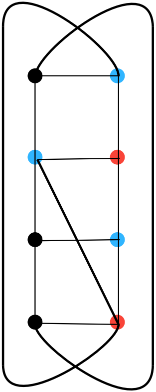

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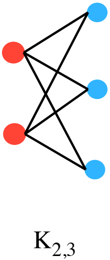

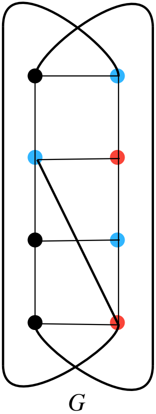

Therefore I_G is not complete intersection.

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($\Leftarrow$) By computer.

Complete Intersection : general case

Theorem Let M be a matroid without loops or coloops and with $n > r + 1$. Then, I_M is a complete intersection if and only if $n = 4$ and M is the matroid whose set of bases is :

1. $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\}\}$,
2. $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\}\}$, or
3. $\mathcal{B} = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\},\{2,3\}\}$, i.e., $M = U_{2,4}$.

Detecting minors

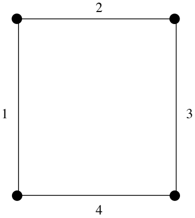
We consider the following binary equivalence relation $\sim$ on the set of pairs of bases :

$$\{B_1, B_2\} \sim \{B_3, B_4\} \iff B_1 \cup B_2 = B_3 \cup B_4 \text{ as multisets,}$$

and we denote by $\Delta_{\{B_1,B_2\}}$ the cardinality of the equivalence class of $\{B_1, B_2\}$.

Detecting minors

We consider the graph



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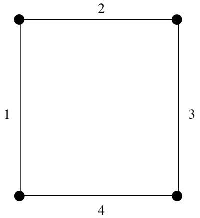

Therefore,
$\mathcal{B}(M(G)) = \{B_1 = \{123\}, B_2 = \{124\}, B_3 = \{134\}, B_4 = \{234\}\}.$

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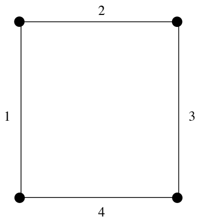

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It can be checked that the equivalent class of $\{B_i, B_j\}$ is $\{B_i, B_j\}$, that is, $\Delta_{\{B_i,B_j\}} = 1$ for any pair $1 \leq i \neq j \leq 4$.

Detecting minors

Lemma (bounds) For every $B_1, B_2 \in \mathcal{B}$, then $2^{d-1} \leq \Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$, where $d := |B_1 \setminus B_2|$.

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We set $A := B_1 \cap B_2$, $C := B_1 \triangle B_2$ and take $e \in B_1 \setminus B_2$. Take $B_3, B_4 \in \mathcal{B}$ such that $B_1 \cup B_2 = B_3 \cup B_4$ as multisets and assume that $e \in B_4$. Then, $B_3 \setminus A \subset C \setminus \{e\}$ with $|B_3| = |B_1 \setminus B_2| = d$ elements ; thus, $\Delta_{\{B_1,B_2\}} \leq \binom{2d-1}{d}$.

Detecting minors

Lemma Let $B_1, B_2 \in \mathcal{B}$ of a matroid M and consider the matroid $M' := (M/(B_1 \cap B_2))|_{(B_1 \triangle B_2)}$ on the ground set $B_1 \triangle B_2$. Then, the number of bases-cobases of M' is equal to $2\Delta_{\{B_1,B_2\}}$.

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Theorem If M has a minor $M' \simeq U_{d,2d}$ for some $d \geq 2$, then there exist $B_1, B_2 \in \mathcal{B}$ such that $\Delta_{\{B_1,B_2\}} = \binom{2d-1}{d}$.

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Proposition Let $\{g_1, \ldots, g_s\}$ be a minimal set of binomial generators of I_M. Then,
$\Delta_{\{B_1,B_2\}} = 1 + |\{g_i = y_{B_{i_1}} y_{B_{i_2}} - y_{B_1} y_{B_2} \mid B_{i_1} \cup B_{i_2} = B_1 \cup B_2$ as a multiset $\}|$ for every $B_1, B_2 \in \mathcal{B}$.

System of generators

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1. $\mu(I_M) \geq (b^2 - b - 2s)/2$, where $b := |\mathcal{B}|$, and
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Moreover, in both cases equality holds whenever I_M is generated by quadratics.

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Question Can we characterize those matroids M with $\nu(I_M) = 1$?

The basis graph of a matroid M is the undirected graph G_M with vertex set $\mathcal{B}$ and edges $\{B, B'\}$ such that $|B \setminus B'| = 1$. The diameter of a graph is the maximum distance between two vertices of the graph.

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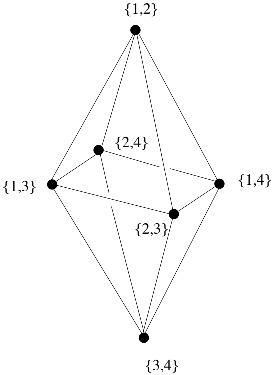
Basis graph $G_{U_{2,4}}$



System of generators

Theorem Let M be a rank $r \geq 2$ matroid. Then, $\nu(I_M) = 1$ if and only if M is binary and the diameter of G_M is at most 2.

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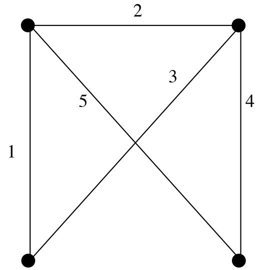
($\Leftarrow$) More complicated.

Example

Matroid $M(G)$ associated to graph G.



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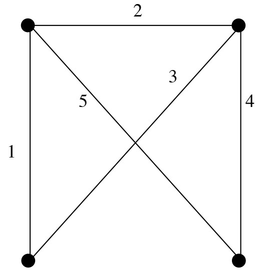
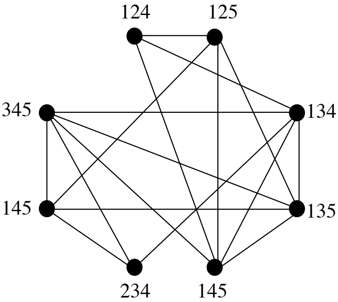


$\mathcal{B}(M(G)) = \{B_1 = \{124\}, B_2 = \{125\}, B_3 = \{134\}, B_4 = \{135\}, B_5 = \{145\}, B_6 = \{234\}, B_7 = \{235\}, B_8 = \{345\}\}$

Example

The base graph $G_{M(G)}$



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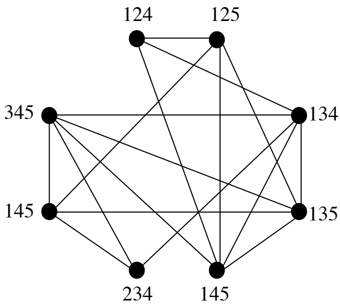

Since diameter of $G_{M(G)}$ is at most two, and $M(G)$ is binary then $\nu(I_M) = 1$.