Oriented Matroids II

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February 2016

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• An oriented matroid M is called acyclic if it does not contain positive circuits.

• Let M be an oriented matroid. We say that an element e of M is interior if there is a circuit $C = (C^+, C^-)$ of M with $C^+ = \{e\}$.

McMullen's problem

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Oriented matroid version Determine the largest integer $n = g(r)$ such that for any uniform rank r oriented matroid M there is an acyclic reorientation of M without interior elements.

Known results

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Strategy We contruct a representable oriented matroid m of rank $r \geq 3$ with $2(r - 1) + \lfloor \frac{r}{2} \rfloor$ elements such that any acyclic reorientation of M has at least one interior element.

Lawrence oriented matroids

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A chirotope χ correspond to a Lawrence oriented matroids M_A iff there exists a matrix $A = (a_{i,j})$ with entries from $\{+1, -1\}$ where the i-th row correspond to the chirotope of M_i such that

$$\chi(B) = \prod_{i=1}^{r} a_{i,j_i}$$

where $B = \{j_1 \leq \cdots \leq j_r\}$ is an ordered base.

Properties

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• The opposite chirotope $-\chi$ is obtained by inversing the sign of all the coefficients of a line of A.

• The oriented matroid $-_c M$ is obtained by inversing the sign of all the coefficients of column c of A.

Chess board

Let $A = (a_{i,j})$, $1 \leq i \leq r, 1 \leq j \leq n$ be a matrix with entries from $\{+1, -1\}$. The chess board $B[A]$ is a chess board of size $(r-1) \times (n-1)$ and a square is white if the product of its corresponding corners is $+1$, black otherwise.

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Observation The chess board is inviarant under reversing the signs of the coefficient of a given column.

Top and Bottom Travels

(1) TT (BT) starts at $a_{1,1}$ (at $a_{r,n}$)

(2) Suppose that TT (BT) arrives at $a_{i,j}$. Let s (s') be the minimum (maximal) integer $j < s \leq n$ ($1 < s' \leq j$) such that $a_{i,j} = -a_{i,s}$ ($a_{i,j} = -a_{i,s'}$).

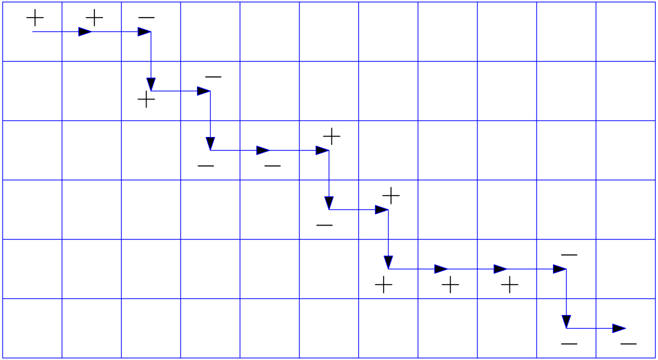
(3) **If** s (s') does not exists **then** TT goes horizontally to $a_{i,n}$ and stops (BT goes horizontally to $a_{i,1}$ and stops)

(4) **else**

   (a) **if** $1 \leq i \leq r-1$ ($2 \leq i \leq r$) **then**
   TT goes horizontally to $a_{i,s}$ and then goes vertically to $a_{i+1,s}$
   (BT goes horizontally to $a_{i,s'}$ and then goes vertically to $a_{i-1,s'}$)

   (a) **else** TT goes horizontally to $a_{r,s}$ and stops
   (BT goes horizontally to $a_{1,s'}$ and stops)

Example of a Top Travel



Three key results

Lemma 1 Let M_A be a Lawrence oriented matroid and A the matrix associated $A = (a_{i,j})$ with $1 \leq i \leq r$, $1 \leq j \leq n$ and entries from $\{+1, -1\}$. Then the following conditions are equivalent.

(a) M_A is cyclic,

(b) TT ends at $a_{r,s}$ for some $1 \leq s < n$,

(c) BT ends at $a_{1,s'}$ for some $1 < s \leq n$.

Three key results

We say that TT and BT are parallel at column k with $2 \leq k \leq n-1$ in A if $TT = (a_{1,1}, \ldots, a_{i,k-1}, a_{i,k}, a_{i,k+1}, \ldots)$ and either $BT = (a_{r,n}, \ldots, a_{i,k+1}, a_{i,k}, a_{i,k-1}, \ldots)$ or $BT = (a_{r,n}, \ldots, a_{i+1,k+1}, a_{i+1,k}, a_{i+1,k-1}, \ldots)$, $1 \leq i \leq r$.

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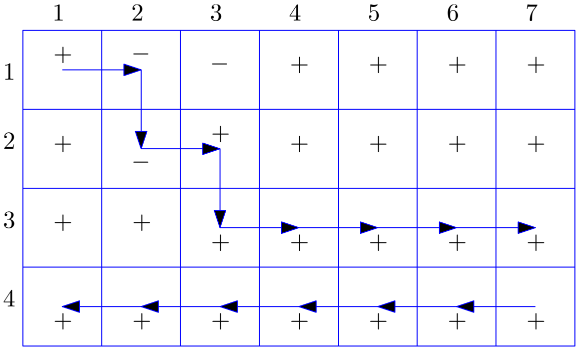
Lemma 2 Let M_A be a Lawrence oriented matroid and A the matrix associated $A = (a_{i,j})$ with $1 \leq i \leq r$, $1 \leq j \leq n$ and entries from $\{+1, -1\}$. Then k is an interior element of M_A if and only if

(a) $BT = (a_{r,n}, \ldots, a_{1,2}, a_{1,1})$ for $k = 1$,

(b) $TT = (a_{1,1}, \ldots, a_{r,n-1}, a_{r,n})$ for $k = n$,

(c) TT and BT are parallel at k for $2 \leq k \leq n-1$.

Let M_A be the Lawrence oriented matroid associated to the matrix A given below



M_A is acyclic and 4, 5 and 6 are interior elements

Three key results

A plain travel T on the entries of A is formed by horizontal and vertical mouvements such that T starts with $a_{1,1}, a_{1,2}$ and T cannot make two consecutive vertical mouvements.

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Lemma 3 Let $A = (a_{i,j})$, $1 \leq i \leq r$, $1 \leq j \leq n$ be a matrix with entries from $\{+1, -1\}$. Then, there exists a natural bijection between the set of all plain travels of A and the set of all acyclic reorientations of M_A.

Construction

It is sufficient to contruct a matrix A of size $r \times 2(r-1) + \lfloor \frac{r}{2} \rfloor$, $r \geq 3$ such that for any given plain travel T of A the corresponding Top Travel in the matrix A' (obtained from A sucht that T is transformed in TT of A') has at least one interior elements.

Good matrix from a chess board