Theory of matroids and applications : I

J.L. Ramírez Alfonsín

I3M, Université Montpellier 2

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Independents

A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set $(E = \{1, \ldots, n\})$ and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \backslash I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

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$$|I_2| \leq dim(W) \leq |I_1| < |I_2| \quad !!!$$

Representable Matroids

Let A be the following matrix with coefficients in $\mathbb{R}$.

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$\{\emptyset, \{1\}, \{2\}, \{4\}, \{4\}, \{5\}, \{1, 2\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{4, 5\}\} \subseteq \mathcal{I}(M)$

A matroid obtained form a matrix A with coefficients in $\mathbb{F}$ is denoted by $M(A)$ and is called representable over $\mathbb{F}$ or $\mathbb{F}$-representable.

Circuits

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$\mathcal{C}$ is the set of circuits of a matrid on E if and only if $\mathcal{C}$ verifies the following properties :

$(C1)$ $\emptyset \notin \mathcal{C}$,

$(C2)$ $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$,

$(C3)$ *(elimination property)* If $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq \{C_1 \cup C_2\} \backslash \{e\}$.

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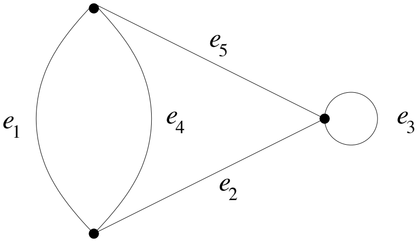
A subset of edges $I\subset\{e_1,\ldots,e_n\}$ of G is independent if the graph induced by I does not contain a cycle.

Graphic Matroid



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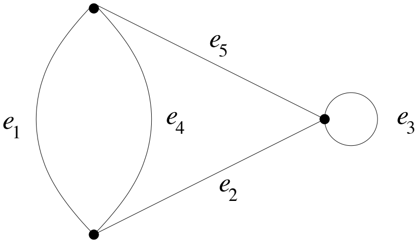

It can be checked that $M(G)$ is isomorphic to $M(A)$ (under the bijection $e_i \to i$).

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
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Proof (idea) Let $G = (V, E)$ be an oriented graph and let $\{x_i, \ i \in V\}$ be the canonical base of $\mathbb{R}$.

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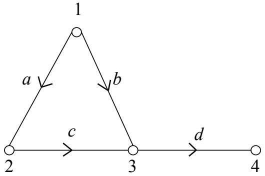
Proof (idea) Let $G = (V, E)$ be an oriented graph and let $\{x_i,\ i \in V\}$ be the canonical base of $\mathbb{R}$.

Exercice : Verify that the graph $G = (V, E)$ gives the same matroid that the one given by the set of vectors $y_e = x_i - x_j$ where $e = (i, j) \in E$.

Graphic Matroid

G



$$A = \begin{array}{c} \begin{array}{cccc} y_a & y_b & y_c & y_d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

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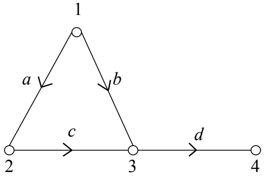
G 

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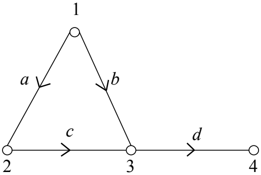
G 

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The cycle formed by the edges $a = \{1,2\}, b = \{1,3\}$ et $c = \{2,3\}$ in the graph correspond to the linear dependency $y_b - y_a = y_c$.

Bases

A base of a matroid is a maximal independent set. We denote by $\mathcal{B}$ the set of all bases of a matroid.

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The family $\mathcal{B}$ verifies the following conditions :

$(B1)$ $\mathcal{B} \neq \emptyset$,

$(B2)$ *(exchange propety)* $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

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If $\mathcal{I}$ is the family of subsets contained in a set of $\mathcal{B}$ then $(E, \mathcal{I})$ is a matroid.

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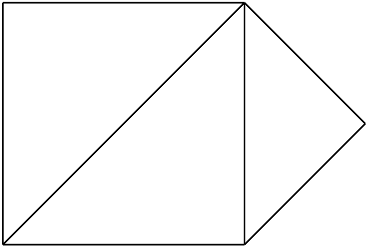

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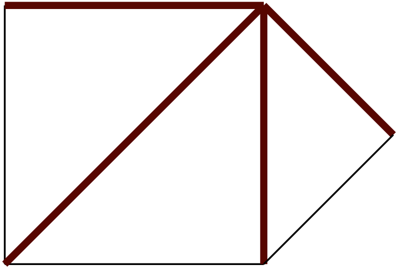

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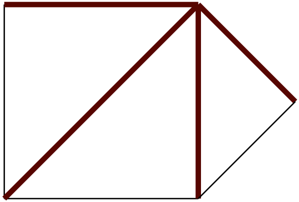

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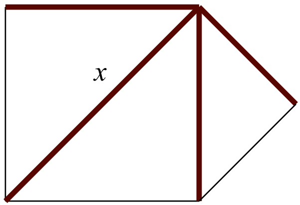

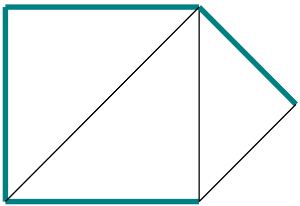

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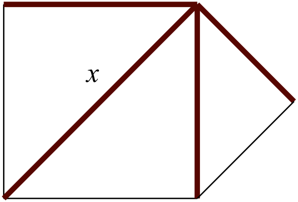

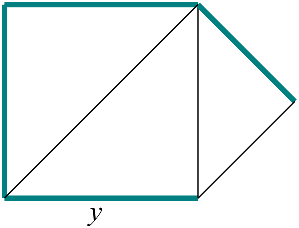

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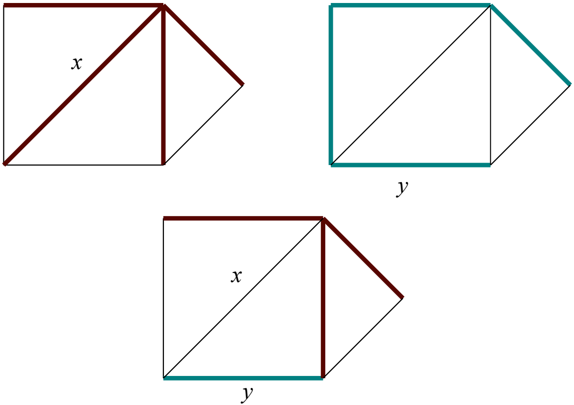

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$r = r_M$ is the rank function of a matroid $(E, \mathcal{I})$ (where $\mathcal{I} = \{I \subseteq E : r(I) = |I|\}$) if and only if r verifies the following conditions :

$(R1)$ $0 \leq r(X) \leq |X|$, for all $X \subseteq E$,

$(R2)$ $r(X) \leq r(Y)$, for all $X \subseteq Y$,

$(R3)$ *(sub-modulairity)* $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ for all $X, Y \subset E$.

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$B_{X\cup Y} \cap X$ is an independent set of $\mathcal{M}|_X$ and $|B_{X\cup Y} \cap X| \leq r(X)$
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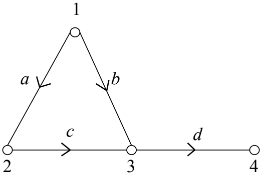
$= |B_{X\cup Y} \cap (X \cup Y)| + |B_{X\cup Y} \cap (X \cap Y)| = |B_{X\cup Y}| + |B_{X\cap Y}|$

Bases

Proof (necessity) : Let $r(X) = \max\{|Y| | Y \subseteq X, Y \in \mathcal{I}\}$.
Properties $(R1)$ and $(R2)$ trivial.
For $(R3)$, let $B_{X\cap Y}$ be a base of $X \cap Y$. Then, $B_{X\cap Y}$ is a maximal independent in $\mathcal{M}|_{(X\cap Y)}$ and thus $B_{X\cap Y}$ is contained in a base $B_{X\cup Y}$ of $\mathcal{M}|_{(X\cup Y)}$. We have that
$B_{X\cup Y} \cap X$ is an independent set of $\mathcal{M}|_X$ and $|B_{X\cup Y} \cap X| \leq r(X)$
$B_{X\cup Y} \cap Y$ is an independent set of $\mathcal{M}|_Y$ and $|B_{X\cup Y} \cap Y| \leq r(Y)$
By using the fact that $|A| + |B| = |A \cup B| + |A \cap B|$ we obtain
$r(X) + r(Y) \geq |B_{X\cup Y} \cap X| + |B_{X\cup Y} \cap Y|$
$= |(B_{X\cup Y} \cap X) \cup (B_{X\cup Y} \cap Y)| + |(B_{X\cup Y} \cap X) \cap (B_{X\cup Y} \cap Y)|$
$= |B_{X\cup Y} \cap (X \cup Y)| + |B_{X\cup Y} \cap (X \cap Y)| = |B_{X\cup Y}| + |B_{X\cap Y}|$
$= r(X \cup Y) + r(X \cap Y)$.

Rank

Let M be a graphic matroid obtained from G



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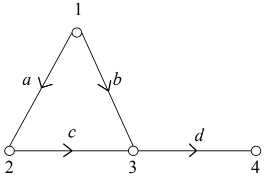

It can be verified that :
$r_M(\{a,b,c\}) = r_M(\{c,d\}) = r_M(\{a,d\}) = 2$ et
$r(M(G)) = r_M(\{a,b,c,d\}) = 3$.

Greedy Algorithm

Let $\mathcal{I}$ be a set of subsets of E verifying $(I1)$ and $(I2)$. Let $w : E \to \mathbb{R}$, and let $w(X) = \sum_{x \in X} w(x), X \subseteq E, w(\emptyset) = 0$.

Greedy Algorithm

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An optimization problem consist of finding a maximal set B of $\mathcal{I}$ with maximal weight (or minimal).

Greedy algorithm for $(\mathcal{I}, w)$

$X_0 = \emptyset$
$j = 0$
While $e \in E \setminus X_j \ : \ X_j \cup \{e\} \in \mathcal{I}$ **do**
    Choose an element e_{j+1} of maximal weight
    $X_{j+1} \leftarrow X_j \cup \{e_{j+1}\}$
    $j \leftarrow j + 1$
$B_G \leftarrow X_j$
Return B_G

Greedy Algorithm

Theorem $(\mathcal{I}, E)$ is a matroid if and only if the following conditions are verified :

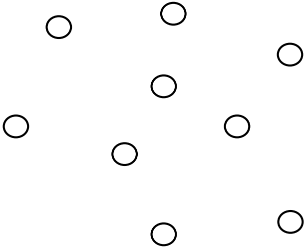
$(I1)$ $\emptyset \in \mathcal{I}$,

$(I2)$ $I \in \mathcal{I}, I' \subseteq I \Rightarrow I' \in \mathcal{I}$,

(G) For any function $w : E \to \mathbb{R}$, the greedy algorithm gives a maximal set of $\mathcal{I}$ of maximal weight.

Application 1 : Spanning tree of minimal weight

We want to construct a network (of minimal cost) connecting the 9 cities.



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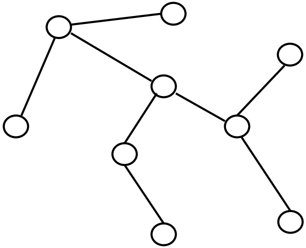

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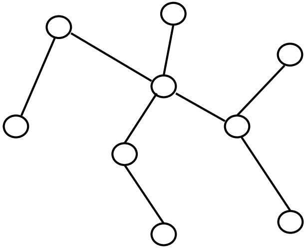

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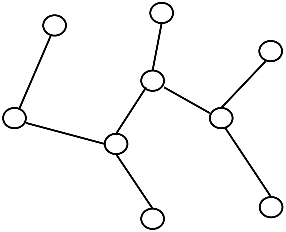

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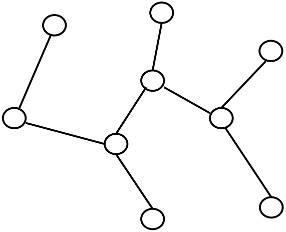

Theorem (Caley) There exist n^{n-2} labeled trees on n vertices.

Application 1 : Spanning tree of minimal weight

Theorem (Kruskal) Given a complete graph with weights on the edges there exist a polynomial time algorithm that finds a spanning tree of minimal weight.

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Theorem (Kruskal) Given a complete graph with weights on the edges there exist a polynomial time algorithm that finds a spanning tree of minimal weight.

Indeed, the greedy algorithm returns a base (maximal independent) of minimal weight by considering the graphic matroid associated to a complete graph and $w(e)$, $e \in E(G)$ is the the weight of each edge.