Theory of matroids and applications : I

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A matroid M is an ordered pair $(E, \mathcal{I})$ where E is a finite set ($E = \{1, \ldots, n\}$) and $\mathcal{I}$ is a family of subsets of E verifying the following conditions :

($I1$) $\emptyset \in \mathcal{I}$,

($I2$) If $I \in \mathcal{I}$ and $I' \subset I$ then $I' \in \mathcal{I}$,

($I3$) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$ then there exists $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members in $\mathcal{I}$ are called the independents of M. A subset in E not belonging to $\mathcal{I}$ is called dependent.

Representable Matroids

Theorem (Whitney 1935) Let $\{e_1, \ldots, e_n\}$ a set of columns (vectors) of a matrix with coefficients in a field $\mathbb{F}$. Let $\mathcal{I}$ be the family of subsets $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} = E$ such that the columns $\{e_{i_1}, \ldots, e_{i_m}\}$ are linearly independent in $\mathbb{F}$. Then, $(E, \mathcal{I})$ is a matroid.

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$$|I_2| \leq dim(W) \leq |I_1| < |I_2| \quad !!!$$

Let A be the following matrix with coefficients in $\mathbb{R}$.

$$A = \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array}$$
$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$\{\emptyset, \{1\}, \{2\}, \{4\}, \{4\}, \{5\}, \{1,2\}, \{1,5\}, \{2,4\}, \{2,5\}, \{4,5\}\} \subseteq \mathcal{I}(M)$

A matroid obtained form a matrix A with coefficients in $\mathbb{F}$ is denoted by $M(A)$ and is called representable over $\mathbb{F}$ or $\mathbb{F}$-representable.

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Circuits

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We denote by $\mathcal{C}$ the set of circuits of a matroid.

$\mathcal{C}$ is the set of circuits of a matrid on E if and only if $\mathcal{C}$ verifies the following properties :

(C1) $\emptyset \notin \mathcal{C}$,

(C2) $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$,

(C3) *(elimination property)* If $C_1, C_2 \in \mathcal{C}, C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq \{C_1 \cup C_2\} \backslash \{e\}$.

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A subset of edges $I \subset \{e_1, \ldots, e_n\}$ of G is independent if the graph induced by I does not contain a cycle.
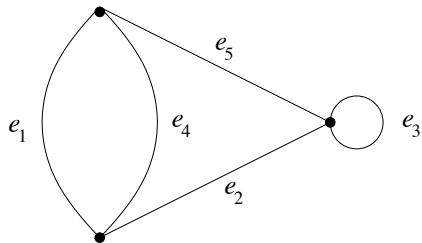
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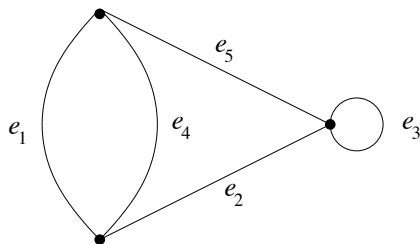
It can be checked that $M(G)$ is isomorphic to $M(A)$ (under the bijection $e_i \rightarrow i$).

$$A = \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 \\ & 1 & 0 & 0 & 1 & 1 \\ & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

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Exercice : Verify that the graph $G = (V, E)$ gives the same matroid that the one given by the set of vectors $y_e = x_i - x_j$ where $e = (i, j) \in E$.
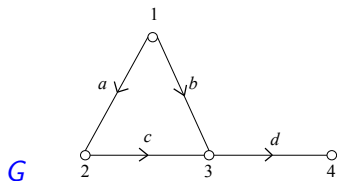
$$G \qquad A = \begin{pmatrix} y_a & y_b & y_c & y_d \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

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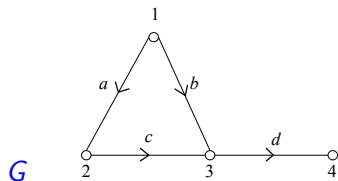
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$M(G)$ is isomorphic to $M(A)$ ($a \to y_a, b \to y_b, c \to y_c, d \to y_d$).

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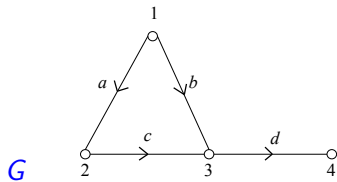
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$M(G)$ is isomorphic to $M(A)$ ($a \to y_a, b \to y_b, c \to y_c, d \to y_d$).

The cycle formed by the edges $a = \{1,2\}, b = \{1,3\}$ et $c = \{2,3\}$ in the graph correspond to the linear dependency $y_b - y_a = y_c$.

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The family $\mathcal{B}$ verifies the following conditions :

$(B1)$ $\mathcal{B} \neq \emptyset$,

$(B2)$ *(exchange propety)* $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \backslash B_2$ then there exist $y \in B_2 \backslash B_1$ such that $(B_1 \backslash x) \cup y \in \mathcal{B}$.

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If $\mathcal{I}$ is the family of subsets contained in a set of $\mathcal{B}$ then $(E, \mathcal{I})$ is a matroid.

Bases

Proof of $(B2)$: Let B_1, B_2 two bases. Then $B_1 \backslash x \in \mathcal{I}$ with $|B_1 \backslash x| < |B_2|$.

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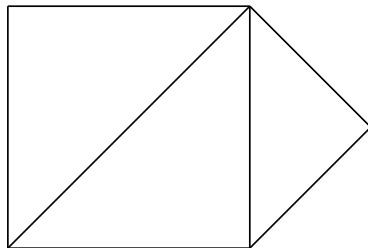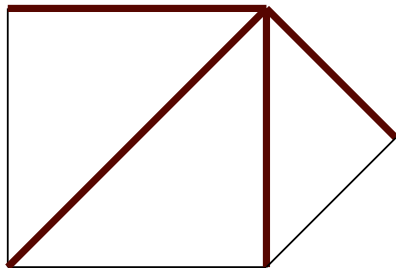
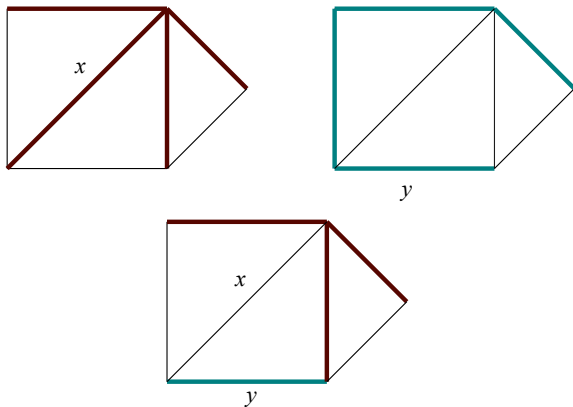# Bases
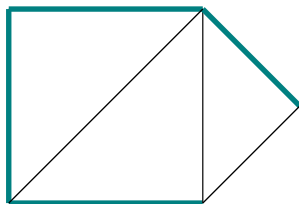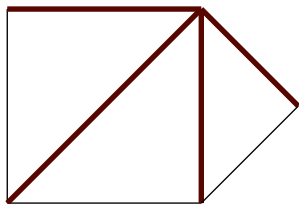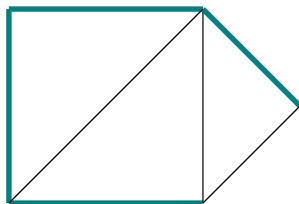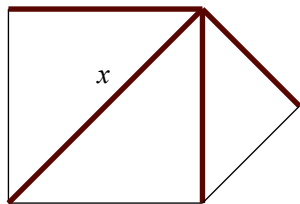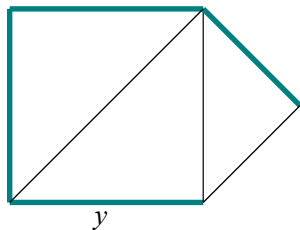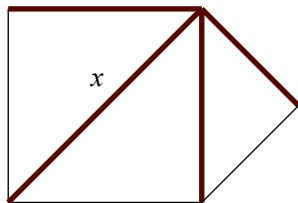
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Rank

The rank of a set $X \subseteq E$ is defined by

$$r_M(X) = \max\{|Y| : Y \subseteq X, Y \in \mathcal{I}\}.$$
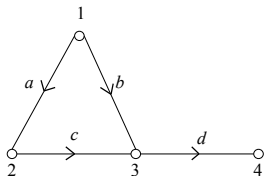
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$r = r_M$ is the rank function of a matroid $(E, \mathcal{I})$ (where $\mathcal{I} = \{I \subseteq E : r(I) = |I|\}$) if and only if r verifies the following conditions :

$(R1)$ $0 \leq r(X) \leq |X|$, for all $X \subseteq E$,

$(R2)$ $r(X) \leq r(Y)$, for all $X \subseteq Y$,

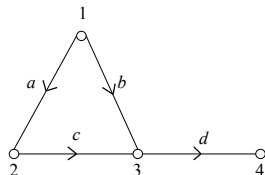$(R3)$ *(sub-modulairity)* $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$ for all $X, Y \subset E$.

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Rank

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It can be verified that :
$r_M(\{a, b, c\}) = r_M(\{c, d\}) = r_M(\{a, d\}) = 2$ et
$r(M(G)) = r_M(\{a, b, c, d\}) = 3$.

Let $\mathcal{I}$ be a set of subsets of E verifying $(I1)$ and $(I2)$. Let $w : E \to \mathbb{R}$, and let $w(X) = \sum_{x \in X} w(x), X \subseteq E, w(\emptyset) = 0$.

Greedy Algorithm

Let $\mathcal{I}$ be a set of subsets of E verifying $(I1)$ and $(I2)$. Let $w : E \to \mathbb{R}$, and let $w(X) = \sum_{x \in X} w(x), X \subseteq E, w(\emptyset) = 0$.

An optimization problem consist of finding a maximal set B of $\mathcal{I}$ with maximal weight (or minimal).

Greedy algorithm for $(\mathcal{I}, w)$

$X_0 = \emptyset$

$j = 0$

While $e \in E \backslash X_j \; : \; X_j \cup \{e\} \in \mathcal{I}$ **do**

      Choose an element e_{j+1} of maximal weight

      $X_{j+1} \leftarrow X_j \cup \{e_{j+1}\}$

      $j \leftarrow j + 1$

$B_G \leftarrow X_j$

Return B_G

Greedy Algorithm

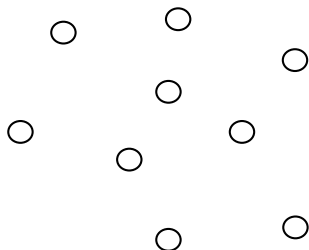Theorem $(\mathcal{I}, E)$ is a matroid if and only if the following conditions are verified :

(*I*1) $\emptyset \in \mathcal{I}$,

(*I*2) $I \in \mathcal{I}, I' \subseteq I \Rightarrow I' \in \mathcal{I}$,

(*G*) For any function $w : E \to \mathbb{R}$, the greedy algorithm gives a maximal set of $\mathcal{I}$ of maximal weight.
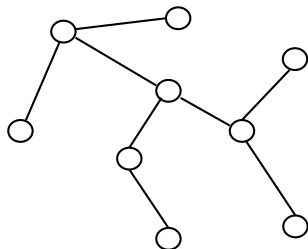
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9 cities.

Theorem (Caley) There exist n^{n-2} labeled trees on n vertices.

Theorem (Kruskal) Given a complete graph with weights on the edges there exist a polynomial time algorithm that finds a spanning tree of minimal weight.

Application 1 : Spanning tree of minimal weight

Theorem (Kruskal) Given a complete graph with weights on the edges there exist a polynomial time algorithm that finds a spanning tree of minimal weight.

Indeed, the greedy algorithm returns a base (maximal independent) of minimal weight by considering the graphic matroid associated to a complete graph and $w(e)$, $e \in E(G)$ is the the weight of each edge.

Let $S = \{e_1, \ldots, e_n\}$ and let $\mathcal{A} = \{A_1, \ldots, A_k\}, A_i \subseteq S, n \geq k$.

Transversal Matroid

Let $S = \{e_1, \ldots, e_n\}$ and let $\mathcal{A} = \{A_1, \ldots, A_k\}$, $A_i \subseteq S$, $n \geq k$.

A transversal of $\mathcal{A}$ is a subset $\{e_{j_1}, \ldots, e_{j_k}\}$ of S such that $e_{j_i} \in A_i$.

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A transversal of $\mathcal{A}$ is a subset $\{e_{j_1}, \ldots, e_{j_k}\}$ of S such that $e_{j_i} \in A_i$.

A set $X \subseteq S$ is called partial transversal of $\mathcal{A}$ if there exists $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, k\}$ such that X is a transversal of $\{A_{i_1}, \ldots, A_{i_l}\}$.

Let $G = (S, \mathcal{A}; E)$ be a bipartite graph constructed from $S = \{e_1, \ldots, e_n\}$ and $\mathcal{A} = \{A_1, \ldots, A_k\}$ and two vertices $e_i \in S$, $A_j \in \mathcal{A}$ are adjacent if and only if $e_i \in A_j$.

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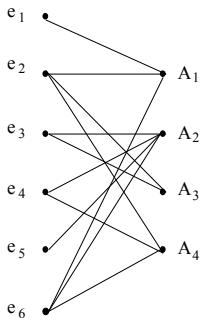A matching in a graph is a set of edges without common vertices.

Partial transversal $\mathcal{A}$ correspond to matchings in $G = (S, \mathcal{A}; E)$.

$E = \{e_1, \ldots, e_6\}$ et $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ with $A_1 = \{e_1, e_2, e_6\}$,
$A_2 = \{e_3, e_4, e_5, e_6\}$, $A_3 = \{e_2, e_3\}$ and $A_4 = \{e_2, e_4, e_6\}$.

Transversal Matroid

$E = \{e_1, \ldots, e_6\}$ et $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ with $A_1 = \{e_1, e_2, e_6\}$, $A_2 = \{e_3, e_4, e_5, e_6\}$, $A_3 = \{e_2, e_3\}$ and $A_4 = \{e_2, e_4, e_6\}$.
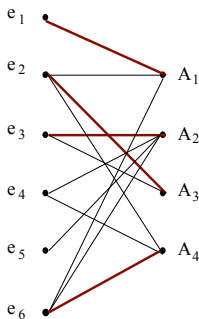
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$\{e_1, e_3, e_2, e_6\}$ is a transversal of $\mathcal{A}$.
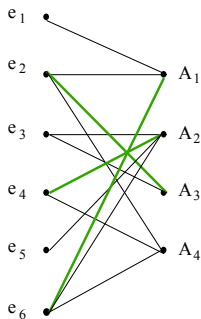
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$X = \{e_6, e_4, e_2\}$ is a partial transversal of $\mathcal{A}$ since X is a transversal of $\{A_1, A_2, A_3\}$.

Theorem Let $S = \{e_1, \ldots, e_n\}$ and $\mathcal{A} = \{A_1, \ldots, A_k\}, A_i \subseteq S$. Then, the set of partial transversals of $\mathcal{A}$ is the set of independents of a matroid.

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Proof Exercise.

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Proof Exercise.

Such matroid is called transversal matroid.

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Problem : Assign the tasks to the agents in an optimal way (maximizing the priorities).

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- Transversal Matroid $M = (\mathcal{I}, \{t_1, t_2, t_3, t_4\})$ where $\mathcal{I}$ is given by the set of matchings of the bipartite graph $G = (U, V; E)$ with $U = \{t_1, t_2, t_3, t_4\}$, $V = \{e_1, e_2, e_3\}$.

Application 2 : Assignment problem

- tasks : $\{t_1, t_2, t_3, t_4\}$.
- priorities : $w(t_1) = 10, w(t_2) = 3, w(t_3) = 3$ et $w(t_4) = 5$.
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e_1 able to perform tasks t_1 et t_2,
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- Transversal Matroid $M = (\mathcal{I}, \{t_1, t_2, t_3, t_4\})$ where $\mathcal{I}$ is given by the set of matchings of the bipartite graph $G = (U, V; E)$ with $U = \{t_1, t_2, t_3, t_4\}$, $V = \{e_1, e_2, e_3\}$.

- By applying the greedy algorithm to M we have
$X_0 = \emptyset, X_1 = \{t_1\}, X_2 = \{t_1, t_4\}$ and $X_3 = \{t_1, t_4, t_2\}$.