# Theory of matroids and applications : II

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# Duality

Let $M$ be a matroid on the ground set $E$ and let $\mathcal{B}$ the set of bases of $M$. Then,

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The matroid on $E$ having $\mathcal{B}^*$ as set of bases, denoted by $M^*$, is called the dual of $M$.

A base of $M^*$ is also called cobase of $M$.

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- The rank function of $M^*$ is given by

$$r_{M^*}(X) = |X| + r_M(E \backslash X) - r_M,$$

for $X \subset E$.

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The matroid obtained on this way is called the matroid of cocycle of $G$ or bond matroid, denoted by $B(G)$.

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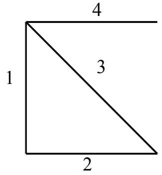

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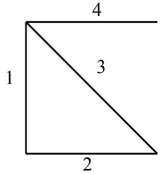


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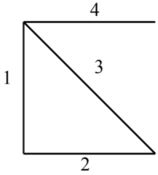


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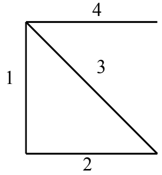


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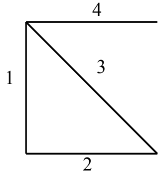


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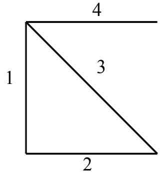


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$\mathcal{C}(M^*(G)) = \{\{4\},\{1,2\},\{1,3\},\{2,3\}\}$ that are precisely the cocycles of $G$.

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**Theorem** If $G$ is planar then $M^*(G) = M(G^*)$.

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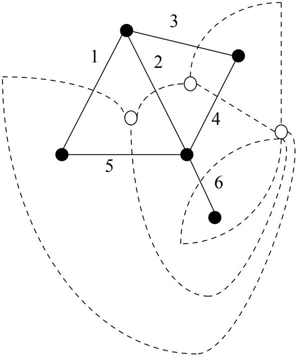

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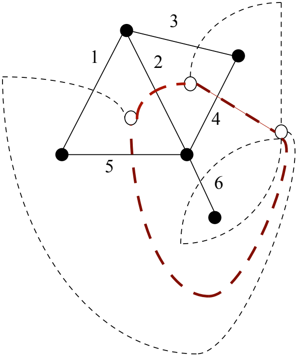

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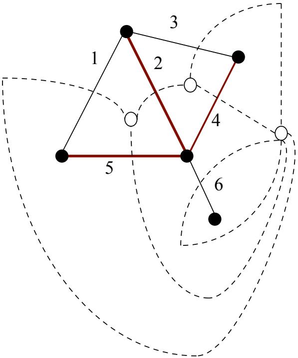

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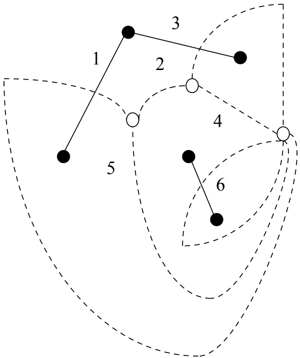


**Remark** The dual of a graphic matroid is not necessarly graphic.

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where $I_r$ is the identity $r \times r$ and $A$ is a matrix of size $r \times (n-r)$.

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(Exercise) $M^*$ can be obtained from the set of columns of the matrix

$$(-{}^tA \mid I_{n-r})$$

where $I_{n-r}$ is the identity $(n-r) \times (n-r)$ and ${}^tA$ is the transpose of $A$.

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Let $V$ be a subspace of $\mathbb{F}^n$ where $n = |E|$. We recall that the orthogonal space $V^\perp$ is defined from the canonical scalar product $\langle u, v \rangle = \sum_{e \in E} u(e)v(e)$ by

$$V^\perp = \{v \in \mathbb{F}^n \mid \langle u, v \rangle = 0 \text{ for any } u \in V\}.$$

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The orthogonal space of the space generated by the columns of $(I \mid A)$ is given by the space generated by the columns of $(-{}^tA \mid I_{n-r})$.

# Operation : deletion

Let $M$ be a matroid on the set $E$ and let $A \subset E$. Then,

$$\{X \subset E \backslash A \mid X \text{ is independent in } M\}$$

is a set of independent of a matroid on $E \backslash A$.

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Proposition

($i$) The circuits of $M \setminus A$ are the circuits of $M$ contained in $E \setminus A$.

($ii$) For $X \subset E \setminus A$ we have $r_{M \setminus A}(X) = r_M(X)$.

# Operation : contraction

Let $M$ be a matroid on the set $E$ and let $A \subset E$.
Let $M|_A = \{X \subseteq A | X \in \mathcal{I}(M)\}$ and $X \subseteq E \setminus A$. Then,

$\{X \subseteq E \setminus A|$ there exists a base $B$ of $M|_A$ such that $X \cup B \in \mathcal{I}(M)\}$

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$(i)$ The circuits of $M/A$ are the non-empty minimal (by inclusion) sets of the form $C \setminus A$ where $C$ is a circuit of $M$.

$(ii)$ For $X \subset E \setminus A$ we have $r_{M/A}(X) = r_M(X \cup A) - r_M(A)$.

# Operations : deletion and contraction

Properties
$(i)\ (M\backslash A)\backslash A' = M\backslash(A \cup A')$
$(ii)\ (M/A)/A' = M/(A \cup A')$
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Proof : $(i)$ and $(ii)$ are immediate by using the rank function.
For $(iii)$, we show that $r_{(M/A)\backslash A'} = r_{(M\backslash A')/A}$. Let $X \subset E \setminus (A \cup A')$, then

$$\begin{aligned} r_{(M/A)\backslash A'}(X) &= r_{(M/A)}(X) = r_M(X \cup A) - r_M(A) \\ &= r_{M\backslash A'}(X \cup A) - r_M(A) = r_{(M\backslash A')/A}. \end{aligned}$$

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Question : Is it true that any family of matroids is closed under deletions/contractions operations ?

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Proof Deletion : let $T \subseteq E$ with $|T| = t$. Then,

$$U_{n,r} \backslash T = \begin{cases} U_{n-t,n-t} & \text{if} \;\; n \geq t \geq n-r \\ U_{n-t,r} & \text{if} \;\; t < n-r. \end{cases}$$

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Contraction : it follows by using duality.

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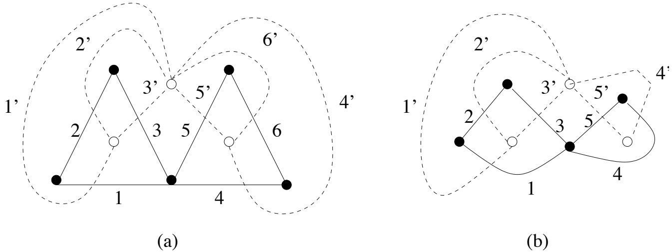


Contracting element 6

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- If we change the nonzero component we obtain another representation of $M/a$.
- If $v_a = \overline{0}$ then $a$ is a loop of $M$ and thus $M/a = M \setminus a$.

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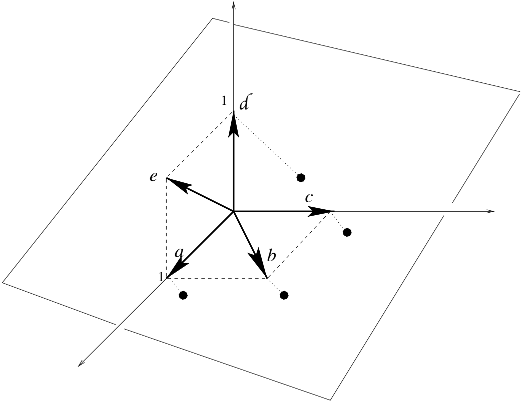


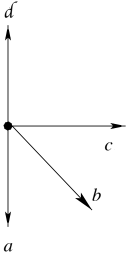

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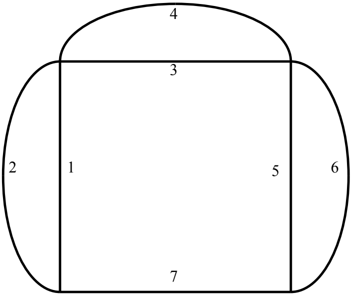
The matroid $M(G)$ is transversal (with $A_1 = \{1, 2, 7\}$, $A_2 = \{3, 4, 7\}, A_3 = \{5, 6, 7\}$). However, $M(G/7)$ is not transversal.



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For $\mathbb{F} = GF(2) = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (binary matroids) : the list has only one matroid $U_{2,4}$ (3 pages proof)

$$\mathcal{B}(U_{2,4}) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

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For $\mathbb{F} = GF(4)$ : the list has 8 matroids explicitly given (50 pages proof)

Theorem A matroid is graphic if and only if has neither $U_{2,4}, F_7, F_7^*, M^*(K_5) = B(K_5)$ nor $M^*(K_{3,3}) = B(K_{3,3})$ as minors.

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For $\mathbb{F} = GF(3) = \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$ (ternary matroids) : the list has 4 matroids $F_7$ $F_7^*$, $U_{2,5}$ $U_{3,5}$ (10 pages proof)

For $\mathbb{F} = GF(4)$ : the list has 8 matroids explicitly given (50 pages proof)

Theorem A matroid is graphic if and only if has neither $U_{2,4}, F_7, F_7^*, M^*(K_5) = B(K_5)$ nor $M^*(K_{3,3}) = B(K_{3,3})$ as minors.

Theorem A matroid is cographic if and only if has neither $U_{2,4}, F_7, F_7^*, M(K_5)$ nor $M(K_{3,3})$ as minors.

# Regular Matroids

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Example : Graphic matroids are regulars.

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• $M$ is built with bricks (graphic, cographic and $R_{10}$) via 3 operations :

*1-sum* : direct sum of two matroids

*2-sum* : patching two matroids on one common element

*3-sum* : patching two binary matroids on 3 common elements forming a 3-circuit in each matroid.

# Regular Matroids - Applications

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Remark Most of the combinatorial optimization problems can be realized as a unimodular linear programming.

# Regular Matroids - Applications

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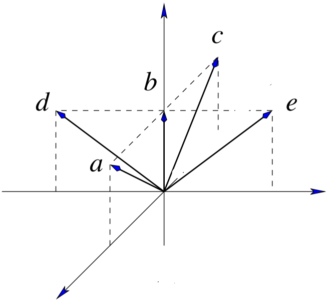
Let $A = \{v_1, \ldots, v_k\}$ be a finite set of elements of $\mathbb{R}^d$.

A zonotope, generated by $A$ and denoted by $Z(A)$, is a polytope formed by the Minkowski's sum of line segments

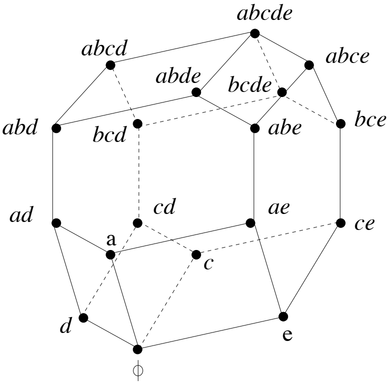
$$Z(A) = \{\alpha_1 + \cdots + \alpha_k | \alpha_i \in [-v_i, v_i]\}.$$

# Regular Matroids - Applications



# Regular Matroids - Applications

## Permutahedron



# Regular Matroids - Applications

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Voronoï's result : there exist exactly 5 regular matroids of rank 3.

## Non Representable Matroids

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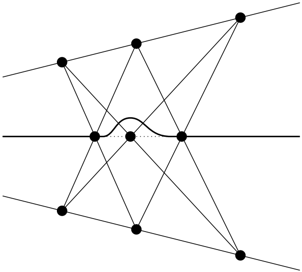
Example (classic) : the rank 3 matroid on 9 elements obtained from the Non-Pappus configuration



# Direct sum

Let $M$ be a matroid on $E = E_1 + E_2$.

We say that $M$ is the direct sum of $M(E_1)$ and $M(E_2)$ if it verifies one of the following equivalent properties :

- $r_M(E) = r_{M_1}(E_1) + r_{M_2}(E_2)$.
- If $X \subset E_1$ et $Y \subset E_2$ are independent in $M$ the $X \cup Y$ is also independent.
- for any circuit $C$ of $M$ we have $C \subset E_1$ or $C \subset E_1$.

# Connectivity

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Theorem Let $G$ be a graph without loops and isolate vertices. If $|V(G)| \geq 3$ then $M(G)$ is connected if and only if $G$ is 2-connected.