Theory of matroids and applications III

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Tutte Polynomial

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$$\begin{aligned} t(U_{2,3};x,y) &= \sum_{X\subseteq E,\ |X|=0}(x-1)^{2-0}(y-1)^{0-0}+\sum_{X\subseteq E,\ |X|=1}(x-1)^{2-1}(y-1)^{1-1} \\ &+\sum_{X\subseteq E,\ |X|=2}(x-1)^{2-2}(y-1)^{2-2}+\sum_{X\subseteq E,\ |X|=3}(x-1)^{2-2}(y-1)^{3-2} \\ &=(x-1)^2+3(x-1)+3(1)+y-1 \\ &=x^2-2x+1+3x-3+3+y-1=x^2+x+y. \end{aligned}$$

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The Tutte polynomial can be expressed recursively as follows

$$t(M;x,y) = \begin{cases} t(M\setminus e;x,y) + t(M/e;x,y) & \text{if } e \neq \text{isthmus, loop,} \\ x \cdot t(M\setminus e;x,y) & \text{if } e \text{ is an isthmus,} \\ y \cdot t(M/e;x,y) & \text{if } e \text{ is a loop.} \end{cases}$$

Acyclic Orientations

Let $G = (V, E)$ be a connected graph. An orientation of G is an orientation of the edges of G.

We say that the orientation is acyclic if the oriented graph do not contain an oriented cycle (i.e., a cycle where all its edges are oriented clockwise or anti-clockwise).

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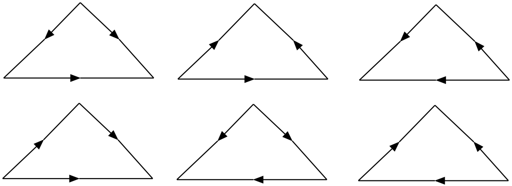
Theorem The number of acyclic orientations of G is equals to

$$t(M(G); 2, 0).$$

Acyclic Orientations

Example : There are 6 acyclic orientations of C_3



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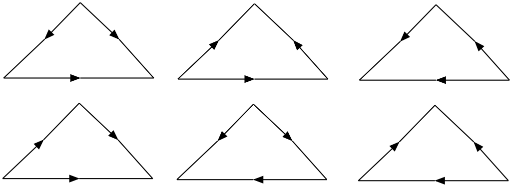

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Since $t(U_{2,3}; x, y) = x^2 + x + y$ then the number of acyclic orientations of C_3 is $t(U_{2,3}; 2, 0) = 2^2 + 2 + 0 = 6$.

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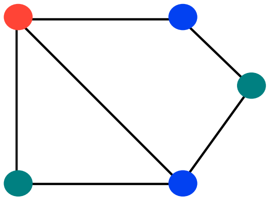

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Theorem $\chi(G, \lambda)$ is a polynomial on λ. Moreover

$$\chi(G, \lambda) = \sum_{X \subseteq E} (-1)^{|X|} \lambda^{\omega(G[X])}$$

where $\omega(G[X])$ denote the number of connected components of the subgraph generated by X.

Proof (idea) By using the inclusion-exclusion formula.

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Theorem If G is a graph with $\omega(G)$ connected components. Then,

$$\chi(G,\lambda) = \lambda^{\omega(G)}(-1)^{|V(G)|-\omega(G)}t(M(G);1-\lambda,0).$$

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Example : $\chi(K_3, 3) = 3^1(-1)^{3-1} t(K_3; 1-3, 0)$
$= 3 \cdot 1 \cdot t(U_{2,3}; -2, 0) = 3((-2)^2 - 2 + 0) = 6.$

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A polytope is called integer if all its vertices have integer coordinates.
Ehrhart studied the function i_P that counts the number of integer points in the polytope P *dilated* by a factor of t

$$\begin{aligned} i_P: \quad & \mathbb{N} \longrightarrow \mathbb{N}^* \\ & t \mapsto |tP \cap \mathbb{Z}^d| \end{aligned}$$

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Theorem (Ehrhart) i_P is a polynomial on t of degree d,

$$i_P(t) = c_d t^d + c_{d-1} t^{d-1} + \cdots + c_1 t + c_0.$$

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All others coefficients remain a mystery ! !

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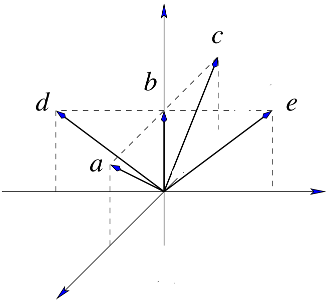
Let $A = \{v_1, \ldots, v_k\}$ be a finite set of elements of $\mathbb{R}^d$.
A zonotope generated by A, denoted by $Z(A)$, is a polytope formed by the Minkowski's sum of line segments

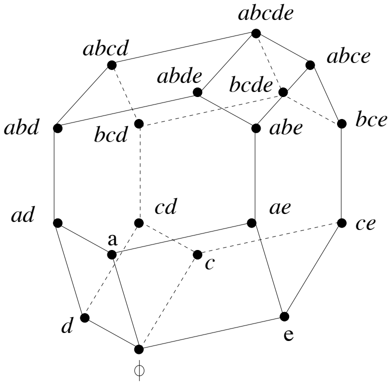
$$Z(A) = \{\alpha_1 + \cdots + \alpha_k | \alpha_i \in [-v_i, v_i]\}.$$

Ehrhart Polynomial



Ehrhart Polynomial

Permutahedron



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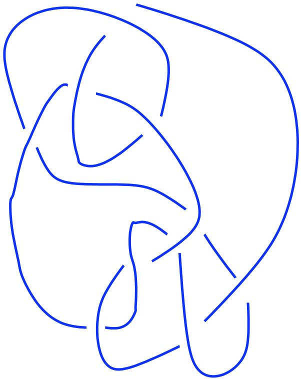
Theorem Let M be a regular matroid and let A be one of its representation matrix. Then, the Ehrhart polynomial associated to the zonotope $Z(A)$ is given by

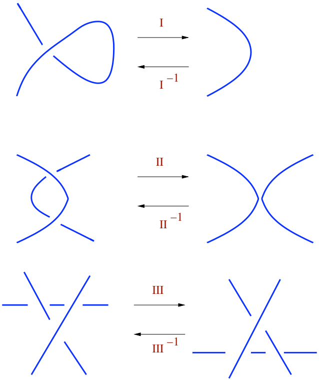
$$i_{Z(A)}(q) = q^{r(M)} t\left(M; 1 + \frac{1}{q}, 1\right).$$

Knots



Knots

Reidemeister moves



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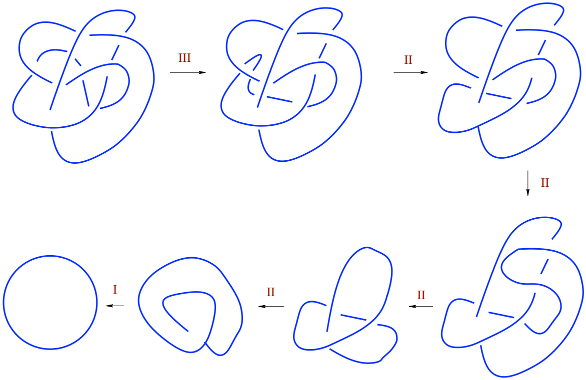

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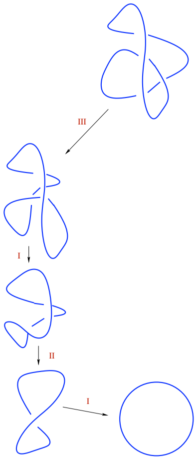

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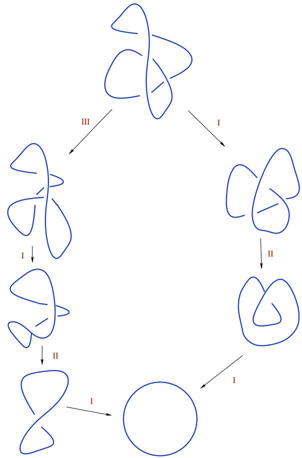

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i) $\langle U \rangle = 1$

ii) $\langle U + D \rangle = -(A^2 + A^{-2}) \langle D \rangle$

iii) 

Knots

Theorem For any link L the bracket polynomial is independent of the order in which rules $(i)-(iii)$ are applied to the crossings. Further, it is invariant under the Reidemeister moves II and III

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The writhe of an oriented link diagram D is the sum of the signs at the crossings of D (denoted by $\omega(D)$).

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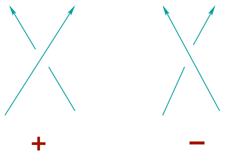

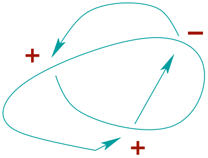

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$$f_D(A) = (-A^3)^{\omega(D)} < L >$$

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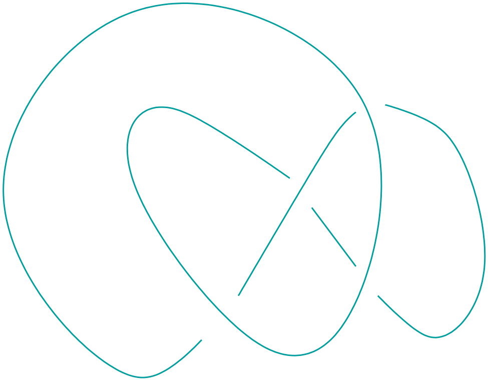
Now, define for any link L

$$V_L(z) = f_D(z^{-1/4})$$

where D is any diagram representing L. Then $V_L(z)$ is the Jones polynomial of the oriented link L.

Knots



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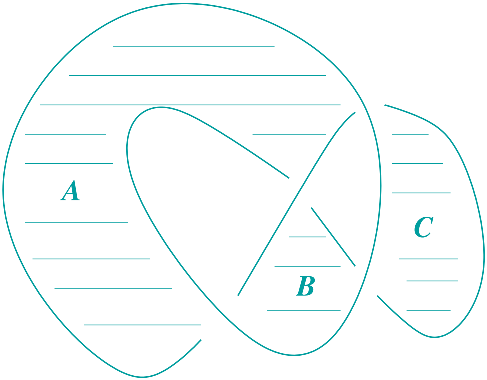

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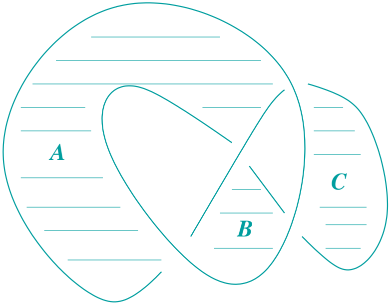

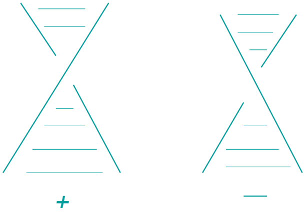

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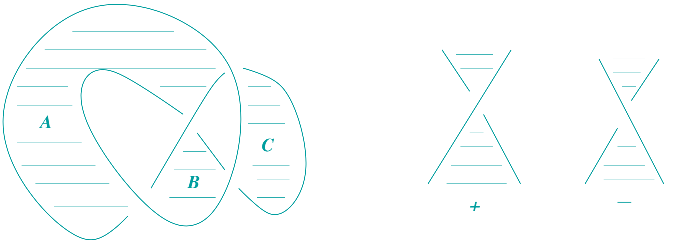

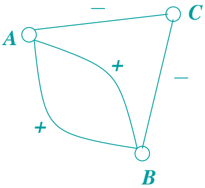

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Theorem (Thistlethwaite 1987) If D is an oriented alternating link diagram then

$$V_L(z) = (z^{-1/4})^{3\omega(D)-2} t(M(G); -z, -z^{-1})$$

where G is the graph associated to the knot diagram.

Simplicial complex

Let $V = \{v_1, \ldots, v_n\}$ be a set of distincts elements. A collection Δ of subsets of V is called a simplicial complex if for every $F \in \Delta$ and $G \subseteq F, G \in \Delta$.

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If $\{v\} \in \Delta$ then we call v a vertex of Δ.

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v is called the apex of $C\Delta$.

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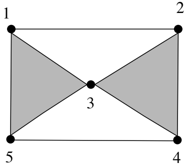
- Typically, we will describe a simplicial complex by listing its facets.

Example

Simplicial complexe Δ of dimension 2



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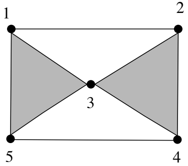

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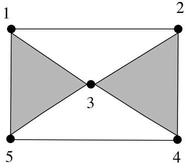

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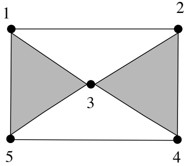

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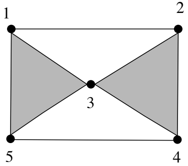

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• The deletion of 3 has facets 12, 24, 45 and 15. The deletion of 5 has facets 234, 13 and 12.

Matroid complex

Recall that axioms $(I1), (I2)$ for the independent set $\mathcal{I}(M)$ of a matroid M on a set V are equivalent to $\mathcal{I}$ being an abstract simplicial complex on V.

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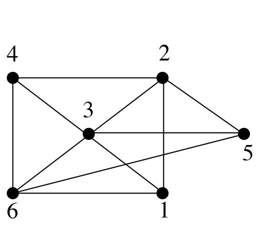
Examples

Two 1-dimensional simplicial complexes.



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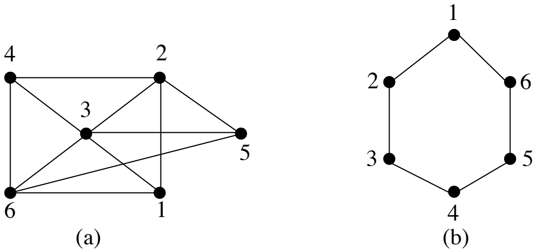

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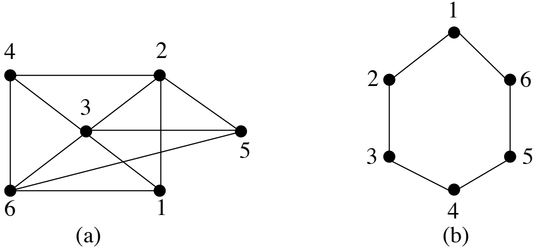

(a) Matroid complex (this can be checked by verifying that every $A \subseteq \{1, \ldots, 6\}$, Δ_A is pure).

(b) is not a matroid complex since it admits a restriction that is not pure, for instance, the facets of $\Delta_{1,3,4}$ are $\{1\}$ and $\{3,4\}$ as facets so the restriction is not pure.

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A matroid complex Δ_M is a cone if and only if M has a coloop (or isthme), which corresponds to the apex defined above.

Stanley-Reisner ideal

Let k be a field. We can associate to a simplicial complex Δ, a square free monomial ideal in $S = k[x_1, \ldots, x_n]$,

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The ideal I_Δ is called the Stanley-Reisner ideal of Δ and S/I_Δ the Stanley-Reisner ring of Δ.

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- Hilbert function

$$h_{S/I_\Delta}(h) = dim_k[S/I_\Delta]_h$$

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$h(\Delta) = (h_0, \ldots, h_d)$ is known as the h-vector of Δ.

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In particular, for any $j = 0, \ldots, d$, we have

$$f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-1} h_i$$

$$h_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-1} f_{i-1}.$$

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Remark v_j is externally passive in B if it is internally passive in $E \setminus B$ in M^*.

h-vector of simplicial complexes

Bjorner proved that

$$\sum_{i=0}^{d} h_j t^j = \sum_{B \in \mathcal{B}(M)} t^{ip(B)}$$

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Alternatively,

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Remarks

• Since the f-numbers (and hence the h-numbers) of a matroid depend only on its independent sets, then above equations hold for any ordering of the ground set of M.

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• h-vector of a matroid complex Δ_M is actually a specialization of the Tutte polynomial of the corresponding matroid ; precisely we have $T(M; x, 1) = h_0 x^d + h_1 x^{d_1} + \cdots + h_d$

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Obtaining that $h(\Delta) = (1,1,1)$.

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A vector $\mathbf{h} = (h_0, \ldots, h_d)$ is a pure O-sequence if there is a pure ideal $\mathcal{O}$ such that $\mathbf{h} = F(\mathcal{O})$.

Example

The pure monomial order ideal (inside $k[x, y, z]$ with maximal monomials $\mathbf{xy^3z}$ and $\mathbf{x^2z^3}$ is :

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Hence the h-vector of X is the pure O-sequence $h = (1, 3, 6, 7, 5, 2)$.

Stanley's conjecture

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Conjecture hold for several families of matroid complexes

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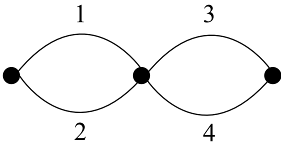
We consider the matroid complexe Δ associated to the rank 2 matroid induced by the graph G



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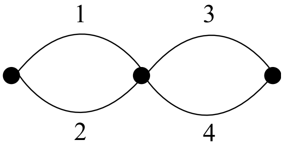

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Obtaining the h-vector $h(1,2,1)$. Since $\mathcal{O} = (1, x_1, x_2, x_1x_2)$ is an order ideal then $h(1,2,1)$ is pure O-sequence.