Theory of matroids and applications : IV

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Operation : deletion

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Proposition

(i) The circuits of $M \setminus A$ are the circuits of M contained in $E \setminus A$.

(ii) For $X \subset E \setminus A$ we have $r_{M \setminus A}(X) = r_M(X)$.

Operation : contraction

Let M be a matroid on the set E and let $A \subset E$.
Let $M|_A = \{X \subseteq A | X \in \mathcal{I}(M)\}$ and $X \subseteq E \setminus A$. Then,

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Proposition

(i) The circuits of M/A are the non-empty minimal (by inclusion) sets of the form $C \setminus A$ where C is a circuit of M.

(ii) For $X \subset E \setminus A$ we have $r_{M/A}(X) = r_M(X \cup A) - r_M(A)$.

Operations : deletion and contraction

Properties

(i) $(M\backslash A)\backslash A' = M\backslash(A \cup A')$

(ii) $(M/A)/A' = M/(A \cup A')$

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Proof : (i) and (ii) are immediate by using the rank function. For (iii), we show that $r_{(M/A) \backslash A'} = r_{(M \backslash A')/A}$. Let $X \subset E \setminus (A \cup A')$, then

$$\begin{aligned} r_{(M/A)\backslash A'}(X) &= r_{(M/A)}(X) = r_M(X \cup A) - r_M(A) \\ &= r_{M \backslash A'}(X \cup A) - r_M(A) = r_{(M \backslash A')/A}. \end{aligned}$$

Operations : deletion and contraction

The operations deletion and contraction are duals, that is,

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Question : Is it true that any family of matroids is closed under deletions/contractions operations ?

Minors - uniform matroids

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Proof Deletion : let $T \subseteq E$ with $|T| = t$. Then,

$$U_{n,r} \setminus T = \begin{cases} U_{n-t,n-t} & \text{if } n \geq t \geq n-r \\ U_{n-t,r} & \text{if } t < n-r. \end{cases}$$

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Contraction : it follows by using duality.

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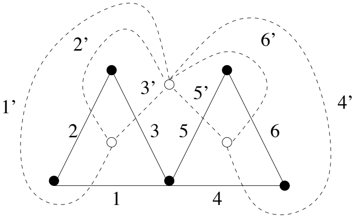
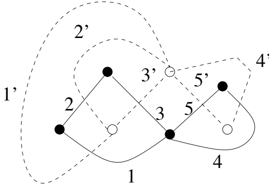


(a)



(b)

Contracting element 6

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- If $v_a = \overline{0}$ then a is a loop of M and thus $M/a = M\setminus a$.

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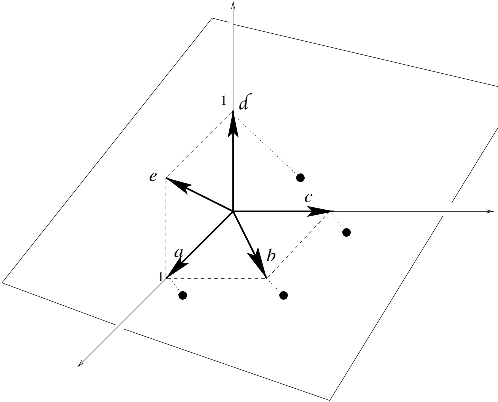

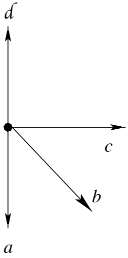

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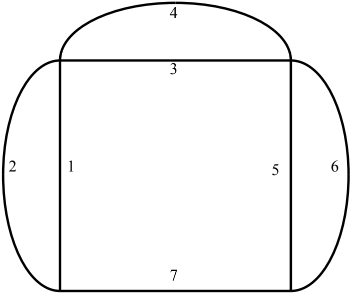
The matroid $M(G)$ is transversal (with $A_1 = \{1, 2, 7\}$, $A_2 = \{3, 4, 7\}, A_3 = \{5, 6, 7\}$). However, $M(G/7)$ is not transversal.



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For $\mathbb{F} = GF(2) = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ (binary matroids) : the list has only one matroid $U_{2,4}$ (3 pages proof)

$$\mathcal{B}(U_{2,4}) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$$

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Theorem A matroid is cographic if and only if has neither $U_{2,4}, F_7, F_7^*, M(K_5)$ nor $M(K_{3,3})$ as minors.

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Example : Graphic matroids are regulars.

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• M is built with bricks (graphic, cographic and R_{10}) via 3 operations :

1-sum : direct sum of two matroids

2-sum : patching two matroids on one common element

3-sum : patching two binary matroids on 3 common elements forming a 3-circuit in each matroid.

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Remark Most of the combinatorial optimization problems can be realized as a unimodular linear programming.

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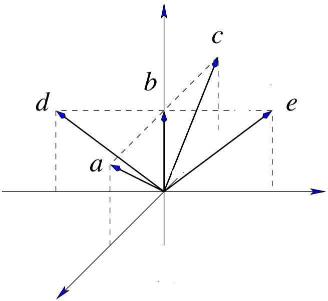
Let $A = \{v_1, \ldots, v_k\}$ be a finite set of elements of $\mathbb{R}^d$.

A zonotope, generated by A and denoted by $Z(A)$, is a polytope formed by the Minkowski's sum of line segments

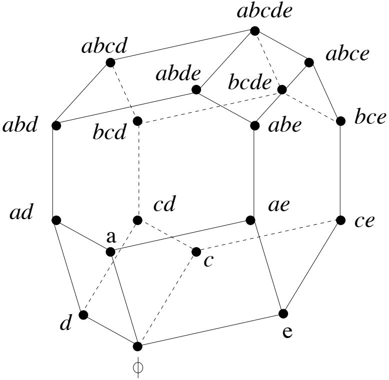
$$Z(A) = \{\alpha_1 + \cdots + \alpha_k | \alpha_i \in [-v_i, v_i]\}.$$

Regular Matroids - Applications



Regular Matroids - Applications

Permutahedron



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Voronoï's result : there exist exactly 5 regular matroids of rank 3.

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Example (classic) : the rank 3 matroid on 9 elements obtained from the Non-Pappus configuration