ON THE MÖBIUS FUNCTION OF THE LOCALLY FINITE POSET ASSOCIATED WITH A NUMERICAL SEMIGROUP

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ABSTRACT. Let S be a numerical semigroup and let $(\mathbb{Z}, \leq_S)$ be the (locally finite) poset induced by S on the set of integers $\mathbb{Z}$ defined by $x \leq_S y$ if and only if $y - x \in S$ for all integers x and y. In this paper, we investigate the *Möbius function* associated to $(\mathbb{Z}, \leq_S)$ when S is an *arithmetic* semigroup.

Keywords: Möbius function, poset, numerical semigroup, arithmetic semigroup

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1. INTRODUCTION

The *Möbius function* is an important concept associated to (*locally finite*) posets. Möbius function can be considered as a generalization of the classical Möbius arithmetic function on the integers (given by the Möbius function of the poset obtained from the positive integers partially ordered by the divisibility). Möbius function has been extremely useful to investigate many different problems. For instance, the *inclusion-exclusion principle* can be retrieved by considering the set of all subsets of a finite set partially ordered by inclusion. We refer the reader to [5] for a large number of applications of the Möbius function.

In this paper, we investigate the Möbius function associated to posets arising naturally from numerical semigroups as follows. Let $a_1, a_2, \ldots, a_n$ be $n \geq 1$ relatively prime positive integers and let $S = \langle a_1, a_2, \ldots, a_n \rangle$ denote the *numerical semigroup* generated by $a_1, a_2, \ldots, a_n$, that is,

$$S = \langle a_1, a_2, \ldots, a_n \rangle = \{x_1 a_1 + x_2 a_2 + \cdots + x_n a_n \mid x_1, x_2, \ldots, x_n \in \mathbb{N}\}.$$

Throughout this paper, we consider the structure of the poset induced by S on the set of integers $\mathbb{Z}$, whose partial order $\leq_S$ is defined by

$$x \leq_S y \iff y - x \in S,$$

for all integers x and y. This (locally finite) poset will be denoted by $(\mathbb{Z}, \leq_S)$.

We denote by μ_S the Möbius function associated to $(\mathbb{Z}, \leq_S)$. As far as we are aware, the only known result concerning μ_S is an old theorem due to Deddens [1] that determines the value of μ_S when S has exactly two generators. Here, we shall introduce and develop a new approach to investigate μ_S when S is an *arithmetic semigroup*, that is, when $S = \langle a, a + d, \ldots, a + kd \rangle$ for some integers a, d and $k \leq a - 1$.

This is a self-contained paper and it is organized as follows. In the next section, we review some classic notions of the Möbius function and present some results needed for the rest of the paper. In Section 3, we give a new direct proof of Deddens' result, shorter than the original one (based on a recursive case-by-case analysis). In Section 4, we discuss results about arithmetic semigroups, in particular, we prove the existence of *unique* representations. The

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latter is a key result that will be used, in Section 5, to give a recursive formula for μ_S when $S = \langle a, a + d, \ldots, a + kd \rangle$. Finally, in Section 6, we propose an explicit formula for μ_S (based on the *multiplicity function* of a multiset) in the case when $k = 2$ and a is even.

Background information on numerical semigroups can be found in the books [2, 4].

2. Möbius function

Let $(P, \leq)$ be a partially ordered set, or *poset* for short. The *strict partial order* $<_P$ is the reduction of $\leq_P$ given by, $a <_P b$ if and only if $a \leq_P b$ and $a \neq b$. For any a and b in the poset P, the *segments* between a and b are defined by

$$[a, b]_P = \{c \in P \mid a \leq_P c \leq_P b\}, \qquad]a, b]_P = \{c \in P \mid a <_P c \leq_P b\},$$

$$[a, b[_P = \{c \in P \mid a \leq_P c <_P b\}, \qquad]a, b[_P = \{c \in P \mid a <_P c <_P b\}.$$

A poset is said to be *locally finite* if every segment has finite cardinality. In this paper, we only consider locally finite posets.

Let a and b be elements of the poset P. A *chain* of length $l \geq 0$ between a and b is a subset of $[a, b]_P$ containing a and b, with cardinality $l + 1$ and totally ordered by $<$, that is $\{a_0, a_1, \ldots, a_l\} \subset [a, b]_P$ such that

$$a = a_0 <_P a_1 <_P a_2 <_P \cdots <_P a_{l-1} <_P a_l = b.$$

For any nonnegative integer l, we denote by $C_l(a, b)$ the set of all chains of length l between a and b. The cardinality of $C_l(a, b)$ is denoted by $c_l(a, b)$. This number is always finite because the poset P is supposed to be locally finite. For instance, the number of chains $c_2(2, 12)$, where the poset is the set $\mathbb{N}$ partially ordered by divisibility, is equal to 2. Indeed, there are exactly 2 chains of length 2 between 2 and 12 in $[2, 12]_{\mathbb{N}} = \{2, 4, 6, 12\}$, which are $\{2, 4, 12\}$ and $\{2, 6, 12\}$.

For any locally finite poset P, the *Möbius function* μ_P is the integer-valued function on $P \times P$ defined by

$$\mu_P(a, b) = \sum_{l \geq 0} (-1)^l c_l(a, b), \tag{1}$$

for all elements a and b of the poset P. One can remark that this sum is always finite because, for a and b given, there exists a maximal length of a possible chain between a and b since the segment $[a, b]_P$ has finite cardinality.

The concept of Möbius function for a locally finite poset $(P, \leq)$ was introduced by Rota in [5] as the inverse of the zeta function in the incidence algebra of a locally finite poset. Let us see this with more detail. Consider the set $\mathcal{I}(P)$ of all real-valued functions $f : P \times P \longrightarrow \mathbb{R}$ for which $f(a, b) = 0$ if $a \not\leq_P b$. The sum $+$ and the multiplication by scalars . are defined as usual in $\mathcal{I}(P)$. The product of two functions f and g in $\mathcal{I}(P)$ is defined by

$$(f \times g)(a, b) = \sum_{c \in [a, b]_P} f(a, c) g(c, b),$$

for all $(a, b) \in P \times P$. Then $(\mathcal{I}(P), +, ., \times)$ appears as an associative algebra over $\mathbb{R}$. This is the *incidence algebra* of P. The *Kronecker delta function* $\delta \in \mathcal{I}(P)$, defined by

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases}$$

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for all $(a, b) \in P \times P$, is the identity element of $\mathcal{I}(P)$. The *zeta function* $\zeta_P \in \mathcal{I}(P)$ is defined by

$$\zeta_P(a, b) = \begin{cases} 1 & \text{if } a \leq_P b, \\ 0 & \text{otherwise,} \end{cases}$$

for all $(a, b) \in P \times P$.

Rota [5] proved that the zeta function ζ_P (called the *inverse function*) is invertible in $\mathcal{I}(P)$ and showed that μ_P is recursively defined as follows: for all $(a, b) \in P \times P$, by

$$\mu_P(a, a) = 1 \quad \text{and} \quad \mu_P(a, b) = - \sum_{c \in [a, b[_P} \mu_P(a, c) \quad \text{if } a <_P b. \tag{2}$$

Let us see that both definitions of μ_S given by (1) and by (2) are equivalent. For, let a and b be two elements of the locally finite poset P such that $a <_P b$. Then,

$$c_l(a, b) = \sum_{c \in [a, b[_P} c_{l-1}(a, c) = \sum_{c \in]a, b]_P} c_{l-1}(c, b), \tag{3}$$

for all positive integers l. Indeed, every chain $\{a_0, a_1, \ldots, a_l\} \in C_l(a, b)$ can be seen like an extension of a chain of $C_{l-1}(a, a_{l-1})$ or of $C_{l-1}(a_1, b)$.

Obviously, the identity $\mu_S(a, a) = 1$ directly comes from (1) since $c_0(a, a) = 1$ and $c_l(a, a) = 0$ for all $l \geq 1$. By combining (3) and (1), for all $a <_P b$, we obtain that

$$\mu_P(a, b) = \sum_{l \geq 0} (-1)^l c_l(a, b) = c_0(a, b) + \sum_{l \geq 1} (-1)^l \sum_{c \in [a, b[_P} c_{l-1}(a, c).$$

Finally, since $a \neq b$, it follows that $c_0(a, b) = 0$ and thus

$$\mu_P(a, b) = \sum_{c \in [a, b[_P} \sum_{l \geq 0} (-1)^{l+1} c_l(a, c) = - \sum_{c \in [a, b[_P} \mu_P(a, c).$$

Similarly, using the second identity of (3), we can also prove that, whenever $a <_P b$, we have

$$\mu_P(a, b) = - \sum_{c \in]a, b]_P} \mu_P(c, b).$$

Therefore the two definitions of the Möbius function (for a locally finite posets) are the same. All the results presented in this paper are derived from the recursive formula presented in (2).

2.1. **Poset of integers induced by a numerical semigroup.** Let S be a numerical semigroup and $(\mathbb{Z}, \leq_S)$ its associated poset. Observe that $(\mathbb{Z}, \leq_S)$ is a locally finite poset since $\left| [x, y]_{(\mathbb{Z}, \leq_S)} \right| \leq y - x$, for all $x, y \in \mathbb{Z}$. It is easy to see that μ_S can be considered as a univariable function of $\mathbb{Z}$. Indeed, for all $x, y \in \mathbb{Z}$ and for all $p \geq 0$, we have

$$c_l(x, y) = c_l(0, y - x). \tag{4}$$

The above follows since the set $C_l(x, y)$ is in bijection with $C_l(0, y - x)$. Indeed the map that assigns the chain $\{x_0, x_1, \ldots, x_l\} \in C_l(x, y)$ to the chain $\{0, x_1 - x_0, \ldots, x_l - x_0\} \in C_l(0, y - x)$ is clearly a bijection. Thus, by definition of μ_S and equality (4) we obtain

$$\mu_S(x, y) = \mu_S(0, y - x)$$

for all $x, y \in \mathbb{Z}$.

In the sequel of this paper we shall only consider the reduced Möbius function $\mu_S : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by

$$\mu_S(x) = \mu_S(0, x), \quad \text{for all } x \in \mathbb{Z}.$$

This recursive formula given by (2) can be more easily presented when the locally finite poset is $(\mathbb{Z}, \leq_S)$.

Proposition 1. *Let S be a numerical semigroup and let $x \in \mathbb{Z} \setminus \{0\}$. Then,*

$$\mu_S(x) = - \sum_{y \in S \setminus \{0\}} \mu_S(x - y) \quad \Longleftrightarrow \quad \sum_{y \in S} \mu_S(x - y) = 0.$$

Proof. From (2), we deduce that

$$\mu_S(x) = - \sum_{y \in [0, x[_{(\mathbb{Z}, \leq_S)}} \mu_S(y) = - \sum_{\substack{y \in S \\ x - y \in S \setminus \{0\}}} \mu_S(y) = - \sum_{\substack{x - y \in S \\ y \in S \setminus \{0\}}} \mu_S(x - y).$$

The result follows since, by definition of μ_S, $\mu_S(x - y) = 0$ unless $x - y \in S$. $\qquad \square$

3. Deddens' result : new proof

In [1], Deddens proved the following.

Theorem 1. [1] *Let a and b be two relatively positive integers and let $S = \langle a, b \rangle$. Then, for all $x \in \mathbb{Z}$, we have*

$$\mu_S(x) = \begin{cases} 1 & \text{if } x \geq 0 \text{ and } x \equiv 0 \text{ or } a + b \pmod{ab}, \\ -1 & \text{if } x \geq 0 \text{ and } x \equiv a \text{ or } b \pmod{ab}, \\ 0 & \text{otherwise.} \end{cases}$$

Dedden's proof was based on a recursive argument and a case-by-case analysis. We may give the following direct proof of Theorem 1.

Proof of Theorem 1. We shall prove that

$$\mu_S(x) = \mu_S(x - ab) \tag{5}$$

for every $x \in \mathbb{Z} \setminus \{0, a, b, a + b\}$. The result then follows since $\mu_S(x) = 0$ for all $x < 0$, $\mu_S(0) = 1$, $\mu_S(a) = \mu_S(b) = -1$ and $\mu_S(a + b) = c_2(0, a + b) - c_1(0, a + b) = 2 - 1 = 1$.

Let us prove then equality (5). Let $S = \langle a, b \rangle = \{m_a a + m_b b \mid m_a, m_b \in \mathbb{N}\}$ and let $x \in \mathbb{Z} \setminus \{0\}$. By Proposition 1 we already know that

$$\mu_S(x) = - \sum_{y \in S \setminus \{0\}} \mu_S(x - y) = - \sum_{\substack{y \in S \setminus \{0\} \\ y - a \in S}} \mu_S(x - y) - \sum_{\substack{y \in S \setminus \{0\} \\ y - a \notin S}} \mu_S(x - y).$$

Since

$$\sum_{\substack{y \in S \setminus \{0\} \\ y - a \in S}} \mu_S(x - y) = \sum_{z \in S} \mu_S((x - a) - z) = 0, \quad \text{for } x - a \neq 0,$$

then

$$\mu_S(x) = - \sum_{\substack{y \in S \setminus \{0\} \\ y - a \notin S}} \mu_S(x - y), \quad \text{for } x \in \mathbb{Z} \setminus \{0, a\}.$$

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Moreover since $\{y \in S \setminus \{0\} \mid y - a \notin S\} = \{m_b b \mid m_b \in \{1, 2, \dots, a - 1\}\}$ then

$$\mu_S(x) = -\sum_{m_b=1}^{a-1} \mu_S(x - m_b b), \quad \text{for } x \in \mathbb{Z} \setminus \{0, a\}. \tag{6}$$

By applying (6) for $x - b \in \mathbb{Z} \setminus \{0, a\}$, that is, $x \in \mathbb{Z} \setminus \{b, a + b\}$ we obtain that

$$\mu_S(x - b) = -\sum_{m_b=2}^{a} \mu_S(x - m_b b), \quad \text{for } x - b \in \mathbb{Z} \setminus \{0, a\}. \tag{7}$$

By combining (6) and (7), for $x \in \mathbb{Z} \setminus \{0, a, b, a + b\}$, we obtain that

$$\begin{aligned}
\mu_S(x) &= -\sum_{m_b=1}^{a-1} \mu_S(x - m_b b) = -\mu_S(x - b) - \sum_{m_b=2}^{a-1} \mu_S(x - m_b b) \\
&= \sum_{m_b=2}^{a} \mu_S(x - m_b b) - \sum_{m_b=2}^{a-1} \mu_S(x - m_b b) \\
&= \mu_S(x - ab)
\end{aligned}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

4. ARITHMETIC SEMIGROUPS : PRELIMINARY RESULTS

Let S be a numerical semigroup. The *Apéry set* of S with respect with $m \in S$ is defined as

$$Ap(S; m) = \{x \in S \mid x - m \notin S\}.$$

It is known that $Ap(S; m)$ constitutes a complete set a of residues $\bmod\, m$.

Roberts [3] has proved that if $S = \langle a, a + d, \dots, a + kd \rangle$ with $\gcd(a, d) = 1$ and $k \in \{1, 2, \dots, a - 1\}$ then

$$Ap(S; a) = \left\{ \left\lceil \frac{i}{k} \right\rceil a + id \;\middle|\; i \in \{0, 1, \dots, a - 1\} \right\}. \tag{8}$$

The following result gives a unique representation of elements in arithmetic semigroups.

Lemma 1. *Let $x \in S = \langle a, a + d, \dots, a + kd \rangle$ with $2 \le k \le a - 1$. Then, there exists a unique triplet $(x_0, x_i, x_k) \in \mathbb{N} \times \{0, 1\} \times \{0, \dots, \lceil \frac{a}{k} \rceil\}$ such that*

$$x = x_0 a + x_i(a + id) + x_k(a + kd)$$

for some $1 \le i \le k - 1$ with $ix_i + kx_k < a$.

Proof. Let $x \in S$ and $x_0, \dots, x_k \in \mathbb{N}$ such that

$$x = x_0 a + x_1(a + d) + x_2(a + 2d) + \cdots + x_k(a + kd). \tag{9}$$

Existence: Let j_1 and j_2 be two integers such that $0 \le j_1, j_2 \le k$. We notice that $(a + j_1 d) + (a + j_2 d)$ can be expressed as either

$$(a + j_1 d) + (a + j_2 d) = a + (a + (j_1 + j_2)d), \text{ for } 0 \le j_1 + j_2 \le k$$

or

$$(a + j_1 d) + (a + j_2 d) = (a + kd) + (a + (j_1 + j_2 - k)d), \text{ for } k \le j_1 + j_2 \le 2k.$$

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So, by repeatedly adding consecutive terms from the expression of x in (9), we obtain that there exists a triplet (x_0, x_i, x_k) such that

$$x = x_0 a + x_i(a + id) + x_k(a + kd),$$

with $1 \leq i \leq k - 1$ and $x_i \in \{0, 1\}$. Moreover, we may suppose that $0 \leq x_k \leq \lfloor a/k \rfloor$. Otherwise, we use the following equality

$$
\begin{aligned}
\left(\left\lfloor \frac{a}{k} \right\rfloor + 1 \right)(a + kd) &= \left(\left\lfloor \frac{a}{k} \right\rfloor + 1 \right) a + \left(\left\lfloor \frac{a}{k} \right\rfloor k + k \right) d \\
&= \left(\left\lfloor \frac{a}{k} \right\rfloor + d + 1 \right) a + \left(\left\lfloor \frac{a}{k} \right\rfloor k + k - a \right) d \\
&= \left(\left\lfloor \frac{a}{k} \right\rfloor + d \right) a + \left(a + \left(\left\lfloor \frac{a}{k} \right\rfloor k + k - a \right) d \right),
\end{aligned}
$$

where $1 \leq \lfloor a/k \rfloor k + k - a \leq k$.

Finally, if $ix_i + kx_k \geq a$ then we consider the following representation

$$
\begin{aligned}
x &= x_0 a + x_i(a + id) + x_k(a + kd) \\
&= (x_0 + x_i + x_k) a + (ix_i + kx_k) d \\
&= (x_0 + x_i + x_k + d) a + (ix_i + kx_k - a) d \\
&= (x_0 + x_i + x_k + d - 1) a + (a + (ix_i + kx_k - a) d),
\end{aligned}
$$

where $0 \leq ix_i + kx_k - a \leq ix_i + k \lfloor a/k \rfloor - a \leq k - 1 + \lfloor a/k \rfloor k - a \leq k - 1$. Obtaining the desired decomposition.

Uniqueness: Let us suppose that there exist two triplets of non-negative integers (x_0, x_i, x_k) and (y_0, y_j, y_k) such that

$$x_0 a + x_i(a + id) + x_k(a + kd) = y_0 a + y_j(a + jd) + y_k(a + kd),$$

with $1 \leq i, j \leq k - 1$, $x_i, y_j \in \{0, 1\}$, $ix_i + kx_k < a$ and $jy_j + ky_k < a$. It follows that

$$(ix_i + kx_k)d \equiv (jy_j + ky_k)d \pmod{a},$$

and since $\gcd(a, d) = 1$ then

$$ix_i + x_k k \equiv jy_j + ky_k \pmod{a}$$

Moreover, since $ix_i + kx_k < a$ and $jy_j + ky_k < a$, then

$$ix_i + kx_k = jy_j + ky_k \text{ or equivalently } ix_i - jy_j = k(y_k - x_k).$$

We have four cases.

Case 1) if $x_i = 0$ and $y_j = 1$ then $-j$ would be a multiple of k which is impossible since $-1 \geq -j \geq -k$

Case 2) if $x_i = 1$ and $y_j = 0$ then i would be a multiple of k which is impossible since $1 \leq i \leq k$

Case 3) if $x_i = y_j = 1$ then $i - j$ would be a multiple of k but since $-k + 2 \leq i - j \leq k - 2$ then $i - j = 0$ implying that $x_k = y_k$ and thus $x_0 = y_0$.

Case 4) if $x_i = y_j = 0$ then $k(x_k - y_k) = 0$ and since $k \geq 1$ then $x_k = y_k$ and thus $x_0 = y_0$. $\square$

Let $(x_0, x_i, x_k) \in \mathbb{N} \times \{0,1\} \times \{0, \ldots, \lfloor a/k \rfloor\}$ with $1 \le i \le k-1$ and $ix_i + kx_k < a$. We shall denote by $[x_0, x_i, x_k]$ the element in S given by the representation of Lemma 1.

5. Recursive formula

We shall now present a recursive formula for μ_S when $S = \langle a, a+d, \ldots, a+kd \rangle$. The following key remark led us to guess such recursion. If $x = m_a a + m_d d$ such that $m_a \ge 0$ and $0 \le m_d \le a-1$ then,

$$x \in S \quad \Longleftrightarrow \quad m_a \ge \left\lceil \frac{m_d}{k} \right\rceil.$$

Theorem 2. *Let $S = \langle a, a+d, \ldots, a+kd \rangle$ with $\gcd(a,d) = 1$ and let $a = qk + r$ with $0 \le r < k$. Let $x \in \mathbb{Z} \setminus \{0, a, a+kd, a+(a+kd)\}$, then*

$$\mu_S(x) = \begin{cases} \mu_S(x - q(a+kd)) + \displaystyle\sum_{i=1}^{k-1} \mu_S(x - (a+id) - q(a+kd)) \\[2mm] \quad -\mu_S(x - (a+id)) & \text{if } r = 0, \\[4mm] \mu_S(x - (q+1)(a+kd)) + \displaystyle\sum_{i=r}^{k-1} \mu_S(x - (a+id) - q(a+kd)) \\[2mm] \quad -\displaystyle\sum_{i=1}^{k-1} \mu_S(x - (a+id)) & \text{if } r = 1, \\[4mm] \mu_S(x - (q+1)(a+kd)) + \displaystyle\sum_{i=1}^{r-1} \mu_S(x - (a+id) - (q+1)(a+kd)) \\[2mm] \quad + \displaystyle\sum_{i=r}^{k-1} \mu_S(x - (a+id) - q(a+kd)) - \displaystyle\sum_{i=1}^{k-1} \mu_S(x - (a+id)) & \text{if } r \ge 2. \end{cases}$$

Proof. Let $x \in \mathbb{Z} \setminus \{0\}$. As for the proof of Theorem 1, we have

$$\mu_S(x) = -\sum_{y \in S \setminus \{0\}} \mu_S(x-y)$$

$$= -\sum_{\substack{y \in S \setminus \{0\} \\ y - a \in S}} \mu_S(x-y) - \sum_{\substack{y \in S \setminus \{0\} \\ y - a \notin S}} \mu_S(x-y)$$

$$= -\sum_{z \in S} \mu_S((x-a) - z) - \sum_{\substack{y \in S \setminus \{0\} \\ y - a \notin S}} \mu_S(x-y)$$

$$= -\sum_{\substack{y \in S \setminus \{0\} \\ y - a \notin S}} \mu_S(x-y) \quad \text{if} \quad x - a \ne 0.$$

Let us now determine the set

$$\{y \in S \setminus \{0\} \mid y - a \notin S\} = Ap(S; a) \setminus \{0\}.$$

For, we consider the set $Ap(S; a)$ given by (8) in function of the unique representation of Lemma 1. We have three cases.

Case a) If $r = 0$ then

$$Ap(S, a) \setminus \{0\} = \{y_k(a+kd) \mid y_k \in \{1, \ldots, q-1\}\}$$
$$\bigcup \left\{ (a+id) + y_k(a+kd) \;\middle|\; \begin{array}{l} y_k \in \{0, \ldots, q-1\} \\ i \in \{1, \ldots, k-1\} \end{array} \right\}$$

Case b) If $r = 1$ then

$$Ap(S, a) \setminus \{0\} = \{y_k(a + kd) \mid y_k \in \{1, \ldots, q\}\}$$

$$\bigcup \left\{ (a + id) + y_k(a + kd) \;\middle|\; \begin{array}{l} y_k \in \{0, \ldots, q - 1\} \\ i \in \{r, \ldots, k - 1\} \end{array} \right\}$$

Case c) If $r \geq 2$ then

$$Ap(S, a) \setminus \{0\} = \{y_k(a + kd) \mid y_k \in \{1, \ldots, q\}\}$$

$$\bigcup \left\{ (a + id) + y_k(a + kd) \;\middle|\; \begin{array}{l} y_k \in \{0, \ldots, q\} \\ i \in \{1, \ldots, r - 1\} \end{array} \right\}$$

$$\bigcup \left\{ (a + id) + y_k(a + kd) \;\middle|\; \begin{array}{l} y_k \in \{0, \ldots, q - 1\} \\ i \in \{r, \ldots, k - 1\} \end{array} \right\}$$

Suppose that $r = 0$, i.e. $a = qk$. For $x \in \mathbb{Z} \setminus \{0, a\}$, we have

$$\mu_S(x) = - \sum_{y_k=1}^{q-1} \mu_S(x - y_k(a + kd)) - \sum_{i=1}^{k-1} \sum_{y_k=0}^{q-1} \mu_S(x - (a + id) - y_k(a + kd)). \qquad (10)$$

By applying (10) to $x - (a + kd) \in \mathbb{Z} \setminus \{0, a\}$, that is, $x \in \mathbb{Z} \setminus \{a + kd, a + (a + kd)\}$ we obtain

$$\mu_S(x - (a + kd)) = - \sum_{y_k=2}^{q} \mu_S(x - y_k(a + kd)) - \sum_{i=1}^{k-1} \sum_{y_k=1}^{q} \mu_S(x - (a + id) - y_k(a + kd)). \qquad (11)$$

By combining (10) and (11) for $x \in \mathbb{Z} \setminus \{0, a, a + kd, a + (a + kd)\}$, we obtain

$$\mu_S(x) = \mu_S(x - q(a + kd)) + \sum_{i=1}^{k-1} \mu_S(x - (a + id) - q(a + kd)) - \mu_S(x - (a + id)).$$

The cases when $r \geq 1$ are similar to Case a (and it is left to the reader as an exercise). $\qquad \square$

6. Case $\langle 2q, 2q + d, 2q + 2d \rangle$

The *multiplicity function* of a multiset A of $\mathbb{N}$ is the function

$$\mathfrak{m}_A : \mathbb{N} \longrightarrow \mathbb{N}$$

which assigns to each element $x \in \mathbb{N}$ its multiplicity, that is, the number of times that x appears in the multiset A.

Let $a = 2q$ and $d \in \mathbb{N}^*$ such that $\gcd(a, d) = \gcd(q, d) = 1$. For each $i \in \{-1, 0, 1\}$, we consider the following multisets.

$$A_i = \{m(q + d) + i \mid m \in \mathbb{N}\},$$

$$B_i = \{m(q + d) - nd + i \mid m \in \mathbb{N} \setminus \{0, 1\}, n \in \{1, 2, \ldots, \lfloor m/2 \rfloor\}\},$$

$$C_i = A_i \bigcup B_i.$$

As we mentioned above, given a triple $(x_0, x_1, x_2) \in \mathbb{N} \times \{0, 1\} \times \{0, \ldots, q - 1\}$, we denote by $[x_0, x_1, x_2]$ the element in S given by the representation in Lemma 1. In the sequel of this section, we shall consider this representation for all $x_0 \in \mathbb{Z}$, i.e.,

$$[x_0, x_1, x_2] = x_0 a + x_1(a + d) + x_2(a + 2d)$$

for all $(x_0, x_1, x_2) \in \mathbb{Z} \times \{0,1\} \times \{0,\ldots,q-1\}$. In this case, it is clear that

$$\text{if } (x_0, x_1, x_2) \in \mathbb{Z} \times \{0,1\} \times \{0,\ldots,q-1\} \text{ then } [x_0, x_1, x_2] \in S \iff x_0 \in \mathbb{N}.$$

The latter will be used in the proofs below.

Theorem 3. *Let $S =< 2q, 2q + d, 2q + 2d >$. Let $(x_0, x_1, x_2) \in \mathbb{Z} \times \{0,1\} \times \{0,\ldots,q-1\}$. Then,*

$$\mu_S([x_0, x_1, x_2]) = \begin{cases} (-1)^{x_1} \left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0) & \text{if } x_2 = 0, \\ (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right)(x_0 - x_2) & \text{if } x_2 \geq 1. \end{cases}$$

We notice that if $x_0 - x_2$ is a constant then we should have the same value for $\mu_S([x_0, 0, x_2])$. The latter is illustrated by the first values of $\mu_S([x_0, 0, x_2])$, listed in Table 1 given at the end of the section, for the case when $a = 22$ and $d = 5$. Indeed, we can see appearing diagonals (corresponding to $x_0 - x_2$ constant) with the same value.

Before proving Theorem 3, we need two lemmas and the following refinement of Theorem 2 when $k = 2$ and a even.

Proposition 2. *Let $(x_0, x_1, x_2) \in \mathbb{Z} \times \{0,1\} \times \{0,\ldots,q-1\}$ with $(x_0, x_2) \notin \{0,1\} \times \{0,1\}$. Then,*

$$\mu_S([x_0, x_1, 0]) = \mu_S([x_0 - (q+d), x_1, 0])$$
$$+\mu_S([x_0 - (q+d) - 1, x_1, q-1])$$
$$-\mu_S([x_0 - 2(q+d) - 1, x_1, q-1])$$

and

$$\mu_S([x_0, x_1, x_2]) = \mu_S([x_0 - (q+d), x_1, x_2])$$
$$+\mu_S([x_0 - 1, x_1, x_2 - 1])$$
$$-\mu_S([x_0 - (q+d) - 1, x_1, x_2 - 1])$$

when $x_2 \geq 1$.

Proof. From Theorem 2, we have

$$\mu_S([x_0, x_1, x_2]) = \mu_S([x_0, x_1, x_2] - [q+d, 0, 0]) + \mu_S([x_0, x_1, x_2] - [q+d, 1, 0]) \quad (12)$$
$$-\mu_S([x_0, x_1, x_2] - [0, 1, 0]).$$

Notice that $q(a + 2d) = (q + d)a$ and $a + (a + 2d) = 2(a + d)$.
Case a) If $x_1 = 0$ and $x_2 = 0$ then, from (12) we obtain

$$\mu_S([x_0, 0, 0]) = \mu_S([x_0, 0, 0] - [q+d, 0, 0]) + \mu_S([x_0, 0, 0] - [q+d, 1, 0]) \quad (13)$$
$$-\mu_S([x_0, 0, 0] - [0, 1, 0]).$$
$$= \mu_S([x_0 - (q+d), 0, 0]) + \mu_S([x_0 - 2(q+d) - 1, 1, q-1])$$
$$-\mu_S([x_0 - (q+d) - 1, 1, q-1]).$$

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By applying the recursive equality (12) to $\mu_S \left([x_0 - (q + d) - 1, 1, q - 1]\right)$, we obtain

$$\mu_S \left([x_0 - (q + d) - 1, 1, q - 1]\right) = \mu_S \left([x_0 - 2(q + d) - 1, 1, q - 1]\right) \tag{14}$$
$$+ \mu_S \left([x_0 - 2(q + d) - 1, 0, q - 1]\right)$$
$$- \mu_S \left([x_0 - (q + d) - 1, 0, q - 1]\right).$$

Finally, by combining equations (13) and (14) we have

$$\mu_S \left([x_0, 0, 0]\right) = \mu_S \left([x_0 - (q + d), 0, 0]\right)$$
$$+ \mu_S \left([x_0 - (q + d) - 1, 0, q - 1]\right)$$
$$- \mu_S \left([x_0 - 2(q + d) - 1, 0, q - 1]\right).$$

Case b) If $x_1 = 0$ and $x_2 \geq 1$ then, from (12) we obtain

$$\mu_S \left([x_0, 0, x_2]\right) = \mu_S \left([x_0, 0, x_2] - x[q + d, 0, 0]\right) + \mu_S \left([x_0, 0, x_2] - [q + d, 1, 0]\right) \tag{15}$$
$$- \mu_S \left([x_0, 0, x_2] - x[0, 1, 0]\right)$$
$$= \mu_S \left([x_0 - (q + d), 0, x_2]\right) + \mu_S \left([x_0 - (q + d) - 1, 1, x_2 - 1]\right)$$
$$- \mu_S \left([x_0 - 1, 1, x_2 - 1]\right).$$

By applying the recursive equality (12) to $\mu_S \left([x_0 - 1, 1, x_2 - 1]\right)$, we obtain

$$\mu_S \left([x_0 - 1, 1, x_2 - 1]\right) = \mu_S \left([x_0 - (q + d) - 1, 1, x_2 - 1]\right) \tag{16}$$
$$+ \mu_S \left([x_0 - (q + d) - 1, 0, x_2 - 1]\right)$$
$$- \mu_S \left([x_0 - 1, 0, x_2 - 1]\right).$$

Finally, by combining equations (14) and (16) we have

$$\mu_S \left([x_0, 0, x_2]\right) = \mu_S \left([x_0 - (q + d), 0, x_2]\right)$$
$$+ \mu_S \left([x_0 - 1, 0, x_2 - 1]\right)$$
$$- \mu_S \left([x_0 - (q + d) - 1, 0, x_2 - 1]\right).$$

This concludes the proof for $x_1 = 0$. The proof for the case $x_1 = 1$ is similar as the above case and it is left to the reader. □

Lemma 2. *Let $i \in \{-1, 0, 1\}$. For all $x \in \mathbb{Z} \setminus \{i\}$, $\mathfrak{m}_{A_i}(x) = \mathfrak{m}_{A_i}(x - (q + d))$.*

Proof. By definition of the sets A_i, for any integer $x \leq q + d + i - 1$ such that $x \neq i$, we have

$$\mathfrak{m}_{A_i}(x) = \mathfrak{m}_{A_i}(x - (q + d)) = 0.$$

For any integer $x \geq q + d + i$, we obtain

$$x \in A_i \iff \text{there exists } m \in \mathbb{N} \text{ with } x = m(q + d) + i$$
$$\iff \text{there exists } m \in \mathbb{N} \text{ with } x - (q + d) = (m - 1)(q + d) + i$$
$$\iff x - (q + d) \in A_i.$$

This completes the proof. □

Lemma 3. *Let $i \in \{-1, 0, 1\}$. For all $x \in \mathbb{Z}$, $\mathfrak{m}_{B_i}(x) = \mathfrak{m}_{C_i}(x - (2q + d))$.*

Proof. By definition of the multisets B_i and C_i, for any integer $x \le 2q + d + i - 1$, we have

$$\mathfrak{m}_{B_i}(x) = \mathfrak{m}_{C_i}(x - (2q + d)) = 0.$$

For any integer $x \ge 2q + d + i$, we obtain

$$
\begin{aligned}
x \in B_i \iff & \text{ there exists } m \in \mathbb{N} \setminus \{0, 1\},\ 1 \le n \le \lfloor m/2 \rfloor \text{ with } x = m(q + d) - nd + i \\
\iff & \text{ there exists } m \in \mathbb{N} \setminus \{0, 1\},\ 1 \le n \le \lfloor m/2 \rfloor \text{ with } x - (2q + d) = (m - 2)(q + d) \\
& \hspace{9.5cm} -(n - 1)d + i \\
\iff & \text{ there exists } m \in \mathbb{N},\ 0 \le n \le \lfloor m/2 \rfloor \text{ with } x - (2q + d) = m(q + d) - nd + i \\
\iff & x - (2q + d) \in C_i.
\end{aligned}
$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We may now prove Theorem 3.

Proof of Theorem 3. By double induction on x_0 and x_2.

For $x_0 < 0$, since $[x_0, x_1, x_2] \notin S$ and $C_i \cap (\mathbb{Z} \setminus \mathbb{N}) = \emptyset$ for all $i \in \{-1, 0, 1\}$, it follows that

$$\mu_S([x_0, x_1, 0]) = (-1)^{x_1} (\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1})(x_0) = 0,$$

and

$$\mu_S([x_0, x_1, x_2]) = (-1)^{x_1} (2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1})(x_0 - x_2) = 0,$$

for all $x_2 \in \{1, \ldots, q - 1\}$.

Now, for $x_0 \in \mathbb{N}$, we suppose that the theorem is true for all values lesser than x_0 and all $x_2 \in \{0, \ldots, q - 1\}$. We distinguish different cases according to the values of x_2.

Case a) $x_2 = 0$.

For $x_0 = 0$, since $\mu_S([0, 0, 0]) = \mu_S(0) = 1$ and $\mu_S([0, 1, 0]) = \mu_S(a + d) = -1$, it follows, by definition of the multisets A_i and B_i, that

$$\mu_S([0, x_1, 0]) = (-1)^{x_1} = (-1)^{x_1}(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1})(0).$$

For $x_0 = 1$, since $\mu_S([1, 0, 0]) = \mu_S(a) = -1$ and $\mu_S([1, 1, 0]) = \mu_S(a + (a + d)) = c_2(0, 2a + d) - c_1(0, 2a + d) = 2 - 1 = 1$, it follows, by definition of the multisets A_i and B_i, that

$$\mu_S([1, x_1, 0]) = (-1)^{x_1 + 1} = (-1)^{x_1}(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1})(1).$$

Suppose now that $x_0 \ge 2$. From Proposition 2, we have

$$\mu_S([x_0, x_1, 0]) = \mu_S([x_0 - (q+d), x_1, 0]) + \mu_S([x_0 - (q+d) - 1, x_1, q-1]) - \mu_S((x_0 - 2(q+d), x_1, q-1]).$$

By induction hypothesis, we have

$$\mu_S([x_0 - (q + d), x_1, 0]) = (-1)^{x_1} \left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right) (x_0 - (q + d)),$$

$$\mu_S([x_0 - (q + d) - 1, x_1, q - 1]) = (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - (2q + d))$$

and

$$\mu_S([x_0 - 2(q + d) - 1, x_1, q - 1]) = (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - (3q + 2d)).$$

By Lemma 2, since $x_0 \ge 2$, we already know that

$$\mathfrak{m}_{A_0}(x_0 - (q + d)) = \mathfrak{m}_{A_0}(x_0) \quad \text{and} \quad \mathfrak{m}_{A_1}(x_0 - (q + d)) = \mathfrak{m}_{A_1}(x_0).$$

Moreover, by Lemma 3, we have

$$\mathfrak{m}_{C_i}(x_0 - (2q + d)) = \mathfrak{m}_{B_i}(x_0) \quad \text{and} \quad \mathfrak{m}_{C_i}(x_0 - (3q + 2d)) = \mathfrak{m}_{B_i}(x_0 - (q + d)),$$

for all $i \in \{-1, 0, 1\}$. Therefore,

$$
\begin{aligned}
\mu_S([x_0, x_1, 0]) &= (-1)^{x_1} \left[(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1})(x_0 - (q + d)) \right. \\
&\quad + \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - (q + d)) \\
&\quad + \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right)(x_0 - (2q + d)) \\
&\quad \left. - \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right)(x_0 - (3q + 2d)) \right] \\
&= (-1)^{x_1} \left[(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1})(x_0) \right. \\
&\quad + \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - (q + d)) \\
&\quad + \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0) \\
&\quad \left. - \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - (q + d)) \right] \\
&= (-1)^{x_1} \left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0).
\end{aligned}
$$

Case b) $x_2 = 1$.

For $x_0 = 0$, since $\mu_S([0, 0, 1]) = \mu_S(a + 2d) = -1$ and $\mu_S([0, 1, 1]) = \mu_S((a + d) + (a + 2d)) = c_2(0, 2a + 3d) - c_1(0, 2a + 3d) = 2 - 1 = 1$, it follows, by definition of the multisets C_i, that

$$
\mu_S([0, x_1, 1]) = (-1)^{x_1 + 1} = (-1)^{x_1}(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1})(-1).
$$

For $x_0 = 1$, since $\mu_S([1, 0, 1]) = \mu_S(a + (a + 2d)) = c_2(0, 2a + 2d) - c_1(0, 2a + 2d) = 3 - 1 = 2$ and $\mu_S([1, 1, 1]) = \mu_S(a + (a + d) + (a + 2d)) = -c_3(0, 3a + 3d) + c_2(0, 3a + 3d) - c_1(0, 3a + 3d) = -7 + 10 - 1 = 2$, it follows, by definition of the multisets C_i, that

$$
\mu_S([1, x_1, 1]) = (-1)^{x_1} 2 = (-1)^{x_1}(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1})(0).
$$

Suppose now that $x_0 \geq 2$. From Proposition 2, we have

$$
\mu_S([x_0, +x_1, 1]) = \mu_S([x_0 - (q + d), x_1, 1]) + \mu_S([x_0 - 1, x_1, 0]) - \mu_S([x_0 - (q + d) - 1, x_1, 0]).
$$

By using Lemmas 2 and 3 and the induction hypothesis, we have

$$
\mu_S([x_0 - (q + d), x_1, 1]) = (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right)(x_0 - (q + d) - 1),
$$

$$
\mu_S([x_0 - 1, x_1, 0]) = (-1)^{x_1} \left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - 1)
$$

and

$$
\mu_S([x_0 - (q + d) - 1, x_1, 0) = (-1)^{x_1} \left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} + 2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - (q + d) - 1).
$$

First, since the multiset difference $C_i \setminus B_i$ is equal to A_i for all $i \in \{-1, 0, 1\}$, it follows that

$$
\mathfrak{m}_{C_i} - \mathfrak{m}_{B_i} = \mathfrak{m}_{A_i}.
$$

Therefore,

$$
\begin{aligned}
\mu_S([x_0, x_1, 1]) &= (-1)^{x_1} \left[(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1})(x_0 - 1) - (\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1})(x_0 - (q + d) - 1) \right. \\
&\quad + \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right)(x_0 - (q + d) - 1) \\
&\quad - \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - (q + d) - 1) \\
&\quad \left. + \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - 1) \right] \\
&= (-1)^{x_1} \left[(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1})(x_0 - 1) - (\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1})(x_0 - (q + d) - 1) \right. \\
&\quad + \left(2\mathfrak{m}_{A_0} - \mathfrak{m}_{A_{-1}} - \mathfrak{m}_{A_1} \right)(x_0 - (q + d) - 1) \\
&\quad \left. + \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - 1) \right] \\
&= (-1)^{x_1} \left[(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1})(x_0 - 1) + \left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_{-1}} \right)(x_0 - (q + d) - 1) \right. \\
&\quad \left. + \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right)(x_0 - 1) \right].
\end{aligned}
$$

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Moreover, by Lemma 2, since $x_0 \geq 2$, we know that

$$\mathfrak{m}_{A_0}(x_0 - (q+d) - 1) = \mathfrak{m}_{A_0}(x_0 - 1) \quad \text{and} \quad \mathfrak{m}_{A_{-1}}(x_0 - (q+d) - 1) = \mathfrak{m}_{A_{-1}}(x_0 - 1).$$

Finally, in this case, we obtain

$$
\begin{aligned}
\mu_S([x_0, x_1, 1]) &= (-1)^{x_1} \left[\left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_1} \right) (x_0 - 1) + \left(\mathfrak{m}_{A_0} - \mathfrak{m}_{A_{-1}} \right) (x_0 - (q+d) - 1) \right. \\
&\qquad \left. + \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right) (x_0 - 1) \right]. \\
&= (-1)^{x_1} \left[\left(2\mathfrak{m}_{A_0} - \mathfrak{m}_{A_{-1}} - \mathfrak{m}_{A_1} \right) (x_0 - 1) + \left(2\mathfrak{m}_{B_0} - \mathfrak{m}_{B_{-1}} - \mathfrak{m}_{B_1} \right) (x_0 - 1) \right] \\
&= (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - 1).
\end{aligned}
$$

Case c) $x_2 \geq 2$.

From Proposition 2, we have

$$\mu_S([x_0, x_1, x_2]) = \mu_S([x_0 - (q+d), x_1, x_2]) + \mu_S([x_0 - 1, x_1, x_2 - 1]) - \mu_S([x_0 - (q+d) - 1, x_1, x_2 - 1]).$$

By induction, we have

$$\mu_S([x_0 - (q+d), x_1 x_2]) = (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - x_2 - (q+d)),$$

$$\mu_S([x_0 - 1, x_1, x_2 - 1]) = (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - x_2)$$

and

$$\mu_S([x_0 - (q+d) - 1, x_1, x_2 - 1]) = (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - x_2 - (q+d)).$$

Therefore,

$$
\begin{aligned}
\mu_S([x_0, x_1, x_2]) &= (-1)^{x_1} \left[\left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - x_2 - (q+d)) \right. \\
&\qquad + \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - x_2) \\
&\qquad \left. - \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - x_2 - (q+d)) \right] \\
&= (-1)^{x_1} \left(2\mathfrak{m}_{C_0} - \mathfrak{m}_{C_{-1}} - \mathfrak{m}_{C_1} \right) (x_0 - x_2).
\end{aligned}
$$

This completes the proof of Theorem 3. $\qquad\qquad\square$

Table 1: First values of $\mu_S([x_0, 0, x_2])$ for $q = 11$ and $d = 5$.

x_0 \ x_2	0	1	2	3	4	5	6	7	8	9	10
0	**1**	**−1**	0	0	0	0	0	0	0	0	0
1	**−1**	**2**	**−1**	0	0	0	0	0	0	0	0
2	0	**−1**	**2**	**−1**	0	0	0	0	0	0	0
	0	0	**−1**	**2**	**−1**	0	0	0	0	0	0
	0	0	0	**−1**	**2**	**−1**	0	0	0	0	0
	0	0	0	0	**−1**	**2**	**−1**	0	0	0	0
	0	0	0	0	0	**−1**	**2**	**−1**	0	0	0
	0	0	0	0	0	0	**−1**	**2**	**−1**	0	0
	0	0	0	0	0	0	0	**−1**	**2**	**−1**	0
	0	0	0	0	0	0	0	0	**−1**	**2**	**−1**
$q-1$	0	0	0	0	0	0	0	0	0	**−1**	**2**
q	0	0	0	0	0	0	0	0	0	0	**−1**
$q+1$	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0
$q+d-1$	**0**	0	0	0	0	0	0	0	0	0	0
$q+d$	**1**	**−1**	0	0	0	0	0	0	0	0	0
$q+d+1$	**−1**	**2**	**−1**	0	0	0	0	0	0	0	0

Table 1 – continued from previous page

x_0 \ x_2	0	1	2	3	4	5	6	7	8	9	10
$q+d+2$	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
	0	0	0	−1	2	−1	0	0	0	0	0
	0	0	0	0	−1	2	−1	0	0	0	0
	0	0	0	0	0	−1	2	−1	0	0	0
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$2q+d-1$	−1	0	0	0	0	0	0	0	0	−1	2
$2q+d$	2	−1	0	0	0	0	0	0	0	0	−1
$2q+d+1$	−1	2	−1	0	0	0	0	0	0	0	0
	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
$2q+2d-1$	0	0	0	−1	2	−1	0	0	0	0	0
$2q+2d$	1	−1	0	0	−1	2	−1	0	0	0	0
$2q+2d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$2q+2d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1
	0	0	0	0	−1	2	−1	0	0	−1	2
	0	0	0	0	0	−1	2	−1	0	0	−1
	0	0	0	0	0	0	−1	2	−1	0	0
	0	0	0	0	0	0	0	−1	2	−1	0
	0	0	0	0	0	0	0	0	−1	2	−1
$3q+2d-1$	−1	0	0	0	0	0	0	0	0	−1	2
$3q+2d$	2	−1	0	0	0	0	0	0	0	0	−1
$3q+2d+1$	−1	2	−1	0	0	0	0	0	0	0	0
	0	−1	2	−1	0	0	0	0	0	0	0
	0	0	−1	2	−1	0	0	0	0	0	0
$3q+3d-1$	0	0	0	−1	2	−1	0	0	0	0	0
$3q+3d$	1	−1	0	0	−1	2	−1	0	0	0	0
$3q+3d+1$	−1	2	−1	0	0	−1	2	−1	0	0	0
$3q+3d+2$	0	−1	2	−1	0	0	−1	2	−1	0	0
	0	0	−1	2	−1	0	0	−1	2	−1	0
	0	0	0	−1	2	−1	0	0	−1	2	−1

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