NUMERICAL SEMIGROUP TREE: TYPE-REPRESENTATION

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Abstract. In this paper, we introduce a new depicting of the so-called numerical semigroup tree $\mathcal{T}$. By exploring computationally this improved picture, relying on the type notion of a semigroup, we found that the number of semigroups of genus g and type t is constante when t is close to g while g grows. We also study the unimodularity of various sequences as well as the behavior of the leaves in $\mathcal{T}$. We put forward several conjectures that are supported by various computational experiments.

1. Introduction

A *numerical semigroup* Λ is a subset of the non-negative integers $\mathbb{N}_0$, containing 0, closed under addition and with finite complement in $\mathbb{N}_0$. The elements of $\mathbb{N}_0$ that are not in Λ are called the *gaps* and the number of gaps $g(\Lambda)$ is the *genus* of the numerical semigroup. A relevant invariant of Λ is the so-called *Frobenius number* $F(\Lambda)$ defined as the largest gap in Λ (we refer the reader to the book [12] for a thoroughly discussion on the Frobenius number).

The *numerical semigroup tree*, first introduced in Rosales' thesis [14] (and discussed in [15, Introduction]), is the infinite tree $\mathcal{T}$ whose root is the semigroup $\mathbb{N}_0$ and in which the *parent* of a numerical semigroup Λ is the numerical semigroup Λ' obtained by adjoining to Λ its Frobenius number. In turn, the *descendants* of a numerical semigroup Λ are the numerical semigroups obtained by taking away one by one the generators that are larger than $F(\Lambda)$. For instance, the descendants of $\Lambda = \langle 3, 7, 8\rangle$ are $\Lambda' = \langle 3, 7, 11\rangle$ and $\Lambda'' = \langle 3, 8, 10\rangle$ (or equivalently, the parent of Λ' and Λ'' is Λ). Indeed, since $7, 8 > F(\Lambda) = 5$ are generators of Λ then $\Lambda' = \Lambda \setminus \{8\}$ and $\Lambda'' = \Lambda \setminus \{7\}$ (or equivalently, $\Lambda = \Lambda' \cup F(\Lambda') = \Lambda'' \cup F(\Lambda'')$ with $F(\Lambda') = 8$ and $F(\Lambda'') = 7$). We notice that all numerical semigroups are in $\mathcal{T}$ and the parent of a numerical semigroup of genus g has genus $g-1$, see Figure 1.

The tree $\mathcal{T}$ has been the object of many investigations in different contexts: Frobenius pseudo-varieties [13], number of numerical semigroups of given genus [1, 4, 5], Wilf's conjecture [9, 10, 11, 19], etc.

In this paper, we introduce a new approach to study $\mathcal{T}$. We put forward a new way to represent $\mathcal{T}$ by using the notion of *type* of a numerical semigroup. This representation outcomes unexpected behaviors of numerical semigroups according with both their type and genus. We hope (and expect) that this investigation leads not only to a better comprehension of $\mathcal{T}$ but also to open new methods to attack well-known open problems.

The type of a numerical semigroup Λ is defined in [6] as the cardinality of the set of its pseudo-Frobenius numbers. The former also equals the Cohen-Macaulay type of the semigroup ring $K[\Lambda]$, where K is any field. In general, it is difficult to estimate the type in terms of the minimal generators of the semigroup, see [18].

In the next section, we introduce the type-representation depicting $\mathcal{T}$ and present some preliminary results on the parent-descendants types relationship.

Let $n(g,t)$ be the number of numerical semigroups of genus g and type t. In Section 3, we investigate a mysterious stabilizer behavior of the value $n(g,t)$ when t is close to g while g grows, as indicated by the data in Table 4. We show that the value $n(g,t)$ becomes constant when g is large enough and $t \geqslant (2g-1)/3$ (Theorem 3.7). We also study the shape of the generators of these *stabilizers* semigroups (Theorem 3.5, Propositions 3.8 and 3.9) allowing us to obtain the

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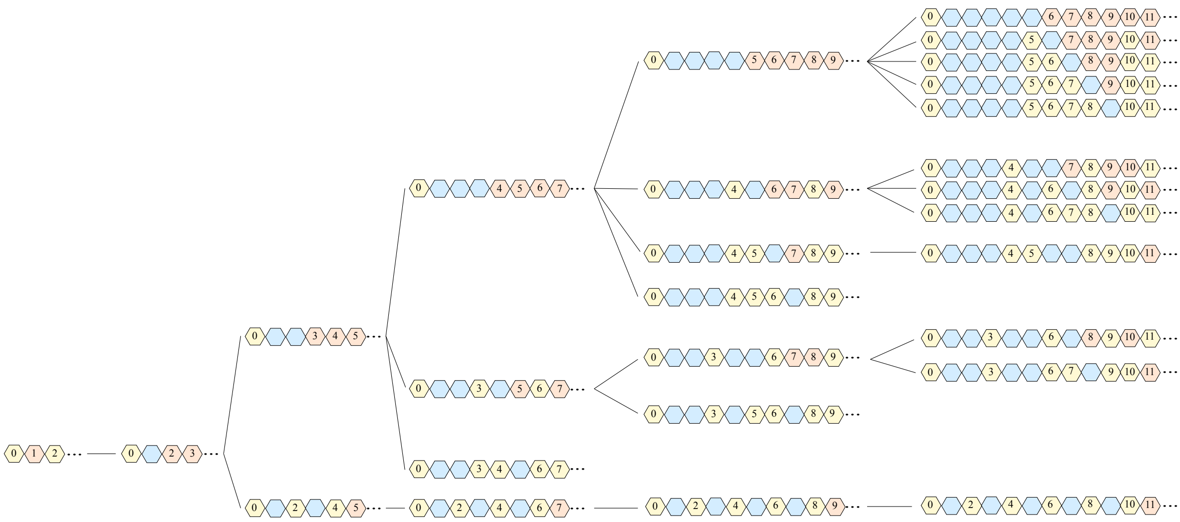

Figure 1. First levels of tree $\mathcal{T}$ where yellow, blue and red hexagons denote elements, gaps and generators greater than the Frobenius number of the corresponding semigroup respectively.

lower bound $n(g, g-\ell) \geqslant \ell^2 - 3\ell + 10$ for $\ell \geqslant 6$ and $g \geqslant 3\ell - 1$ (Theorem 3.10). One tool is to represent the gaps relevant for deciding the type of the semigroup in terms of what we call *gap vectors*. These are (0/1)-vectors of length $2(g-t)$ with equal number of 0's and 1's. When $t \geqslant (2g-1)/3$ we show that the semigroup is essentially determined by its genus and the gap vector (Proposition 3.3).

In Section 4, we focus our attention to the study of the unimodularity property of various sequences, for instance, the one obtained by the number of semigroups with fixed genus and increasing type that are leaves in $\mathcal{T}$. We put forward a number of conjectures/problems supported by various experimental computations that uses the `numericalsgps package` [3] in `GAP` [8].

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2. Type-representation

2.1. Some preliminaries. The *type* $t(\Lambda)$ of Λ is the cardinality of the set

$$PF(\Lambda) = \{x \in \mathbb{Z} \setminus \Lambda \mid x + \ell \in \Lambda \text{ for all } \ell \in \Lambda \setminus \{0\}\}.$$

The elements of $PF(\Lambda)$ are called *pseudo-Frobenius* numbers. This concept was introduced in [6]. Notice that $F(\Lambda) = \max\{p \mid p \in PF(\Lambda)\}$. The partial order $\leqslant_\Lambda$ induced by Λ on the integers is defined as follows: $x \leqslant_\Lambda y$ if $y - x \in \Lambda$. It is well-known [6] that

1) $PF(\Lambda) = \text{Maximals}_{\leqslant_\Lambda}(\mathbb{Z} \setminus \Lambda)$
2) $x \in \mathbb{Z} \setminus \Lambda$ if and only if $f - x \in \Lambda$ for some $f \in PF(\Lambda)$.

It is known that each numerical semigroup has a unique minimal system of generators. The cardinality of the minimal system of generators is called the *embedding dimension* and we denote it by $e(\Lambda)$. The *multiplicity* of Λ, denoted by $m(\Lambda)$, is the least positive element in Λ.

If Λ is a numerical semigroup with positive genus, that is, $\Lambda \subsetneq \mathbb{N}$, then $\Lambda \cup \{F(\Lambda)\}$ is also a numerical semigroup whose embedding dimension is one more that of Λ. Also,

$$m(\Lambda \cup \{F(\Lambda)\}) = \begin{cases} m(\Lambda) - 1 & \text{if } t(\Lambda) = g(\Lambda), \\ m(\Lambda) & \text{otherwise.} \end{cases}$$

We have, by construction, that $g(\Lambda) > g(\Lambda \cup F(\Lambda)) = g(\Lambda) - 1$.

2.2. **New depicting of $\mathcal{T}$.** Let Λ be a numerical semigroup where x and y are two minimal generators with $y, x > F(\Lambda)$. By convention, when depicting the tree $\mathcal{T}$, if $\Lambda' = \Lambda \setminus \{x\}, \Lambda'' = \Lambda \setminus \{y\}$ are descendants of Λ then Λ' comes before Λ'' if $x < y$.

We propose a different depicting of $\mathcal{T}$ according with the types. The *type-representation* of $\mathcal{T}$ is given by ordering each level of $\mathcal{T}$ by increasing type order; see Figure 2.

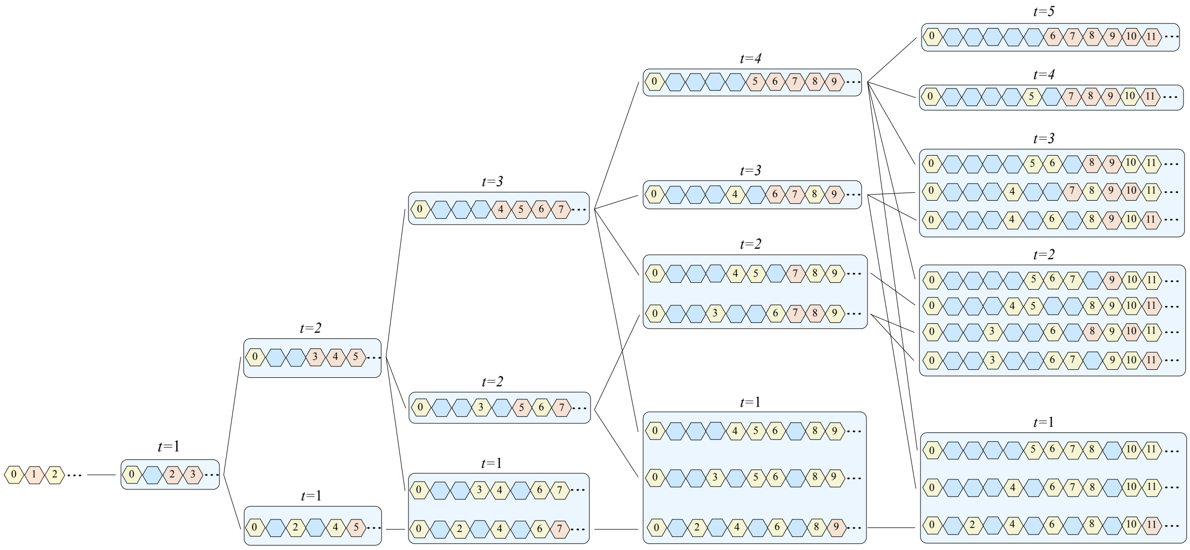

FIGURE 2. First levels of the type-representation of $\mathcal{T}$. In each level, semigroups of the same type t are grouped together in a box.

2.3. **Parent-descendants types.** We present a few results relating $t(\Lambda)$ and $t(\Lambda')$ where Λ' is a descendant of Λ. We have the following

Proposition 2.1. *Let Λ' be a numerical semigroup of genus $g > 0$. Let $\Lambda = \Lambda' \cup \{F(\Lambda')\}$. Then,*

(i) $PF(\Lambda') \setminus \{F(\Lambda')\} \subseteq PF(\Lambda)$, *with equality if and only if* $t(\Lambda') = g(\Lambda')$.
(ii) $t(\Lambda) \geqslant t(\Lambda') - 1$, *with equality if and only if* $t(\Lambda') = g(\Lambda')$.
(iii) *If* $m(\Lambda') < F(\Lambda')$ *then* $t(\Lambda) \geqslant t(\Lambda')$.

We shall prove this proposition below. We first notice that, by definition, $PF(\Lambda) \subseteq Gaps(\Lambda)$ and thus obtaining the obvious bound $t(\Lambda) \leqslant g(\Lambda)$. The equality can be characterized as follows.

Proposition 2.2. *Let Λ be a numerical semigroup. Then, the following statements are equivalent.*

(i) $t(\Lambda) = g(\Lambda)$.
(ii) $m(\Lambda) = F(\Lambda) + 1$.
(iii) $F(\Lambda) < m(\Lambda)$.
(iv) $\Lambda = \langle c, c+1, \ldots, 2c-1 \rangle$ *for some integer* $c \geqslant 1$.

If these hold then $g(\Lambda) = t(\Lambda) = F(\Lambda) = c - 1$.

Proof. The equality $t(\Lambda) = g(\Lambda)$ means that all gaps in Λ are pseudo-Frobenius numbers, that is, these are not comparable with respect to the $\leqslant_\Lambda$ partial order. In particular, for any gap $x < m(\Lambda)$ we get that $x + m(\Lambda)$ is not a gap. Thus (i) is equivalent to the fact that $1, \ldots, m(\Lambda) - 1$ are all the gaps in Λ, which implies (ii). Clearly, $(ii) \implies (iii) \implies (iv) \implies (i)$, and from the latter we get that $c - 1 = t(\Lambda) = g(\Lambda) = F(\Lambda)$. $\square$

Proof of Proposition 2.1. For (i) we distinguish two cases. First, assume $t(\Lambda') = g(\Lambda')$. By Proposition 2.2 we get $\Lambda' = \langle c, c+1, \ldots, 2c-1 \rangle$ with $c = g(\Lambda') + 1 = F(\Lambda') + 1$, and $\Lambda = \langle c-1, c, \ldots, 2c-3 \rangle$. Hence, $PF(\Lambda') \setminus \{F(\Lambda')\} = \{1, \ldots, c-2\} = PF(\Lambda)$.

Now, assume $t(\Lambda') < g(\Lambda')$, that is, $m(\Lambda') < F(\Lambda')$ by Proposition 2.2. We verify that $F(\Lambda') - m(\Lambda') \in PF(\Lambda) \setminus PF(\Lambda')$. Indeed, $F(\Lambda') - m(\Lambda') <_{\Lambda'} F(\Lambda') \in PF(\Lambda')$, hence $F(\Lambda') -$

$m(\Lambda') \notin PF(\Lambda')$. Also, for any $0 \neq h \in \Lambda$ we have $F(\Lambda') - m(\Lambda') + h \geqslant F(\Lambda')$, thus $F(\Lambda') - m(\Lambda') \in PF(\Lambda)$.

Let $a \in PF(\Lambda') \setminus \{F(\Lambda')\}$. Therefore, $a \notin \Lambda' \cup \{F(\Lambda')\}$ and $a + (\Lambda' \setminus \{0\}) \subset \Lambda'$. As $a + F(\Lambda') \geqslant F(\Lambda') + 1$, we obtain $a + F(\Lambda') \in \Lambda$. This implies that $a + (\Lambda \setminus \{0\}) \subset \Lambda$, that is, $a \in PF(\Lambda)$. Consequently, $PF(\Lambda') \setminus \{F(\Lambda')\} \subsetneq PF(\Lambda)$. This finishes the proof of part (i).

Part (ii) follows at once from (i), and also part (iii) by using Proposition 2.2. □

3. Stabilizer sequence

Let $n(g,t)$ be the number of numerical semigroups of genus g and type t. In Table 4 are computed with GAP ([8], [3]) the values of $n(g,t)$ for each $1 \leqslant t \leqslant g \leqslant 33$. We notice that the right diagonals (bold numbers) in Table 4 become constant at certain threshold. These values are given in Table 1.

$g \geqslant$	2	5	8	11	14	17	20	23	26	29	32
t	$g-1$	$g-2$	$g-3$	$g-4$	$g-5$	$g-6$	$g-7$	$g-8$	$g-9$	$g-10$	$g-11$
$n(g,t)$	1	3	7	15	35	78	161	367	757	1632	3436

Table 1. Some values of $n(g,t)$.

The values in Table 1 suggest that for g large enough and t close to g then $n(g,t)$ remains constant. Before establishing the latter formally in Theorem 3.7 below, we discuss some connections between the type and genus of a numerical semigroup. We first show the following inequality. This might be known, nevertheless, we include a proof for completeness.

Proposition 3.1. *For any numerical semigroup Λ we have*

$$F(\Lambda) + t(\Lambda) \leqslant 2g(\Lambda).$$

Proof. Set $g = g(\Lambda)$ and $t = t(\Lambda)$. We denote $A = \Lambda \cap [1, \ldots, F(\Lambda)]$ and $B = Gaps(\Lambda) \setminus PF(\Lambda)$. Clearly, $|B| = g - t$, and since $PF(\Lambda) \cup A \cup B$ is a partition of $[1, \ldots, F(\Lambda)]$ it follows that $|A| = F(\Lambda) - g$. The map

$$\varphi : (PF(\Lambda) \setminus \{F(\Lambda)\}) \cup A \to Gaps(\Lambda) \setminus \{F(\Lambda)\}$$

given by $\varphi(x) = F(\Lambda) - x$ is well defined and injective, hence

$$|(PF(\Lambda) \setminus \{F(\Lambda)\}) \cup A| \leqslant |Gaps(\Lambda) \setminus \{F(\Lambda)\}|,$$

that is $(t-1) + (F(\Lambda) - g) \leqslant g - 1$, leading to the desired inequality. □

Let Λ be a numerical semigroup with genus $g(\Lambda) > 0$ and type $t(\Lambda) = g(\Lambda) - \ell$, for $0 \leqslant \ell \leqslant g-1$. The case $\ell = 0$ is fully understood by Proposition 2.2.

Assume now $\ell > 0$. Since $m(\Lambda) \geqslant t(\Lambda) + 1$ (cf. [16, Corollary 2.23]) and, by Proposition 3.1, we have

$$F(\Lambda) \leqslant 2g(\Lambda) - t(\Lambda) = g(\Lambda) + \ell,$$

we obtain

$$\{1, \ldots, g - \ell\} \subseteq Gaps(S) \subseteq \{1, \ldots, g + \ell\}.$$

Motivated by these inclusions, we may attach to Λ the (0/1)-vector $v(\Lambda) = (v_1, \ldots, v_{2\ell})$, where

$$v_i = \begin{cases} 1 \text{ if } g - \ell + i \in Gaps(\Lambda), \\ 0 \text{ otherwise.} \end{cases}$$

It is clear that ℓ of the entries in $v(\Lambda)$ are equal 1, and the rest of ℓ entries are 0.

We call $v(\Lambda)$ the *gap vector* of Λ.

For any (0/1)-vector $v = (v_1, \ldots, v_{2\ell})$ and any integer $g > \ell$ we denote

$$\begin{aligned} \mathcal{G}_{g,v} &= \{1, \ldots, g - \ell\} \cup \{g - \ell + i : 1 \leqslant i \leqslant 2\ell \text{ and } v_i = 1\}, \\ \Lambda_{g,v} &= \mathbb{N}_0 \setminus \mathcal{G}_{g,v}. \end{aligned}$$

The semigroup Λ and its gaps can be recovered from $v(\Lambda)$ and $g(\Lambda)$ since

$$Gaps(\Lambda) = \mathcal{G}_{g(\Lambda),v(\Lambda)} \text{ and } \Lambda = \Lambda_{g(\Lambda),v(\Lambda)}.$$

Remark 3.2. (a) It is not always true that the set $\mathcal{G}_{g,v}$ is the set of gaps of a numerical semigroup. Indeed, for $v = (0, 1, \ldots, 0, 1) \in \mathbb{Z}^{2\ell}$ and $g = 3\ell - 2$ the set

$$\mathcal{G}_{3\ell-2,v} = \{1, \ldots, g - \ell = 2\ell - 2\} \cup \{2\ell, 2\ell + 2, 2\ell + 4, \ldots, g + \ell = 4\ell - 2\}$$

is not the set of gaps of a numerical semigroup since $g - \ell + 1 = 2\ell - 1 \notin \mathcal{G}_{3\ell-2,v}$, yet $2(2\ell - 1) = 4\ell - 2 \in \mathcal{G}_{3\ell-2,v}$.

b) Eventhough $\mathcal{G}_{g,v}$ is the set of gaps of a numerical semigroup, its type may be different from ℓ. Indeed, for the vector $v = (1, \ldots, 1, 0, \ldots, 0)$ with ℓ 1's and ℓ 0's, and any $g > \ell > 1$ one has

$$\mathcal{G}_{g,v} = \{1, 2, \ldots, g\},$$

and its complement $\Lambda_{g,v} = \{0, g + 1, g + 2, \ldots\}$ is a numerical semigroup of type equal to g, by Proposition 2.1.

Proposition 3.3. *Let $\ell \geqslant 1$ and $g \geqslant 3\ell - 1$ be positive integers and let $v = (v_1, \ldots, v_{2\ell})$ be any vector where half of the entries are 0 and the other half are 1. Then,*

(i) *$\Lambda_{g,v}$ is a numerical semigroup of genus g.*
(ii) *$Gaps(\Lambda_{g,v}) \setminus PF(\Lambda_{g,v}) = \{j - i : 1 \leqslant i < j \leqslant 2\ell, v_i = 0, v_j = 1\}$.*
(iii) *$t(\Lambda_{g,v}) = g - |\{j - i : 1 \leqslant i < j \leqslant 2\ell, v_i = 0, v_j = 1\}|$.*

Proof. We write $\Lambda = \Lambda_{g,v}$, for short.

(i)] We clearly have that $2 \min(\Lambda \setminus \{0\}) \geqslant 2(g - \ell + 1)$ and $\max \mathcal{G}_{g,v} \leqslant g + \ell$. The hypothesis $g \geqslant 3\ell - 1$ gives $2(g - \ell + 1) \geqslant g + \ell + 1$, and therefore Λ is a numerical semigroup with $Gaps(\Lambda) = \mathcal{G}_{g,v}$ and genus $g(\Lambda) = g$.

(ii)] Let $a \in Gaps(\Lambda) \setminus PF(\Lambda)$. Then there exists $b \in Gaps(\Lambda)$ and $0 \neq \lambda \in \Lambda$ with $a + \lambda = b$. Since $g + \ell \geqslant b > \lambda \geqslant m(\Lambda) \geqslant g - \ell + 1$, we may write $b = g - \ell + j$, $\lambda = g - \ell + i$ with $1 \leqslant i < j \leqslant 2\ell$, $v_j = 1$ and $v_i = 0$. Therefore, $a = b - \lambda = j - i$.

Conversely, if $a = j - i$ with $1 \leqslant i < j \leqslant 2\ell$ such that $v_j = 1$ and $v_i = 0$, we may write $a = (g + \ell + j) - (g + \ell + i)$ with $g + \ell + j \in Gaps(\Lambda)$ and $g + \ell + i \in \Lambda$.

We clearly have that (iii) is a straight consequence of (ii). $\square$

The previous result motivates the following definition. For any (0/1)-vector $v = (v_1, \ldots, v_{2\ell})$, its *cotype* is the integer

$$cotype(v) = |\{j - i : 1 \leqslant i < j \leqslant 2\ell \text{ with } v_i = 0 \text{ and } v_j = 1\}|.$$

Under the assumptions of Proposition 3.3, we have

$$type(\Lambda_{g,v}) = g - cotype(v). \tag{1}$$

Example 3.4. *The following vectors have $\ell \geqslant 1$ entries equal to 1 and ℓ entries equal to 0.*

(i) $cotype(1, \ldots, 1, 0, \ldots, 0) = |\emptyset| = 0$.
(ii) $cotype(0, \ldots, 0, 1, \ldots, 1) = |\{1, \ldots, \ell, \ell + 1, \ldots, 2\ell - 1\}| = 2\ell - 1$.
(iii) $cotype(0, 1, 0, 1, \ldots, 0, 1) = |\{1, 3, \ldots, 2\ell - 1\}| = \ell$.

Theorem 3.5. *Let $g \geqslant t > 0$ be integers with $t \geqslant (2g - 1)/3$ (or equivalently $g \leqslant (3t + 1)/2$).*

(a) If Λ is any numerical semigroup of genus g and type t, then

$$\Gamma = \{0\} \cup \{x + 1 : 0 \neq x \in \Lambda\}$$

is a numerical semigroup of genus $g + 1$ and type $t + 1$.

(b) If Γ is any numerical semigroup of genus $g + 1$ and type $t + 1$, then

$$\Lambda = \{0\} \cup \{x - 1 : 0 \neq x \in \Gamma\}$$

is a numerical semigroup of genus g and type t.

We give two proofs of Theorem 3.5. The first one is quite straight, however, it does not give much information about the shape of the generators of these stabilizer semigroups. The second one allows us to understand better such generators.

First proof of Theorem 3.5. First, suppose that Λ is a numerical semigroup of genus g and type t. Let γ_1 and γ_2 be two positive elements of Γ. Since $\gamma_1 - 1$ and $\gamma_2 - 1$ are in Λ and since $m(\Lambda) \geqslant t+1$ (cf. [16, Corollary 2.23]), we have that

$$\gamma_1 + \gamma_2 - 1 = (\gamma_1 - 1) + (\gamma_2 - 1) + 1 \geqslant 2m(\Lambda) + 1 \geqslant 2t + 3.$$

Moreover, since $3t+1 \geqslant 2g$ by hypothesis and since $F(\Lambda) + t \leqslant 2g$ by Proposition 3.1, we obtain that

$$\gamma_1 + \gamma_2 - 1 \geqslant 2t + 3 \geqslant 2g - t + 2 \geqslant F(\Lambda) + 2.$$

It follows that $\gamma_1 + \gamma_2 - 1 \in \Lambda$ and thus $\gamma_1 + \gamma_2 \in \Gamma$. Therefore Γ is closed under addition. Since $|\mathbb{N}_0 \setminus \Gamma| = g+1$ by definition, we have that Γ is a numerical semigroup of genus $g+1$.

Conversely, suppose that Γ is a numerical semigroup of genus $g+1$ and type $t+1$. Let λ_1 and λ_2 be two positive elements of Λ. Since $\lambda_1 + 1$ and $\lambda_2 + 1$ are in Γ and since $m(\Gamma) \geqslant t+2$ (cf. [16, Corollary 2.23]), we have that

$$\lambda_1 + \lambda_2 + 1 = (\lambda_1 + 1) + (\lambda_2 + 1) - 1 \geqslant 2m(\Gamma) - 1 \geqslant 2t + 3.$$

Moreover, since $3t+1 \geqslant 2g$ by hypothesis and since $F(\Gamma) + t + 1 \leqslant 2g + 2$ by Proposition 3.1, we obtain that

$$\lambda_1 + \lambda_2 + 1 \geqslant 2t + 3 \geqslant 2g - t + 2 \geqslant F(\Gamma) + 1.$$

It follows that $\lambda_1 + \lambda_2 + 1 \in \Gamma$ and thus $\lambda_1 + \lambda_2 \in \Lambda$. Therefore Λ is closed under addition. Since $|\mathbb{N}_0 \setminus \Lambda| = g$ by definition, we have that Λ is a numerical semigroup of genus g.

For the types, we can show that

$$\mathit{Gaps}(\Lambda) \setminus PF(\Lambda) = \mathit{Gaps}(\Gamma) \setminus PF(\Gamma). \tag{2}$$

Let $a \in \mathit{Gaps}(\Lambda) \setminus PF(\Lambda)$. Since $a \notin PF(\Lambda)$, there exists $b \in \Lambda^*$ such that $a + b \in \mathit{Gaps}(\Lambda)$. Since $a + b + 1 \in \mathit{Gaps}(\Gamma)$ with $b + 1 \in \Gamma^*$, it follows that $a \in \mathit{Gaps}(\Gamma) \setminus PF(\Gamma)$.

Conversely, let $a \in \mathit{Gaps}(\Gamma) \setminus PF(\Gamma)$. Since $a \notin PF(\Gamma)$, there exists $b \in \Gamma^*$ such that $a + b \in \mathit{Gaps}(\Gamma)$. If $b = 1$, then $\Gamma = \mathbb{N}_0$ of genus 0, in contradiction with the hypothesis that Γ is of genus $g+1$. Since $a + b - 1 \in \mathit{Gaps}(\Lambda)$ with $b - 1 \in \Lambda^*$, it follows that $a \in \mathit{Gaps}(\Lambda) \setminus PF(\Lambda)$.

From (2), we obtain that

$$g - t(\Lambda) = |\mathit{Gaps}(\Lambda) \setminus PF(\Lambda)| = |\mathit{Gaps}(\Gamma) \setminus PF(\Gamma)| = g + 1 - t(\Gamma).$$

Therefore, we have $t(\Lambda) = t(\Gamma) - 1$. This completes the proof. □

Second proof of Theorem 3.5. Set $\ell = g - t$. The condition $t \geqslant (2g-1)/3$ is equivalent to $g \geqslant 3\ell - 1$.

(a): If $\ell = 0$, then by Proposition 2.2 we have $\Lambda = \{0, g+1, g+2, \dots\}$. Hence $\Gamma = \{0, g+2, g+3, \dots\}$ has genus and type equal to $g+1$, again by Proposition 2.2.

Assume $\ell > 0$. Let $v = (v_1, \dots, v_{2\ell})$ the gap vector of Γ. So, $\Lambda = \Lambda_{g,v}$ and $\mathit{Gaps}(\Lambda) = \mathcal{G}_{g,v}$. By construction, the set $\mathbb{N}_0 \setminus \Gamma = \{1\} \cup \{x + 1 : x \in \mathit{Gaps}(\Lambda)\}$ has cardinality $g+1$ and the former equals $\mathcal{G}_{g+1,v}$. Therefore, $\Gamma = \Lambda_{g+1,v}$. As $g + 1 > g \geqslant 3\ell - 1$, by Proposition 3.3 we obtain that Γ is a numerical semigroup of genus $g+1$ and using (1) twice we get

$$t(\Gamma) = g + 1 - \mathit{cotype}(v) = (g - \mathit{cotype}(v)) + 1 = t(\Lambda_{g,v}) + 1 = t(\Lambda) + 1 = t + 1.$$

(b): Let w be the gap vector of Γ. Then $\Gamma = \Lambda_{g+1,w}$ and $\Lambda = \Lambda_{g,w}$. As $\ell = g - t = (g+1) - (t+1)$, and $g + 1 > g > 3\ell - 1$, from Proposition 3.3 applied twice we have that Λ is a numerical semigroup of genus g and $t(\Lambda) = g - \mathit{cotype}(w) = g - (g + 1 - t(\Gamma)) = t(\Gamma) - 1 = t$. □

For integers $g, t \geqslant 1$, we denote by $\mathcal{L}(g,t)$ the set of numerical semigroups of genus g and type t. We notice that the second proof of Theorem 3.5 relies on the fact that the numerical semigroups Γ and Λ have the same gap vector. We have the following

Corollary 3.6. *Let ℓ and $g \geqslant 3\ell - 1$ be integers. Then, the set*
$$\left\{v \in \mathbb{Z}^{2\ell} : v = v(\Lambda) \text{ for some } \Lambda \in \mathcal{L}(g, g-\ell)\right\}$$
does not depend on g.

The vector $v \in \mathbb{Z}^{2\ell}$ is called a *stable gap vector* if there exists Λ a numerical semigroup of genus g and type $g - \ell$ with $v = v(\Lambda)$ for some (and hence for all) $g \geqslant 3\ell - 1$. We denote $\mathcal{V}(\ell)$ the set of stable gap vectors in $\mathbb{Z}^{2\ell}$.

Theorem 3.7. *Let ℓ and $g \geqslant 3\ell - 1$ be integers. Then,*
$$n(g, g-\ell) = |\mathcal{V}(\ell)|$$
where $\mathcal{V}(\ell)$ denotes the set of stable gap vectors in $\mathbb{Z}^{2\ell}$.

Proof. Consider the map $\varphi : \mathcal{L}(g, g-\ell) \to \mathcal{V}(\ell)$ where $\varphi(\Lambda) = v(\Lambda)$. By Corollary 3.6, the image of φ does not depend on g, thus φ is surjective. We note that if $\varphi(\Lambda) = v$ then $\Lambda = \Lambda_{g,v}$. This proves that φ is injective. □

It would be interesting to determine in terms of ℓ the stabilized value of $n(g, g-\ell)$ and the set $\mathcal{L}(g, g-\ell)$ for $g \gg 0$. According to Theorem 3.7, this process goes hand in hand with finding the set $\mathcal{V}(\ell)$ of stable gap vectors for any ℓ. The following two propositions are on this direction.

Proposition 3.8. *Let Λ be a numerical semigroup of genus $g > 0$.*

(a) $t(\Lambda) = g$ if and only if $\Lambda = \langle g+1, \ldots, 2g+1\rangle$, that is,
$$Gaps(\Lambda) = \{1, \ldots, g\}.$$
(b) If $g \geqslant 2$ then, $t(\Lambda) = g - 1$ if and only if $\Lambda = \langle g, g+2, g+3, \ldots, 2g-1, 2g+1\rangle$, that is,
$$v(\Lambda) = (0,1) \text{ and } Gaps(\Lambda) = \{1, \ldots, g-1, \star, g+1\}.$$
(c) If $g \geqslant 5$ then, $t(\Lambda) = g - 2$ if and only if
$$\begin{aligned} v(\Lambda) = (1,0,0,1) \text{ and } Gaps(\Lambda) &= \{1, \ldots, g-1, \star, \star, g+2\}, \text{ or} \\ v(\Lambda) = (0,1,1,0) \text{ and } Gaps(\Lambda) &= \{1, 2, \ldots, g-2, \star, g, g+1\}, \text{ or} \\ v(\Lambda) = (0,1,0,1) \text{ and } Gaps(\Lambda) &= \{1, 2, \ldots, g-2, \star, g, \star, g+2\}. \end{aligned}$$
(d) If $g \geqslant 8$ then, $t(\Lambda) = g - 3$ if and only if
$$\begin{aligned} v(\Lambda) = (1,1,0,0,0,1) \text{ and } Gaps(\Lambda) &= \{1, \ldots, g-1, \star, \star, \star, g+3\}, \text{ or} \\ v(\Lambda) = (1,0,1,0,0,1) \text{ and } Gaps(\Lambda) &= \{1, \ldots, g-2, \star, g, \star, \star, g+3\}, \text{ or} \\ v(\Lambda) = (1,0,0,1,1,0) \text{ and } Gaps(\Lambda) &= \{1, \ldots, g-2, \star, \star, g+1, g+2\}, \text{ or} \\ v(\Lambda) = (0,1,1,1,0,0) \text{ and } Gaps(\Lambda) &= \{1, \ldots, g-3, \star, g-1, g, g+1\}, \text{ or} \\ v(\Lambda) = (0,1,1,0,1,0) \text{ and } Gaps(\Lambda) &= \{1, \ldots, g-3, \star, g-1, g, \star, g+2\}, \text{ or} \\ v(\Lambda) = (0,1,1,0,0,1) \text{ and } Gaps(\Lambda) &= \{1, \ldots, g-3, \star, g-1, g, \star, \star, g+3\}, \text{ or} \\ v(\Lambda) = (0,1,0,1,0,1) \text{ and } Gaps(\Lambda) &= \{1, \ldots, g-3, \star, g-1, \star, g+1, \star, g+3\}. \end{aligned}$$

In this description, the $\star$ means that the corresponding element is missing in the sequence.

Proof. Set $\ell = g - t$. The case (a) when $\ell = 0$ is covered by Proposition 2.2. For the rest, by Proposition 3.3, it suffices to find the vectors in $v \in \{0,1\}^{2\ell}$ having ℓ entries equal to 0 and $cotype(v) = \ell$, for $\ell = 1, 2, 3$.

When $\ell = 1$, the only possible gap vectors are $(0,1)$ and $(1,0)$. Since $cotype(0,1) = 1$ and $cotype(1,0) = 0$ then $(1,0)$ is the only stable gap vector.

When $\ell = 2$, there are 6 possible gap vectors:
$$(1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1).$$
Their respective cotypes are equal $0, 1, 2, 2, 2$ and 3. Therefore, the three stable gap vectors are $(1,0,0,1), (0,1,1,0), (0,1,0,1)$.

When $\ell = 3$, it is routine to check that among the 20 possible (0/1)-vectors of length 6 with 3 zero entries, only the 7 vectors listed at part (d), have cotype equal to 3. □

In the next proposition we describe for each integer $\ell > 0$ several families of stable gap vectors in $\{0,1\}^{2\ell}$.

Proposition 3.9. *Let $\ell \geqslant 1$ be an integer and let $v = (v_1, \ldots, v_{2\ell})$ be a vector with ℓ entries equal to* 0 *and ℓ entries equal to* 1.

(i) Assume $v_i = 0$ for a unique $1 \leqslant i \leqslant \ell$. Then, v is a stable gap vector if and only if
  (a) $i = 1$, or
  (b) $2 \leqslant i \leqslant \ell$ and $v_{2\ell} = 1$.

(ii) Assume $v_1 = \cdots = v_{\ell-2} = 1$ and that $v_{\ell+m} = v_{\ell+n} = 1$ for $0 \leqslant m < n \leqslant \ell$. Then, v is a stable gap vector if and only if
  (a) $n = \ell - 1$ and $\ell/2 - 1 \leqslant m \leqslant \ell - 2$, or
  (b) $n = \ell$ and $0 \leqslant m \leqslant \ell/2 - 1$.

(iii) Assume $v_2 = v_4 = \cdots = v_{2\ell} = 1$. Then, v is a stable gap vector.

(iv) Assume $\ell \geqslant 5, v_2 = \cdots = v_{\ell-2} = v_\ell = 1$, and that $v_{\ell+m} = v_{\ell+n} = 1$ for some $1 \leqslant m < n \leqslant \ell$. Then, v is a stable gap vector if and only if
  (a) $1 \leqslant m < n \leqslant \ell - 2$ with $m \neq \ell - 3$ and $n \neq \ell - 3$, or
  (b) $m = 1$ and $n = \ell - 1$, or
  (c) $\ell \geqslant 6$, $m = 2$ and $n = \ell$.

(v) Assume $v_2 = \cdots = v_{\ell-1} = 1$, and that $v_{\ell+m} = v_{\ell+n} = 1$ for some $1 \leqslant m < n \leqslant \ell$. Then, v is a stable gap vector if and only if
  (a) $2 \leqslant m < n \leqslant \ell - 2$, or
  (b) $m = 1$ and $n \neq \ell - 1$.

Proof. For each of the formats analyzed, we have to chararacterize when $cotype(v) = \ell$. We denote $\mathcal{D} = \{j - i : 1 \leqslant i < j \leqslant 2\ell$ with $v_i = 0$ and $v_j = 1\}$, so $cotype(v) = |\mathcal{D}|$.

(i)] It follows that $v_k = 1$ for all $1 \leqslant k \leqslant \ell$ with $k \neq i$. Since there are ℓ entries equal to 1 in v, we have $v_{\ell+m} = 1$ for a unique $1 \leqslant m \leqslant \ell$, and $v_{\ell+n} = 0$ for all $1 \leqslant n \leqslant \ell$ where $n \neq m$. So $\mathcal{D}$ contains $\ell + m - i$, the numbers $1, \ldots, \ell - i$ if $i < \ell$, and $1, \ldots, m - 1$ if $m \geqslant 2$. Hence, $|\mathcal{D}| \leqslant 1 + \max\{\ell - i, m - 1\} \leqslant \ell$. Therefore, if v is a stable gap vector then $\ell - i = \ell - 1$, or $m - 1 = \ell - 1$.

If $i = 1$ then $\mathcal{D} = \{1, \ldots, \ell - 1, \ell - 1 + m\}$, and v is a stable gap vector for all $1 \leqslant m \leqslant \ell$.

If $i > 1$ and $m = \ell$, then $\mathcal{D} = \{1, \ldots, \ell - 1, 2\ell - i\}$, and v is a stable gap vector.

(ii)] From the hypothesis $v_{\ell-1} = 0$. So, $\mathcal{D} = \{1, \ldots, m + 1\} \cup (\{1, \ldots, n + 1\} \setminus \{n - m\})$.

If $n - m \leqslant m + 1$, that is, $n \leqslant 2m + 1$, then $|\mathcal{D}| = n + 1$. Hence v is a stable gap vector if and only if $n = \ell - 1$ and $\ell/2 - 1 \leqslant m < n = \ell - 1$.

If $n - m > m + 1$, that is, $n \geqslant 2m + 2$, then $|\mathcal{D}| = n$. Hence v is a stable gap vector if and only if $n = \ell$ and $0 \leqslant m \leqslant (n - 2)/2$.

(iii)] This follows from Example 3.4 (iii).

(iv)] Clearly, $v_1 = v_{\ell-1} = 0$. Considering the differences $j - i$ in $\mathcal{D}$ where $i < j \leqslant \ell$ we obtain that $\{1, \ldots, \ell - 3, \ell - 1\} \subseteq \mathcal{D}$. When $j \in \{\ell + m, \ell + n\}$ and $i \in \{1, \ell - 1\}$ we get that also $m + 1, n + 1, \ell + m - 1, \ell + n - 1 \in \mathcal{D}$. We already notice ℓ distinct values in $\mathcal{D}$. So, v is a stable gap vector if and only if

$$\mathcal{D} = \{1, \ldots, \ell - 3, \ell - 1, \ell + m - 1, \ell + n - 1\}. \tag{3}$$

If $n \leqslant \ell - 2$ then for any difference $j - i$ in $\mathcal{D}$ with $\ell < i < j \leqslant 2\ell$ and $v_j = 1, v_i = 0$ we have $j - i \leqslant \ell - 3$. So, in this case we have that $cotype(v) = \ell$ if and only if $m \neq \ell - 3$ and $n \neq \ell - 3$.

Assume $n = \ell - 1$. If $cotype(v) = \ell$, then $v_{\ell+1} = 1$ and $m = 1$, otherwise $\ell - 2 = (2\ell - 1) - (\ell + 1) \in \mathcal{D}$, a contradiction with (3). Conversely, if $m = 1$ it is immediate to check that (3) holds.

Assume $n = \ell$. If $cotype(v) = \ell$ then $v_{\ell+2} = 1$ and $m = 2$, arguing as before. Note that if $\ell = 5$ then $m + 1 = 3 = \ell - 2$, so (3) does not hold. However, it is straightforward to check that once $\ell > 5$ and $m = 2$ then (3) is verified.

(v)] Clearly, $v_1 = v_\ell = 0$. As in the proof of (v), considering the differences $j - i$ where $i \leqslant \ell$ and $v_i = 0$, $v_j = 1$, we obtain that $\{1, 2, \dots, \ell - 2, m, n, \ell + m - 1, \ell + n - 1\} \subseteq \mathcal{D}$. So, v is a stable gap vector if and only if

$$\mathcal{D} = \{1, 2, \dots, \ell - 2, \ell + m - 1, \ell + n - 1\}. \tag{4}$$

Assume $m > 1$. Then $\ell < \ell + m - 1 < \ell + n - 1$. So, if (4) holds then $m < n \leqslant \ell - 2$. Conversely, it is easy to verify that (4) holds when $n \leqslant \ell - 2$.

If $m = 1$ and (4) holds, then $n \neq \ell - 1$. Conversely, it is easy to check that whenever $2 \leqslant n \leqslant \ell$ and $n \neq \ell - 1$, equation 4 holds. □

Theorem 3.10. *Let $\ell \geqslant 6$ and $g \geqslant 3\ell - 1$ be integers. Then, $n(g, g - \ell) \geqslant \ell^2 - 3\ell + 10$.*

Proof. We count the stable gap vectors presented in Proposition 3.9. There are ℓ vectors for $(i)(a)$, $\ell - 1$ vectors for $(i)(b)$ and 1 vector for (iii).

For (ii), we note that for each $0 \leqslant m \leqslant \ell - 2$ and $m \neq \ell/2$ we may find a unique n in $\{\ell - 1, \ell\}$ such that v is a stable gap vector. If ℓ is even, then for $m = \ell/2$, n can be any of ℓ or $\ell - 1$ in a stable gap vector. Therefore, for (ii) we identified $\ell - 1$, or ℓ vectors depending on ℓ being odd, or even, respectively.

For (iv), $\binom{\ell-3}{2} + 2$ stable gap vectors are described, while for (v) $\binom{\ell-3}{2} + \ell - 2$ more vectors are given.

The only overlap between the cases in Proposition 3.9 is the vector which occurs at $(i)(b)$ for $i = \ell - 1$ and at the same time at $(ii)(b)$ for $m = 0, n = \ell$. Hence, if $\ell \geqslant 6$ and $g \geqslant 3\ell - 1$ then

$$\begin{aligned} n(g, g - \ell) &\geqslant 2\ell - 1 + (\ell - 1) + 1 + \left(\binom{\ell-3}{2} + 2\right) + \left(\binom{\ell-3}{2} + \ell - 2\right) - 1 \\ &= \ell^2 - 3\ell + 10. \end{aligned}$$

□

4. Unimodularity and leaves

A sequence $a_0, \dots, a_n$ of real numbers is said to be *unimodal* if for some $0 \leqslant j \leqslant n$ we have $a_0 \leqslant a_1 \leqslant \cdots \leqslant a_j \geqslant a_{j+1} \geqslant \cdots \geqslant a_n$.

Theorem 4.1. *The sequence $n(g, 1), n(g, 2), \dots, n(g, g)$ is unimodal for each $1 \leqslant g \leqslant 33$.*

Proof. The result can be checked from values in Table 4. □

We notice that for fixed g the maximum of $n(g, t)$ is attained when t is around $\lceil \frac{g}{4} \rceil$; see circled values in Table 4.

Conjecture 4.2. *Let $1 \leqslant t \leqslant g$ be integers. Then, the sequence $n(g, 1), n(g, 2), \dots, n(g, g)$ is unimodal.*

Let $n(g)$ be the number of numerical semigroups of genus g. There is a great deal of literature to understand the asymptotic behavior of the sequence $n(g)$. Bras-Amorós [1] conjectured that

$$\text{for each } g \geqslant 1,\ n(g-2) + n(g-1) \leqslant n(g).$$

In fact, the following far weaker version of the conjecture is still open

$$\text{for each } g \geqslant 1,\ n(g) < n(g+1).$$

On this direction, we put forward the following type-version

Conjecture 4.3. *For each $2 \leqslant t \leqslant g$, $n(g, t) < n(g + 1, t)$.*

Table 4 supports this conjecture for each $2 \leqslant t \leqslant 33$. Notice that the sequence $n(g, 1)$ (corresponding to symmetric semigroups) is not increasing, for instance, $n(22, 1) > n(23, 1)$.

Let $v_i(\Lambda)$ be the number of descendants of Λ of type i. One may wonder whether the sequence $v_i(\Lambda)$ is increasing (or unimodal). The latter is not true in general. Indeed, consider $\Lambda = \langle 7, 9, 10, 12, 13, 15 \rangle$ with $t(\Lambda) = 5, g(\Lambda) = 8$ and $F(\Lambda) = 11$. The descendants of Λ are :

$$\Lambda' = \Lambda \setminus \{12\} = \langle 7, 9, 10, 13, 15 \rangle \text{ with } t(\Lambda') = 4,$$

$$\Lambda'' = \Lambda \setminus \{13\} = \langle 7, 9, 10, 12, 15\rangle \text{ with } t(\Lambda'') = 4,$$
$$\Lambda''' = \Lambda \setminus \{15\} = \langle 7, 9, 10, 12, 13\rangle \text{ with } t(\Lambda''') = 2.$$

We thus have that $v_1(\Lambda) = 0, v_2(\Lambda) = 1, v_3(\Lambda) = 0$ and $v_4(\Lambda) = 2$.

Moreover, if $\mathcal{F}_{g,t}$ denotes the family of numerical semigroups of type t and genus g then the sequence $w_i(\mathcal{F}_{g,t}) = \sum\limits_{\Lambda\in\mathcal{F}_{g,t}} v_i(\Lambda)$ is not necessarily either increasing or unimodal, for instance, $w_1(\mathcal{F}_{8,3}) = 1, w_2(\mathcal{F}_{8,3}) = 1, w_3(\mathcal{F}_{8,3}) = 3, w_4(\mathcal{F}_{8,3}) = 2, w_5(\mathcal{F}_{8,3}) = 4, w_6(\mathcal{F}_{8,3}) = 5, w_7(\mathcal{F}_{8,3}) = 0, w_8(\mathcal{F}_{8,3}) = 0$ and $w_9(\mathcal{F}_{8,3}) = 0$.

4.1. **Leaves.** In [2, 17] the infinite chains in $\mathcal{T}$ were investigated. For the latter, it is thus natural to study the nature of the semigroups that are leaves of $\mathcal{T}$. The semigroups $\langle 2, 2n+1\rangle, n \geqslant 1$ are symmetric (or equivalently of type 1). They are called *hyperelliptic* semigroups. In [2, Lemma 3] was proved that non-hyperelliptic symmetric semigroups are leaves. Nevertheless, there are non-symmetric semigroups that are also leaves, for instance, $\Lambda = \langle 6, 7, 8, 9, 11\rangle$ is of genus 6 and of type 2 (its gap set is $\{1, 2, 3, 4, 5, 10\}$ where the pseudo-Frobenius numbers are 5 and 10).

On this direction, we study the type of semigroups that are leaves. Let Λ be a numerical semigroup of genus g. If Λ is a leaf, what can we say about $t(\Lambda)$? Our calculations outcome the values in Table 2

g	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$t \leqslant$	3	3	3	4	5	5	5	6	7	7	7	8	9	9	9	10	11	11	11	12	13	13	13	14

Table 2. Bounds on the type t of semigroups of genus g that are leaves.

These values support the following

Conjecture 4.4. *Let $g \geqslant 2$ be an integer such that $g = 4q+r,\ 0 \leqslant r < 4,\ q \geqslant 0$. If a numerical semigroup Λ of genus g is a leaf then*

$$t(\Lambda) \leqslant \begin{cases} 2q-1 & \text{if } r = 0, 1, \\ 2q & \text{if } r = 2, \\ 2q+1 & \text{if } r = 3. \end{cases}$$

Let $\ell(g, t)$ be the number of semigroups of genus g and type t that are leaves.

Conjecture 4.5. *Let $1 \leqslant t \leqslant g$ be integers. Then, the sequence $\ell(g, 1), \ell(g, 2), \ldots, \ell(g, g)$ is unimodal.*

Table 5 supports this conjecture for each $1 \leqslant g \leqslant 33$.

Let $\ell(g)$ be the number of semigroups of genus g that are leaves. It is intriguing that the ratio $\frac{\ell(g)}{n(g)}$ increases very slowly as g grows, see Table 3.

Problem 4.6. *Investigate the asymptotic behavior of $\frac{\ell(g)}{n(g)}$. More precisely, is it true that $\lim\limits_{g\to\infty} \frac{\ell(g)}{n(g)}$ is finite? More ambitious, is it strictly less than $\frac{1}{2}$?*

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g	$\ell(g)$	$n(g)$	$\ell(g)/n(g)$
1	0	1	0
2	0	2	0
3	1	4	0,25
4	2	7	0,2857
5	2	12	0,1667
6	8	23	0,3478
7	12	39	0,3077
8	20	67	0,2985
9	37	118	0,3136
10	72	204	0,3529
11	110	343	0,3207
12	211	592	0,3564
13	350	1001	0,3497
14	590	1693	0,3485
15	1021	2857	0,3574
16	1742	4806	0,3625
17	2901	8045	0,3606
18	4927	13467	0,3659
19	8244	22464	0,3670
20	13751	37396	0,3677
21	22959	62194	0,3692
22	38356	103246	0,3715
23	63560	170963	0,3718
24	105479	282828	0,3729
25	174673	467224	0,3739
26	288699	770832	0,3745
27	476507	1270267	0,3751
28	786075	2091030	0,3759
29	1293751	3437839	0,3763
30	2127773	5646773	0,3768
31	3495791	9266788	0,3772
32	5738253	15195070	0,3776
33	9409859	24896206	0,3780

TABLE 3. Values of $\ell(g)$, $n(g)$ and $\ell(g)/n(g)$ for each $1 \leqslant g \leqslant 33$.

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genus \ type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
1	1																																
2	1	**1**																															
3	2	**1**	**1**																														
4	3	2	**1**	**1**																													
5	3	4	**3**	**1**	**1**																												
6	6	6	6	**3**	**1**	**1**																											
7	8	9	12	5	**3**	**1**	**1**																										
8	7	17	20	11	**7**	**3**	**1**	**1**																									
9	15	21	35	21	14	**7**	**3**	**1**	**1**																								
10	20	32	56	40	32	12	**7**	**3**	**1**	**1**																							
11	18	47	94	69	60	28	**15**	**7**	**3**	**1**	**1**																						
12	36	63	136	132	108	57	33	**15**	**7**	**3**	**1**	**1**																					
13	44	88	217	210	202	111	70	32	**15**	**7**	**3**	**1**	**1**																				
14	45	136	322	343	361	219	137	68	**35**	**15**	**7**	**3**	**1**	**1**																			
15	83	159	470	565	619	405	289	130	75	**35**	**15**	**7**	**3**	**1**	**1**																		
16	109	227	672	897	1039	754	533	289	152	72	**35**	**15**	**7**	**3**	**1**	**1**																	
17	101	329	1025	1309	1730	1364	1026	546	328	147	**78**	**35**	**15**	**7**	**3**	**1**	**1**																
18	174	413	1376	2090	2778	2392	1879	1103	651	312	159	**78**	**35**	**15**	**7**	**3**	**1**	**1**															
19	246	562	1938	3067	4441	4087	3426	2122	1298	626	358	153	**78**	**35**	**15**	**7**	**3**	**1**	**1**														
20	227	812	2810	4422	6967	6909	6024	4043	2523	1307	709	342	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**													
21	420	948	3798	6706	10613	11318	10684	7399	4871	2617	1474	684	361	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**												
22	546	1331	5179	9644	16265	18321	18134	13642	9226	5168	2956	1440	733	360	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**											
23	498	1827	7463	13360	24522	29297	30564	24245	17344	10052	5927	2922	1546	728	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**										
24	926	2200	9672	19844	36037	45754	50845	42662	31726	19539	11522	5966	3198	1519	750	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**									
25	1182	3031	13210	27564	53279	70929	82681	73752	57493	36975	22589	11895	6476	3151	1607	742	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**								
26	1121	4207	18579	37735	77889	109057	132168	125194	102947	69067	43209	23899	12970	6457	3323	1585	**757**	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**							
27	2015	4874	24100	54527	111716	163904	211078	208858	180140	127562	82539	46611	26054	13089	6892	3258	1625	**757**	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**						
28	2496	6774	32223	74629	161352	245919	328811	344606	311816	231314	155044	90898	51330	26707	13900	6767	3407	1612	**757**	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**					
29	2436	9096	44777	99780	230203	366597	508326	558563	531771	413415	288413	174711	100730	53306	28428	13785	7078	3367	**1632**	**757**	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**				
30	4350	10965	56778	142264	322409	536007	780577	896660	890706	730168	528017	332219	196498	105970	56741	28499	14478	6985	3425	**1632**	**757**	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**			
31	5602	14520	76367	191216	457167	779733	1181200	1419227	1476568	1266870	958160	623766	377946	209567	113636	57428	29860	14305	7175	3418	**1632**	**757**	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**		
32	5317	20329	105381	252790	639033	1139142	1763982	2221672	2415743	2169709	1709518	1161149	722149	409209	225282	116301	60364	29614	14745	7148	**3436**	**1632**	**757**	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**	
33	8925	23282	133674	355362	882069	1625034	2645417	3435192	3894557	3668138	3016976	2127885	1367186	794338	443863	233104	121816	60317	30733	14631	7214	**3436**	**1632**	**757**	**367**	**161**	**78**	**35**	**15**	**7**	**3**	**1**	**1**

TABLE 4. Values of $n(g,t)$ for each $1 \leqslant t \leqslant g \leqslant 33$. First largest value in each row is circled and stability numbers are bolded.

genus \ type	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
1	0																																
2	0	0																															
3	1	0	0																														
4	2	0	0	0																													
5	2	0	0	0	0																												
6	5	3	0	0	0	0																											
7	7	4	1	0	0	0	0																										
8	6	11	3	0	0	0	0	0																									
9	14	14	9	0	0	0	0	0	0																								
10	19	25	26	2	0	0	0	0	0	0																							
11	17	37	47	8	1	0	0	0	0	0	0																						
12	35	56	83	33	4	0	0	0	0	0	0	0																					
13	43	76	148	70	13	0	0	0	0	0	0	0	0																				
14	44	123	228	143	51	1	0	0	0	0	0	0	0	0																			
15	82	146	362	283	141	6	1	0	0	0	0	0	0	0	0																		
16	108	216	540	518	311	45	4	0	0	0	0	0	0	0	0	0																	
17	100	312	853	818	656	147	15	0	0	0	0	0	0	0	0	0	0																
18	173	400	1194	1426	1242	423	68	1	0	0	0	0	0	0	0	0	0	0															
19	245	541	1714	2256	2216	1033	230	8	1	0	0	0	0	0	0	0	0	0	0														
20	226	794	2535	3366	3936	2159	682	49	4	0	0	0	0	0	0	0	0	0	0	0													
21	419	926	3484	5339	6559	4195	1813	212	12	0	0	0	0	0	0	0	0	0	0	0	0												
22	545	1313	4838	7982	10661	8041	4101	789	84	2	0	0	0	0	0	0	0	0	0	0	0	0											
23	497	1796	7036	11291	17266	14311	8642	2418	294	7	2	0	0	0	0	0	0	0	0	0	0	0	0										
24	925	2179	9207	17264	26553	24839	17124	6252	1069	62	5	0	0	0	0	0	0	0	0	0	0	0	0	0									
25	1181	3006	12681	24497	40881	41798	32492	14280	3594	238	25	0	0	0	0	0	0	0	0	0	0	0	0	0	0								
26	1120	4178	17932	34006	62198	68824	58382	30926	9929	1094	107	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0							
27	2014	4842	23402	49962	91855	109834	103543	61905	24560	4166	408	14	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0						
28	2495	6747	31462	69240	136027	173885	176834	118288	56267	13328	1412	78	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0					
29	2435	9055	43844	93362	199051	270115	294755	219955	118964	36206	5644	334	31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0				
30	4349	10936	55785	134633	283435	410920	483129	394911	239431	89875	18821	1360	186	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0			
31	5601	14477	75255	182259	408813	616662	775025	685918	464957	204831	55432	5946	590	22	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0		
32	5316	20294	104066	242269	580380	927332	1212984	1170758	866410	437022	146484	22736	2041	149	12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
33	8924	23235	132277	342815	810110	1356021	1900504	1943772	1567615	888897	353474	73589	8033	507	85	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 5. Values of $\ell(g,t)$ for each $1 \leqslant t \leqslant g \leqslant 33$.

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