Complete Kneser Transversals

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joint work with
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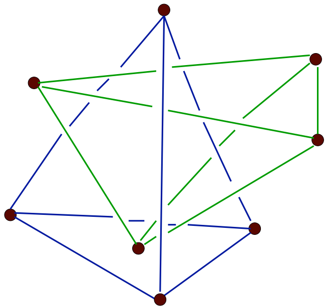
Let us consider 8 points in $\mathbb{R}^3$ general position.



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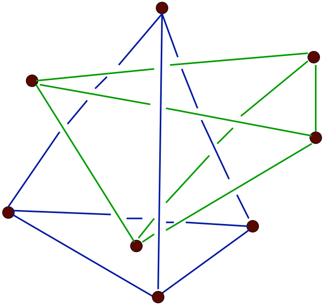
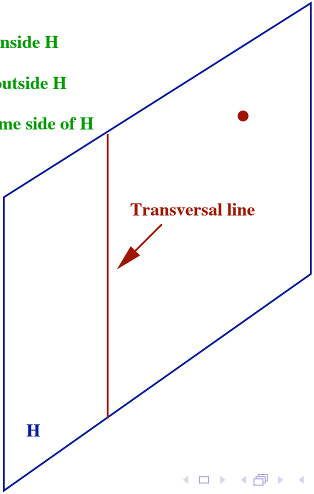


Question : Is there a transversal line to all tetrahedra ?

NEVER

- There are at most 3 points inside H
- There are at least 5 points outside H
- There are 3 points in the same side of H



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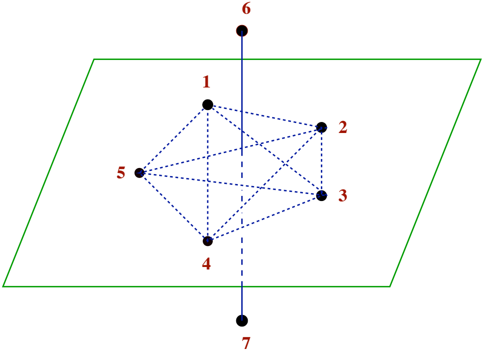
Question : Let A be a set of 7 points in $\mathbb{R}^3$ in general position. Is there a transversal line to all tetrahedra of A?

Sometimes YES



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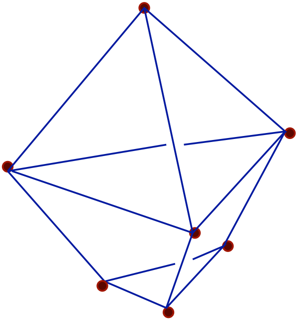

Kneser Transversal

Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

$m(k, d, \lambda) \stackrel{\text{def}}{=}$ the maximum positive integer n such that every set X of n points (not necessarily in general position) in $\mathbb{R}^d$ has the property that the convex hull of all k-set of X have a transversal $(d-\lambda)$-plane (called Kneser Transversal).

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$M(k, d, \lambda) \stackrel{\text{def}}{=}$ the minimum positive integer n such that for every set of n points in general position in $\mathbb{R}^d$ the convex hull of the k-sets does not have a transversal $(d - \lambda)$-plane.

- $m(k, d, \lambda) < M(k, d, \lambda)$.

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- $m(4, 3, 2) = 6$ and $M(4, 3, 2) = 8$.

- $m(k,d,\lambda) < M(k,d,\lambda)$.
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Theorem (Arocha, Bracho, Montejano, R.A., 2011)

$$M(k,d,\lambda) = \begin{cases} d + 2(k-\lambda) + 1 & \text{if } k \geq \lambda, \\ k + (d-\lambda) + 1 & \text{if } k \leq \lambda. \end{cases}$$

Kneser hypergraphs

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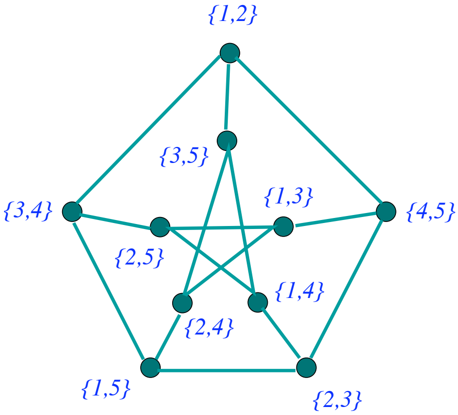
Remark Kneser graphs are obtained when $\lambda = 1$.

Kneser hypergraph when $n = 5$, $k = 2$ and $\lambda = 1$ (Petersen graph)



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A collection of vertices $\{S_1, \ldots, S_\rho\}$ of $K^{\lambda+1}(n,k)$ are in the same color class if and only if either

a) $\rho \leq \lambda + 1$ and $S_1 \cap \cdots \cap S_\rho \neq \emptyset$ or

b) $\rho > \lambda + 1$ and any $(\lambda+1)$-subfamily $\{S_{i_1}, \ldots, S_{i_{\lambda+1}}\}$ of $\{S_1, \ldots, S_\rho\}$ is such that $S_{i_1} \cap \cdots \cap S_{i_{\lambda+1}} \neq \emptyset$.

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Corollary (Arocha, Bracho, Montejano, R.A., 2011) $d-\lambda+k+\lceil \frac{k}{\lambda} \rceil - 1 \leq m(k,d,\lambda)$.

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Theorem (Lovász) $\chi(K^2(n,k)) = n-2k+2$.

Conjecture $m(k, d, \lambda) = d - \lambda + k + \lceil \frac{k}{\lambda} \rceil - 1.$

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Theorem (Arocha, Bracho, Montejano, R.A., 2011)
The conjecture is true if either
a) $d = \lambda$ or
b) $\lambda = 1$ or
c) $k \leq \lambda$ or
d) $\lambda = k - 1$ or
e) $k = 2, 3.$

Rado's central point theorem

Rado's theorem If X is a bounded measurable set in $\mathbb{R}^d$ then there exists a point $x \in \mathbb{R}^d$ such that

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A generalization of the discrete version of Rado's result.

Theorem (Arocha, Bracho, Montejano, R.A. 2011)
Let X be a finite set of n points in $\mathbb{R}^d$. Then, there is a $(d-\lambda)$-plane L such that any closed half-space H through L contains at least $\lfloor \frac{n-d+2\lambda}{\lambda+1} \rfloor + (d-\lambda)$ points of X.

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We clearly have that

$$m^*(k,d,\lambda) \leq m(k,d,\lambda)$$

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- We assume $k \geq \lambda + 1$.
- It turns out that the function m^* has two different behaviours :

$$\alpha(d,\lambda) = \frac{\lambda-1}{\lceil \frac{d}{2} \rceil} \geq 1$$

$$\alpha(d,\lambda) = \frac{\lambda-1}{\lceil \frac{d}{2} \rceil} < 1$$

Radon's theorem Let X be a set of $d + 2$ points in $\mathbb{R}^d$ in general position. Then, there exists a unique partition $X = X_1 \cup X_2$ such that $conv(X_1) \cap conv(X_2) \neq \emptyset$.

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Lemma Let X be any set of $d+2$ distinct points in $\mathbb{R}^d$ and let $\lfloor \frac{d+2}{2} \rfloor \leq t \leq d+1$. Then, X can be partitioned into disjoint sets S and T such that $|T| = t$ and $conv(S) \cap aff(T) \neq \emptyset$.

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- By Lemma, the set Y can be partitioned into disjoint sets S and T such that $|T| = d - \lambda + 1$ and $conv(S) \cap aff(T) \neq \emptyset$.
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Proof : Let X be a collection of $d-\lambda+1+k$ points in $\mathbb{R}^d$.

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- Since $\alpha(d,\lambda)<1$ then $\lfloor\frac{d+2}{2}\rfloor\leq d-\lambda+1\leq d+1$.
- By Lemma, the set Y can be partitioned into disjoint sets S and T such that $|T|=d-\lambda+1$ and $conv(S)\cap aff(T)\neq\emptyset$.
- We claim that $aff(T)$ is a complete Kneser transversal for X.

Since $|X|=d-\lambda+1+k$ then there is exactly one k-set not intersected by T. But this k-set contains S for which $conv(S)\cap aff(T)\neq\emptyset$.

Cyclic polytope

The cyclic polytope is the convex hull of a finite set of points in the moment curve in $\mathbb{R}^d$ (defined as the map $\gamma : \mathbb{R} \to \mathbb{R}^d, t \mapsto (t, t^2, \ldots, t^d)$).

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Let $k, d, \lambda \geq 1$ be integers with $d \geq \lambda$.

$\eta(k, d, \lambda) \stackrel{\text{def}}{=}$ the maximum number of vertices that the cyclic polytope in $\mathbb{R}^d$ can have, so that it has a complete Kneser $(d-\lambda)$-transversal to the convex hull of its k-sets of vertices.

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- $m^*(k,d,\lambda) \leq \eta(k,d,\lambda)$

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Let $\beta(\lambda,j) = \frac{j+\lambda-1}{2}$ for each j with $j+\lambda$ odd.

Theorem If $\alpha(d,\lambda)\geq 1$ then $m^*(k,d,\lambda)=d-\lambda+1=\eta(k,d,\lambda)$.

Let $\beta(\lambda,j)=\frac{j+\lambda-1}{2}$ for each j with $j+\lambda$ odd.

$$z(k,d,\lambda)\stackrel{\text{def}}{=}d-\lambda+1+\max_{\substack{j\in\{\lambda+1,\dots,d-\lambda+2\}\\ j+\lambda \text{ is odd}}}\left(\left\lfloor\frac{k-1}{\beta(\lambda,j)}\right\rfloor\right)\cdot j+(k-1)_{mod\,\beta(\lambda,j)}$$

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Asymptotics

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Corollary If $\alpha(d,\lambda) < 1$ then $m^*(k,d,2) < m(k,d,2)$ for k large enough and $d \geq 3$.

Question : Is the existence of a Kneser Transversal invariant of the order type ?

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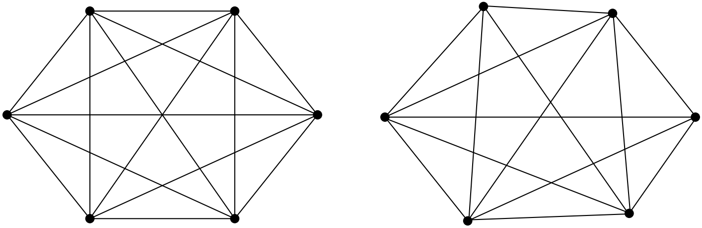

Stability and instability

A Kneser transversal is said to be stable (resp. instable) if the given set of points can be slightly perturbated (move each point to, not more than $\epsilon > 0$ distance of their original position) such that the new configuration of points admits (if there is any) only complete Kneser transversals (resp. the new configuration of points does not admit a Kneser transversal).

Codimension 2 and 3

Theorem Let $X = \{x_1, x_2, \ldots x_n\}$ be a collection of $n = d + 2(k - \lambda)$ points in general position in $\mathbb{R}^d$. Suppose that L is a $(d - \lambda)$-plane transversal to the convex hulls of all k-sets of X with $\lambda = 2, 3$ and $k \geq \lambda + 2$ and $d \geq 2(\lambda - 1)$. Then, either

(1) L is a complete Kneser transversal (i.e., it contains $d - \lambda + 1$ points of X) or

(2) $\mid L \cap X \mid = d - 2(\lambda - 1)$ and the other $2(k - 1)$ points of X are matched in $k - 1$ pairs in such a way that L intersects the corresponding closed segments determined by them.

Theorem Let $\epsilon > 0$ and let $X = \{x_1, \ldots, x_n\}$ be a finite collection of points in $\mathbb{R}^d$. Suppose that $n = d + 2(k - \lambda)$, $k - \lambda \geq 2$ and $\lambda = 2, 3$. Then, there exists $X' = \{x'_1, \ldots, x'_n\}$, a collection of points in $\mathbb{R}^d$ in general position such that $\mid x_i - x'_i \mid < \epsilon$, for every $i = 1, \ldots, n$, and with the property that every transversal $(d - \lambda)$-plane to the convex hull of the k-sets of X' is complete (i.e., it contains $d - \lambda + 1$ points of X').

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Theorem Let $\lambda = 2, 3$, $k - \lambda \geq 2$ and $d \geq 2(\lambda - 1)$. Then,

$$m(k, d, \lambda) < d + 2(k - \lambda).$$

Some computational results

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Question What about transversal lines to all tetrahedra in configurations of 7 points in $\mathbb{R}^3$?

Complete Kneser lines : determined by oriented matroids

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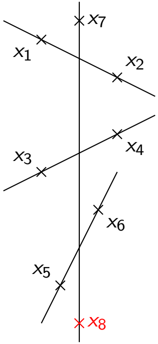
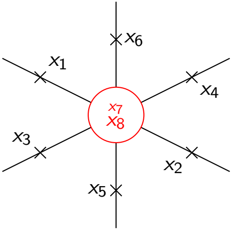
Kneser lines : a bit more complicated



Representation in $\mathbb{R}^3$



Projection in $\mathbb{R}^2$

Theorem Among the 246 different order types of 7 points in general position in $\mathbb{R}^3$ there are :

$A = 124$ admitting a complete Kneser line to the tetrahedra

$B = 124$ admitting a representation for which there is non-complete Kneser line to the tetrahedra

We have $|A \cap B| = 46, |A \setminus B| = |B \setminus A| = 78$ and $|\overline{A \cup B}| = 44$. Moreover, for each of the 78 order types of $B \setminus A$ there exists a representation for which there is no Kneser transversal line.